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Sato, Junichi
Graduate School of Mathematics, Kyushu University

Kawasaki, Hidefumi
Faculty of Mathematics, Kyushu University

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J. Sato & H. Kawasaki

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

DISCRETE FIXED POINT THEOREMS AND THEIR APPLICATION TO NASH EQUILIBRIUM*

JUN-ICHI SATO† AND HIDEFUMI KAWASAKI‡

Abstract. Fixed point theorems are powerful tools in not only mathematics but also economic. In some economic problems, we need not real-valued but integer-valued equilibriums. However, classical fixed point theorems guarantee only real-valued equilibria. So we need discrete fixed point theorems in order to get discrete equilibria. In this paper, we first provide discrete fixed point theorems, next apply them to a non-cooperative game and prove the existence of a Nash equilibrium of pure strategies.

Key words. discrete fixed point theorem, pure strategy, Nash equilibrium, n-person non-cooperative game, bimatrix game.

AMS subject classifications. 47H10, 91A10, 91A05, 91A06, 91B50

1. Introduction. Existence theorem of Nash equilibrium is one of the most important applications of fixed point theorems such as Bouwer's, Kakutani's, and so on. In economics, we often encounter the situation that the equilibrium is not real-valued but integer-valued. For example, it is nonsense to assert that the equilibrium is to product 1.5 cars. In order to deal with such a case, we need a discrete fixed point theorem. The aims of this paper are to provide discrete fixed point theorems and to apply them to a non-cooperative game.

There are two types of discrete fixed point theorems. Tarski [5] gave some theorems on a lattice. Iimura-Murota-Tamura [2] gave one on an integrally convex set by using Brower's fixed point theorem, and Yang [6] obtained some extensions, see Section 4 for details. On the other hand, our discrete fixed point theorem are based on the following simple idea.

- The base set V is essentially finite, see (i) in Theorem 2.1.
- The mapping $f : V \to V$ reduces the area of candidates for fixed points.

We don't need any convexity assumption.

This paper is organized as follows. In Section 2, we give discrete fixed point theorems. In Section 3, we apply our fixed point theorems to a class of non-cooperative games and obtain some existence theorems of a Nash equilibrium of pure strategies. In Section 4, we compare our discrete fixed point theorems to the conventional one.

Throughout this paper, $(V, \preceqq)$ is a partially ordered set in $\mathbb{Z}^n$ and $f : V \to V$ is a nonempty set-valued mapping. For any $x \in \mathbb{Z}^n$, x_i denotes the i-th component of x. The symbol $x \preceq y$ means $x \preceqq y$ and $x \neq y$. We denote the component-wise order by $\leqq$. Further, $x \leq y$ means $x \leqq y$ and $x \neq y$.

2. Discrete fixed point theorems. In this section, we present discrete fixed point theorems. Although they are elementary, they are useful in Section 3.

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†Graduate School of Mathematics, Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-8581, JAPAN (jun-ichi@math.kyushu-u.ac.jp).

‡Faculty of Mathematics, Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-8581, JAPAN (kawasaki@math.kyushu-u.ac.jp).

Theorem 2.1. *Assume that there exist $x^0 \in V$ and $x^1 \in f(x^0)$ such that $x^0 \leqq\!\!\!\preceq x^1$ and $\{x \in V;\ x^0 \trianglelefteq x\}$ is finite. Further assume that for any $x \in V$ and $y \in f(x)$,*

$$x \preceq y \ \Rightarrow\ \exists z \in f(y) \text{ s.t. } y \trianglelefteq z. \tag{2.1}$$

Then, f has a fixed point x^*, that is, $x^* \in f(x^*)$.

Proof. Assume that f has no fixed points. Then, $x^0 \preceq x^1$. So, by (ii), there exists $x^2 \in f(x^1)$ such that $x^1 \trianglelefteq x^2$. Since f has no fixed points, we are led to $x^1 \preceq x^2$. Repeating this procedure, we have a sequence $\{x^m\}_{m\in\mathbb{N}}$ satisfying $x^m \preceq x^{m+1}$, which contradicts that $\{x \in V;\ x^0 \trianglelefteq x\}$ is finite. □

In particular, when V has a minimum element x^0, the first assumption in Theorem 2.1 is trivially satisfied. Further, we can easily weaken the assumptions of Theorem 2.1 as follows. Since the proof is trivial, we omit it.

