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Constructive error estimates of finite element approximations for non-coercive elliptic problems and its applications

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Abstract

In this paper, we show constructive a priori and a posteriori error estimates of finite element approximations for not necessary coercive linear second order Dirichlet problems. Here, 'constructive' means we can get the error bounds in which all constants included are explicitly given or represented as a numerically computable form. Using the invertibility condition of concerning elliptic operator, constructive a priori and a posteriori error estimates are formulated. This kind of estimates plays essential and important roles in the numerical verification of solitions for nonlinear elliptic problems. Several numerical examples that confirm the actual effectiveness of the method are presented.

Key words: Constructive a priori and a posteriori error estimates, linear elliptic problem.
Classifications: 35J25, 35J60, 65N25.

1 Introduction

In this paper, we consider the constructive a priori and a posteriori error estimates for the general linear elliptic boundary value problem of the form

$$\begin{aligned} \mathcal{L}u \equiv -\Delta u + b\cdot\nabla u + cu &= f \ \text{ in } \ \Omega, \\ u &= 0 \ \text{ on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $f \in L^2(\Omega)$. Here, for $n = 1, 2, 3$, we assume that $b \in (W^1_\infty(\Omega))^n$, $c \in L^\infty(\Omega)$, where $\Omega \subset R^n$ is a bounded open domain with piecewise smooth boundary. In this paper, we use the terminology 'constructive error estimates' means an error estimation that by some numerical computations based on the estimates, we can obtain the true error bounds between the exact solution and its approximation in mathematically rigorous sense, even if the concerning problem (1.1) is not coericive. This kind of estimations should be useful when the existence or uniqueness of solutions are not a priori assured, e.g., in case that the coefficient function c is not nonnegative. And it also be important for the numerical verification of solitions for nonlinear boundary value problems(e.g., [2] [4] [5]etc.).

Now, we denote the usual k-th order Sobolev space on Ω by $H^k(\Omega)$ and define $(\cdot,\cdot)_{L^2}$ as the L^2 inner product. And we set $H_0^1(\Omega) \equiv \{v \in H^1(\Omega)\ ;\ v = 0 \text{ on } \partial\Omega\}$ with the inner product $(\nabla u, \nabla v)_{L^2}$ for $u, v \in H_0^1(\Omega)$. Also, define $X(\Omega) \equiv \{v \in H^1(\Omega)\ ;\ \Delta v \in L^2(\Omega)\}$.
We now introduce the finite dimensional subspace S_h of $H_0^1(\Omega)$ depending on the parameter h with nodal functions $\{\phi_i\}_{1\le i\le N}$. For each $v \in H_0^1(\Omega)$, define the H_0^1-projection $P_h v \in S_h$ by

$$(\nabla(v - P_h v), \nabla\phi_h)_{L^2} = 0, \qquad \forall \phi_h \in S_h.$$

Also, corresponding to the usual finite element approximations of a solution u in (1.1), we define the $\mathcal{L}$-projection $P_{\mathcal{L}} v \in S_h$, whose existence is assumed, by

$$a(v - P_{\mathcal{L}} v, \phi_h)_{L^2} = 0, \qquad \forall \phi_h \in S_h, \tag{1.2}$$

where $a(u, v) \equiv (\nabla u, \nabla v)_{L^2} + (b \cdot \nabla u, v)_{L^2} + (cu, v)_{L^2}$. Further, we assume that there exists a positive constant $C(h)$ which can be numerically estimated satisfying, for any $u \in H_0^1(\Omega) \cap X(\Omega)$,

$$\|u - P_h u\|_{H_0^1} \le C(h)\|\Delta u\|_{L^2}. \tag{1.3}$$

Note that (1.3) is equivalent to the following estimation.

$$\|u - P_h u\|_{L^2} \le C(h)\|u - P_h u\|_{H_0^1}. \tag{1.4}$$

Then our main purpose of this paper is to determine explicitly a priori constants $K_0(h)$ and $K_1(h)$ satisfying

$$\|u - P_{\mathcal{L}} u\|_{L^2} \le K_0(h)\|\mathcal{L}u\|_{L^2}, \tag{1.5}$$
$$\|u - P_{\mathcal{L}} u\|_{H_0^1} \le K_1(h)\|\mathcal{L}u\|_{L^2}, \tag{1.6}$$

respectively. Also we show a constant $K(h)$ satisfying

$$\|u - P_{\mathcal{L}} u\|_{L^2} \le K(h)\|u - P_{\mathcal{L}} u\|_{H_0^1}. \tag{1.7}$$

