Guaranteed error bounds for finite element approximations of noncoercive elliptic problems and their applications

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Constructive error estimates of finite element approximations for non-coercive elliptic problems and its applications

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Abstract

In this paper, we show constructive a priori and a posteriori error estimates of finite element approximations for not necessary coercive linear second order Dirichlet problems. Here, 'constructive' means we can get the error bounds in which all constants included are explicitly given or represented as a numerically computable form. Using the invertibility condition of concerning elliptic operator, constructive a priori and a posteriori error estimates are formulated. This kind of estimates plays essential and important roles in the numerical verification of solutions for nonlinear elliptic problems. Several numerical examples that confirm the actual effectiveness of the method are presented.

Key words: Constructive a priori and a posteriori error estimates, linear elliptic problem.

1 Introduction

In this paper, we consider the constructive a priori and a posteriori error estimates for the general linear elliptic boundary value problem of the form

\[ \mathcal{L}u \equiv -\Delta u + b \cdot \nabla u + cu = f \quad \text{in} \quad \Omega, \]
\[ u = 0 \quad \text{on} \quad \partial\Omega, \]

(1.1)

where \( f \in L^2(\Omega) \). Here, for \( n = 1, 2, 3 \), we assume that \( b \in (W^{\infty}_1(\Omega))^n \), \( c \in L^\infty(\Omega) \), where \( \Omega \subset R^n \) is a bounded open domain with piecewise smooth boundary. In this paper, we use the terminology 'constructive error estimates'
means an error estimation that by some numerical computations based on the estimates, we can obtain the true error bounds between the exact solution and its approximation in mathematically rigorous sense, even if the concerning problem (1.1) is not coercive. This kind of estimations should be useful when the existence or uniqueness of solutions are not a priori assured, e.g., in case that the coefficient function $c$ is not nonnegative. And it also be impor-
tant for the numerical verification of solutions for nonlinear boundary value problems(e.g., [2] [4] [5]etc.).

Now, we denote the usual $k$-th order Sobolev space on $\Omega$ by $H^k(\Omega)$ and define $(\cdot, \cdot)_{L^2}$ as the $L^2$ inner product. And we set $H^1_0(\Omega) \equiv \{v \in H^1(\Omega) ; v = 0 \text{ on } \partial \Omega\}$ with the inner product $(\nabla u , \nabla v)_{L^2}$ for $u, v \in H^1_0(\Omega)$ . Also, define $X(\Omega) \equiv \{v \in H^1(\Omega) ; \Delta v \in L^2(\Omega)\}$.

We now introduce the finite dimensional subspace $S_h$ of $H^1_0(\Omega)$ depending on the parameter $h$ with nodal functions $\{\phi_i\}_{1 \leq i \leq N}$. For each $v \in H^1_0(\Omega)$, define the $H^1_0$-projection $P_h v \in S_h$ by

$$(\nabla (v - P_h v), \nabla \phi_h)_{L^2} = 0, \quad \forall \phi_h \in S_h.$$  

Also, corresponding to the usual finite element approximations of a solution $u$ in (1.1), we define the $L$-projection $P_L v \in S_h$, whose existence is assumed, by

$$a(v - P_L v, \phi_h)_{L^2} = 0, \quad \forall \phi_h \in S_h,$$

where $a(u, v) \equiv (\nabla u, \nabla v)_{L^2} + (b \cdot \nabla u, v)_{L^2} + (cu, v)_{L^2}$. Further, we assume that there exists a positive constant $C(h)$ which can be numerically estimated satisfying, for any $u \in H^1_0(\Omega) \cap X(\Omega)$ ,

$$\|u - P_h u\|_{H^1_0} \leq C(h) \|\Delta u\|_{L^2}.$$  

Note that (1.3) is equivalent to the following estimation.

