Guaranteed error bounds for finite element approximations of noncoercive elliptic problems and their applications

Nakao, Mitsuhiro T.
Faculty of Mathematics, Kyushu University

Hashimoto, Kouji
Graduate School of Informatics, Kyoto University

http://hdl.handle.net/2324/3405

Faculty of Mathematics, Kyushu University
バージョン：accepted
権利関係：
Constructive error estimates
of finite element approximations
for non-coercive elliptic problems
and its applications

M. T. Nakao & K. Hashimoto

MHF 2007-5

( Received January 19, 2007 )
Constructive error estimates of finite element approximations for non-coercive elliptic problems and its applications

Mitsuhiro T. Nakao† and Kouji Hashimoto‡

mtnakao@math.kyushu-u.ac.jp
†Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan
‡Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan

Abstract

In this paper, we show constructive a priori and a posteriori error estimates of finite element approximations for not necessary coercive linear second order Dirichlet problems. Here, 'constructive' means we can get the error bounds in which all constants included are explicitly given or represented as a numerically computable form. Using the invertibility condition of concerning elliptic operator, constructive a priori and a posteriori error estimates are formulated. This kind of estimates plays essential and important roles in the numerical verification of solutions for nonlinear elliptic problems. Several numerical examples that confirm the actual effectiveness of the method are presented.

Key words: Constructive a priori and a posteriori error estimates, linear elliptic problem.

1 Introduction

In this paper, we consider the constructive a priori and a posteriori error estimates for the general linear elliptic boundary value problem of the form

\[ \mathcal{L}u \equiv -\Delta u + b \cdot \nabla u + cu = f \text{ in } \Omega, \]
\[ u = 0 \text{ on } \partial \Omega, \]

(1.1)

where \( f \in L^2(\Omega) \). Here, for \( n = 1, 2, 3 \), we assume that \( b \in (W^{1}_\infty(\Omega))^n \), \( c \in L^\infty(\Omega) \), where \( \Omega \subset R^n \) is a bounded open domain with piecewise smooth boundary. In this paper, we use the terminology 'constructive error estimates'.
means an error estimation that by some numerical computations based on the estimates, we can obtain the true error bounds between the exact solution and its approximation in mathematically rigorous sense, even if the concerning problem (1.1) is not coercive. This kind of estimations should be useful when the existence or uniqueness of solutions are not a priori assured, e.g., in case that the coefficient function $c$ is not nonnegative. And it also be important for the numerical verification of solutions for nonlinear boundary value problems(e.g., [2] [4] [5] etc.).

Now, we denote the usual $k$-th order Sobolev space on $\Omega$ by $H^k(\Omega)$ and define $(\cdot, \cdot)_{L^2}$ as the $L^2$ inner product. And we set $H^1_0(\Omega) \equiv \{ v \in H^1(\Omega) ; v = 0 \text{ on } \partial \Omega \}$ with the inner product $(\nabla u, \nabla v)_{L^2}$ for $u, v \in H^1_0(\Omega)$ . Also, define $X(\Omega) \equiv \{ v \in H^1(\Omega) ; \Delta v \in L^2(\Omega) \}$.

We now introduce the finite dimensional subspace $S_h$ of $H^1_0(\Omega)$ depending on the parameter $h$ with nodal functions $\{ \phi_i \}_{1 \leq i \leq N}$. For each $v \in H^1_0(\Omega)$, define the $H^1_0$-projection $P_h v \in S_h$ by

$$(\nabla(v - P_h v), \nabla \phi_h)_{L^2} = 0, \quad \forall \phi_h \in S_h.$$  

Also, corresponding to the usual finite element approximations of a solution $u$ in (1.1), we define the $L$-projection $P_L v \in S_h$, whose existence is assumed, by

$$a(v - P_L v, \phi_h)_{L^2} = 0, \quad \forall \phi_h \in S_h,$$  

where $a(u, v) \equiv (\nabla u, \nabla v)_{L^2} + (b \cdot \nabla u, v)_{L^2} + (cu, v)_{L^2}$. Further, we assume that there exists a positive constant $C(h)$ which can be numerically estimated satisfying, for any $u \in H^1_0(\Omega) \cap X(\Omega)$ ,

$$\| u - P_h u \|_{H^1_0} \leq C(h) \| \Delta u \|_{L^2}.$$  

(1.3)

Note that (1.3) is equivalent to the following estimation.

$$\| u - P_h u \|_{L^2} \leq C(h) \| u - P_h u \|_{H^1_0}.$$  

(1.4)

Then our main purpose of this paper is to determine explicitly a priori constants $K_0(h)$ and $K_1(h)$ satisfying

$$\| u - P_L u \|_{L^2} \leq K_0(h) \| Lu \|_{L^2},$$  

(1.5)

$$\| u - P_L u \|_{H^1_0} \leq K_1(h) \| Lu \|_{L^2},$$  

(1.6)

respectively. Also we show a constant $K(h)$ satisfying

$$\| u - P_L u \|_{L^2} \leq K(h) \| u - P_L u \|_{H^1_0}.$$  

(1.7)
Defining the compact operator \( A : H_0^1 \to H_0^1 \) by \( Au := \Delta^{-1}(b \cdot \nabla u + cu) \), where \( \Delta^{-1} \) stands for the solution operator of the Poisson equation with homogeneous boundary condition, the invertibility of the elliptic operator \( \mathcal{L} \) defined in (1.1) is equivalent to the unique solvability of the following fixed point equation:

\[
u = Au.\]

As the preliminary, we define \( N \times N \) matrices \( G = (G_{i,j}) \) and \( D = (D_{i,j}) \) by

\[
G_{i,j} = (\nabla \phi_j, \nabla \phi_i)_{L^2} + (b \cdot \nabla \phi_j, \phi_i)_{L^2} + (c \phi_j, \phi_i)_{L^2},
\]

\[
D_{i,j} = (\nabla \phi_j, \nabla \phi_i)_{L^2},
\]