Theorem 2.2. *Assume that there exists a sequence $\{x^m\}_{m\geq 0}$ in V such that $x^m \trianglelefteq x^{m+1} \in f(x^m)$ for any $m \geq 0$ and $\{x \in V;\ x^0 \trianglelefteq x\}$ is finite. Then, f has a fixed point $x^* \in f(x^*)$.*

When $\trianglelefteq$ is the component-wise order $\leqq$ or $\geqq$, Theorem 2.4 below shows a way to find x^0 and x^1 such that $x^0 \leqq x^1 \in f(x^0)$ in the case where V is a finite set.

Definition 2.3. *For any $k \in \{1,\ldots,n\}$ and $x \in \mathbb{Z}^n$, we denote $K := \{1,\ldots,k\}$, $x_K := (x_i)_{i\in K}$, $K+1 := \{1,\ldots,k+1\}$, and $x_{K+1} := (x_i)_{i\in K+1}$. We call $x \in V$ a fixed point of f w.r.t. $\mathbb{Z}^k$ if $x_K \in f_K(x) := \{y_K\,;\ y \in f(x)\}$.*

Theorem 2.4. *Let V be a finite set in $\mathbb{Z}^n$, and assume that for any $k \in \{1,\ldots,n\}$, $x \in V$ and $y \in f(x)$,*

$$x_K \leq (resp.\ \geq)\, y_K \Rightarrow \exists z \in f(y) \text{ s.t. } y_K \leq (resp.\ \geq)\, z_K.$$

Then, f has a fixed point $x^* \in f(x^*)$.

Proof. (By induction on k) Assume that f has no fixed points w.r.t. $\mathbb{Z}$. Then, taking arbitrary points $x \in V$ and $y \in f(x)$, we have either $x_1 < y_1$ or $x_1 > y_1$. Without loss of generality we may assume that $x_1 < y_1$. By the assumption, there exists $z^1 \in f(y)$ such that $y_1 \leq z^1_1$. Since f has no fixed points w.r.t. $\mathbb{Z}$, we are led to $y_1 < z^1_1$. Repeating this procedure, we obtain a sequence $\{z^m\}_{m\in\mathbb{N}}$ such that $z^m_1 < z^{m+1}_1$, which contradicts that V is finite.

Next, assume that f has a fixed point, say, x^0, w.r.t. $\mathbb{Z}^k$ and no fixed points w.r.t. $\mathbb{Z}^{k+1}$. Then, since $x^0_K \in f_K(x^0)$, there exists $y^0 \in f(x^0)$ such that $x^0_K = y^0_K$. However, since f has no fixed points w.r.t. $\mathbb{Z}^{k+1}$, we are led to $x^0_{K+1} \leq y^0_{K+1}$ or $x^0_{K+1} \geq y^0_{K+1}$. Here we may assume that $x^0_{K+1} \leq y^0_{K+1}$. So by the assumption of the theorem, there exists $z^1 \in f(y^0)$ such that $y^0_{K+1} \leqq z^1_{K+1}$. Since f has no fixed points w.r.t. $\mathbb{Z}^{k+1}$, we have $y^0_{K+1} \leq z^1_{K+1}$. Repeating this procedure, we obtain a sequence $\{z^m\}_{m\in\mathbb{N}}$ such that $z^m_K \leq z^{m+1}_{K+1}$ for any $m \in \mathbb{N}$, which contradicts that V is finite. □

Remark 2.1. Let σ be an arbitrary permutation of order n. Then, it is evident that one can replace K of Theorem 2.4 by $\{\sigma(1),\ldots,\sigma(k)\}$.

3. Nash equilibrium of pure strategies. As an application of our discrete fixed point theorems, we shall present a class of non-cooperative games that have a Nash equilibrium of pure strategies. We consider the following non-cooperative n-parson game $G = (N, \{S_i\}_{i\in N}, \{p_i\}_{i\in N})$, where

- $N := \{1,\ldots,n\}$ is the set of all players.
- For any $i \in N$, S_i denotes the set of player i's strategies. Its element is denoted by s_i. We assume that each S_i is a finite subset of $\mathbb{Z}$.
- $s_{-i} := (s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n)$.