$$\|u - P_h u\|_{L^2} \leq C(h) \|u - P_h u\|_{H^1_0}.$$  

Then our main purpose of this paper is to determine explicitly a priori constants $K_0(h)$ and $K_1(h)$ satisfying

$$\|u - P_L u\|_{L^2} \leq K_0(h) \|Lu\|_{L^2},$$

$$\|u - P_L u\|_{H^1_0} \leq K_1(h) \|Lu\|_{L^2},$$

respectively. Also we show a constant $K(h)$ satisfying

$$\|u - P_L u\|_{L^2} \leq K(h) \|u - P_L u\|_{H^1_0}.$$
Defining the compact operator $A : H^1_0 \rightarrow H^1_0$ by $Au := \Delta^{-1}(b \cdot \nabla u + cu)$, where $\Delta^{-1}$ stands for the solution operator of the Poisson equation with homogeneous boundary condition, the invertibility of the elliptic operator $\mathcal{L}$ defined in (1.1) is equivalent to the unique solvability of the following fixed point equation:

$$u = Au.$$ 

As the preliminary, we define $N \times N$ matrices $G = (G_{i,j})$ and $D = (D_{i,j})$ by

$$G_{i,j} = (\nabla \phi_j, \nabla \phi_i)_{L^2} + (b \cdot \nabla \phi_j, \phi_i)_{L^2} + (c \phi_j, \phi_i)_{L^2},$$

$$D_{i,j} = (\nabla \phi_j, \nabla \phi_i)_{L^2},$$

Note that $D$ is symmetric and positive definite. We denote the matrix norm by $\| \cdot \|_E$ induced from the Euclidean norm $| \cdot |_E$. Also, we define the following constants:

$$C_1 = C_p \text{div } b + C_b, \quad C_3 = C_b + C_p C_c,$$

$$C_2 = C_p C_c, \quad C_4 = C_b + C(h) C_c,$$

$$C_{\text{div } b} = \| \text{div } b \|_{L^\infty}, \quad C_b = \| |b|_E \|_{L^\infty}, \quad C_c = \| c \|_{L^\infty},$$

where $\| \cdot \|_{L^\infty}$ means $L^\infty$ norm on $\Omega$ and $C_p$ is a Poincaré constant such that $\| \phi \|_{L^2} \leq C_p \| \phi \|_{H^1_0}$ for an arbitrary $\phi \in H^1_0(\Omega)$.

In [5], authors show the following results.

**Theorem 1** If the matrix $G$ is nonsingular, and for the constants defined above,

$$\kappa(h) \equiv C(h)\left(C(h)M_h(C_1 + C_2)C_3 + C_4\right) < 1$$

holds, then the operator $\mathcal{L}$ defined in (1.1) is invertible. Here, $M_h \equiv \| D^{1/2} G^{-1} D^{1/2} \|_E$ and $C(h)$ is the same constant as in (1.3).

Moreover, we have the following a priori estimate for the $H^1_0$-projection.

**Theorem 2** Assuming that same conditions in Theorem 1, let $u \in H^1_0(\Omega) \cap X(\Omega)$ be a unique solution of (1.1). Then we have

$$\| u - P_h u \|_{H^1_0} \leq C(h)\sigma \| f \|_{L^2},$$

where the constant $\sigma$ is given by $\sigma = (1 + C_p M_h C_3)(1 - \kappa(h))^{-1}$.

When the coefficient vector function $b$ in (1.1) is not differentiable, we have the following alternative results.

**Corollary 3** Let $b \in (L^\infty(\Omega))^n$. If

$$\hat{\kappa}(h) \equiv C(h)\left(M_h(\hat{C}_1 + C(h) C_2)C_3 + C_4\right) < 1$$
holds, then the operator $\mathcal{L}$ defined in (1.1) is invertible. Here, $\hat{C}_1 = C_p C_b$.