Note that \( D \) is symmetric and positive definite. We denote the matrix norm by \( \| \cdot \|_E \) induced from the Euclidean norm \( | \cdot |_E \). Also, we define the following constants:

\[
C_1 = C_p \text{div } b + C_b, \quad C_3 = C_b + C_p C_c,
\]

\[
C_2 = C_p C_c, \quad C_4 = C_b + C(h) C_c,
\]

\[
C_{\text{div } b} = \| \text{div } b \|_{L^\infty}, \quad C_b = \| |b|_E \|_{L^\infty}, \quad C_c = \| c \|_{L^\infty},
\]

where \( \| \cdot \|_{L^\infty} \) means \( L^\infty \) norm on \( \Omega \) and \( C_p \) is a Poincaré constant such that \( \| \phi \|_{L^2} \leq C_p \| \phi \|_{H_0^1} \) for an arbitrary \( \phi \in H_0^1(\Omega) \).

In [5], authors show the following results.

**Theorem 1** If the matrix \( G \) is nonsingular, and for the constants defined above,

\[
\kappa(h) \equiv C(h) \left( C(h) M_h (C_1 + C_2) C_3 + C_4 \right) < 1
\]

holds, then the operator \( \mathcal{L} \) defined in (1.1) is invertible. Here, \( M_h \equiv \| D^{\frac{1}{2}} G^{-1} D^{\frac{1}{2}} \|_E \) and \( C(h) \) is the same constant as in (1.3).

Moreover, we have the following a priori estimate for the \( H_0^1 \)-projection.

**Theorem 2** Assuming that same conditions in Theorem 1, let \( u \in H_0^1(\Omega) \cap X(\Omega) \) be a unique solution of (1.1). Then we have

\[
\| u - P_h u \|_{H_0^1} \leq C(h) \sigma \| f \|_{L^2},
\]

where the constant \( \sigma \) is given by \( \sigma = (1 + C_p M_h C_3)(1 - \kappa(h))^{-1} \).

When the coefficient vector function \( b \) in (1.1) is not differentiable, we have the following alternative results.

**Corollary 3** Let \( b \in (L^\infty(\Omega))^n \). If

\[
\tilde{\kappa}(h) \equiv C(h) \left( \tilde{M}_h (\tilde{C}_1 + C(h) \tilde{C}_2) C_3 + C_4 \right) < 1
\]

3
holds, then the operator $L$ defined in (1.1) is invertible. Here, $C_1 = C_p C_b$. Also we have
\[ \|u - P_h u\|_{H^1_0} \leq C(h) \hat{\sigma} \|f\|_{L^2}, \]
for a unique solution of $Lu = f$, where the constant $\hat{\sigma}$ is given by
\[ \hat{\sigma} = (1 + C_p M_h C_3)(1 - \hat{\kappa}(h))^{-1}. \]

2 Main results

In this section, we show the constructive a priori and a posteriori error estimates of finite element approximations (1.2) for linear elliptic problems (1.1). Note that the existence of the inverse $L^{-1} : L^2(\Omega) \to X(\Omega)$ is equivalent to the invertibility of $I - A$, where $I$ denotes the identity operator in $H^1_0(\Omega)$. Using this fact, we first show the a priori error estimate between a solution of our problems and its $H^1_0$-projection. First, we show the following lemma.

**Lemma 4** (cf.[5]) For an arbitrary $v \in H^1_0(\Omega)$, we have
\[ \|Av\|_{H^1_0} \leq (C_1 + C_2)\|v\|_{L^2}, \]
\[ \|(I - P_h)Av\|_{H^1_0} \leq C(h) \left( C_3\|P_h v\|_{H^1_0} + C_4\|v - P_h v\|_{H^1_0} \right). \]

**Proof:** Let $\psi := -Av = -\Delta^{-1}(b \cdot \nabla + c)v \in H^1_0(\Omega) \cap X(\Omega)$. Then we have
\[ \|\psi\|^2_{H^1_0} = (-\Delta \psi, \psi)_{L^2} = (v, \text{div} (b \psi))_{L^2} + (v, c \psi)_{L^2} \]
\[ \leq \left( \|\text{div} (b \psi)\|_{L^2} + \|c \psi\|_{L^2} \right)\|v\|_{L^2} \]
\[ \leq C(h) \left( \|\text{div} b\|_{L^\infty} \|\psi\|_{L^2} + \|b\|_{L^\infty} \|\psi\|_{H^1_0} + \|c\|_{L^\infty} \|\psi\|_{L^2} \right)\|v\|_{H^1_0}, \]
where we have used (1.4). Moreover, we have
\[ \|(I - P_h)Av\|_{H^1_0} = \|(I - P_h)\Delta^{-1}(b \cdot \nabla + c)v\|_{H^1_0} \]
\[ \leq C(h)\|(b \cdot \nabla + c)v\|_{L^2} \]
\[ \leq C(h) \left( \|b\|_{L^\infty} \|v\|_{H^1_0} + \|c\|_{L^\infty} \|v\|_{L^2} \right), \]
where we have used (1.3). Therefore, this proof is completed.

For the $L$-projection, we have the following one of the main results of this paper.
\textbf{Theorem 5} For an arbitrary $v \in H_0^1(\Omega)$, if $G$ is nonsingular, then for the same constants in Theorem 1, we have

$$
\|v - P_e v\|_{H_0^1} \leq \alpha \|v - P_h v\|_{H_0^1},
$$

$$
\|v - P_e v\|_{L^2} \leq C(h)\beta \|v - P_h v\|_{H_0^1} \leq C(h)\beta \|v - P_e v\|_{H_0^1},
$$

where $\alpha \equiv \sqrt{1 + \left( C(h)M_h(C_1 + C_2) \right)^2}$, $\beta \equiv 1 + C_pM_h(C_1 + C_2)$.