- $S = \prod_{j=1}^{n} S_j$ is equipped with a signed component- wise order, that is, N is divided into two subsets (possibly empty) N_+ and N_-, $\varepsilon_j = \pm 1$ are allocated to $j \in N_+$ and $j \in N_-$, respectively, and $s \leqq t$ is defined by $\varepsilon_j s_j \leqq \varepsilon_j t_j$ for any $j \in N$. $S_{-i} = \prod_{j \ neqi} S_j$ is also equipped with the signed component-wise order.
- $p_i : S \to \mathbb{R}$ denotes the payoff function of player i.
- For any given $s_{-i} \in S_{-i}$, player i maximizes $p_i(s_i; s_{-i})$. We denote by $f_i(s_{-i})$ the set of best responses of player i, that is,

$$f_i(s_{-i}) := \left\{ s_i \in S_i \,;\, p_i(s_i, s_{-i}) = \max_{t_i \in S_i} p_i(t_i, s_{-i}) \right\}.$$

- $f(s) := f_1(s_{-1}) \times \cdots \times f_n(s_{-n})$ for any $s = (s_1, \ldots, s_n)$.

We call an n-tuple of pure strategies a Nash equilibrium if $s^* \in f(s^*)$.

Definition 3.1. (Monotone game) *We say a game G monotone if, for any $i \in N$, $s^0_{-i}, s^1_{-i} \in S_{-i}$ with $s^0_{-i} \preceq s^1_{-i}$ and for any $t^1_i \in f_i(s^0_{-i})$, there exists $t^2_i \in f_i(s^1_{-i})$ such that $\varepsilon_i t^1_i \leqq \varepsilon_i t^2_i$.*

Theorem 3.2. *Any monotone n-person non-cooperative game G has a Nash equilibrium of pure strategies.*

Proof. We apply Theorem 2.1 to G. Since S is a product set, it has a minimum element. So it suffices to show that for any $s^0 \in S$ and $s^1 \in f(s^0)$ satisfying $s^0 \preceq s^1$, there exists $s^2 \in f(s^1)$ such that $s^1 \leqq s^2$. Now, assume that $s^0 \in S$ and $s^1 \in f(s^0)$ satisfy $s^0 \preceq s^1$, and define $N_1 := \{i \in N;\ s^0_{-i} = s^1_{-i}\}$ and $N_2 := \{i \in N;\ s^0_{-i} \preceq s^1_{-i}\}$. Then, $N_1 \cap N_2 = \emptyset$, $N = N_1 \cup N_2$, N_1 has at most one element, and $\varepsilon_i s^0_i < \varepsilon_i s^1_i$ for $i \in N_1$. Thus, by taking $s^2_i := s^1_i$ for $i \in N_1$, we have $s^2_i = s^1_i \in f_i(s^0_{-i}) = f_i(s^1_{-i})$. On the other hand, by definition of N_2, we have $s^0_{-i} \preceq s^1_{-i}$ and $s^1_i \in f_i(s^0_{-i})$ for any $i \in N_2$. Hence, by monotonicity, there exists $s^2_i \in f_i(s^1_{-i})$ such that $\varepsilon_i s^1_i \leqq \varepsilon_i s^2_i$. Therefore, $s^2 := (s^2_1, \ldots, s^2_n)$ belongs to $f(s^1)$ and $s^1 \leqq s^2$. So, by Theorem 2.1, f has a fixed point. □

As a special case of game G, let us consider the following bimatrix game.