Also we have

$$
\|u - P_h u\|_{H_0^1} \leq C(h) \hat{\sigma} \|f\|_{L^2},
$$

for a unique solution of $\mathcal{L}u = f$, where the constant $\hat{\sigma}$ is given by

$$
\hat{\sigma} = (1 + C_p M_h C_3)(1 - \hat{\kappa}(h))^{-1}.
$$

2 Main results

In this section, we show the constructive a priori and a posteriori error estimates of finite element approximations (1.2) for linear elliptic problems (1.1). Note that the existence of the inverse $\mathcal{L}^{-1} : L^2(\Omega) \rightarrow X(\Omega)$ is equivalent to the invertibility of $I - A$, where $I$ denotes the identity operator in $H_0^1(\Omega)$. Using this fact, we first show the a priori error estimate between a solution of our problems and its $H_0^1$-projection. First, we show the following lemma.

Lemma 4 (cf. [5]) For an arbitrary $v \in H_0^1(\Omega)$, we have

$$
\|Av\|_{H_0^1} \leq (C_1 + C_2) \|v\|_{L^2},
$$

$$
\|(I - P_h) Av\|_{H_0^1} \leq C(h) \left( C_3 \|P_h v\|_{H_0^1} + C_4 \|v - P_h v\|_{H_0^1} \right).
$$

Proof: Let $\psi := -Av = -\Delta^{-1}(b \cdot \nabla + c)v \in H_0^1(\Omega) \cap X(\Omega)$. Then we have

$$
\|\psi\|_{H_0^1}^2 = (-\Delta \psi, \psi)_{L^2}
$$

$$
= (v, \text{div} (b \psi))_{L^2} + (v, c \psi)_{L^2}
$$

$$
\leq \left( \|\text{div} (b \psi)\|_{L^2} + \|c \|_{L^2} \right) \|v\|_{L^2}
$$

$$
\leq C(h) \left( \|\text{div} b\|_{L^\infty} \|\psi\|_{L^2} + \|b\|_{L^\infty} \|\psi\|_{H_0^1} + \|c\|_{L^\infty} \|\psi\|_{L^2} \right) \|v\|_{H_0^1},
$$

where we have used (1.4). Moreover, we have

$$
\|(I - P_h) Av\|_{H_0^1} = \|(I - P_h) \Delta^{-1}(b \cdot \nabla + c)v\|_{H_0^1}
$$

$$
\leq C(h) \|(b \cdot \nabla + c)v\|_{L^2}
$$

$$
\leq C(h) \left( \|b\|_{L^\infty} \|v\|_{H_0^1} + \|c\|_{L^\infty} \|v\|_{L^2} \right),
$$

where we have used (1.3). Therefore, this proof is completed.  

For the $\mathcal{L}$-projection, we have the following one of the main results of this paper.
**Theorem 5** For an arbitrary \( v \in H_0^1(\Omega) \), if \( G \) is nonsingular, then for the same constants in Theorem 1, we have

\[
\|v - P_L v\|_{H_0^1} \leq \alpha \|v - P_h v\|_{H_0^1},
\]
\[
\|v - P_L v\|_{L^2} \leq C(h)\beta \|v - P_h v\|_{H_0^1} \leq C(h)\beta \|v - P_L v\|_{H_0^1},
\]

where \( \alpha \equiv \sqrt{1 + (C(h)M_h(C_1 + C_2))^2} \), \( \beta \equiv 1 + C_pM_h(C_1 + C_2) \).

**Proof:** From the property of the \( H_0^1 \)- and \( L \)-projections, we can obtain

\[
\|v - P_L v\|_{H_0^1}^2 = \|v - P_h v\|_{H_0^1}^2 + \|P_L v - P_h v\|_{H_0^1}^2,
\]

(2.1)

for an arbitrary \( v \in H_0^1(\Omega) \). Let \( e \equiv v - P_h v \).

