

## Guaranteed error bounds for finite element approximations of noncoercive elliptic problems and their applications

Nakao, Mitsuhiro T.  
Faculty of Mathematics, Kyushu University

Hashimoto, Kouji  
Graduate School of Informatics, Kyoto University

<https://hdl.handle.net/2324/3405>

---

出版情報 : Journal of Computational and Applied Mathematics. 218 (1), pp.106-115, 2008-08-15.  
Faculty of Mathematics, Kyushu University  
バージョン :  
権利関係 :



# MHF Preprint Series

Kyushu University  
21st Century COE Program  
Development of Dynamic Mathematics with  
High Functionality

## Constructive error estimates of finite element approximations for non-coercive elliptic problems and its applications

M. T. Nakao & K. Hashimoto

MHF 2007-5

( Received January 19, 2007 )

Faculty of Mathematics  
Kyushu University  
Fukuoka, JAPAN

# Constructive error estimates of finite element approximations for non-coercive elliptic problems and its applications

Mitsuhiro T. Nakao<sup>†</sup> and Kouji Hashimoto<sup>‡</sup>

*mtnakao@math.kyushu-u.ac.jp*

<sup>†</sup>*Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan*

<sup>‡</sup>*Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan*

---

## Abstract

In this paper, we show constructive a priori and a posteriori error estimates of finite element approximations for not necessary coercive linear second order Dirichlet problems. Here, 'constructive' means we can get the error bounds in which all constants included are explicitly given or represented as a numerically computable form. Using the invertibility condition of concerning elliptic operator, constructive a priori and a posteriori error estimates are formulated. This kind of estimates plays essential and important roles in the numerical verification of solutions for nonlinear elliptic problems. Several numerical examples that confirm the actual effectiveness of the method are presented.

*Key words:* Constructive a priori and a posteriori error estimates, linear elliptic problem.

*Classifications:* 35J25, 35J60, 65N25.

---

## 1 Introduction

In this paper, we consider the constructive a priori and a posteriori error estimates for the general linear elliptic boundary value problem of the form

$$\begin{aligned}\mathcal{L}u &\equiv -\Delta u + b \cdot \nabla u + cu = f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega,\end{aligned}\tag{1.1}$$

where  $f \in L^2(\Omega)$ . Here, for  $n = 1, 2, 3$ , we assume that  $b \in (W_\infty^1(\Omega))^n$ ,  $c \in L^\infty(\Omega)$ , where  $\Omega \subset \mathbb{R}^n$  is a bounded open domain with piecewise smooth boundary. In this paper, we use the terminology 'constructive error estimates'

means an error estimation that by some numerical computations based on the estimates, we can obtain the true error bounds between the exact solution and its approximation in mathematically rigorous sense, even if the concerning problem (1.1) is not coercive. This kind of estimations should be useful when the existence or uniqueness of solutions are not a priori assured, e.g., in case that the coefficient function  $c$  is not nonnegative. And it also be important for the numerical verification of solitions for nonlinear boundary value problems(e.g., [2] [4] [5]etc.).

Now, we denote the usual  $k$ -th order Sobolev space on  $\Omega$  by  $H^k(\Omega)$  and define  $(\cdot, \cdot)_{L^2}$  as the  $L^2$  inner product. And we set  $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) ; v = 0 \text{ on } \partial\Omega\}$  with the inner product  $(\nabla u, \nabla v)_{L^2}$  for  $u, v \in H_0^1(\Omega)$ . Also, define  $X(\Omega) \equiv \{v \in H^1(\Omega) ; \Delta v \in L^2(\Omega)\}$ .

We now introduce the finite dimensional subspace  $S_h$  of  $H_0^1(\Omega)$  depending on the parameter  $h$  with nodal functions  $\{\phi_i\}_{1 \leq i \leq N}$ . For each  $v \in H_0^1(\Omega)$ , define the  $H_0^1$ -projection  $P_h v \in S_h$  by

$$(\nabla(v - P_h v), \nabla \phi_h)_{L^2} = 0, \quad \forall \phi_h \in S_h.$$

Also, corresponding to the usual finite element approximations of a solution  $u$  in (1.1), we define the  $\mathcal{L}$ -projection  $P_{\mathcal{L}} v \in S_h$ , whose existence is assumed, by

$$a(v - P_{\mathcal{L}} v, \phi_h)_{L^2} = 0, \quad \forall \phi_h \in S_h, \quad (1.2)$$

where  $a(u, v) \equiv (\nabla u, \nabla v)_{L^2} + (b \cdot \nabla u, v)_{L^2} + (cu, v)_{L^2}$ . Further, we assume that there exists a positive constant  $C(h)$  which can be numerically estimated satisfying, for any  $u \in H_0^1(\Omega) \cap X(\Omega)$ ,

$$\|u - P_h u\|_{H_0^1} \leq C(h) \|\Delta u\|_{L^2}. \quad (1.3)$$

Note that (1.3) is equivalent to the following estimation.

$$\|u - P_h u\|_{L^2} \leq C(h) \|u - P_h u\|_{H_0^1}. \quad (1.4)$$

Then our main purpose of this paper is to determine explicitly a priori constants  $K_0(h)$  and  $K_1(h)$  satisfying

$$\|u - P_{\mathcal{L}} u\|_{L^2} \leq K_0(h) \|\mathcal{L}u\|_{L^2}, \quad (1.5)$$

$$\|u - P_{\mathcal{L}} u\|_{H_0^1} \leq K_1(h) \|\mathcal{L}u\|_{L^2}, \quad (1.6)$$

respectively. Also we show a constant  $K(h)$  satisfying

$$\|u - P_{\mathcal{L}} u\|_{L^2} \leq K(h) \|u - P_{\mathcal{L}} u\|_{H_0^1}. \quad (1.7)$$