\textbf{Proof:} From the property of the $H_0^1$- and $L$-projections, we can obtain

$$
\|v - P_e v\|_{H_0^1}^2 = \|v - P_h v\|_{H_0^1}^2 + \|P_e v - P_h v\|_{H_0^1}^2, \quad (2.1)
$$

for an arbitrary $v \in H_0^1(\Omega)$. Let $e \equiv v - P_h v$. Then since $P_e v - P_h v = P_e(v - P_h v)$, for all $\phi_h \in S_h$, we have

$$
a(P_e e, \phi_h) = (\nabla e, \nabla \phi_h)_{L^2} + ((b \cdot \nabla + c)e, \phi_h)_{L^2}
$$

$$
= (b \cdot \nabla e + ce, \phi_h)_{L^2}
$$

$$
= (\nabla P_h \psi, \nabla \phi_h)_{L^2},
$$

where we set $\psi \equiv -Ae = -\Delta^{-1}(b \cdot \nabla + c)e$. It implies that

$$
G\vec{e}_h = D\vec{\psi}_h,
$$

where $\vec{e}_h$ and $\vec{\psi}_h$ are coefficient vectors of $P_e e$ and $P_h \psi$, respectively. Thus in the similar way to the proof of Lemma 4, we can obtain the following estimate since $\|P_e e\|_{H_0^1} = \|D^{\frac{3}{2}}\vec{e}_h\|_E$, $\|P_h \psi\|_{H_0^1} = \|D^{\frac{3}{2}}\vec{\psi}_h\|_E$ and $\|P_h \psi\|_{H_0^1} \leq \|\psi\|_{H_0^1}$ for any $\psi \in H_0^1(\Omega)$.

$$
\|P_e v - P_h v\|_{H_0^1} = \|P_e e\|_{H_0^1} \leq M_h\|P_h \psi\|_{H_0^1}
$$

$$
\leq M_h\|A(v - P_h v)\|_{H_0^1}
$$

$$
\leq C(h)M_h(C_1 + C_2)\|v - P_h v\|_{H_0^1}.
$$

Moreover, we have

$$
\|P_e v - P_h v\|_{L^2} \leq C_p\|P_e v - P_h v\|_{H_0^1}
$$

$$
\leq C(h)C_pM_h(C_1 + C_2)\|v - P_h v\|_{H_0^1}.
$$

Hence we can obtain the following estimate.

$$
\|v - P_e v\|_{L^2} \leq \|v - P_h v\|_{L^2} + \|P_e v - P_h v\|_{L^2}
$$

$$
\leq C(h)\|v - P_h v\|_{H_0^1} + C(h)C_pM_h(C_1 + C_2)\|v - P_h v\|_{H_0^1},
$$
where we have used (1.4). Therefore, the proof is completed from (2.1).

Note that the constant $\alpha$ in Theorem 5 tends to 1 if $h \to 0$ as illustrated in Figure 1.

![Diagram of $H^1_0$- and $L$-projections]

Fig. 1. Image of the $H^1_0$- and $L$-projections

Now, as in [7], let $S_h^*$ be an appropriate finite element subspace of $H^1(\Omega)$ satisfying $S_h \subset S_h^*$, and let define $(\nabla u_h) \equiv (P_0 \nabla_x u_h, P_0 \nabla_y u_h, P_0 \nabla_z u_h) \in (S_h^*)^n$, where $P_0 : L^2(\Omega) \to S_h^*$ means the $L^2$-projection defined by, for each $v \in L^2(\Omega)$,

$$(v - P_0 v, \phi_h^*)_{L^2} = 0 \quad \text{for any } \phi_h^* \in S_h^*.$$  

Also note that, for the problem (1.1), the finite element solution $u_h$ defined by

$$(\nabla u_h, \nabla \phi_h)_{L^2} + (b \cdot \nabla u_h + cu_h, \phi_h)_{L^2} = (f, \phi_h)_{L^2}, \quad \forall \phi_h \in S_h, \quad (2.2)$$

coincides with the $L$-projection $P_L u$.

Now, by using Theorems 1, 2 and 5, we have the following constructive a priori and a posteriori error estimates for linear elliptic problems.

**Theorem 6** Assuming that Theorem 1 holds, then for a unique solution of $Lu = f$, we have

$$\|u - P_L u\|_{H^1_0} \leq C(h)\alpha \sigma \|f\|_{L^2},$$

$$\|u - P_L u\|_{L^2} \leq C(h)^2 \beta \sigma \|f\|_{L^2}.$$  

And we have the following a posteriori error estimate for the finite element solution $u_h$ defined by (2.2).

$$\|u - u_h\|_{H^1_0} \leq \|R\|_{L^2} + C(h)\beta \|S\|_{L^2} + C(h)^2 \beta \sigma (C_6 + C(h)C_7 \beta) \|f\|_{L^2}, \quad (2.3)$$

where $R \equiv \nabla u_h - (\nabla u_h)$ and $S \equiv f + \text{div}(\nabla u_h) - b \cdot \nabla u_h - cu_h$. 

6
Proof: From Theorems 2 and 5, we can easily obtain the following inequalities.

\[ \| u - P_L u \|_{H^1_0} \leq C(h) \alpha \sigma \| f \|_{L^2}, \]
\[ \| u - P_L u \|_{L^2} \leq C(h) \beta \sigma \| f \|_{L^2}. \]

Thus we consider the a posteriori error estimate below.