Then since \( P_L v - P_h v = P_L(v - P_h v) \), for all \( \phi_h \in S_h \), we have

\[
a(P_L e, \phi_h) = (\nabla e, \nabla \phi_h)_{L^2} + ((b \cdot \nabla + c)e, \phi_h)_{L^2}
\]
\[
= (b \cdot \nabla e + ce, \phi_h)_{L^2}
\]
\[
= (\nabla P_h \psi, \nabla \phi_h)_{L^2},
\]

where we set \( \psi \equiv -Ae = -\Delta^{-1}(b \cdot \nabla + c)e \). It implies that

\[
G \vec{e}_h = D\vec{\psi}_h,
\]

where \( \vec{e}_h \) and \( \vec{\psi}_h \) are coefficient vectors of \( P_L e \) and \( P_h \psi \), respectively. Thus in the similar way to the proof of Lemma 4, we can obtain the following estimate since \( \|P_L e\|_{H_0^1} = \|D^\frac{1}{2} \vec{e}_h\|_E \), \( \|P_h \psi\|_{H_0^1} = \|D^\frac{1}{2} \vec{\psi}_h\|_E \) and \( \|P_h \psi\|_{H_0^1} \leq \|\psi\|_{H_0^1} \) for any \( \psi \in H_0^1(\Omega) \).

\[
\|P_L v - P_h v\|_{H_0^1} = \|P_L e\|_{H_0^1} \leq M_h\|P_h \psi\|_{H_0^1}
\]
\[
\leq M_h\|A(v - P_h v)\|_{H_0^1}
\]
\[
\leq C(h)M_h(C_1 + C_2)\|v - P_h v\|_{H_0^1}.
\]

Moreover, we have

\[
\|P_L v - P_h v\|_{L^2} \leq C_p\|P_L v - P_h v\|_{H_0^1}
\]
\[
\leq C(h)C_pM_h(C_1 + C_2)\|v - P_h v\|_{H_0^1}.
\]

Hence we can obtain the following estimate.

\[
\|v - P_L v\|_{L^2} \leq \|v - P_h v\|_{L^2} + \|P_L v - P_h v\|_{L^2}
\]
\[
\leq C(h)\|v - P_h v\|_{H_0^1} + C(h)C_pM_h(C_1 + C_2)\|v - P_h v\|_{H_0^1}.
\]
where we have used (1.4). Therefore, the proof is completed from (2.1).

Note that the constant $\alpha$ in Theorem 5 tends to 1 if $h \to 0$ as illustrated in Figure 1.

![Image](https://via.placeholder.com/150)

Fig. 1. Image of the $H^1_0$- and $L$-projections

Now, as in [7], let $S^*_h$ be an appropriate finite element subspace of $H^1(\Omega)$ satisfying $S_h \subset S^*_h$, and let define $(\nabla u_h) \equiv (P_0 \nabla x u_h, P_0 \nabla y u_h, P_0 \nabla z u_h) \in (S^*_h)^n$, where $P_0 : L^2(\Omega) \rightarrow S^*_h$ means the $L^2$-projection defined by, for each $v \in L^2(\Omega)$,

$$(v - P_0 v, \phi^*_h)_{L^2} = 0 \quad \text{for any } \phi^*_h \in S^*_h.$$ 

Also note that, for the problem (1.1), the finite element solution $u_h$ defined by

$$(\nabla u_h, \nabla \phi_h)_{L^2} + (b \cdot \nabla u_h + cu_h, \phi_h)_{L^2} = (f, \phi_h)_{L^2}, \quad \forall \phi_h \in S_h, \quad (2.2)$$

coincides with the $L$-projection $P_L u$.

Now, by using Theorems 1, 2 and 5, we have the following constructive a priori and a posteriori error estimates for linear elliptic problems.

**Theorem 6** Assuming that Theorem 1 holds, then for a unique solution of $\mathcal{L} u = f$, we have

$$\|u - P_L u\|_{H^1_0} \leq C(h)\alpha \sigma \|f\|_{L^2},$$

$$\|u - P_L u\|_{L^2} \leq C(h)^2 \beta \sigma \|f\|_{L^2}.$$ 

And we have the following a posteriori error estimate for the finite element solution $u_h$ defined by (2.2).

$$\|u - u_h\|_{H^1_0} \leq \|R\|_{L^2} + C(h)\beta \|S\|_{L^2} + C(h)^2 \beta \sigma \left(C_b + C(h)C_c\beta\right) \|f\|_{L^2},$$

where $R \equiv \nabla u_h - (\nabla u_h)$ and $S \equiv f + \text{div}(\nabla u_h) - b \cdot \nabla u_h - cu_h$. 