Defining the compact oprator $A : H_0^1 \longrightarrow H_0^1$ by $Au := \Delta^{-1}(b\cdot\nabla u + cu)$, where Δ^{-1} stands for the solution operator of the Poisson equation with homogeneous boundary condition, the invertibility of the elliptic operator $\mathcal{L}$ defined in (1.1) is equivalent to the unique solvability of the following fixed point equation:

$$u = Au.$$

As the preliminary, we define $N \times N$ matrices $\mathbf{G} = (\mathbf{G}_{i,j})$ and $\mathbf{D} = (\mathbf{D}_{i,j})$ by

$$\begin{aligned}
\mathbf{G}_{i,j} &= (\nabla\phi_j, \nabla\phi_i)_{L^2} + (b\cdot\nabla\phi_j, \phi_i)_{L^2} + (c\phi_j, \phi_i)_{L^2},\\
\mathbf{D}_{i,j} &= (\nabla\phi_j, \nabla\phi_i)_{L^2},
\end{aligned}$$

Note that $\mathbf{D}$ is symmetric and positive definite. We denote the matrix norm by $\|\cdot\|_E$ induced from the Euclidean norm $|\cdot|_E$. Also, we define the following constants:

$$\begin{aligned}
C_1 &= C_p C_{\mathrm{div}\, b} + C_b, & C_3 &= C_b + C_p C_c,\\
C_2 &= C_p C_c, & C_4 &= C_b + C(h) C_c,
\end{aligned}$$
$$C_{\mathrm{div}\, b} = \|\mathrm{div}\, b\|_{L^\infty},\ C_b = \|\, |b|_E \,\|_{L^\infty},\ C_c = \|c\|_{L^\infty},$$

where $\|\cdot\|_{L^\infty}$ means L^∞ norm on Ω and C_p is a Poincaré constant such that $\|\phi\|_{L^2} \le C_p\|\phi\|_{H_0^1}$ for an arbitrary $\phi \in H_0^1(\Omega)$.

In [5], authors show the following results.

Theorem 1 *If the matrix* $\mathbf{G}$ *is nonsingular, and for the constants defined above,*

$$\kappa(h) \equiv C(h)\Big(C(h)M_h(C_1 + C_2)C_3 + C_4\Big) < 1$$

holds, then the operator $\mathcal{L}$ *defined in (1.1) is invertible. Here,* $M_h \equiv \|\mathbf{D}^{\frac{1}{2}}\mathbf{G}^{-1}\mathbf{D}^{\frac{1}{2}}\|_E$ *and* $C(h)$ *is the same constant as in (1.3).*

Moreover, we have the following a priori estimate for the H_0^1-projection.

Theorem 2 *Assuming that same conditions in Theorem 1, let* $u \in H_0^1(\Omega) \cap X(\Omega)$ *be a unique solution of (1.1). Then we have*

$$\|u - P_h u\|_{H_0^1} \le C(h)\sigma\|f\|_{L^2},$$

where the constant σ *is given by* $\sigma = (1 + C_p M_h C_3)(1 - \kappa(h))^{-1}$.

When the coefficient vector function b in (1.1) is not differentiable, we have the following alternative results.

Corollary 3 *Let* $b \in (L^\infty(\Omega))^n$. *If*

$$\hat{\kappa}(h) \equiv C(h)\Big(M_h(\hat{C}_1 + C(h)C_2)C_3 + C_4\Big) < 1$$

holds, then the operator $\mathcal{L}$ defined in (1.1) is invertible. Here, $\hat{C}_1 = C_p C_b$. Also we have

$$\|u - P_h u\|_{H_0^1} \le C(h)\hat{\sigma}\|f\|_{L^2},$$

for a unique solution of $\mathcal{L}u = f$, where the constant $\hat{\sigma}$ is given by

$$\hat{\sigma} = (1 + C_p M_h C_3)(1 - \hat{\kappa}(h))^{-1}.$$

2 Main results

In this section, we show the constructive a priori and a posteriori error estimates of finite element approximations (1.2) for linear elliptic problems (1.1). Note that the existence of the inverse $\mathcal{L}^{-1} : L^2(\Omega) \longrightarrow X(\Omega)$ is equivalent to the invertibility of $I - A$, where I denotes the identity operator in $H_0^1(\Omega)$. Using this fact, we first show the a priori error estimate between a solution of our problems and its H_0^1-projection. First, we show the following lemma.