Let \( e \equiv u - u_h \).

\[ \| u - u_h \|_{H^1_0}^2 = (\nabla e, \nabla u)_{L^2} - (\nabla e, \nabla u_h)_{L^2} \]
\[ = (e, f)_{L^2} - (e, b \cdot \nabla u + cu)_{L^2} - (\nabla e, \nabla u_h)_{L^2} \]
\[ = (e, f - b \cdot \nabla u_h + cu_h)_{L^2} - (e, b \cdot \nabla e + ce)_{L^2} - (\nabla e, \nabla u_h)_{L^2}. \]

Since \( ((\nabla u_h), \nabla v)_{L^2} = -\text{div}(\nabla u_h, v)_{L^2} \) for any \( v \in H^1_0(\Omega) \), taking as \( v = e \), it implies that

\[ \| u - u_h \|_{H^1_0}^2 = (e, S)_{L^2} - (e, b \cdot \nabla e + ce)_{L^2} - (\nabla e, R)_{L^2} \]
\[ \leq \| e \|_{L^2} \| S \|_{L^2} + \| e \|_{L^2} \| b \cdot \nabla e + ce \|_{L^2} + \| e \|_{H^1_0} \| R \|_{L^2}. \]

Moreover, using Lemma 4, we have

\[ \| b \cdot \nabla e + ce \|_{L^2} \leq \| b \|_{E \infty} \| e \|_{H^1_0} + \| c \|_{L^\infty} \| e \|_{L^2}. \]

Hence using the fact \( \| e \|_{L^2} \leq C(h) \beta \| e \|_{H^1_0} \) in Theorem 5, we have the estimate (2.3). Therefore, this proof is completed.

Remark. The last term in the estimates (2.3) looks like an a priori estimation. However, since the order \( C(h)^2 \) is higher than the usual optimal estimation in \( H^1 \) norm, combining it with the first and second terms, this estimates can be considered as a kind of a posteriori error estimates.

From Theorems 5 - 6, we can take the constants \( K_0(h), K_1(h) \) and \( K(h) \) as

\[ K_0(h) := C(h)^2 \beta \sigma, \quad K_1(h) := C(h) \alpha \sigma, \quad K(h) := C(h) \beta. \]

Also we have the following estimates corresponding to Corollary 3.

Corollary 7 Let \( b \in (L^\infty(\Omega))^n \). Under the same assumptions in Corollary 3, we have

\[ \| u - P_L u \|_{H^1_0} \leq C(h) \hat{\alpha} \hat{\sigma} \| f \|_{L^2}, \]
for a unique solution of \( Lu = f \), where \( \hat{\alpha} \equiv \sqrt{1 + \left(M_h(C_1 + C(h)C_2)C_3\right)^2} \).

For usual finite element approximations in the one dimensional case, we can get the better estimates, even if the function \( b \) has no smoothness.
Lemma 8 Let $S_h$ be a finite element subspace of $H^1_0(\Omega)$, where $\Omega = (p, q)$ is an interval in $\mathbb{R}^1$, comprising piecewise polynomials with the mesh

$$p = x_0 < x_1 < \cdots < x_N < x_{N+1} = q.$$ 

For an arbitrary $v \in H^1_0(\Omega)$, if $b \in \Lambda^N_{i=0} W^1_\infty(I_i) \subset L^\infty(\Omega)$ then we have

$$\|A(v - P_h v)\|_{H^1_0} \leq (D_1 + C_2)\|v - P_h v\|_{L^2},$$

where $D_1 = C_p D_{\text{div}} b + C_b$, $D_{\text{div}} b = \max_{0 \leq i \leq N} \|b\|_{W^1_\infty(I_i)}$ and $I_i := (x_i, x_{i+1})$.

Proof: Let $\psi \equiv -\Delta^{-1}(be + ce)$, where $e := v - P_h v$. Then it implies that

$$\|\psi\|_{H^1_0} = (\psi', \psi')_{L^2} = (be' + ce, \psi)_{L^2} = (e', b\psi)_{L^2} + (e, c\psi)_{L^2}$$

Note that the $H^1_0$-projection satisfies $e(x_i) = 0$ for $i = 0, \cdots, N + 1$. Hence we have

$$(e', b\psi)_{L^2} = \sum_i (e, (b\psi)')_{L^2(I_i)}$$

$$\leq \sum_i \|e\|_{L^2(I_i)} \|(b\psi)'\|_{L^2(I_i)}$$

$$\leq \sum_i \|e\|_{L^2(I_i)} \left(\|b\|_{W^1_\infty(I_i)} \|\psi\|_{L^2(I_i)} + \|b\|_{L^\infty(I_i)} \|\psi'\|_{L^2(I_i)}\right)$$

$$\leq D_{\text{div}} b \sum_i \|e\|_{L^2(I_i)} \|\psi\|_{L^2(I_i)} + C_b \sum_i \|e\|_{L^2(I_i)} \|\psi'\|_{L^2(I_i)}$$

$$\leq \left(D_{\text{div}} b \|\psi\|_{L^2} + C_b \|\psi\|_{H^1_0}\right)\|e\|_{L^2}$$

and $(e, c\psi)_{L^2} \leq C_c \|e\|_{L^2} \|\psi\|_{L^2}$. Therefore, the proof is completed. \hfill \blacksquare

Applying similar arguments in Theorems 5 - 6 with the above lemma, we have the following results for a special case.

Theorem 9 Under the same assumption in Lemma 8, if $G$ is nonsingular then we have

$$\|v - P_\mathcal{L} v\|_{H^1_0} \leq \hat{\alpha} \|v - P_h v\|_{H^1_0},$$

$$\|v - P_\mathcal{L} v\|_{L^2} \leq C(h) \hat{\beta} \|v - P_h v\|_{H^1_0},$$

where $\hat{\alpha} \equiv \sqrt{1 + \left(C(h) M_h (D_1 + C_2)\right)^2}$ and $\hat{\beta} \equiv 1 + C_p M_h (D_1 + C_2)$. Moreover, if

$$\hat{\kappa}(h) \equiv C(h) \left(C(h) M_h (D_1 + C_2) C_3 + C_4\right) < 1$$

holds, then the operator $\mathcal{L}$ is invertible, and we have the following a priori error estimate for a unique solution of $\mathcal{L} u = f$.

$$\|u - P_\mathcal{L} u\|_{H^1_0} \leq C(h) \hat{\alpha} \hat{\sigma} \|f\|_{L^2},$$

$$\|u - P_\mathcal{L} u\|_{L^2} \leq C(h)^2 \hat{\beta} \hat{\sigma} \|f\|_{L^2},$$

8
\[
\dot{\sigma} = (1 + C_p M_h C_3)(1 - \kappa(h))^{-1}.
\]

3 Numerical examples

In this section, we show several numerical results for linear elliptic problems. In the below, the 1-dimensional problems are presented in the examples 1-3 and 2-dimensional cases in 4-5.