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Proof: From Theorems 2 and 5, we can easily obtain the following inequalities.

\[
\|u - P_L u\|_{H^1_0} \leq C(h)\alpha \sigma \|f\|_{L^2},
\]

\[
\|u - P_L u\|_{L^2} \leq C(h)^2 \beta \sigma \|f\|_{L^2}.
\]

Thus we consider the a posteriori error estimate below.

Let \(e \equiv u - u_h\).

\[
\|u - u_h\|_{H^1_0}^2 = (\nabla e, \nabla u)_{L^2} - (\nabla e, \nabla u_h)_{L^2}
\]

\[
= (e, f)_{L^2} - (e, b \cdot \nabla u + cu)_{L^2} - (\nabla e, \nabla u_h)_{L^2}
\]

\[
= (e, f - b \cdot \nabla u_h + cu_h)_{L^2} - (e, b \cdot \nabla e + ce)_{L^2} - (\nabla e, \nabla u_h)_{L^2}.
\]

Since 
\[
((\nabla u_h), \nabla v)_{L^2} = (-\text{div}(\nabla u_h), v)_{L^2}
\]

for any \(v \in H^1_0(\Omega)\), taking as \(v = e\), it implies that

\[
\|u - u_h\|_{H^1_0}^2 = (e, S)_{L^2} - (e, b \cdot \nabla e + ce)_{L^2} - (\nabla e, R)_{L^2}
\]

\[
\leq \|e\|_{L^2} \|S\|_{L^2} + \|e\|_{L^2} \|b \cdot \nabla e + ce\|_{L^2} + \|e\|_{H^1_0} \|R\|_{L^2}.
\]

Moreover, using Lemma 4, we have

\[
\|b \cdot \nabla e + ce\|_{L^2} \leq \|b|_E \|L_\infty\| e\|_{H^1_0} + \|c\|_{L_\infty}\| e\|_{L^2}.
\]

Hence using the fact \(\|e\|_{L^2} \leq C(h)^2 \|e\|_{H^1_0}\) in Theorem 5, we have the estimate (2.3). Therefore, this proof is completed.

Remark. The last term in the estimates (2.3) looks like an a priori estimation. However, since the order \(C(h)^2\) is higher than the usual optimal estimation in \(H^1\) norm, combining it with the first and second terms, this estimates can be considered as a kind of a posteriori error estimates.

From Theorems 5 - 6, we can take the constants \(K_0(h), K_1(h)\) and \(K(h)\) as

\[
K_0(h) := C(h)^2 \beta \sigma,\quad K_1(h) := C(h)\alpha \sigma,\quad K(h) := C(h)^2 \beta.
\]

Also we have the following estimates corresponding to Corollary 3.

Corollary 7 Let \(b \in (L_\infty(\Omega))^n\). Under the same assumptions in Corollary 3, we have

\[
\|u - P_L u\|_{H^1_0} \leq C(h)\alpha \hat{\sigma} \|f\|_{L^2},
\]

for a unique solution of \(Lu = f\), where \(\hat{\alpha} \equiv \sqrt{1 + \left(M_h(\hat{C}_1 + C(h)C_2)C_3\right)^2}\).

For usual finite element approximations in the one dimensional case, we can get the better estimates, even if the function \(b\) has no smoothness.
Lemma 8  Let $S_h$ be a finite element subspace of $H^1_0(\Omega)$, where $\Omega = (p,q)$ is an interval in $\mathbb{R}^1$, comprising piecewise polynomials with the mesh $p = x_0 < x_1 < \cdots < x_N < x_{N+1} = q$.