Lemma 4 *(cf.[5]) For an arbitrary $v \in H_0^1(\Omega)$, we have*

$$\|Av\|_{H_0^1} \le (C_1 + C_2)\|v\|_{L^2},$$
$$\|(I - P_h)Av\|_{H_0^1} \le C(h)\left(C_3\|P_h v\|_{H_0^1} + C_4\|v - P_h v\|_{H_0^1}\right).$$

Proof: Let $\psi := -Av = -\Delta^{-1}(b \cdot \nabla + c)v \in H_0^1(\Omega) \cap X(\Omega)$. CThen we have

$$\begin{aligned}
\|\psi\|_{H_0^1}^2 &= (-\Delta\psi, \psi)_{L^2} \\
&= (v, \mathrm{div}\,(b\psi))_{L^2} + (v, c\psi)_{L^2} \\
&\le \left(\|\mathrm{div}\,(b\psi)\|_{L^2} + \|c\psi\|_{L^2}\right)\|v\|_{L^2} \\
&\le C(h)\Big(\|\mathrm{div}\,b\|_{L^\infty}\|\psi\|_{L^2} + \|\,|b|_E\,\|_{L^\infty}\|\psi\|_{H_0^1} + \|c\|_{L^\infty}\|\psi\|_{L^2}\Big)\|v\|_{H_0^1},
\end{aligned}$$

where we have used (1.4). Moreover, we have

$$\begin{aligned}
\|(I - P_h)Av\|_{H_0^1} &= \|(I - P_h)\Delta^{-1}(b \cdot \nabla + c)v\|_{H_0^1} \\
&\le C(h)\|(b \cdot \nabla + c)v\|_{L^2} \\
&\le C(h)\Big(\|\,|b|_E\,\|_{L^\infty}\|v\|_{H_0^1} + \|c\|_{L^\infty}\|v\|_{L^2}\Big),
\end{aligned}$$

where we have used (1.3). Therefore, this proof is completed. ∎

For the $\mathcal{L}$-projection, we have the following one of the main results of this paper.

Theorem 5 *For an arbitrary $v \in H_0^1(\Omega)$, if $\mathbf{G}$ is nonsingular, then for the same constants in Theorem 1, we have*

$$\begin{aligned}
&\|v - P_{\mathcal{L}}v\|_{H_0^1} \le \alpha \|v - P_h v\|_{H_0^1}, \\
&\|v - P_{\mathcal{L}}v\|_{L^2} \le C(h)\beta \|v - P_h v\|_{H_0^1} \le C(h)\beta \|v - P_{\mathcal{L}}v\|_{H_0^1},
\end{aligned}$$

where $\alpha \equiv \sqrt{1 + \left(C(h)M_h(C_1 + C_2)\right)^2}$, $\beta \equiv 1 + C_p M_h(C_1 + C_2)$.

Proof: From the property of the H_0^1- and $\mathcal{L}$-projections, we can obtain

$$\|v - P_{\mathcal{L}}v\|_{H_0^1}^2 = \|v - P_h v\|_{H_0^1}^2 + \|P_{\mathcal{L}}v - P_h v\|_{H_0^1}^2, \tag{2.1}$$

for an arbitrary $v \in H_0^1(\Omega)$. Let $e \equiv v - P_h v$.
Then since $P_{\mathcal{L}}v - P_h v = P_{\mathcal{L}}(v - P_h v)$, for all $\phi_h \in S_h$, we have

$$\begin{aligned}
a(P_{\mathcal{L}}e, \phi_h) &= (\nabla e, \nabla \phi_h)_{L^2} + ((b \cdot \nabla + c)e, \phi_h)_{L^2} \\
&= (b \cdot \nabla e + ce, \phi_h)_{L^2} \\
&= (\nabla P_h \psi, \nabla \phi_h)_{L^2},
\end{aligned}$$

where we set $\psi \equiv -Ae = -\Delta^{-1}(b \cdot \nabla + c)e$. It implies that

$$\mathbf{G}\vec{e}_h = \mathbf{D}\vec{\psi}_h,$$

where $\vec{e}_h$ and $\vec{\psi}_h$ are coefficient vectors of $P_{\mathcal{L}}e$ and $P_h\psi$, respectively. Thus in the similar way to the proof of Lemma 4, we can obtain the following estimate since $\|P_{\mathcal{L}}e\|_{H_0^1} = \|\mathbf{D}^{\frac{1}{2}}\vec{e}_h\|_E$, $\|P_h\psi\|_{H_0^1} = \|\mathbf{D}^{\frac{1}{2}}\vec{\psi}_h\|_E$ and $\|P_h\psi\|_{H_0^1} \le \|\psi\|_{H_0^1}$ for any $\psi \in H_0^1(\Omega)$.