Example 1 (nearly singular problem)

\[-u'' + cu = 1 \text{ in } \Omega = (0, 1),
\]
\[u = 0 \text{ on } \partial\Omega,
\]
where \(c = \pm 10\). Note that if \(c = -\pi^2 = -9.8696 \cdots\) then this example has no solution.

Example 2 (linearized Burgers equation)

\[-u'' + \lambda(\tilde{\phi}_h + 2x - 1)u' + \lambda(\tilde{\phi}_h + 2x - 1)'u = 1 \text{ in } \Omega = (0, 1),
\]
\[u = 0 \text{ on } \partial\Omega,
\]
where \(\lambda = 10\) and \(\tilde{\phi}_h \in S_h\) is an approximation of the following Burgers equation.

\[
\phi'' = \lambda \phi \phi' \text{ in } \Omega,
\]
\[\phi(0) = -1, \quad \phi(1) = 1.
\]

Moreover, as a special case, we consider the following example.

Example 3 (discontinuous coefficient)

\[-u'' + bu' = 1 \text{ in } \Omega = (0, 1),
\]
\[u = 0 \text{ on } \partial\Omega,
\]
where \(b \in L^\infty(\Omega)\) is given by

\[
b \equiv b(x) = \begin{cases} 
4(8x^2 - x)' = 4(16x - 1) \text{ if } x \in (0, 0.25), \\
2(16x^2 - 14x + 3)' = 4(16x - 7) \text{ if } x \in (0.25, 0.5), \\
2(2x - 1)' = 4 \quad \text{ if } x \in (0.5, 0.75), \\
4(1 - x)' = -4 \quad \text{ if } x \in (0.75, 1). 
\end{cases}
\]
In above examples, we take the finite element subspace $S_h$ as piecewise quadratic functions with uniform mesh. Then it can be taken as $C(h) = (2\pi)^{-1} h$ ([3]) for piecewise quadratic functions on $\Omega = (0, 1)$ and $C_p = \pi^{-1}$.

We show validated numerical results using interval techniques ([1]) for Examples 1, 2 and 3 in Tables 1, 2 and 3, respectively.

Table 1
Numerical results for Example 1

<table>
<thead>
<tr>
<th>$h^{-1}$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\kappa(h)$</th>
<th>$M_h$</th>
<th>$C_{\text{div}b}$</th>
<th>$C_b$</th>
<th>$C_c$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1.0000</td>
<td>2.0132</td>
<td>2.0133</td>
<td>5.09e-5</td>
<td>0.9999</td>
<td>0.0</td>
<td>0.0</td>
<td>10</td>
<td>+10</td>
</tr>
<tr>
<td>200</td>
<td>1.0000</td>
<td>2.0132</td>
<td>2.0132</td>
<td>1.27e-5</td>
<td>1.0000</td>
<td>0.0</td>
<td>0.0</td>
<td>10</td>
<td>+10</td>
</tr>
<tr>
<td>400</td>
<td>1.0000</td>
<td>2.0135</td>
<td>2.0135</td>
<td>3.18e-6</td>
<td>1.0003</td>
<td>0.0</td>
<td>0.0</td>
<td>10</td>
<td>+10</td>
</tr>
<tr>
<td>800</td>
<td>1.0000</td>
<td>2.0248</td>
<td>2.0248</td>
<td>8.01e-7</td>
<td>1.0114</td>
<td>0.0</td>
<td>0.0</td>
<td>10</td>
<td>+10</td>
</tr>
<tr>
<td>100</td>
<td>1.0709</td>
<td>77.69</td>
<td>77.84</td>
<td>1.96e-3</td>
<td>75.69</td>
<td>0.0</td>
<td>0.0</td>
<td>10</td>
<td>−10</td>
</tr>
<tr>
<td>200</td>
<td>1.0182</td>
<td>77.71</td>
<td>77.75</td>
<td>4.92e-4</td>
<td>75.71</td>
<td>0.0</td>
<td>0.0</td>
<td>10</td>
<td>−10</td>
</tr>
<tr>
<td>400</td>
<td>1.0046</td>
<td>78.05</td>
<td>78.06</td>
<td>1.23e-4</td>
<td>76.04</td>
<td>0.0</td>
<td>0.0</td>
<td>10</td>
<td>−10</td>
</tr>
<tr>
<td>800</td>
<td>1.0013</td>
<td>83.72</td>
<td>83.72</td>
<td>3.31e-5</td>
<td>81.64</td>
<td>0.0</td>
<td>0.0</td>
<td>10</td>
<td>−10</td>
</tr>
</tbody>
</table>

Table 2
Numerical results for Example 2

<table>
<thead>
<tr>
<th>$h^{-1}$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\kappa(h)$</th>
<th>$M_h$</th>
<th>$C_{\text{div}b}$</th>
<th>$C_b$</th>
<th>$C_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1.4245</td>
<td>203.91</td>
<td>134.09</td>
<td>5.85e-2</td>
<td>14.94</td>
<td>51.30</td>
<td>10.00</td>
<td>51.30</td>
</tr>
<tr>
<td>200</td>
<td>1.1212</td>
<td>203.88</td>
<td>128.61</td>
<td>1.86e-2</td>
<td>14.94</td>
<td>51.28</td>
<td>10.00</td>
<td>51.28</td>
</tr>
<tr>
<td>400</td>
<td>1.0318</td>
<td>204.35</td>
<td>127.36</td>
<td>6.64e-3</td>
<td>14.97</td>
<td>51.28</td>
<td>10.00</td>
<td>51.28</td>
</tr>
<tr>
<td>800</td>
<td>1.0092</td>
<td>219.33</td>
<td>136.13</td>
<td>2.70e-3</td>
<td>16.08</td>
<td>51.28</td>
<td>10.00</td>
<td>51.28</td>
</tr>
</tbody>
</table>

