Hypergeometric solutions to the q-Painlevé equation of {type A^(1)}_4

Hamamoto, Taro
Graduate School of Mathematics, Kyushu University

Kajiwara, Kenji
Graduate School of Mathematics, Kyushu University

Witte, Nicholas S.

https://hdl.handle.net/2324/3399

KYUSHU UNIVERSITY

Hypergeometric solutions
to the q-Painlevé equation
of type $A_4^{(1)}$

T. Hamamoto & K. Kajiwara

Faculty of Mathematics

Kyushu University
Fukuoka, JAPAN

Hypergeometric solutions to the q-Painlevé equation of type $A_4^{(1)}$

Taro Hamamoto and Kenji Kajiwara

Graduate School of Mathematics, Kyushu University, 6-10-1 Hakozaki, Fukuoka 812-8581, Japan

Abstract. We consider the q-Painlevé equation of type $A_4^{(1)}$ (a version of q-Painlevé V equation) and construct a family of solutions expressible in terms of certain basic hypergeometric series. We also present the determinant formula for the solutions.

1. Introduction

In this article we consider the q-difference equation

$$
\begin{cases}
\overline{g}g = \dfrac{qt}{b_2}\,\dfrac{(f+b_3)(f+1)}{f+\dfrac{1}{s}}, \\[3ex]
\overline{f}f = \dfrac{1}{s}\,\dfrac{\left(\overline{g}+\dfrac{1}{b_2}\right)\left(\overline{g}+\dfrac{1}{b_1 b_2}\right)}{\overline{g}+\dfrac{qt}{b_2}},
\end{cases}
\tag{1.1}
$$

$$
\overline{b}_i = b_i\ (i=1,2,3), \quad \overline{t}=qt, \quad s=\frac{1}{qb_1 b_2 b_3 t},
$$

where q is a constant and $^{-}$ denotes the discrete time evolution. (1.1) can be also expressed as

$$
\begin{cases}
\overline{y}y = \dfrac{\left(x+\dfrac{a_1}{z}\right)\left(x+\dfrac{1}{a_1 z}\right)}{1+a_3 x}, \\[3ex]
x\underline{x} = \dfrac{\left(y+\dfrac{a_2}{\rho}\right)\left(y+\dfrac{1}{a_2 \rho}\right)}{1+\dfrac{y}{a_3}},
\end{cases}
\tag{1.2}
$$

$$
\overline{a}_i = a_i\ (i=1,2,3), \quad \overline{z}=qz, \quad z=q^{\frac{1}{2}}\rho,
$$

where the variables are related as

$$
b_1=a_2^2, \quad b_2=\frac{1}{q^{\frac{1}{2}}a_1 a_2 a_3^2}, \quad b_3=a_1^2, \quad t=\frac{a_3}{q^{\frac{1}{2}}a_2}z, \quad f=a_1 zx, \quad g=a_1 a_3^2 zy. \tag{1.3}
$$

(1.2) was first derived and identified as one of the discrete Painlevé equations with a continuous limit to the Painlevé V equation in [21]. Sakai has classified (1.1) as the discrete dynamical system on the rational surface of type $A_4^{(1)}$ which admits the symmetry of affine Weyl group of type $A_4^{(1)}$ [31]. Geometrical structure of the τ functions on the A_4 weight lattice has been investigated in [28] as well as various Bäcklund transformations. In this article, we denote (1.1) (or (1.2)) as $\mathrm{dP}(A_4^{(1)})$ following the notation that was adopted in [23]. We also write (1.1) as $\mathrm{dP}(A_4^{(1)})[b_1, b_2, b_3]$ when it is necessary to specify the values of parameters explicitly.

It is well-known that the Painlevé and discrete Painlevé equations admit two classes of particular solutions; hypergeometric solutions and algebraic solutions. In particular, the determinant formula for the hypergeometric solutions play an important role in applications, for example, to the area related to matrix integration, such as random matrix theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 25, 32]. The simplest hypergeometric solution to $\mathrm{dP}(A_4^{(1)})$ has been obtained in [15, 16, 29]. The purpose of this article is to construct hypergeometric solutions to $\mathrm{dP}(A_4^{(1)})$ (1.1) and present the determinant formula. In section 2, we construct the simplest hypergeometric solution through the Riccati equation which is reduced from (1.1) by imposing a condition on the parameters. By applying a Bäcklund transformation we construct complex hypergeometric solutions and present the determinant formula in section 3. We give the proof in section 4.