$$\begin{aligned}
\|P_{\mathcal{L}}v - P_h v\|_{H_0^1} = \|P_{\mathcal{L}}e\|_{H_0^1} &\le M_h \|P_h \psi\|_{H_0^1} \\
&\le M_h \|A(v - P_h v)\|_{H_0^1} \\
&\le C(h) M_h (C_1 + C_2) \|v - P_h v\|_{H_0^1}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\|P_{\mathcal{L}}v - P_h v\|_{L^2} &\le C_p \|P_{\mathcal{L}}v - P_h v\|_{H_0^1} \\
&\le C(h) C_p M_h (C_1 + C_2) \|v - P_h v\|_{H_0^1}.
\end{aligned}$$

Hence we can obtain the following estimate.

$$\begin{aligned}
\|v - P_{\mathcal{L}}v\|_{L^2} &\le \|v - P_h v\|_{L^2} + \|P_{\mathcal{L}}v - P_h v\|_{L^2} \\
&\le C(h)\|v - P_h v\|_{H_0^1} + C(h) C_p M_h (C_1 + C_2)\|v - P_h v\|_{H_0^1},
\end{aligned}$$

where we have used (1.4). Therefore, the proof is completed from (2.1). ∎

Note that the constant α in Theorem 5 tends to 1 if $h \to 0$ as illustrated in Figure 1.

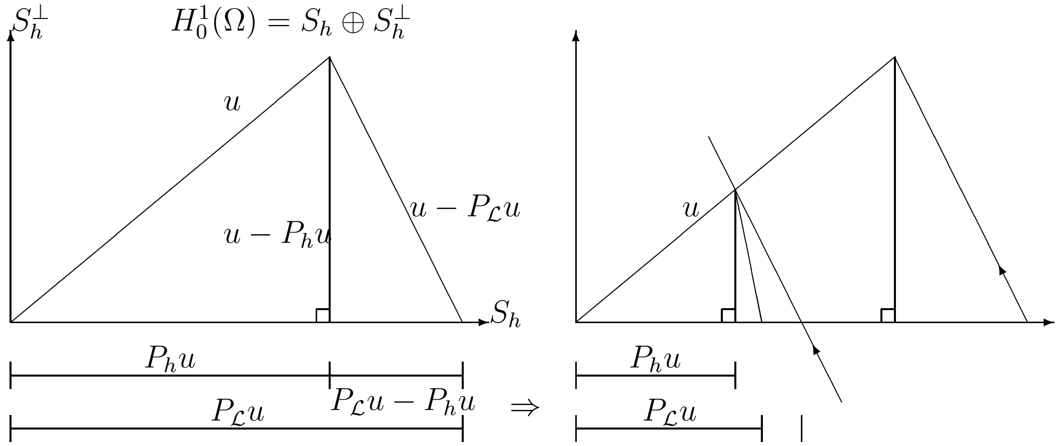

Fig. 1. Image of the H_0^1- and $\mathcal{L}$-projections

Now, as in [7], let S_h^* be an appropriate finite element subspace of $H^1(\Omega)$ satisfying $S_h \subset S_h^*$, and let define $(\overline{\nabla} u_h) \equiv (P_0 \nabla_x u_h, P_0 \nabla_y u_h, P_0 \nabla_z u_h) \in (S_h^*)^n$, where $P_0 : L^2(\Omega) \longrightarrow S_h^*$ means the L^2-projection defined by, for each $v \in L^2(\Omega)$,

$$(v - P_0 v, \phi_h^*)_{L^2} = 0 \quad \text{for any } \phi_h^* \in S_h^*.$$

Also note that, for the problem (1.1), the finite element solution u_h defined by

$$(\nabla u_h, \nabla \phi_h)_{L^2} + (b \cdot \nabla u_h + c u_h, \phi_h)_{L^2} = (f, \phi_h)_{L^2}, \qquad \forall \phi_h \in S_h, \tag{2.2}$$

coincides with the $\mathcal{L}$-projection $P_{\mathcal{L}} u$.
Now, by using Theorems 1, 2 and 5, we have the following constructive a priori and a posteriori error estimates for linear elliptic problems.