Table 3
Numerical results for Example 3

<table>
<thead>
<tr>
<th>$h^{-1}$</th>
<th>$\dot{\alpha}$</th>
<th>$\dot{\beta}$</th>
<th>$\dot{\sigma}$</th>
<th>$\dot{\kappa}(h)$</th>
<th>$M_h$</th>
<th>$D_{\text{div}b}$</th>
<th>$C_b$</th>
<th>$C_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1.0260</td>
<td>23.97</td>
<td>9.9857</td>
<td>4.69e-2</td>
<td>2.2296</td>
<td>64.00</td>
<td>12.00</td>
<td>0.0</td>
</tr>
<tr>
<td>200</td>
<td>1.0065</td>
<td>23.97</td>
<td>9.7242</td>
<td>2.12e-2</td>
<td>2.2298</td>
<td>64.00</td>
<td>12.00</td>
<td>0.0</td>
</tr>
<tr>
<td>400</td>
<td>1.0016</td>
<td>23.97</td>
<td>9.6146</td>
<td>1.00e-2</td>
<td>2.2298</td>
<td>64.00</td>
<td>12.00</td>
<td>0.0</td>
</tr>
<tr>
<td>800</td>
<td>1.0004</td>
<td>23.99</td>
<td>9.5719</td>
<td>4.91e-3</td>
<td>2.2318</td>
<td>64.00</td>
<td>12.00</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Next we consider the following 2-dimensional problems.

**Example 4** (*linearized Emden’s equation*)

\[-\Delta u - 2\tilde{\phi}_h u = \frac{\sqrt{5}}{2} \text{ in } \Omega = (0, 1)^2 \setminus [0, \frac{1}{5}]^2,\]

\[u = 0 \text{ on } \partial \Omega,\]
where \( \tilde{\phi}_h \in S_h \) is an approximation of the following Emden’s equation.

\[
-\Delta \phi = \phi^2 \quad \text{in} \quad \Omega,
\]

\[
\phi = 0 \quad \text{on} \quad \partial \Omega.
\]

**Example 5**

\[
-\Delta u + \tilde{u}_h(\nabla \tilde{u}_h) \cdot \nabla u - \left( \lambda - \frac{1}{2} |\nabla \tilde{u}_h|^2 \right) u = 1 \quad \text{in} \quad \Omega = (0,1)^2,
\]

\[
u = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( \lambda = 40 \) and \( \tilde{u}_h \in S_h \) is an approximation of Plum’s example.

\[
-\Delta \phi = \phi \left( \lambda - \frac{1}{2} |\nabla \phi|^2 \right) \quad \text{in} \quad \Omega,
\]

\[
\phi = 0 \quad \text{on} \quad \partial \Omega.
\]

In this example, we considered two cases for the coefficient vector function \( b \), that is, in case of \( \nabla \tilde{u}_h \equiv \nabla \tilde{u}_h \), discontinuous, and \( \nabla \tilde{u}_h \equiv (P_0 \nabla_x \tilde{u}_h, P_0 \nabla_y \tilde{u}_h) \), where \( \tilde{u}_h \) is an approximate solution in \( S_h \) and \( P_0 \) stands for the \( L^2 \)-projection into \( S_h^* \) defined in Section 2.

In above two examples, we take the finite element subspace \( S_h \) as piecewise bi-linear functions with uniform mesh. Note that we can take the constant \( C_p \) for \( \Omega = (0,1)^2 \setminus [0,\frac{1}{5}]^2 \) and \( \Omega = (0,1)^2 \) as \( C_p = \sqrt{10}^{-1} \) and \( C_p = (\sqrt{2\pi})^{-1} \), respectively. Moreover, we can obtain the a priori constant \( C(h) \) for the \( L \)-shaped domain by techniques in [7], and it is taken as \( C(h) = \pi^{-1} h \) for bi-linear functions on \( \Omega = (0,1)^2 \). We show validated numerical results for Example 4 in Table 4. Also, for Example 5, we illustrate several numerical results for \( \nabla \tilde{u}_h = \nabla \tilde{u}_h \) and \( \nabla \tilde{u}_h = (P_0 \nabla_x \tilde{u}_h, P_0 \nabla_y \tilde{u}_h) \) in Tables 5 and 6, respectively.

As shown in these tables, the capability for the verification of invertibility seems to be influenced by the smoothness of the function \( b \).

All computations in these tables are carried out on the Dell Precision 650 Workstation Intel Xeon CPU 3.20GHz using INTLAB, a tool box in MATLAB developed by Rump [6] for self-validating algorithms.

**References**


### Table 4
Numerical results for Example 4

<table>
<thead>
<tr>
<th>$h^{-1}$</th>
<th>$C(h)$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\kappa(h)$</th>
<th>$M_h$</th>
<th>$C_{\text{div} , b}$</th>
<th>$C_b$</th>
<th>$C_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.8433$\pi^{-1} h$</td>
<td>3.4498</td>
<td>18.79</td>
<td>-</td>
<td>Fail</td>
<td>4.0656</td>
<td>2.8320</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>20</td>
<td>2.2063$\pi^{-1} h$</td>
<td>2.2159</td>
<td>18.80</td>
<td>-</td>
<td>Fail</td>
<td>1.4244</td>
<td>2.8994</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>30</td>
<td>2.4772$\pi^{-1} h$</td>
<td>1.7862</td>
<td>18.80</td>
<td>91.57</td>
<td>7.94e-1</td>
<td>2.9118</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>40</td>
<td>2.6992$\pi^{-1} h$</td>
<td>1.5718</td>
<td>18.85</td>
<td>40.33</td>
<td>5.32e-1</td>
<td>2.9159</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