Defining the compact oprator  $A : H_0^1 \longrightarrow H_0^1$  by  $Au := \Delta^{-1}(b \cdot \nabla u + cu)$ , where  $\Delta^{-1}$  stands for the solution operator of the Poisson equation with homogeneous boundary condition, the invertibility of the elliptic operator  $\mathcal{L}$  defined in (1.1) is equivalent to the unique solvability of the following fixed point equation:

$$u = Au.$$

As the preliminary, we define  $N \times N$  matrices  $\mathbf{G} = (\mathbf{G}_{i,j})$  and  $\mathbf{D} = (\mathbf{D}_{i,j})$  by

$$\begin{aligned}\mathbf{G}_{i,j} &= (\nabla \phi_j, \nabla \phi_i)_{L^2} + (b \cdot \nabla \phi_j, \phi_i)_{L^2} + (c \phi_j, \phi_i)_{L^2}, \\ \mathbf{D}_{i,j} &= (\nabla \phi_j, \nabla \phi_i)_{L^2},\end{aligned}$$

Note that  $\mathbf{D}$  is symmetric and positive definite. We denote the matrix norm by  $\|\cdot\|_E$  induced from the Euclidean norm  $|\cdot|_E$ . Also, we define the following constants:

$$\begin{aligned}C_1 &= C_p C_{\text{div } b} + C_b, & C_3 &= C_b + C_p C_c, \\ C_2 &= C_p C_c, & C_4 &= C_b + C(h) C_c, \\ C_{\text{div } b} &= \|\text{div } b\|_{L^\infty}, & C_b &= \| |b|_E \|_{L^\infty}, & C_c &= \|c\|_{L^\infty},\end{aligned}$$

where  $\|\cdot\|_{L^\infty}$  means  $L^\infty$  norm on  $\Omega$  and  $C_p$  is a Poincaré constant such that  $\|\phi\|_{L^2} \leq C_p \|\phi\|_{H_0^1}$  for an arbitrary  $\phi \in H_0^1(\Omega)$ .

In [5], authors show the following results.

**Theorem 1** *If the matrix  $\mathbf{G}$  is nonsingular, and for the constants defined above,*

$$\kappa(h) \equiv C(h) \left( C(h) M_h (C_1 + C_2) C_3 + C_4 \right) < 1$$

*holds, then the operator  $\mathcal{L}$  defined in (1.1) is invertible. Here,  $M_h \equiv \|\mathbf{D}^{\frac{1}{2}} \mathbf{G}^{-1} \mathbf{D}^{\frac{1}{2}}\|_E$  and  $C(h)$  is the same constant as in (1.3).*

Moreover, we have the following a priori estimate for the  $H_0^1$ -projection.

**Theorem 2** *Assuming that same conditions in Theorem 1, let  $u \in H_0^1(\Omega) \cap X(\Omega)$  be a unique solution of (1.1). Then we have*

$$\|u - P_h u\|_{H_0^1} \leq C(h) \sigma \|f\|_{L^2},$$

*where the constant  $\sigma$  is given by  $\sigma = (1 + C_p M_h C_3)(1 - \kappa(h))^{-1}$ .*

When the coefficient vector function  $b$  in (1.1) is not differentiable, we have the following alternative results.

**Corollary 3** *Let  $b \in (L^\infty(\Omega))^n$ . If*

$$\hat{\kappa}(h) \equiv C(h) \left( M_h (\hat{C}_1 + C(h) C_2) C_3 + C_4 \right) < 1$$

holds, then the operator  $\mathcal{L}$  defined in (1.1) is invertible. Here,  $\hat{C}_1 = C_p C_b$ . Also we have

$$\|u - P_h u\|_{H_0^1} \leq C(h) \hat{\sigma} \|f\|_{L^2},$$

for a unique solution of  $\mathcal{L}u = f$ , where the constant  $\hat{\sigma}$  is given by

$$\hat{\sigma} = (1 + C_p M_h C_3)(1 - \hat{\kappa}(h))^{-1}.$$

## 2 Main results

In this section, we show the constructive a priori and a posteriori error estimates of finite element approximations (1.2) for linear elliptic problems (1.1). Note that the existence of the inverse  $\mathcal{L}^{-1} : L^2(\Omega) \rightarrow X(\Omega)$  is equivalent to the invertibility of  $I - A$ , where  $I$  denotes the identity operator in  $H_0^1(\Omega)$ . Using this fact, we first show the a priori error estimate between a solution of our problems and its  $H_0^1$ -projection. First, we show the following lemma.

**Lemma 4** (cf. [5]) *For an arbitrary  $v \in H_0^1(\Omega)$ , we have*

$$\begin{aligned} \|Av\|_{H_0^1} &\leq (C_1 + C_2)\|v\|_{L^2}, \\ \|(I - P_h)Av\|_{H_0^1} &\leq C(h) \left( C_3 \|P_h v\|_{H_0^1} + C_4 \|v - P_h v\|_{H_0^1} \right). \end{aligned}$$

**Proof:** Let  $\psi := -Av = -\Delta^{-1}(b \cdot \nabla + c)v \in H_0^1(\Omega) \cap X(\Omega)$ . Then we have

$$\begin{aligned} \|\psi\|_{H_0^1}^2 &= (-\Delta\psi, \psi)_{L^2} \\ &= (v, \operatorname{div}(b\psi))_{L^2} + (v, c\psi)_{L^2} \\ &\leq \left( \|\operatorname{div}(b\psi)\|_{L^2} + \|c\psi\|_{L^2} \right) \|v\|_{L^2} \\ &\leq C(h) \left( \|\operatorname{div} b\|_{L^\infty} \|\psi\|_{L^2} + \| |b|_E \|_{L^\infty} \|\psi\|_{H_0^1} + \|c\|_{L^\infty} \|\psi\|_{L^2} \right) \|v\|_{H_0^1}, \end{aligned}$$

where we have used (1.4). Moreover, we have

$$\begin{aligned} \|(I - P_h)Av\|_{H_0^1} &= \|(I - P_h)\Delta^{-1}(b \cdot \nabla + c)v\|_{H_0^1} \\ &\leq C(h) \|(b \cdot \nabla + c)v\|_{L^2} \\ &\leq C(h) \left( \| |b|_E \|_{L^\infty} \|v\|_{H_0^1} + \|c\|_{L^\infty} \|v\|_{L^2} \right), \end{aligned}$$

where we have used (1.3). Therefore, this proof is completed. ■

For the  $\mathcal{L}$ -projection, we have the following one of the main results of this paper.