For an arbitrary $v \in H^1_0(\Omega)$, if $b \in \Lambda^N_{i=0} W^1_\infty(I_i) \subset L^\infty(\Omega)$ then we have

$$\|A(v - P_h v)\|_{H^1_0} \leq (D_1 + C_2)\|v - P_h v\|_{L^2},$$

where $D_1 = C_p D_{\text{div}} + C_b$, $D_{\text{div}} b = \max_{0 \leq i \leq N} \|b\|_{W^1_\infty(I_i)}$ and $I_i := (x_i, x_{i+1})$.

Proof: Let $\psi \equiv \Delta^{-1}(be' + ce)$, where $e := v - P_h v$. Then it implies that

$$\|\psi\|^2_{H^1_0} = (\psi', \psi')_{L^2} = (be' + ce, \psi)_{L^2} = (e', b\psi)_{L^2} + (e, c\psi)_{L^2}$$

Note that the $H^1_0$-projection satisfies $e(x_i) = 0$ for $i = 0, \cdots, N + 1$. Hence we have

$$(e', b\psi)_{L^2} = \sum_i (e, (b\psi)'_{L^2(I_i)})$$

$$\leq \sum_i \|e\|_{L^2(I_i)} \|(b\psi)'\|_{L^2(I_i)}$$

$$\leq \sum_i \|e\|_{L^2(I_i)} \left(\|b\|_{W^1_{\infty}(I_i)} \|\psi\|_{L^2(I_i)} + \|[b]\|_{L^\infty(I_i)} \|\psi'\|_{L^2(I_i)}\right)$$

$$\leq D_{\text{div}} b \sum_i \|e\|_{L^2(I_i)} \|\psi\|_{L^2(I_i)} + C_b \sum_i \|e\|_{L^2(I_i)} \|\psi'\|_{L^2(I_i)}$$

$$\leq \left( D_{\text{div}} b \|e\|_{L^2} + C_b \|\psi\|_{H^1_0} \right) \|e\|_{L^2}$$

$$(e, c\psi)_{L^2} \leq C_c \|e\|_{L^2} \|\psi\|_{L^2}.$$  Therefore, the proof is completed.

Applying similar arguments in Theorems 5 - 6 with the above lemma, we have the following results for a special case.

Theorem 9  Under the same assumption in Lemma 8, if $G$ is nonsingular then we have

$$\|v - P_L v\|_{H^1_0} \leq \hat{\alpha} \|v - P_h v\|_{H^1_0},$$

$$\|v - P_L v\|_{L^2} \leq C(h) \hat{\beta} \|v - P_h v\|_{H^1_0},$$

where $\hat{\alpha} \equiv \sqrt{1 + \left( C(h) M_h (D_1 + C_2) \right)^2}$ and $\hat{\beta} \equiv 1 + C_p M_h (D_1 + C_2)$. Moreover, if

$$\hat{\kappa}(h) \equiv C(h) \left( C(h) M_h (D_1 + C_2) C_3 + C_4 \right) < 1$$

holds, then the operator $L$ is invertible, and we have the following a priori error estimate for a unique solution of $Lu = f$.

$$\|u - P_L u\|_{H^1_0} \leq C(h) \hat{\alpha} \hat{\sigma} \|f\|_{L^2},$$

$$\|u - P_L u\|_{L^2} \leq C(h)^2 \hat{\beta} \hat{\sigma} \|f\|_{L^2},$$
where $\dot{\sigma} = (1 + C_p M_h C_3)(1 - \dot{k}(h))^{-1}$.

3 Numerical examples

In this section, we show several numerical results for linear elliptic problems. In the below, the 1-dimensional problems are presented in the examples 1-3 and 2-dimensional cases in 4-5.

Example 1 (nearly singular problem)

$$-u'' + cu = 1 \text{ in } \Omega = (0,1),$$

$$u = 0 \text{ on } \partial\Omega,$$

where $c = \pm 10$. Note that if $c = -\pi^2 = -9.8696 \cdots$ then this example has no solution.