2. Riccati solution

We first recall the definition of the basic hypergeometric series[11]

$$
{}_r\phi_s \left[\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; \ q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n (q; q)_n} \left[(-1)^n q^{\frac{n(n-1)}{2}} \right]^{1+s-r} z^n, \qquad (2.1)
$$

where

$$
(a_1, \ldots, a_r; q)_n = \prod_{i=1}^{r} (a_i; q)_n, \quad (a; q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}). \qquad (2.2)
$$

The simplest solution that is expressible in terms of the basic hypergeometric function is constructed by looking for the special case in which $\mathrm{dP}(A_4^{(1)})$ (1.1) is reduced to the Riccati equation. In fact, imposing the condition on (1.1)

$$
b_2 = 1 \qquad (2.3)
$$

then it admits a specialization $1 + f + g = 0$ to yield the Riccati equation

$$
\overline{g} = -qt \frac{g + 1 - b_3}{g + 1 - qtb_1b_3} \ , \ \overline{f} = -(1 + \overline{g}) = \frac{qtb_3(1 - b_1) + (qt - 1)f}{qtb_1b_3 + f}. \qquad (2.4)
$$

Linearizing the Riccati equation (2.4) by the standard technique, one obtains the following solution (see also [29, 15, 16]):

Proposition 2.1 *Let $\psi = \psi(t, b_1, b_3)$ a function satisfying*

$$
\psi(qt, b_1, b_3) = b_1 \psi(t, b_1, b_3) + (1 - b_1) \ \psi(qt, b_1/q, b_3), \qquad (2.5)
$$

$$
b_3 \psi(qt, b_1, b_3) = \psi(t, b_1, b_3) + (b_3 - 1) \ \psi(t, b_1, qb_3), \qquad (2.6)
$$

$$
qtb_1b_3 \psi(qt, b_1, b_3) = (qtb_1 - 1) \ \psi(t, b_1, b_3) + \psi(t, qb_1, b_3), \qquad (2.7)
$$

$$
qt\psi(qt, b_1, b_3) = (qtb_1 - 1) \ \psi(t, b_1, b_3) + \psi(qt, b_1, b_3/q). \qquad (2.8)
$$

Then

$$f = qtb_3(1 - b_1)\frac{\psi(qt, b_1/q, qb_3)}{\psi(t, b_1, qb_3)}, \quad g = -\frac{\psi(t, b_1, b_3)}{\psi(t, b_1, qb_3)}, \tag{2.9}$$

gives a solution of $dP(A_4^{(1)})$ (1.1) with $b_2 = 1$.

It should be remarked that several basic hypergeometric functions satisfy the contiguous relations (2.5)-(2.8)[22]. For example, we have

(i) $\psi(t, b_1, b_3) = {}_2\phi_1 \left[\begin{array}{c} 1/b_1, b_3 \\ 0 \end{array} ; q, qtb_1 \right],$ (2.10)

(ii)

$$\psi(t, b_1, b_3) = \frac{(qt, 1/t, b_3; q)_\infty}{(qtb_1, b_1b_3; q)_\infty} {}_2\phi_1 \left[\begin{array}{c} q/b_3, 0 \\ q/b_1b_3 \end{array} ; q, 1/tb_1 \right] \tag{2.11}$$

$$= \frac{(qt, 1/t, b_3; q)_\infty}{(qtb_1, b_1b_3; q)_\infty} {}_1\phi_1 \left[\begin{array}{c} b_3/q \\ b_1b_3/q \end{array} ; 1/q, 1/qt \right], \tag{2.12}$$

(iii) $\psi(t, b_1, b_3) = \frac{(b_3t, q/b_3t; q)_\infty}{(qtb_1, qb_1, q/b_3; q)_\infty} {}_2\phi_1 \left[\begin{array}{c} b_3/q, 1/qb_1 \\ 0 \end{array} ; 1/q, tb_1 \right].$ (2.13)

In order to prove Proposition 2.1 we use the following Lemma:

Lemma 2.2 *$\psi(t, b_1, b_3)$ satisfy the contiguous relations*

$$qtb_3\psi(qt, b_1, qb_3) = \psi(t, b_1, b_3) - (1 - qtb_1b_3)\ \psi(t, b_1, qb_3), \tag{2.14}$$

$$tb_3\psi(qt, b_1, b_3) = \psi(t/q, qb_1, b_3) - (1 - t)\ \psi(t, b_1, b_3). \tag{2.15}$$

Proof of Lemma 2.2. Eliminating $\psi(t, b_1, b_3)$ from (2.5) and (2.7) we have

$$(qtb_1b_3 - qt + \frac{1}{b_1})\ \psi(qt, b_1, b_3) + (qtb_1 - 1)(\frac{1}{b_1} - 1)\psi(qt, b_1/q, b_3) = \psi(t, qb_1, b_3).(2.16)$$

Similarly, eliminating $\psi(qt, b_1/q, b_3)$ from (2.5) and $(2.7)_{b_1 \to b_1/q}$ we obtain

$$tb_1b_3\psi(qt, b_1, b_3) - (tb_1{}^2b_3 + 1 - b_1)\ \psi(t, b_1, b_3) = (1 - b_1)(tb_1 - 1)\psi(t, b_1/q, b_3). \tag{2.17}$$