**Theorem 5** For an arbitrary  $v \in H_0^1(\Omega)$ , if  $\mathbf{G}$  is nonsingular, then for the same constants in Theorem 1, we have

$$\begin{aligned}\|v - P_{\mathcal{L}}v\|_{H_0^1} &\leq \alpha \|v - P_hv\|_{H_0^1}, \\ \|v - P_{\mathcal{L}}v\|_{L^2} &\leq C(h)\beta \|v - P_hv\|_{H_0^1} \leq C(h)\beta \|v - P_{\mathcal{L}}v\|_{H_0^1},\end{aligned}$$

where  $\alpha \equiv \sqrt{1 + \left(C(h)M_h(C_1 + C_2)\right)^2}$ ,  $\beta \equiv 1 + C_pM_h(C_1 + C_2)$ .

**Proof:** From the property of the  $H_0^1$ - and  $\mathcal{L}$ -projections, we can obtain

$$\|v - P_{\mathcal{L}}v\|_{H_0^1}^2 = \|v - P_hv\|_{H_0^1}^2 + \|P_{\mathcal{L}}v - P_hv\|_{H_0^1}^2, \quad (2.1)$$

for an arbitrary  $v \in H_0^1(\Omega)$ . Let  $e \equiv v - P_hv$ .

Then since  $P_{\mathcal{L}}v - P_hv = P_{\mathcal{L}}(v - P_hv)$ , for all  $\phi_h \in S_h$ , we have

$$\begin{aligned}a(P_{\mathcal{L}}e, \phi_h) &= (\nabla e, \nabla \phi_h)_{L^2} + ((b \cdot \nabla + c)e, \phi_h)_{L^2} \\ &= (b \cdot \nabla e + ce, \phi_h)_{L^2} \\ &= (\nabla P_h\psi, \nabla \phi_h)_{L^2},\end{aligned}$$

where we set  $\psi \equiv -Ae = -\Delta^{-1}(b \cdot \nabla + c)e$ . It implies that

$$\mathbf{G}\vec{e}_h = \mathbf{D}\vec{\psi}_h,$$

where  $\vec{e}_h$  and  $\vec{\psi}_h$  are coefficient vectors of  $P_{\mathcal{L}}e$  and  $P_h\psi$ , respectively. Thus in the similar way to the proof of Lemma 4, we can obtain the following estimate since  $\|P_{\mathcal{L}}e\|_{H_0^1} = \|\mathbf{D}^{\frac{1}{2}}\vec{e}_h\|_E$ ,  $\|P_h\psi\|_{H_0^1} = \|\mathbf{D}^{\frac{1}{2}}\vec{\psi}_h\|_E$  and  $\|P_h\psi\|_{H_0^1} \leq \|\psi\|_{H_0^1}$  for any  $\psi \in H_0^1(\Omega)$ .

$$\begin{aligned}\|P_{\mathcal{L}}v - P_hv\|_{H_0^1} &= \|P_{\mathcal{L}}e\|_{H_0^1} \leq M_h\|P_h\psi\|_{H_0^1} \\ &\leq M_h\|A(v - P_hv)\|_{H_0^1} \\ &\leq C(h)M_h(C_1 + C_2)\|v - P_hv\|_{H_0^1}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\|P_{\mathcal{L}}v - P_hv\|_{L^2} &\leq C_p\|P_{\mathcal{L}}v - P_hv\|_{H_0^1} \\ &\leq C(h)C_pM_h(C_1 + C_2)\|v - P_hv\|_{H_0^1}.\end{aligned}$$

Hence we can obtain the following estimate.

$$\begin{aligned}\|v - P_{\mathcal{L}}v\|_{L^2} &\leq \|v - P_hv\|_{L^2} + \|P_{\mathcal{L}}v - P_hv\|_{L^2} \\ &\leq C(h)\|v - P_hv\|_{H_0^1} + C(h)C_pM_h(C_1 + C_2)\|v - P_hv\|_{H_0^1},\end{aligned}$$

where we have used (1.4). Therefore, the proof is completed from (2.1).  $\blacksquare$

Note that the constant  $\alpha$  in Theorem 5 tends to 1 if  $h \rightarrow 0$  as illustrated in Figure 1.

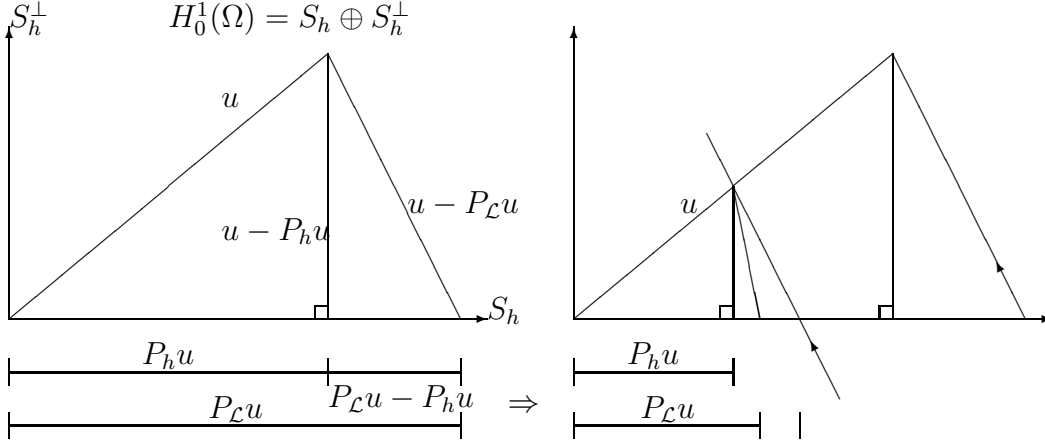


Fig. 1. Image of the  $H_0^1$ - and  $\mathcal{L}$ -projections

Now, as in [7], let  $S_h^*$  be an appropriate finite element subspace of  $H^1(\Omega)$  satisfying  $S_h \subset S_h^*$ , and let define  $(\bar{\nabla} u_h) \equiv (P_0 \nabla_x u_h, P_0 \nabla_y u_h, P_0 \nabla_z u_h) \in (S_h^*)^n$ , where  $P_0 : L^2(\Omega) \rightarrow S_h^*$  means the  $L^2$ -projection defined by, for each  $v \in L^2(\Omega)$ ,

$$(v - P_0 v, \phi_h^*)_{L^2} = 0 \quad \text{for any } \phi_h^* \in S_h^*.$$

Also note that, for the problem (1.1), the finite element solution  $u_h$  defined by

$$(\nabla u_h, \nabla \phi_h)_{L^2} + (b \cdot \nabla u_h + c u_h, \phi_h)_{L^2} = (f, \phi_h)_{L^2}, \quad \forall \phi_h \in S_h, \quad (2.2)$$

coincides with the  $\mathcal{L}$ -projection  $P_{\mathcal{L}} u$ .