Theorem 6 *Assuming that Theorem 1 holds, then for a unique solution of* $\mathcal{L}u = f$, *we have*

$$\|u - P_{\mathcal{L}} u\|_{H_0^1} \le C(h) \alpha \sigma \|f\|_{L^2},$$

$$\|u - P_{\mathcal{L}} u\|_{L^2} \le C(h)^2 \beta \sigma \|f\|_{L^2}.$$

And we have the following a posteriori error estimate for the finite element solution u_h *defined by (2.2).*

$$\|u - u_h\|_{H_0^1} \le \|R\|_{L^2} + C(h)\beta \|S\|_{L^2} + C(h)^2 \beta \sigma \Big(C_b + C(h) C_c \beta \Big) \|f\|_{L^2}, \tag{2.3}$$

where $R \equiv \nabla u_h - (\overline{\nabla} u_h)$ *and* $S \equiv f + \mathrm{div}(\overline{\nabla} u_h) - b \cdot \nabla u_h - c u_h$.

Proof: From Theorems 2 and 5, we can easily obtain the following inequalities.

$$\|u - P_{\mathcal{L}}u\|_{H_0^1} \leq C(h)\alpha\sigma\|f\|_{L^2},$$
$$\|u - P_{\mathcal{L}}u\|_{L^2} \leq C(h)^2\beta\sigma\|f\|_{L^2}.$$

Thus we consider the a posteriori error estimate below.

Let $e \equiv u - u_h$.

$$\begin{aligned}\|u - u_h\|_{H_0^1}^2 &= (\nabla e, \nabla u)_{L^2} - (\nabla e, \nabla u_h)_{L^2} \\ &= (e, f)_{L^2} - (e, b\cdot\nabla u + cu)_{L^2} - (\nabla e, \nabla u_h)_{L^2} \\ &= (e, f - b\cdot\nabla u_h + cu_h)_{L^2} - (e, b\cdot\nabla e + ce)_{L^2} - (\nabla e, \nabla u_h)_{L^2}.\end{aligned}$$

Since $((\overline{\nabla} u_h), \nabla v)_{L^2} = (-\mathrm{div}(\overline{\nabla} u_h), v)_{L^2}$ for any $v \in H_0^1(\Omega)$, taking as $v = e$, it implies that

$$\begin{aligned}\|u - u_h\|_{H_0^1}^2 &= (e, S)_{L^2} - (e, b\cdot\nabla e + ce)_{L^2} - (\nabla e, R)_{L^2} \\ &\leq \|e\|_{L^2}\|S\|_{L^2} + \|e\|_{L^2}\|b\cdot\nabla e + ce\|_{L^2} + \|e\|_{H_0^1}\|R\|_{L^2}.\end{aligned}$$

Moreover, using Lemma 4, we have

$$\|b\cdot\nabla e + ce\|_{L^2} \leq \|\ |b|_E\ \|_{L^\infty}\|e\|_{H_0^1} + \|c\|_{L^\infty}\|e\|_{L^2}.$$

Hence using the fact $\|e\|_{L^2} \leq C(h)\beta\|e\|_{H_0^1}$ in Theorem 5, we have the estimate (2.3). Therefore, this proof is completed. ∎

Remark. The last term in the estimates (2.3) looks like an a priori estimation. However, since the order $C(h)^2$ is higher than the usual optimal estimation in H^1 norm, combining it with the first and second terms, this estimates can be considered as a kind of a posteriori error estimates.

From Theorems 5 - 6, we can take the constants $K_0(h)$, $K_1(h)$ and $K(h)$ as

$$K_0(h) := C(h)^2\beta\sigma, \quad K_1(h) := C(h)\alpha\sigma, \quad K(h) := C(h)\beta.$$

Also we have the following estimates corresponding to Corollary 3.

Corollary 7 *Let $b \in (L^\infty(\Omega))^n$. Under the same assumptions in Corollary 3, we have*

$$\|u - P_{\mathcal{L}}u\|_{H_0^1} \leq C(h)\hat{\alpha}\hat{\sigma}\|f\|_{L^2},$$

for a unique solution of $\mathcal{L}u = f$, where $\hat{\alpha} \equiv \sqrt{1 + \left(M_h(\hat{C}_1 + C(h)C_2)C_3\right)^2}$.

For usual finite element approximations in the one dimensional case, we can get the better estimates, even if the function b has no smoothness.

Lemma 8 *Let S_h be a finite element subspace of $H_0^1(\Omega)$, where $\Omega = (p,q)$ is an interval in $\mathbf{R}^1$, comprising piecewise polynomials with the mesh*

$$p = x_0 < x_1 < \cdots < x_N < x_{N+1} = q.$$

For an arbitrary $v \in H_0^1(\Omega)$, if $b \in \bigwedge_{i=0}^N W_\infty^1(I_i) \subset L^\infty(\Omega)$ then we have

$$\|A(v - P_h v)\|_{H_0^1} \leq (D_1 + C_2)\|v - P_h v\|_{L^2},$$

where $D_1 = C_p D_{\mathrm{div}\, b} + C_b$, $D_{\mathrm{div}\, b} = \max\limits_{0\leq i\leq N} \|b\|_{W_\infty^1(I_i)}$ and $I_i := (x_i, x_{i+1})$.