Example 2 (linearized Burgers equation)

$$-u'' + \lambda(\tilde{\phi}_h + 2x - 1)u' + \lambda(\tilde{\phi}_h + 2x - 1)'u = 1 \text{ in } \Omega = (0,1),$$

$$u = 0 \text{ on } \partial\Omega,$$

where $\lambda = 10$ and $\tilde{\phi}_h \in S_h$ is an approximation of the following Burgers equation.

$$\phi'' = \lambda\phi\phi' \text{ in } \Omega,$$

$$\phi(0) = -1, \quad \phi(1) = 1.$$

Moreover, as a special case, we consider the following example.

Example 3 (discontinuous coefficient)

$$-u'' + bu' = 1 \text{ in } \Omega = (0,1),$$

$$u = 0 \text{ on } \partial\Omega,$$

where $b \in L^\infty(\Omega)$ is given by

$$b \equiv b(x) = \begin{cases} 
4(8x^2 - x)' & = 4(16x - 1) \text{ if } x \in (0,0.25), \\
2(16x^2 - 14x + 3)' & = 4(16x - 7) \text{ if } x \in (0.25,0.5), \\
2(2x - 1)' & = 4 \quad \text{if } x \in (0.5,0.75), \\
4(1 - x)' & = -4 \quad \text{if } x \in (0.75,1).
\end{cases}$$
In above examples, we take the finite element subspace $S_h$ as piecewise quadratic functions with uniform mesh. Then it can be taken as $C(h) = (2\pi)^{-1}h$ ([3]) for piecewise quadratic functions on $\Omega = (0, 1)$ and $C_p = \pi^{-1}$.

We show validated numerical results using interval techniques ([1]) for Examples 1, 2 and 3 in Tables 1, 2 and 3, respectively.

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<td>1.0065</td>
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<tr>
<td>400</td>
<td>1.0016</td>
</tr>
<tr>
<td>800</td>
<td>1.0004</td>
</tr>
</tbody>
</table>

Next we consider the following 2-dimensional problems.

**Example 4** (linearized Emden’s equation)

\[-\Delta u - 2\tilde{\phi}_h u = \frac{\sqrt{5}}{2} \text{ in } \Omega = (0, 1)^2 \setminus [0, \frac{1}{5}]^2,\]

\[u = 0 \text{ on } \partial \Omega,\]
where $\phi_h \in S_h$ is an approximation of the following Emden’s equation.

\[-\Delta \phi = \phi^2 \text{ in } \Omega, \]
\[\phi = 0 \text{ on } \partial \Omega.\]

**Example 5**

\[-\Delta u + \bar{u}_h(\nabla \hat{u}_h) \cdot \nabla u - \left( \lambda - \frac{1}{2} |\nabla \tilde{u}_h|^2 \right) u = 1 \text{ in } \Omega = (0, 1)^2, \]
\[u = 0 \text{ on } \partial \Omega,\]

where $\lambda = 40$ and $\bar{u}_h \in S_h$ is an approximation of Plum’s example.

\[-\Delta \phi = \phi \left( \lambda - \frac{1}{2} |\nabla \phi|^2 \right) \text{ in } \Omega, \]
\[\phi = 0 \text{ on } \partial \Omega.\]

In this example, we considered two cases for the coefficient vector function $b$, that is, in case of $(\nabla \bar{u}_h) \equiv \nabla \hat{u}_h$, discontinuous, and $(\nabla \bar{u}_h) \equiv (P_0 \nabla_x \bar{u}_h, P_0 \nabla_y \bar{u}_h)$, where $\bar{u}_h$ is an approximate solution in $S_h$ and $P_0$ stands for the $L^2$-projection into $S^*_h$ defined in Section 2.

In above two examples, we take the finite element subspace $S_h$ as piecewise bi-linear functions with uniform mesh. Note that we can take the constant $C_p$ for $\Omega = (0, 1)^2 \setminus [0, \frac{1}{5}]^2$ and $\Omega = (0, 1)^2$ as $C_p = \sqrt{10}^{-1}$ and $C_p = (\sqrt{2\pi})^{-1}$, respectively. Moreover, we can obtain the a priori constant $C(h)$ for the L-shaped domain by techniques in [7], and it is taken as $C(h) = \pi^{-1}h$ for bi-linear functions on $\Omega = (0, 1)^2$. We show validated numerical results for Example 4 in Table 4. Also, for Example 5, we illustrate several numerical results for $(\nabla \bar{u}_h) = \nabla \hat{u}_h$ and $(\nabla \bar{u}_h) = (P_0 \nabla_x \bar{u}_h, P_0 \nabla_y \bar{u}_h)$ in Tables 5 and 6, respectively. As shown in these tables, the capability for the verification of invertibility seems to be influenced by the smoothness of the function $b$.