Now, by using Theorems 1, 2 and 5, we have the following constructive a priori and a posteriori error estimates for linear elliptic problems.

**Theorem 6** *Assuming that Theorem 1 holds, then for a unique solution of  $\mathcal{L}u = f$ , we have*

$$\|u - P_{\mathcal{L}} u\|_{H_0^1} \leq C(h) \alpha \sigma \|f\|_{L^2},$$

$$\|u - P_{\mathcal{L}} u\|_{L^2} \leq C(h)^2 \beta \sigma \|f\|_{L^2}.$$

And we have the following a posteriori error estimate for the finite element solution  $u_h$  defined by (2.2).

$$\|u - u_h\|_{H_0^1} \leq \|R\|_{L^2} + C(h) \beta \|S\|_{L^2} + C(h)^2 \beta \sigma (C_b + C(h) C_c \beta) \|f\|_{L^2}, \quad (2.3)$$

where  $R \equiv \nabla u_h - (\bar{\nabla} u_h)$  and  $S \equiv f + \operatorname{div}(\bar{\nabla} u_h) - b \cdot \nabla u_h - c u_h$ .

**Proof:** From Theorems 2 and 5, we can easily obtain the following inequalities.

$$\begin{aligned}\|u - P_{\mathcal{L}}u\|_{H_0^1} &\leq C(h)\alpha\sigma\|f\|_{L^2}, \\ \|u - P_{\mathcal{L}}u\|_{L^2} &\leq C(h)^2\beta\sigma\|f\|_{L^2}.\end{aligned}$$

Thus we consider the a posteriori error estimate below.

Let  $e \equiv u - u_h$ .

$$\begin{aligned}\|u - u_h\|_{H_0^1}^2 &= (\nabla e, \nabla u)_{L^2} - (\nabla e, \nabla u_h)_{L^2} \\ &= (e, f)_{L^2} - (e, b \cdot \nabla u + cu)_{L^2} - (\nabla e, \nabla u_h)_{L^2} \\ &= (e, f - b \cdot \nabla u_h + cu_h)_{L^2} - (e, b \cdot \nabla e + ce)_{L^2} - (\nabla e, \nabla u_h)_{L^2}.\end{aligned}$$

Since  $((\nabla u_h), \nabla v)_{L^2} = (-\operatorname{div}(\nabla u_h), v)_{L^2}$  for any  $v \in H_0^1(\Omega)$ , taking as  $v = e$ , it implies that

$$\begin{aligned}\|u - u_h\|_{H_0^1}^2 &= (e, S)_{L^2} - (e, b \cdot \nabla e + ce)_{L^2} - (\nabla e, R)_{L^2} \\ &\leq \|e\|_{L^2}\|S\|_{L^2} + \|e\|_{L^2}\|b \cdot \nabla e + ce\|_{L^2} + \|e\|_{H_0^1}\|R\|_{L^2}.\end{aligned}$$

Moreover, using Lemma 4, we have

$$\|b \cdot \nabla e + ce\|_{L^2} \leq \| |b|_E \|_{L^\infty} \|e\|_{H_0^1} + \|c\|_{L^\infty} \|e\|_{L^2}.$$

Hence using the fact  $\|e\|_{L^2} \leq C(h)\beta\|e\|_{H_0^1}$  in Theorem 5, we have the estimate (2.3). Therefore, this proof is completed.  $\blacksquare$

*Remark.* The last term in the estimates (2.3) looks like an a priori estimation. However, since the order  $C(h)^2$  is higher than the usual optimal estimation in  $H^1$  norm, combining it with the first and second terms, this estimates can be considered as a kind of a posteriori error estimates.

From Theorems 5 - 6, we can take the constants  $K_0(h)$ ,  $K_1(h)$  and  $K(h)$  as

$$K_0(h) := C(h)^2\beta\sigma, \quad K_1(h) := C(h)\alpha\sigma, \quad K(h) := C(h)\beta.$$

Also we have the following estimates corresponding to Corollary 3.

**Corollary 7** *Let  $b \in (L^\infty(\Omega))^n$ . Under the same assumptions in Corollary 3, we have*

$$\|u - P_{\mathcal{L}}u\|_{H_0^1} \leq C(h)\hat{\alpha}\hat{\sigma}\|f\|_{L^2},$$

for a unique solution of  $\mathcal{L}u = f$ , where  $\hat{\alpha} \equiv \sqrt{1 + \left(M_h(\hat{C}_1 + C(h)C_2)C_3\right)^2}$ .

For usual finite element approximations in the one dimensional case, we can get the better estimates, even if the function  $b$  has no smoothness.

**Lemma 8** Let  $S_h$  be a finite element subspace of  $H_0^1(\Omega)$ , where  $\Omega = (p, q)$  is an interval in  $\mathbf{R}^1$ , comprising piecewise polynomials with the mesh

$$p = x_0 < x_1 < \cdots < x_N < x_{N+1} = q.$$

For an arbitrary  $v \in H_0^1(\Omega)$ , if  $b \in \Lambda_{i=0}^N W_\infty^1(I_i) \subset L^\infty(\Omega)$  then we have

$$\|A(v - P_h v)\|_{H_0^1} \leq (D_1 + C_2)\|v - P_h v\|_{L^2},$$

where  $D_1 = C_p D_{\text{div } b} + C_b$ ,  $D_{\text{div } b} = \max_{0 \leq i \leq N} \|b\|_{W_\infty^1(I_i)}$  and  $I_i := (x_i, x_{i+1})$ .