- $A = (a_{ij})$ is a payoff matrix of player 1 (P1), that is, $p_1(i,j) = a_{ij}$.
- $B = (b_{ij})$ is a payoff matrix of player 2 (P2), that is, $p_2(i,j) = b_{ij}$.
- $S_1 := \{1, \ldots, m_1\}$ is the set of strategies of P1.
- $S_2 := \{1, \ldots, m_2\}$ is the set of strategies of P2.
- $I(j) := \{i \in S_1\,;\, a_{ij} = \max_{i \in S_1} a_{ij}\}$ is the set of best responses of P1 to $j \in S_2$.
- $J(i) := \{j \in S_2\,;\, b_{ij} = \max_{j \in S_2} b_{ij}\}$ is the set of best responses of P2 to $i \in S_1$.
- $f(i,j) := I(j) \times J(i)$ denotes the set of best responses of $(i,j) \in S_1 \times S_2$.
- A pair (i^*, j^*) is a pure strategy Nash equilibrium if $(i^*, j^*) \in f(i^*, j^*)$.

Then Definition 3.1 reduces to Definition 3.3 below.

Definition 3.3. (Monotone bimatrix game) *We say payoff matrix A monotone if for any $j^0, j^1 \in S_2$ such that $\varepsilon_2 j^0 < \varepsilon_2 j^1$ and for any $i^1 \in I(j^0)$, there exists $i^2 \in I(j^1)$ such that $\varepsilon_1 i^1 \leqq \varepsilon_1 i^2$. Also, we say payoff matrix B monotone if for any i^0, i^1 such that $\varepsilon_1 i^0 < \varepsilon_1 i^1$ and for any $j^1 \in J(i^0)$, there exists $j^2 \in J(i^1)$ such that $\varepsilon_2 j^1 \leqq \varepsilon_2 j^2$. When both A and B are monotone, we say the bimatrix game monotone.*

The following corollary is a direct consequence of Theorem 3.2.

Corollary 3.4. *Any monotone bimatrix game has a Nash equilibrium of pure strategies.*

Example 3.1. The following matrices are monotone for $(\varepsilon_1, \varepsilon_2) = (1, 1)$, where framed numbers correspond to best responses, and circled numbers correspond to the Nash equilibrium.

$$A = \left(\begin{array}{c|c|c|c} 5 & \boxed{7} & 1 & \boxed{9} \\ \boxed{8} & 2 & 3 & 5 \\ 4 & \boxed{7} & \boxed{\textcircled{4}} & 8 \\ 7 & 6 & 2 & \boxed{9} \end{array}\right), \quad B = \left(\begin{array}{cccc} \boxed{7} & 2 & 6 & 3 \\ \hline 3 & \boxed{9} & 5 & 4 \\ \hline \boxed{8} & 6 & \boxed{\textcircled{8}} & 5 \\ \hline 1 & \boxed{3} & \boxed{3} & 2 \end{array}\right).$$

Indeed, the following inequalities show that they are monotone.

$$\begin{array}{ccccccc} I(1) = \{2\} & & I(2) = \{1,3\} & & I(3) = \{3\} & & I(4) = \{1,4\} \\ \cup & & \cup & & \cup & & \cup \\ 2 & \leqq & 3 & \leqq & 3 & \leqq & 4 \\ J(1) = \{1\} & & J(2) = \{2\} & & J(3) = \{1,3\} & & J(4) = \{2,3\} \\ \cup & & \cup & & \cup & & \cup \\ 1 & \leqq & 2 & \leqq & 3 & \leqq & 3. \end{array}$$

Moreover, since $(i, j) = (3, 3)$ belongs to the set of best responses to $(3, 3)$, $(3, 3)$ is a pure strategy Nash equilibrium.

Remark 3.1. Suppose that both players test whether each payoff matrix is monotone, respectively. If they answer "yes", then the matrix game has a pure strategy Nash equilibrium. They don't need to answer the set of their best responses. This is an advantage of Theorem 3.4.

Example 3.2. The following matrices are not monotone for $(\varepsilon_1, \varepsilon_2) = (1, 1)$.

$$A' = \left(\begin{array}{c|c|c|c} 5 & 1 & \boxed{7} & \boxed{9} \\ \boxed{8} & 3 & 2 & 5 \\ 7 & 2 & 6 & \boxed{9} \\ 4 & \boxed{\textcircled{4}} & \boxed{7} & 8 \end{array}\right), \quad B' = \left(\begin{array}{cccc} \boxed{7} & 6 & 2 & 3 \\ \hline 3 & 5 & \boxed{9} & 4 \\ \hline 1 & \boxed{3} & \boxed{3} & 2 \\ \hline \boxed{8} & \boxed{\textcircled{8}} & 6 & 5 \end{array}\right)$$

However, we can transform them into monotone matrices. Indeed, by exchanging the second and third columns and the third and fourth rows, A' and B' are transformed into A and B in Example 3.1, respectively. Thus, the original matrix game has a pure strategy Nash equilibrium $(4, 2)$.