All computations in these tables are carried out on the Dell Precision 650 Workstation Intel Xeon CPU 3.20GHz using INTLAB, a tool box in MATLAB developed by Rump [6] for self-validating algorithms.

**References**


Table 4
Numerical results for Example 4

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<tr>
<th>$h^{-1}$</th>
<th>$C(h)$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\kappa(h)$</th>
<th>$M_h$</th>
<th>$C_{\text{div} b}$</th>
<th>$C_b$</th>
<th>$C_c$</th>
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</thead>
<tbody>
<tr>
<td>10</td>
<td>1.8433 $\pi^{-1}h$</td>
<td>3.4498</td>
<td>18.79</td>
<td>Fail</td>
<td>4.0656</td>
<td>2.8320</td>
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<td>0.0</td>
<td>62.83</td>
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<tr>
<td>20</td>
<td>2.2063 $\pi^{-1}h$</td>
<td>2.2159</td>
<td>18.80</td>
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<td>0.0</td>
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Table 5
Numerical results for Example 5 for $(\nabla \tilde{u}_h) = \nabla \tilde{u}_h$

<table>
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<th>$\hat{\kappa}(h)$</th>
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<th>$C_{\text{div} b}$</th>
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<td>5.6214</td>
<td>1.7563</td>
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<td>17.65</td>
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<td>40</td>
<td>5.5576</td>
<td>1.2963</td>
<td>1.3608</td>
<td>17.52</td>
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Table 6
Numerical results for Example 5 for $(\nabla \tilde{u}_h) = (P_0 \nabla_x \tilde{u}_h, P_0 \nabla_y \tilde{u}_h)$

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<tr>
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<th>$\beta$</th>
<th>$\sigma$</th>
<th>$\kappa(h)$</th>
<th>$M_h$</th>
<th>$C_{\text{div} b}$</th>
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<tr>
<td>10</td>
<td>5.5421</td>
<td>39.54</td>
<td>Fail</td>
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<td>Fail</td>
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<td>40.00</td>
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<th>Authors</th>
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<tr>
<td>MHF2005-1</td>
<td>Matrix trace inequalities related to uncertainty principle</td>
<td>Hideki KOSAKI</td>
</tr>
<tr>
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<td>Discrepancy between theory and real computation on the stability of some finite element schemes</td>
<td>Masahisa TABATA</td>
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<td>MHF2005-3</td>
<td>Functional regression modeling via regularized basis expansions and model selection</td>
<td>Yuko ARAKI &amp; Sadanori KONISHI</td>
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<td>Point configurations, Cremona transformations and the elliptic difference Painlevé equations</td>
<td>Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA &amp; Yasuhiko YAMADA</td>
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<td>MHF2005-6</td>
<td>Construction of hypergeometric solutions to the $q$-$\hat{P}$ Painlevé equations</td>
<td>Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA &amp; Yasuhiko YAMADA</td>
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<td>MHF2005-7</td>
<td>Simple estimators for non-linear Markovian trend from sampled data: I. ergodic cases</td>
<td>Hiroki MASUDA</td>
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<td>Masayuki UCHIDA</td>
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<td>Ryo MATSUZAKI &amp; Masayuki UCHIDA</td>
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<td>Estimation for a discretely observed small diffusion process with a linear drift</td>
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<td>MHF2005-12</td>
<td>AIC for ergodic diffusion processes from discrete observations</td>
<td>Masayuki UCHIDA &amp; Nakahiro YOSHIDA</td>
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