**Proof:** Let  $\psi \equiv -\Delta^{-1}(be' + ce)$ , where  $e := v - P_h v$ . Then it implies that

$$\|\psi\|_{H_0^1}^2 = (\psi', \psi')_{L^2} = (be' + ce, \psi)_{L^2} = (e', b\psi)_{L^2} + (e, c\psi)_{L^2}$$

Note that the  $H_0^1$ -projection satisfies  $e(x_i) = 0$  for  $i = 0, \dots, N+1$ . Hence we have

$$\begin{aligned} (e', b\psi)_{L^2} &= \sum_i (e, (b\psi)')_{L^2(I_i)} \\ &\leq \sum_i \|e\|_{L^2(I_i)} \|(b\psi)'\|_{L^2(I_i)} \\ &\leq \sum_i \|e\|_{L^2(I_i)} \left( \|b\|_{W_\infty^1(I_i)} \|\psi\|_{L^2(I_i)} + \|b\|_{L^\infty(I_i)} \|\psi'\|_{L^2(I_i)} \right) \\ &\leq D_{\text{div } b} \sum_i \|e\|_{L^2(I_i)} \|\psi\|_{L^2(I_i)} + C_b \sum_i \|e\|_{L^2(I_i)} \|\psi'\|_{L^2(I_i)} \\ &\leq \left( D_{\text{div } b} \|\psi\|_{L^2} + C_b \|\psi\|_{H_0^1} \right) \|e\|_{L^2} \\ &\leq (C_p D_{\text{div } b} + C_b) \|\psi\|_{H_0^1} \|e\|_{L^2}, \end{aligned}$$

and  $(e, c\psi)_{L^2} \leq C_c \|e\|_{L^2} \|\psi\|_{L^2}$ . Therefore, the proof is completed.  $\blacksquare$

Applying similar arguments in Theorems 5 - 6 with the above lemma, we have the following results for a special case.

**Theorem 9** Under the same assumption in Lemma 8, if  $\mathbf{G}$  is nonsingular then we have

$$\begin{aligned} \|v - P_{\mathcal{L}} v\|_{H_0^1} &\leq \dot{\alpha} \|v - P_h v\|_{H_0^1}, \\ \|v - P_{\mathcal{L}} v\|_{L^2} &\leq C(h) \dot{\beta} \|v - P_{\mathcal{L}} v\|_{H_0^1}, \end{aligned}$$

where  $\dot{\alpha} \equiv \sqrt{1 + \left(C(h)M_h(D_1 + C_2)\right)^2}$  and  $\dot{\beta} \equiv 1 + C_p M_h(D_1 + C_2)$ . Moreover, if

$$\dot{\kappa}(h) \equiv C(h) \left( C(h)M_h(D_1 + C_2)C_3 + C_4 \right) < 1$$

holds, then the operator  $\mathcal{L}$  is invertible, and we have the following a priori error estimate for a unique solution of  $\mathcal{L}u = f$ .

$$\begin{aligned} \|u - P_{\mathcal{L}} u\|_{H_0^1} &\leq C(h) \dot{\alpha} \dot{\sigma} \|f\|_{L^2}, \\ \|u - P_{\mathcal{L}} u\|_{L^2} &\leq C(h)^2 \dot{\beta} \dot{\sigma} \|f\|_{L^2}, \end{aligned}$$

where  $\dot{\sigma} = (1 + C_p M_h C_3)(1 - \dot{\kappa}(h))^{-1}$ .

### 3 Numerical examples

In this section, we show several numerical results for linear elliptic problems. In the below, the 1-dimensional problems are presented in the examples 1-3 and 2-dimensional cases in 4-5.

**Example 1** (*nearly singular problem*)

$$\begin{aligned} -u'' + cu &= 1 \text{ in } \Omega = (0, 1), \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where  $c = \pm 10$ . Note that if  $c = -\pi^2 = -9.8696 \dots$  then this example has no solution.

**Example 2** (*linearized Burgers equation*)

$$\begin{aligned} -u'' + \lambda(\tilde{\phi}_h + 2x - 1)u' + \lambda(\tilde{\phi}_h + 2x - 1)'u &= 1 \text{ in } \Omega = (0, 1), \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where  $\lambda = 10$  and  $\tilde{\phi}_h \in S_h$  is an approximation of the following Burgers equation.

$$\begin{aligned} \phi'' &= \lambda\phi\phi' \text{ in } \Omega, \\ \phi(0) &= -1, \quad \phi(1) = 1. \end{aligned}$$

Moreover, as a special case, we consider the following example.

**Example 3** (*discontinuous coefficient*)

$$\begin{aligned} -u'' + bu' &= 1 \text{ in } \Omega = (0, 1), \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where  $b \in L^\infty(\Omega)$  is given by

$$b \equiv b(x) = \begin{cases} 4(8x^2 - x)' & = 4(16x - 1) \text{ if } x \in (0, 0.25), \\ 2(16x^2 - 14x + 3)' & = 4(16x - 7) \text{ if } x \in (0.25, 0.5), \\ 2(2x - 1)' & = 4 \text{ if } x \in (0.5, 0.75), \\ 4(1 - x)' & = -4 \text{ if } x \in (0.75, 1). \end{cases}$$

In above examples, we take the finite element subspace  $S_h$  as piecewise quadratic functions with uniform mesh. Then it can be taken as  $C(h) = (2\pi)^{-1}h$  ([3]) for piecewise quadratic functions on  $\Omega = (0, 1)$  and  $C_p = \pi^{-1}$ .

We show validated numerical results using interval techniques ([1]) for Examples 1, 2 and 3 in Tables 1, 2 and 3, respectively.