Example 3.3. The following matrices are monotone for $(\varepsilon_1, \varepsilon_2) = (1, -1)$. So the matrix game has a pure strategy Nash equilibrium $(2, 3)$.

$$A = \left(\begin{array}{c|c|c} 4 & 2 & 1 \\ 5 & \boxed{7} & \boxed{\textcircled{4}} \\ \boxed{8} & 6 & 3 \end{array}\right), \quad B = \left(\begin{array}{ccc} 2 & 3 & \boxed{9} \\ \hline 4 & 5 & \boxed{\textcircled{6}} \\ \hline 7 & \boxed{8} & 6 \end{array}\right)$$

Further, we can weaken monotonicity as follows.

Definition 3.5. (Sequentially monotone bimatrix game) *We say payoff matrix A sequentially monotone if there exists a sequence of best responses $j^k \in J(k)$ such that $\varepsilon_2 j^k \leqq \varepsilon_2 j^{k+1}$ for any $k = 1, \ldots, m_1 - 1$. We say payoff matrix B sequentially monotone if there exists a sequence of best responses $i^k \in I(k)$ such that $\varepsilon_1 i^k \leqq \varepsilon_1 i^{k+1}$ for any $k = 1, \ldots, m_2 - 1$. When both A and B are sequentially monotone, we say the bimatrix game sequentially monotone.*

It is obvious that any monotone bimatrix game is sequentially monotone.

COROLLARY 3.6. *Any sequentially monotone bimatrix game has a Nash equilibrium of pure strategies.*

Proof. Define the initial point x^1 by $(\min\{1, m_1\}, \min\{1, m_2\})$. Then x^1 is a minimum point of $S = \{1, \dots, m_1\} \times \{1, \dots, m_2\}$ w.r.t. the signed component-wise order. Suppose that we have obtained $x^k = (i, j) \in S$. Then, by sequential monotonicity, there exist $i^j \in I(j)$ and $j^i \in J(i)$ such that $(\varepsilon_1 i, \varepsilon_2 j) \leqq (\varepsilon_1 i^j, \varepsilon_2 j^i)$, which implies $(i, j) \trianglelefteq (i^j, j^i)$. Define $x^{k+1} := (i^j, j^i)$, then $x^{k+1} \in I(j) \times J(i) = f(i, j) = f(x^k)$ and $x^k \trianglelefteq x^{k+1}$. Therefore, by Theorem 2.2, the bimatrix game has a Nash equilibrium of pure strategies. ▯

Example 3.4. Although matrix A below is not monotone for $(\varepsilon_1, \varepsilon_2) = (1, 1)$, it is sequentially monotone. In fact, asterisked numbers give a sequence of best responses in Definition 3.5.

$$A = \left(\begin{array}{c|c|c|c} 5 & 2 & 1 & \boxed{9} \\ \boxed{8^*} & \boxed{7^*} & \boxed{\textcircled{4}^*} & 5 \\ 4 & \boxed{7} & 3 & 8 \\ \boxed{8} & 6 & 2 & \boxed{9^*} \end{array} \right), \quad B = \left(\begin{array}{cccc} \boxed{7} & 2 & 6 & 3 \\ \hline 3 & 5 & \boxed{\textcircled{9}} & 4 \\ \hline \boxed{8} & 6 & \boxed{8} & 5 \\ \hline 1 & \boxed{3} & \boxed{3} & 2 \end{array} \right).$$

4. Concluding remarks. In this section, we compare our discrete fixed point theorems to Iimura-Murota-Tamura's [2]. Throughout this section, V is a subset of $\mathbb{Z}^n$ and $f : V \to V$ is a nonempty set-valued mapping, and we use the following notation.