Proof: Let $\psi \equiv -\Delta^{-1}(be' + ce)$, where $e := v - P_h v$. Then it implies that

$$\|\psi\|_{H_0^1}^2 = (\psi', \psi')_{L^2} = (be' + ce, \psi)_{L^2} = (e', b\psi)_{L^2} + (e, c\psi)_{L^2}$$

Note that the H_0^1-projection satisfies $e(x_i) = 0$ for $i = 0, \cdots, N+1$. Hence we have

$$\begin{aligned}
(e', b\psi)_{L^2} &= \sum_i (e, (b\psi)')_{L^2(I_i)} \\
&\leq \sum_i \|e\|_{L^2(I_i)} \|(b\psi)'\|_{L^2(I_i)} \\
&\leq \sum_i \|e\|_{L^2(I_i)} \left(\|b\|_{W_\infty^1(I_i)} \|\psi\|_{L^2(I_i)} + \|b\|_{L^\infty(I_i)} \|\psi'\|_{L^2(I_i)} \right) \\
&\leq D_{\mathrm{div}\, b} \sum_i \|e\|_{L^2(I_i)} \|\psi\|_{L^2(I_i)} + C_b \sum_i \|e\|_{L^2(I_i)} \|\psi'\|_{L^2(I_i)} \\
&\leq \left(D_{\mathrm{div}\, b} \|\psi\|_{L^2} + C_b \|\psi\|_{H_0^1} \right) \|e\|_{L^2} \\
&\leq (C_p D_{\mathrm{div}\, b} + C_b) \|\psi\|_{H_0^1} \|e\|_{L^2},
\end{aligned}$$

and $(e, c\psi)_{L^2} \leq C_c \|e\|_{L^2} \|\psi\|_{L^2}$. Therefore, the proof is completed. ∎

Applying similar arguments in Theorems 5 - 6 with the above lemma, we have the following results for a special case.

Theorem 9 *Under the same assumption in Lemma 8, if $\mathbf{G}$ is nonsingular then we have*

$$\begin{aligned}
\|v - P_{\mathcal{L}} v\|_{H_0^1} &\leq \dot{\alpha} \|v - P_h v\|_{H_0^1}, \\
\|v - P_{\mathcal{L}} v\|_{L^2} &\leq C(h) \dot{\beta} \|v - P_{\mathcal{L}} v\|_{H_0^1},
\end{aligned}$$

where $\dot{\alpha} \equiv \sqrt{1 + \Big(C(h) M_h (D_1 + C_2) \Big)^2}$ and $\dot{\beta} \equiv 1 + C_p M_h (D_1 + C_2)$. Moreover, if

$$\dot{\kappa}(h) \equiv C(h) \Big(C(h) M_h (D_1 + C_2) C_3 + C_4 \Big) < 1$$

holds, then the operator $\mathcal{L}$ is invertible, and we have the following a priori error estimate for a unique solution of $\mathcal{L}u = f$.

$$\begin{aligned}
\|u - P_{\mathcal{L}} u\|_{H_0^1} &\leq C(h) \dot{\alpha} \dot{\sigma} \|f\|_{L^2}, \\
\|u - P_{\mathcal{L}} u\|_{L^2} &\leq C(h)^2 \dot{\beta} \dot{\sigma} \|f\|_{L^2},
\end{aligned}$$

where $\dot{\sigma} = (1 + C_p M_h C_3)(1 - \dot{\kappa}(h))^{-1}$.

3 Numerical examples

In this section, we show several numerical results for linear elliptic problems. In the below, the 1-dimensional problems are presented in the examples 1-3 and 2-dimensional cases in 4-5.

Example 1 *(nearly singular problem)*

$$
\begin{aligned}
-u'' + cu &= 1 \text{ in } \Omega = (0,1), \\
u &= 0 \text{ on } \partial\Omega,
\end{aligned}
$$

where $c = \pm 10$. *Note that if* $c = -\pi^2 = -9.8696\cdots$ *then this example has no solution.*

Example 2 *(linearized Burgers equation)*

$$
\begin{aligned}
-u'' + \lambda(\tilde{\phi}_h + 2x - 1)u' + \lambda(\tilde{\phi}_h + 2x - 1)'u &= 1 \text{ in } \Omega = (0,1), \\
u &= 0 \text{ on } \partial\Omega,
\end{aligned}
$$

where $\lambda = 10$ *and* $\tilde{\phi}_h \in S_h$ *is an approximation of the following Burgers equation.*

$$
\begin{aligned}
\phi'' &= \lambda\phi\phi' \text{ in } \Omega, \\
\phi(0) = -1, &\quad \phi(1) = 1.
\end{aligned}
$$

Moreover, as a special case, we consider the following example.