Table 1

Numerical results for Example 1

$h^{-1}$	$\alpha$	$\beta$	$\sigma$	$\kappa(h)$	$M_h$	$C_{\text{div } b}$	$C_b$	$C_c$	$c$
100	1.0000	2.0132	2.0133	5.09e-5	0.9999	0.0	0.0	10	+10
200	1.0000	2.0132	2.0132	1.27e-5	1.0000	0.0	0.0	10	+10
400	1.0000	2.0135	2.0135	3.18e-6	1.0003	0.0	0.0	10	+10
800	1.0000	2.0248	2.0248	8.01e-7	1.0114	0.0	0.0	10	+10
100	1.0709	77.69	77.84	1.96e-3	75.69	0.0	0.0	10	-10
200	1.0182	77.71	77.75	4.92e-4	75.71	0.0	0.0	10	-10
400	1.0046	78.05	78.06	1.23e-4	76.04	0.0	0.0	10	-10
800	1.0013	83.72	83.72	3.31e-5	81.64	0.0	0.0	10	-10

Table 2

Numerical results for Example 2

$h^{-1}$	$\alpha$	$\beta$	$\sigma$	$\kappa(h)$	$M_h$	$C_{\text{div } b}$	$C_b$	$C_c$
100	1.4245	203.91	134.09	5.85e-2	14.94	51.30	10.00	51.30
200	1.1212	203.88	128.61	1.86e-2	14.94	51.28	10.00	51.28
400	1.0318	204.35	127.36	6.64e-3	14.97	51.28	10.00	51.28
800	1.0092	219.33	136.13	2.70e-3	16.08	51.28	10.00	51.28

Table 3

Numerical results for Example 3

$h^{-1}$	$\dot{\alpha}$	$\dot{\beta}$	$\dot{\sigma}$	$\dot{\kappa}(h)$	$M_h$	$D_{\text{div } b}$	$C_b$	$C_c$
100	1.0260	23.97	9.9857	4.69e-2	2.2296	64.00	12.00	0.0
200	1.0065	23.97	9.7242	2.12e-2	2.2298	64.00	12.00	0.0
400	1.0016	23.97	9.6146	1.00e-2	2.2298	64.00	12.00	0.0
800	1.0004	23.99	9.5719	4.91e-3	2.2318	64.00	12.00	0.0

Next we consider the following 2-dimensional problems.

**Example 4** (*linearized Emden's equation*)

$$\begin{aligned}
-\Delta u - 2\tilde{\phi}_h u &= \frac{\sqrt{5}}{2} \text{ in } \Omega = (0, 1)^2 \setminus [0, \frac{1}{5}]^2, \\
u &= 0 \text{ on } \partial\Omega,
\end{aligned}$$

where  $\tilde{\phi}_h \in S_h$  is an approximation of the following Emden's equation.

$$\begin{aligned} -\Delta\phi &= \phi^2 \text{ in } \Omega, \\ \phi &= 0 \text{ on } \partial\Omega. \end{aligned}$$

### Example 5

$$\begin{aligned} -\Delta u + \tilde{u}_h(\bar{\nabla}\hat{u}_h) \cdot \nabla u - \left(\lambda - \frac{1}{2}|\nabla\tilde{u}_h|^2\right) u &= 1 \text{ in } \Omega = (0,1)^2, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where  $\lambda = 40$  and  $\tilde{u}_h \in S_h$  is an approximation of Plum's example.

$$\begin{aligned} -\Delta\phi &= \phi \left(\lambda - \frac{1}{2}|\nabla\phi|^2\right) \text{ in } \Omega, \\ \phi &= 0 \text{ on } \partial\Omega. \end{aligned}$$

In this example, we considered two cases for the coefficient vector function  $b$ , that is, in case of  $(\bar{\nabla}\hat{u}_h) \equiv \nabla\tilde{u}_h$ , discontinuous, and  $(\bar{\nabla}\hat{u}_h) \equiv (P_0\nabla_x\tilde{u}_h, P_0\nabla_y\tilde{u}_h)$ , where  $\tilde{u}_h$  is an approximate solution in  $S_h$  and  $P_0$  stands for the  $L^2$ -projection into  $S_h^*$  defined in Section 2.

In above two examples, we take the finite element subspace  $S_h$  as piecewise bi-linear functions with uniform mesh. Note that we can take the constant  $C_p$  for  $\Omega = (0,1)^2 \setminus [0, \frac{1}{5}]^2$  and  $\Omega = (0,1)^2$  as  $C_p = \sqrt{10}^{-1}$  and  $C_p = (\sqrt{2}\pi)^{-1}$ , respectively. Moreover, we can obtain the a priori constant  $C(h)$  for the  $L$ -shaped domain by techniques in [7], and it is taken as  $C(h) = \pi^{-1}h$  for bi-linear functions on  $\Omega = (0,1)^2$ . We show validated numerical results for Example 4 in Table 4. Also, for Example 5, we illustrate several numerical results for  $(\bar{\nabla}\hat{u}_h) = \nabla\tilde{u}_h$  and  $(\bar{\nabla}\hat{u}_h) = (P_0\nabla_x\tilde{u}_h, P_0\nabla_y\tilde{u}_h)$  in Tables 5 and 6, respectively. As shown in these tables, the capability for the verification of invertibility seems to be influenced by the smoothness of the function  $b$ .

All computations in these tables are carried out on the Dell Precision 650 Workstation Intel Xeon CPU 3.20GHz using INTLAB, a tool box in MATLAB developed by Rump [6] for self-validating algorithms.

### References

- [1] G. Alefeld, J. Herzberger; *Introduction to Interval Computations*, Academic Press, New York, 1983.
- [2] Nakao, M.T., A numerical approach to the proof of existence of solutions for elliptic problems, *Japan Journal of Applied Mathematics* **5** (1988), 313-332.