### Table 5
Numerical results for Example 5 for $(\nabla \tilde{u}_h) = \nabla \tilde{u}_h$

<table>
<thead>
<tr>
<th>$h^{-1}$</th>
<th>$\hat{\alpha}$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{\kappa}(h)$</th>
<th>$M_h$</th>
<th>$C_{\text{div} , b}$</th>
<th>$C_b$</th>
<th>$C_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>6.2448</td>
<td>6.1895</td>
<td>1.3365</td>
<td>19.21</td>
<td>40.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>5.8008</td>
<td>2.7618</td>
<td>1.3556</td>
<td>18.08</td>
<td>40.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>5.6214</td>
<td>1.7563</td>
<td>1.3595</td>
<td>17.65</td>
<td>40.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>5.5576</td>
<td>1.2963</td>
<td>1.3608</td>
<td>17.52</td>
<td>40.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 6
Numerical results for Example 5 for $(\nabla \tilde{u}_h) = (P_0 \nabla_x \tilde{u}_h, P_0 \nabla_y \tilde{u}_h)$

<table>
<thead>
<tr>
<th>$h^{-1}$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\kappa(h)$</th>
<th>$M_h$</th>
<th>$C_{\text{div} , b}$</th>
<th>$C_b$</th>
<th>$C_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5.5421</td>
<td>39.54</td>
<td>Fail</td>
<td>5.6096</td>
<td>1.6630</td>
<td>330.81</td>
<td>19.51</td>
<td>40.00</td>
</tr>
<tr>
<td>20</td>
<td>3.3515</td>
<td>46.23</td>
<td>Fail</td>
<td>1.6841</td>
<td>1.7513</td>
<td>389.06</td>
<td>18.19</td>
<td>40.00</td>
</tr>
<tr>
<td>30</td>
<td>2.4286</td>
<td>47.94</td>
<td>62.61</td>
<td>8.14e-1</td>
<td>1.7723</td>
<td>404.84</td>
<td>17.56</td>
<td>40.00</td>
</tr>
<tr>
<td>40</td>
<td>1.9588</td>
<td>48.64</td>
<td>22.94</td>
<td>4.95e-1</td>
<td>1.7801</td>
<td>410.88</td>
<td>17.41</td>
<td>40.00</td>
</tr>
</tbody>
</table>


List of MHF Preprint Series, Kyushu University
21st Century COE Program
Development of Dynamic Mathematics with High Functionality

MHF2005-1 Hideki KOSAKI
Matrix trace inequalities related to uncertainty principle

MHF2005-2 Masahisa TABATA
Discrepancy between theory and real computation on the stability of some finite element schemes

MHF2005-3 Yuko ARAKI & Sadanori KONISHI
Functional regression modeling via regularized basis expansions and model selection

MHF2005-4 Yuko ARAKI & Sadanori KONISHI
Functional discriminant analysis via regularized basis expansions

MHF2005-5 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Point configurations, Cremona transformations and the elliptic difference Painlevé equations

MHF2005-6 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Construction of hypergeometric solutions to the $q$-Painlevé equations

MHF2005-7 Hiroki MASUDA
Simple estimators for non-linear Markovian trend from sampled data: I. ergodic cases

MHF2005-8 Hiroki MASUDA & Nakahiro YOSHIDA
Edgeworth expansion for a class of Ornstein-Uhlenbeck-based models

MHF2005-9 Masayuki UCHIDA
Approximate martingale estimating functions under small perturbations of dynamical systems

MHF2005-10 Ryo MATSUZAKI & Masayuki UCHIDA
One-step estimators for diffusion processes with small dispersion parameters from discrete observations

MHF2005-11 Junichi MATSUKUBO, Ryo MATSUZAKI & Masayuki UCHIDA
Estimation for a discretely observed small diffusion process with a linear drift

MHF2005-12 Masayuki UCHIDA & Nakahiro YOSHIDA
AIC for ergodic diffusion processes from discrete observations
Hiromichi GOTO & Kenji KAJIWARA
Generating function related to the Okamoto polynomials for the Painlevé IV equation

Masato KIMURA & Shin-ichi NAGATA
Precise asymptotic behaviour of the first eigenvalue of Sturm-Liouville problems with large drift

Daisuke TAGAMI & Masahisa TABATA
Numerical computations of a melting glass convection in the furnace

Raimundas VIDUNAS
Normalized Leonard pairs and Askey-Wilson relations

Raimundas VIDUNAS
Askey-Wilson relations and Leonard pairs

Kenji KAJIWARA & Atsushi MUKAIHIRA
Soliton solutions for the non-autonomous discrete-time Toda lattice equation

Yuu HARIYA
Construction of Gibbs measures for 1-dimensional continuum fields

Yuu HARIYA
Integration by parts formulae for the Wiener measure restricted to subsets in $\mathbb{R}^d$

Yuu HARIYA
A time-change approach to Kotani’s extension of Yor’s formula

Tadahisa FUNAKI, Yuu HARIYA & Mark YOR
Wiener integrals for centered powers of Bessel processes, I

Masahisa TABATA & Satoshi KAIZU
Finite element schemes for two-fluids flow problems

Ken-ichi MARUNO & Yasuhiro OHTA
Determinant form of dark soliton solutions of the discrete nonlinear Schrödinger equation

Alexander V. KITAEV & Raimundas VIDUNAS
Quadratic transformations of the sixth Painlevé equation

Toru FUJII & Sadanori KONISHI
Nonlinear regression modeling via regularized wavelets and smoothing parameter selection

Shuichi INOKUCHI, Kazumasa HONDA, Hyen Yeal LEE, Tatsuro SATO, Yoshihiro MIZOGUCHI & Yasuo KAWAHARA
On reversible cellular automata with finite cell array
MHF2005-28 Toru KOMATSU
Cyclic cubic field with explicit Artin symbols