Example 3 *(discontinuous coefficient)*

$$
\begin{aligned}
-u'' + bu' &= 1 \text{ in } \Omega = (0,1), \\
u &= 0 \text{ on } \partial\Omega,
\end{aligned}
$$

where $b \in L^\infty(\Omega)$ *is given by*

$$
b \equiv b(x) = \begin{cases}
4(8x^2 - x)' & = 4(16x - 1) \text{ if } x \in (0, 0.25), \\
2(16x^2 - 14x + 3)' & = 4(16x - 7) \text{ if } x \in (0.25, 0.5), \\
2(2x - 1)' & = 4 \qquad\qquad\ \text{if } x \in (0.5, 0.75), \\
4(1 - x)' & = -4 \qquad\qquad \text{if } x \in (0.75, 1).
\end{cases}
$$

In above examples, we take the finite element subspace S_h as piecewise quadratic functions with uniform mesh. Then it can be taken as $C(h) = (2\pi)^{-1}h$ ([3]) for piecewise quadratic functions on $\Omega = (0, 1)$ and $C_p = \pi^{-1}$.

We show validated numerical results using interval techniques ([1]) for Examples 1, 2 and 3 in Tables 1, 2 and 3, respectively.

Table 1
Numerical results for Example 1

h^{-1}	α	β	σ	$\kappa(h)$	M_h	$C_{\mathrm{div}\, b}$	C_b	C_c	c
100	1.0000	2.0132	2.0133	5.09e-5	0.9999	0.0	0.0	10	+10
200	1.0000	2.0132	2.0132	1.27e-5	1.0000	0.0	0.0	10	+10
400	1.0000	2.0135	2.0135	3.18e-6	1.0003	0.0	0.0	10	+10
800	1.0000	2.0248	2.0248	8.01e-7	1.0114	0.0	0.0	10	+10
100	1.0709	77.69	77.84	1.96e-3	75.69	0.0	0.0	10	−10
200	1.0182	77.71	77.75	4.92e-4	75.71	0.0	0.0	10	−10
400	1.0046	78.05	78.06	1.23e-4	76.04	0.0	0.0	10	−10
800	1.0013	83.72	83.72	3.31e-5	81.64	0.0	0.0	10	−10

Table 2
Numerical results for Example 2

h^{-1}	α	β	σ	$\kappa(h)$	M_h	$C_{\mathrm{div}\, b}$	C_b	C_c
100	1.4245	203.91	134.09	5.85e-2	14.94	51.30	10.00	51.30
200	1.1212	203.88	128.61	1.86e-2	14.94	51.28	10.00	51.28
400	1.0318	204.35	127.36	6.64e-3	14.97	51.28	10.00	51.28
800	1.0092	219.33	136.13	2.70e-3	16.08	51.28	10.00	51.28

Table 3
Numerical results for Example 3

h^{-1}	$\dot{\alpha}$	$\dot{\beta}$	$\dot{\sigma}$	$\dot{\kappa}(h)$	M_h	$D_{\mathrm{div}\, b}$	C_b	C_c
100	1.0260	23.97	9.9857	4.69e-2	2.2296	64.00	12.00	0.0
200	1.0065	23.97	9.7242	2.12e-2	2.2298	64.00	12.00	0.0
400	1.0016	23.97	9.6146	1.00e-2	2.2298	64.00	12.00	0.0
800	1.0004	23.99	9.5719	4.91e-3	2.2318	64.00	12.00	0.0

Next we consider the following 2-dimensional problems.