Table 4

Numerical results for Example 4

$h^{-1}$	$C(h)$	$\alpha$	$\beta$	$\sigma$	$\kappa(h)$	$M_h$	$C_{\text{div } b}$	$C_b$	$C_c$
10	$1.8433\pi^{-1}h$	3.4498	18.79	Fail	4.0656	2.8320	0.0	0.0	62.83
20	$2.2063\pi^{-1}h$	2.2159	18.80	Fail	1.4244	2.8994	0.0	0.0	61.41
30	$2.4772\pi^{-1}h$	1.7862	18.80	91.57	7.94e-1	2.9118	0.0	0.0	61.15
40	$2.6992\pi^{-1}h$	1.5718	18.85	40.33	5.32e-1	2.9159	0.0	0.0	61.22

Table 5

Numerical results for Example 5 for  $(\bar{\nabla}\hat{u}_h) = \nabla\tilde{u}_h$ 

$h^{-1}$	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\kappa}(h)$	$M_h$	$C_{\text{div } b}$	$C_b$	$C_c$
10	6.2448	Fail	6.1895	1.3365	—	19.21	40.00
20	5.8008	Fail	2.7618	1.3556	—	18.08	40.00
30	5.6214	Fail	1.7563	1.3595	—	17.65	40.00
40	5.5576	Fail	1.2963	1.3608	—	17.52	40.00

Table 6

Numerical results for Example 5 for  $(\bar{\nabla}\hat{u}_h) = (P_0\nabla_x\tilde{u}_h, P_0\nabla_y\tilde{u}_h)$ 

$h^{-1}$	$\alpha$	$\beta$	$\sigma$	$\kappa(h)$	$M_h$	$C_{\text{div } b}$	$C_b$	$C_c$
10	5.5421	39.54	Fail	5.6096	1.6630	330.81	19.51	40.00
20	3.3515	46.23	Fail	1.6841	1.7513	389.06	18.19	40.00
30	2.4286	47.94	62.61	8.14e-1	1.7723	404.84	17.56	40.00
40	1.9588	48.64	22.94	4.95e-1	1.7801	410.88	17.41	40.00

- [3] Nakao, M.T., Yamamoto, N. & Kimura, S., On best constant in the optimal error estimates for the  $H_0^1$ -projection into piecewise polynomial spaces, *Journal of Approximation Theory* **93**, (1998), 491-500.
- [4] Nakao, M.T., Numerical verification methods for solutions of ordinary and partial differential equations, *Numerical Functional Analysis and Optimization* **22** (2001), 321-356.
- [5] M.T. Nakao, K. Hashimoto, Y. Watanabe; A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems, *Computing* **75** (2005), 1-14.
- [6] S.M. Rump; INTLAB-INTerval LABoratory, a Matlab toolbox for verified computations, Inst. Infomatik, Technical University Hamburg-Harburg.
- [7] N. Yamamoto, M.T. Nakao; Numerical verifications of solutions for elliptic equations in nonconvex polygonal domains, *Numer. Math.* **65** (1993), 503-521.

# List of MHF Preprint Series, Kyushu University

## 21st Century COE Program

### Development of Dynamic Mathematics with High Functionality

- MHF2005-1 Hideki KOSAKI  
Matrix trace inequalities related to uncertainty principle
- MHF2005-2 Masahisa TABATA  
Discrepancy between theory and real computation on the stability of some finite element schemes
- MHF2005-3 Yuko ARAKI & Sadanori KONISHI  
Functional regression modeling via regularized basis expansions and model selection
- MHF2005-4 Yuko ARAKI & Sadanori KONISHI  
Functional discriminant analysis via regularized basis expansions
- MHF2005-5 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA  
Point configurations, Cremona transformations and the elliptic difference Painlevé equations
- MHF2005-6 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA  
Construction of hypergeometric solutions to the  $q$ -Painlevé equations
- MHF2005-7 Hiroki MASUDA  
Simple estimators for non-linear Markovian trend from sampled data:  
I. ergodic cases
- MHF2005-8 Hiroki MASUDA & Nakahiro YOSHIDA  
Edgeworth expansion for a class of Ornstein-Uhlenbeck-based models
- MHF2005-9 Masayuki UCHIDA  
Approximate martingale estimating functions under small perturbations of dynamical systems
- MHF2005-10 Ryo MATSUZAKI & Masayuki UCHIDA  
One-step estimators for diffusion processes with small dispersion parameters from discrete observations
- MHF2005-11 Junichi MATSUKUBO, Ryo MATSUZAKI & Masayuki UCHIDA  
Estimation for a discretely observed small diffusion process with a linear drift
- MHF2005-12 Masayuki UCHIDA & Nakahiro YOSHIDA  
AIC for ergodic diffusion processes from discrete observations

- MHF2005-13 Hiromichi GOTO & Kenji KAJIWARA  
Generating function related to the Okamoto polynomials for the Painlevé IV equation
- MHF2005-14 Masato KIMURA & Shin-ichi NAGATA  
Precise asymptotic behaviour of the first eigenvalue of Sturm-Liouville problems with large drift
- MHF2005-15 Daisuke TAGAMI & Masahisa TABATA  
Numerical computations of a melting glass convection in the furnace
- MHF2005-16 Raimundas VIDŪNAS  
Normalized Leonard pairs and Askey-Wilson relations
- MHF2005-17 Raimundas VIDŪNAS  
Askey-Wilson relations and Leonard pairs
- MHF2005-18 Kenji KAJIWARA & Atsushi MUKAIHIRA  
Soliton solutions for the non-autonomous discrete-time Toda lattice equation
- MHF2005-19 Yuu HARIYA  
Construction of Gibbs measures for 1-dimensional continuum fields
- MHF2005-20 Yuu HARIYA  
Integration by parts formulae for the Wiener measure restricted to subsets in  $\mathbb{R}^d$
- MHF2005-21 Yuu HARIYA  
A time-change approach to Kotani's extension of Yor's formula
- MHF2005-22 Tadahisa FUNAKI, Yuu HARIYA & Mark YOR  
Wiener integrals for centered powers of Bessel processes, I
- MHF2005-23 Masahisa TABATA & Satoshi KAIZU  
Finite element schemes for two-fluids flow problems
- MHF2005-24 Ken-ichi MARUNO & Yasuhiro OHTA  
Determinant form of dark soliton solutions of the discrete nonlinear Schrödinger equation
- MHF2005-25 Alexander V. KITAEV & Raimundas VIDŪNAS  
Quadratic transformations of the sixth Painlevé equation
- MHF2005-26 Toru FUJII & Sadanori KONISHI  
Nonlinear regression modeling via regularized wavelets and smoothing parameter selection
- MHF2005-27 Shuichi INOKUCHI, Kazumasa HONDA, Hyen Yeal LEE, Tatsuro SATO, Yoshihiro MIZOGUCHI & Yasuo KAWAHARA  
On reversible cellular automata with finite cell array