MHF2005-29 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Kaori NAGATOU
A computational approach to constructive a priori and a posteriori error estimates for finite element approximations of bi-harmonic problems

MHF2005-30 Kaori NAGATOU, Kouji HASHIMOTO & Mitsuhiro T. NAKAO
Numerical verification of stationary solutions for Navier-Stokes problems

MHF2005-31 Hidefumi KAWASAKI
A duality theorem for a three-phase partition problem

MHF2005-32 Hidefumi KAWASAKI
A duality theorem based on triangles separating three convex sets

MHF2005-33 Takeaki FUCHIKAMI & Hidefumi KAWASAKI
An explicit formula of the Shapley value for a cooperative game induced from the conjugate point

MHF2005-34 Hideki MURAKAWA
A regularization of a reaction-diffusion system approximation to the two-phase Stefan problem

MHF2006-1 Masahisa TABATA
Numerical simulation of Rayleigh-Taylor problems by an energy-stable finite element scheme

MHF2006-2 Ken-ichi MARUNO & G R W QUISPEL
Construction of integrals of higher-order mappings

MHF2006-3 Setsuo TANIGUCHI
On the Jacobi field approach to stochastic oscillatory integrals with quadratic phase function

MHF2006-4 Kouji HASHIMOTO, Kaori NAGATOU & Mitsuhiro T. NAKAO
A computational approach to constructive a priori error estimate for finite element approximations of bi-harmonic problems in nonconvex polygonal domains

MHF2006-5 Hidefumi KAWASAKI
A duality theory based on triangular cylinders separating three convex sets in $\mathbb{R}^n$

MHF2006-6 Raimundas VIDUNAS
Uniform convergence of hypergeometric series

MHF2006-7 Yuji KODAMA & Ken-ichi MARUNO
N-Soliton solutions to the DKP equation and Weyl group actions
MHF2006-8 Toru KOMATSU
Potentially generic polynomial

MHF2006-9 Toru KOMATSU
Generic sextic polynomial related to the subfield problem of a cubic polynomial

MHF2006-10 Shu TEZUKA & Anargyros PAPAGEORGIOU
Exact cubature for a class of functions of maximum effective dimension

MHF2006-11 Shu TEZUKA
On high-discrepancy sequences

MHF2006-12 Raimundas VIDUÑAS
Detecting persistent regimes in the North Atlantic Oscillation time series

MHF2006-13 Toru KOMATSU
Tamely Eisenstein field with prime power discriminant

MHF2006-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCH
Generating function associated with the Hankel determinant formula for the solutions of the Painlevé IV equation

MHF2006-15 Raimundas VIDUÑAS
Darboux evaluations of algebraic Gauss hypergeometric functions

MHF2006-16 Masato KIMURA & Isao WAKANO
New mathematical approach to the energy release rate in crack extension

MHF2006-17 Toru KOMATSU
Arithmetic of the splitting field of Alexander polynomial

MHF2006-18 Hiroki MASUDA
Likelihood estimation of stable Lévy processes from discrete data

MHF2006-19 Hiroshi KAWABI & Michael RÖCKNER
Essential self-adjointness of Dirichlet operators on a path space with Gibbs measures via an SPDE approach

MHF2006-20 Masahisa TABATA
Energy stable finite element schemes and their applications to two-fluid flow problems

MHF2006-21 Yuzuru INAHAMA & Hiroshi KAWABI
Asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths

MHF2006-22 Yoshiyuki KAGEI
Resolvent estimates for the linearized compressible Navier-Stokes equation in an infinite layer
MHF2006-23 Yoshiyuki KAGEI
Asymptotic behavior of the semigroup associated with the linearized compressible Navier-Stokes equation in an infinite layer

MHF2006-24 Akihiro MIKODA, Shuichi INOKUCHI, Yoshihiro MIZOGUCHI & Mitsuhiko FUJIO
The number of orbits of box-ball systems

MHF2006-25 Toru FUJII & Sadanori KONISHI
Multi-class logistic discrimination via wavelet-based functionalization and model selection criteria

MHF2006-26 Taro HAMAMOTO, Kenji KAJIWARA & Nicholas S. WITTE
Hypergeometric solutions to the $q$-Painlevé equation of type $(A_1 + A_1')^{(1)}$

MHF2006-27 Hiroshi KAWABI & Tomohiro MIYOKAWA
The Littlewood-Paley-Stein inequality for diffusion processes on general metric spaces

MHF2006-28 Hiroki MASUDA
Notes on estimating inverse-Gaussian and gamma subordinators under high-frequency sampling

MHF2006-29 Setsuo TANIGUCHI
The heat semigroup and kernel associated with certain non-commutative harmonic oscillators

MHF2006-30 Setsuo TANIGUCHI
Stochastic analysis and the KdV equation

MHF2006-31 Masato KIMURA, Hideki KOMURA, Masayasu MIMURA, Hidenori MIYOSHI, Takeshi TAKAISHI & Daishin UEYAMA
Quantitative study of adaptive mesh FEM with localization index of pattern

MHF2007-1 Taro HAMAMOTO & Kenji KAJIWARA
Hypergeometric solutions to the $q$-Painlevé equation of type $A_1^{(1)}$

MHF2007-2 Kouji HASHIMOTO, Kenta KOBAYASHI & Mitsuhiro T. NAKAO
Verified numerical computation of solutions for the stationary Navier-Stokes equation in nonconvex polygonal domains

MHF2007-3 Kenji KAJIWARA, Marta MAZZOCCHI & Yasuhiro OHTA
A remark on the Hankel determinant formula for solutions of the Toda equation

MHF2007-4 Jun-ichi SATO & Hidefumi KAWASAKI
Discrete fixed point theorems and their application to Nash equilibrium

MHF2007-5 Mitsuhiro T. NAKAO & Kouji HASHIMOTO
Constructive error estimates of finite element approximations for non-coercive elliptic problems and its applications