Example 4 *(linearized Emden's equation)*

$$
\begin{aligned}
-\Delta u - 2\tilde{\phi}_h u &= \tfrac{\sqrt{5}}{2} \ \text{ in } \ \Omega = (0,1)^2 \backslash [0, \tfrac{1}{5}]^2, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
$$

where $\tilde{\phi}_h \in S_h$ is an approximation of the following Emden's equation.

$$
\begin{aligned}
-\Delta\phi &= \phi^2 \text{ in } \quad \Omega, \\
\phi \ &= 0 \ \text{ on } \partial\Omega.
\end{aligned}
$$

Example 5

$$
\begin{aligned}
-\Delta u + \tilde{u}_h(\overline{\nabla}\hat{u}_h) \cdot \nabla u - \left(\lambda - \frac{1}{2}|\nabla \tilde{u}_h|^2\right) u &= 1 \text{ in } \Omega = (0,1)^2, \\
u &= 0 \text{ on } \partial\Omega,
\end{aligned}
$$

where $\lambda = 40$ and $\tilde{u}_h \in S_h$ is an approximation of Plum's example.

$$
\begin{aligned}
-\Delta\phi = \phi\left(\lambda - \frac{1}{2}|\nabla\phi|^2\right) & \text{ in } \quad \Omega, \\
\phi = 0 \qquad\qquad\qquad & \text{on } \partial\Omega.
\end{aligned}
$$

In this example, we considered two cases for the coefficient vector function b, taht is, in case of $(\overline{\nabla}\hat{u}_h) \equiv \nabla\tilde{u}_h$, discontinuous, and $(\overline{\nabla}\hat{u}_h) \equiv (P_0\nabla_x\tilde{u}_h, P_0\nabla_y\tilde{u}_h)$, where $\tilde{u}_h$ is an approximate solution in S_h and P_0 stands for the L^2-projection into S_h^* defined in Section 2.

In above two examples, we take the finite element subspace S_h as piecewise bi-linear functions with uniform mesh. Note that we can take the constant C_p for $\Omega = (0,1)^2 \backslash [0, \frac{1}{5}]^2$ and $\Omega = (0,1)^2$ as $C_p = \sqrt{10}^{-1}$ and $C_p = (\sqrt{2}\pi)^{-1}$, respectively. Moreover, we can obtain the a priori constant $C(h)$ for the L-shaped domain by techniques in [7], and it is taken as $C(h) = \pi^{-1}h$ for bi-linear functions on $\Omega = (0,1)^2$. We show validated numerical results for Example 4 in Table 4. Also, for Example 5, we illustrate several numerical results for $(\overline{\nabla}\hat{u}_h) = \nabla\tilde{u}_h$ and $(\overline{\nabla}\hat{u}_h) = (P_0\nabla_x\tilde{u}_h, P_0\nabla_y\tilde{u}_h)$ in Tables 5 and 6, respectively. As shown in these tables, the capability for the verifivcation of invertibility seems to be influenced by the smoothness of the function b.

All computations in these tables are carried out on the Dell Precision 650 Workstation Intel Xeon CPU 3.20GHz using INTLAB, a tool box in MATLAB developed by Rump [6] for self-validating algorithms.

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Table 4
Numerical results for Example 4

h^{-1}	$C(h)$	α	β	σ	$\kappa(h)$	M_h	$C_{\text{div } b}$	C_b	C_c
10	$1.8433\pi^{-1}h$	3.4498	18.79	Fail	4.0656	2.8320	0.0	0.0	62.83
20	$2.2063\pi^{-1}h$	2.2159	18.80	Fail	1.4244	2.8994	0.0	0.0	61.41
30	$2.4772\pi^{-1}h$	1.7862	18.80	91.57	7.94e-1	2.9118	0.0	0.0	61.15
40	$2.6992\pi^{-1}h$	1.5718	18.85	40.33	5.32e-1	2.9159	0.0	0.0	61.22

Table 5
Numerical results for Example 5 for $(\overline{\nabla}\hat{u}_h) = \nabla\tilde{u}_h$

h^{-1}	$\hat{\alpha}$	$\hat{\sigma}$	$\hat{\kappa}(h)$	M_h	$C_{\text{div } b}$	C_b	C_c
10	6.2448	Fail	6.1895	1.3365	—	19.21	40.00
20	5.8008	Fail	2.7618	1.3556	—	18.08	40.00
30	5.6214	Fail	1.7563	1.3595	—	17.65	40.00
40	5.5576	Fail	1.2963	1.3608	—	17.52	40.00

Table 6
Numerical results for Example 5 for $(\overline{\nabla}\hat{u}_h) = (P_0\nabla_x\tilde{u}_h, P_0\nabla_y\tilde{u}_h)$

h^{-1}	α	β	σ	$\kappa(h)$	M_h	$C_{\text{div } b}$	C_b	C_c
10	5.5421	39.54	Fail	5.6096	1.6630	330.81	19.51	40.00
20	3.3515	46.23	Fail	1.6841	1.7513	389.06	18.19	40.00
30	2.4286	47.94	62.61	8.14e-1	1.7723	404.84	17.56	40.00
40	1.9588	48.64	22.94	4.95e-1	1.7801	410.88	17.41	40.00

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