- MHF2005-28 Toru KOMATSU  
Cyclic cubic field with explicit Artin symbols
- MHF2005-29 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Kaori NAGATOU  
A computational approach to constructive a priori and a posteriori error estimates for finite element approximations of bi-harmonic problems
- MHF2005-30 Kaori NAGATOU, Kouji HASHIMOTO & Mitsuhiro T. NAKAO  
Numerical verification of stationary solutions for Navier-Stokes problems
- MHF2005-31 Hidefumi KAWASAKI  
A duality theorem for a three-phase partition problem
- MHF2005-32 Hidefumi KAWASAKI  
A duality theorem based on triangles separating three convex sets
- MHF2005-33 Takeaki FUCHIKAMI & Hidefumi KAWASAKI  
An explicit formula of the Shapley value for a cooperative game induced from the conjugate point
- MHF2005-34 Hideki MURAKAWA  
A regularization of a reaction-diffusion system approximation to the two-phase Stefan problem
- MHF2006-1 Masahisa TABATA  
Numerical simulation of Rayleigh-Taylor problems by an energy-stable finite element scheme
- MHF2006-2 Ken-ichi MARUNO & G R W QUISPEL  
Construction of integrals of higher-order mappings
- MHF2006-3 Setsuo TANIGUCHI  
On the Jacobi field approach to stochastic oscillatory integrals with quadratic phase function
- MHF2006-4 Kouji HASHIMOTO, Kaori NAGATOU & Mitsuhiro T. NAKAO  
A computational approach to constructive a priori error estimate for finite element approximations of bi-harmonic problems in nonconvex polygonal domains
- MHF2006-5 Hidefumi KAWASAKI  
A duality theory based on triangular cylinders separating three convex sets in  $R^n$
- MHF2006-6 Raimundas VIDŪNAS  
Uniform convergence of hypergeometric series
- MHF2006-7 Yuji KODAMA & Ken-ichi MARUNO  
N-Soliton solutions to the DKP equation and Weyl group actions

- MHF2006-8 Toru KOMATSU  
Potentially generic polynomial
- MHF2006-9 Toru KOMATSU  
Generic sextic polynomial related to the subfield problem of a cubic polynomial
- MHF2006-10 Shu TEZUKA & Anargyros PAPAGEORGIOU  
Exact cubature for a class of functions of maximum effective dimension
- MHF2006-11 Shu TEZUKA  
On high-discrepancy sequences
- MHF2006-12 Raimundas VIDŪNAS  
Detecting persistent regimes in the North Atlantic Oscillation time series
- MHF2006-13 Toru KOMATSU  
Tamely Eisenstein field with prime power discriminant
- MHF2006-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCO  
Generating function associated with the Hankel determinant formula for the solutions of the Painlevé IV equation
- MHF2006-15 Raimundas VIDŪNAS  
Darboux evaluations of algebraic Gauss hypergeometric functions
- MHF2006-16 Masato KIMURA & Isao WAKANO  
New mathematical approach to the energy release rate in crack extension
- MHF2006-17 Toru KOMATSU  
Arithmetic of the splitting field of Alexander polynomial
- MHF2006-18 Hiroki MASUDA  
Likelihood estimation of stable Lévy processes from discrete data
- MHF2006-19 Hiroshi KAWABI & Michael RÖCKNER  
Essential self-adjointness of Dirichlet operators on a path space with Gibbs measures via an SPDE approach
- MHF2006-20 Masahisa TABATA  
Energy stable finite element schemes and their applications to two-fluid flow problems
- MHF2006-21 Yuzuru INAHAMA & Hiroshi KAWABI  
Asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths
- MHF2006-22 Yoshiyuki KAGEI  
Resolvent estimates for the linearized compressible Navier-Stokes equation in an infinite layer

- MHF2006-23 Yoshiyuki KAGEI  
Asymptotic behavior of the semigroup associated with the linearized compressible Navier-Stokes equation in an infinite layer
- MHF2006-24 Akihiro MIKODA, Shuichi INOKUCHI, Yoshihiro MIZOGUCHI & Mitsuhiro FUJIO  
The number of orbits of box-ball systems
- MHF2006-25 Toru FUJII & Sadanori KONISHI  
Multi-class logistic discrimination via wavelet-based functionalization and model selection criteria
- MHF2006-26 Taro HAMAMOTO, Kenji KAJIWARA & Nicholas S. WITTE  
Hypergeometric solutions to the  $q$ -Painlevé equation of type  $(A_1 + A'_1)^{(1)}$
- MHF2006-27 Hiroshi KAWABI & Tomohiro MIYOKAWA  
The Littlewood-Paley-Stein inequality for diffusion processes on general metric spaces
- MHF2006-28 Hiroki MASUDA  
Notes on estimating inverse-Gaussian and gamma subordinators under high-frequency sampling
- MHF2006-29 Setsuo TANIGUCHI  
The heat semigroup and kernel associated with certain non-commutative harmonic oscillators
- MHF2006-30 Setsuo TANIGUCHI  
Stochastic analysis and the KdV equation
- MHF2006-31 Masato KIMURA, Hideki KOMURA, Masayasu MIMURA, Hidenori MIYOSHI, Takeshi TAKAISHI & Daishin UEYAMA  
Quantitative study of adaptive mesh FEM with localization index of pattern
- MHF2007-1 Taro HAMAMOTO & Kenji KAJIWARA  
Hypergeometric solutions to the  $q$ -Painlevé equation of type  $A_4^{(1)}$
- MHF2007-2 Kouji HASHIMOTO, Kenta KOBAYASHI & Mitsuhiro T. NAKAO  
Verified numerical computation of solutions for the stationary Navier-Stokes equation in nonconvex polygonal domains
- MHF2007-3 Kenji KAJIWARA, Marta MAZZOCCO & Yasuhiro OHTA  
A remark on the Hankel determinant formula for solutions of the Toda equation
- MHF2007-4 Jun-ichi SATO & Hidefumi KAWASAKI  
Discrete fixed point theorems and their application to Nash equilibrium
- MHF2007-5 Mitsuhiro T. NAKAO & Kouji HASHIMOTO  
Constructive error estimates of finite element approximations for non-coercive elliptic problems and its applications