- $\lceil \cdot \rceil$ is a rounding up to the nearest integer.
- $\lfloor \cdot \rfloor$ is a rounding down to the nearest integer.
- Let $N(y) := \{z \in \mathbb{Z}^n \,;\, \lfloor y \rfloor \leqq z \leqq \lceil y \rceil\}$ for all $y \in \mathbb{R}^n$. It is called the integral neighbourhood.
- We denote by $\operatorname{co} V$ the convex hull of V.
- $\|y\|_2 := (\sum_{i=1}^n y_i^2)^{1/2}$ and $\|y\|_\infty := \max\{|y_i| \,;\, i \in N\}$ for $y \in \mathbb{R}^n$.
- For $x^1, x^2 \in \mathbb{Z}^n$, $x^1 \simeq x^2$ is defined by $\|x^1 - x^2\|_\infty \leqq 1$.

We say V integrally convex if

$$y \in \operatorname{co}(V \cap N(y)) \text{ for all } y \in \operatorname{co} V,$$

see e.g. [2][3]. For each $x \in V$, $\pi_f(x)$ denotes the projection of x onto $\operatorname{co} f(x)$, that is,

$$\|\pi_f(x) - x\|_2 = \min_{y \in \operatorname{co} f(x)} \|y - x\|_2.$$

We say f direction preserving if for any $x, y \in V$ with $x \simeq y$

$$x_i < (\pi_f(x))_i \;\Rightarrow\; y_i \leqq (\pi_f(y))_i \quad \forall i = 1, 2, \dots, n. \tag{4.1}$$

THEOREM 4.1. ([2, Theorem 2]) *Let V be a nonempty finite integrally convex set. If f is a nonempty- and discretely convex-valued direction preserving set-valued mapping. Then f has a fixed point.*

For the sake of simplicity, we consider the case where f is single-valued. Then $\pi_f(x) = f(x)$ for any x. So (4.1) reduces to

$$x_i < f_i(x) \;\Rightarrow\; y_i \leqq f_i(y) \quad \forall i = 1, 2, \dots, n. \tag{4.2}$$

On the other hand, (2.1) in Theorem 2.1 reduces to

$$x \leq f(x) \;\Rightarrow\; f(x) \leqq f(f(x)). \tag{4.3}$$

In Figures 4.1 and 4.2, V consists of sixteen points, so that it is integrally convex.

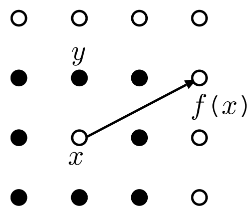

Fig. 4.1.

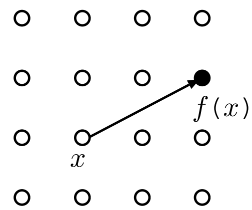

Fig. 4.2.

When we apply Theorem 4.1, we have to test (4.2) for eight solid points in Figure 4.1. On the other hands, when we apply our results, it suffices to test (4.3) only for one point in Figure 4.2. This is an advantages of our results. Another advantage is that we don't impose any convexity assumption on V. Yang [6] extended Theorem 4.1 by introducing a local gross direction preserving correspondence, which is weaker than direction preserving correspondence. However, when we apply his theorem, we also need information on eight solid points in Figure 4.1, see [6, Definition 4.6, Theorem 3.12] for details.

On the other hand, Tarski [5] provided some fixed point theorems on a complete lattice, that is, every subset of the lattice has a least upper bound and a greatest lower bound, see [5, pp. 285, Theorem 1]. Further he assumed that a mapping f is increasing, that is, $x \preceq y$ implies $f(y) \leqq f(y)$. These assumptions seem restrictive. For example, when V is equipped with the component-wise order and has a hole as in Figure 4.3, it is not a complete lattice. In fact, let U consist of two solid points. Then gray points are upper bounds. However, U has no least upper bound. Our theorems

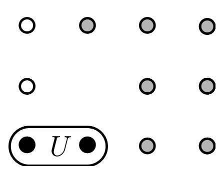

Fig. 4.3.

are more flexible. They can deal with not only the above V but also the case that the best response is not unique. So Examples 3.1, 3.2, and 3.4 are outside of Tarski's scope.

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