Then eliminating $\psi(qt, b_1/q, b_3)$ from (2.16) and $(2.17)_{t \to qt}$ we get

$$(1 - qt)\ \psi(qt, b_1, b_3) + qtb_3\ \psi(q^2t, b_1, b_3) = \psi(t, qb_1, b_3),$$

which is nothing but $(2.15)_{t \to qt}$. Similarly, (2.14) can be derived by eliminating $\psi(qt, b_1, b_3)$ from (2.6) and $(2.8)_{b_3 \to qb_3}$. $\square$

Proposition 2.1 follows immediately from Lemma 2.2. In fact, dividing (2.6) by (2.14) we have

$$\frac{1}{qt}\frac{\psi(qt, b_1, b_3)}{\psi(qt, b_1, qb_3)} = -\frac{\bar{g}}{qt} = \frac{\psi(t, b_1, b_3) + (b_3 - 1)\ \psi(t, b_1, qb_3)}{\psi(t, b_1, b_3) - (1 - qtb_1b_3)\ \psi(t, b_1, qb_3)} = \frac{-g + (b_3 - 1)}{-g - (1 - qtb_1b_3)},$$

which is the first equation of (2.4). The second equation of (2.4) can be derived in similar manner by dividing $(2.15)_{t \to qt, b_1 \to b_1/q, b_3 \to qb_3}$ by $(2.5)_{b_3 \to qb_3}$. $\square$

3. Determinant formula and bilinear equations

3.1. Bäcklund transformations

Sakai constructed the following transformations for the homogeneous variables x, y, z of $\mathbb{P}^2$ and the parameters b_i $(i = 0, 1, 2, 3, 4)$ on the $A_4^{(1)}$ type (*Mul.5*) surface[31]‡:

$$\sigma : \begin{pmatrix} b_1\ b_2\ b_3 \\ \quad b_4\ b_0 \end{pmatrix} ;\ x : y : z \Big) \mapsto$$

$$\begin{pmatrix} b_3\ b_4\ b_0 \\ \quad b_1\ b_2 \end{pmatrix} ;\ b_4xy(z+x) : b_2z(x+y+z)(x+b_4y+z) : x(x+z)^2 \Big), \tag{3.1}$$

$$\sigma' : \begin{pmatrix} b_1\ b_2\ b_3 \\ \quad b_4\ b_0 \end{pmatrix} ;\ x : y : z \Big) \mapsto$$

$$\begin{pmatrix} \frac{1}{b_1}\ \frac{1}{b_0}\ \frac{1}{b_4} \\ \quad \frac{1}{b_3}\ \frac{1}{b_2} \end{pmatrix} ;\ b_2z(x+z)(x+y+z) : y((z+x)(b_0x+b_2z)+b_2yz) : b_0x(x+z)^2 \Big) \tag{3.2}$$

$$w_3 : \begin{pmatrix} b_1\ b_2\ b_3 \\ \quad b_4\ b_0 \end{pmatrix} ;\ x : y : z \Big) \mapsto$$

$$\begin{pmatrix} b_1\ b_2b_3\ \frac{1}{b_3} \\ \quad b_3b_4\ b_0 \end{pmatrix} ;\ b_3x(b_3x+y+b_3z) : y(b_3x+y+z) : b_3z(x+y+z) \Big), \tag{3.3}$$

$$w_1 : \begin{pmatrix} b_1\ b_2\ b_3 \\ \quad b_4\ b_0 \end{pmatrix} ;\ x : y : z \Big) \mapsto \begin{pmatrix} \frac{1}{b_1}\ b_1b_2\ b_3 \\ \quad b_4\ b_1b_0 \end{pmatrix} ;\ x : y : z \Big), \tag{3.4}$$

$$w_2 : \begin{pmatrix} b_1\ b_2\ b_3 \\ \quad b_4\ b_0 \end{pmatrix} ;\ x : y : z \Big) \mapsto \begin{pmatrix} b_1b_2\ \frac{1}{b_2}\ b_2b_3 \\ \quad b_4\ b_0 \end{pmatrix} ;\ x : b_2y : b_2z \Big), \tag{3.5}$$

$$w_4 : \begin{pmatrix} b_1\ b_2\ b_3 \\ \quad b_4\ b_0 \end{pmatrix} ;\ x : y : z \Big) \mapsto \begin{pmatrix} b_1\ b_2\ b_3b_4 \\ \quad \frac{1}{b_4}\ b_4b_0 \end{pmatrix} ;\ x : b_4y : z \Big), \tag{3.6}$$

$$w_0 = \sigma^2 \circ w_1 \circ \sigma^3. \tag{3.7}$$

Introducing the variables f and g by

$$f = \frac{y}{z+x}, \quad g = \frac{z(x+y+z)}{x(z+x)}, \tag{3.8}$$

then (3.1) - (3.7) can be rewritten as

$$\sigma : (b_0, b_1, b_2, b_3, b_4, f, g) \mapsto (b_2, b_3, b_4, b_0, b_1, b_2g, \tfrac{1+b_2g}{b_4f}),$$

$$\sigma' : (b_0, b_1, b_2, b_3, b_4, g) \mapsto (\tfrac{1}{b_1}, \tfrac{1}{b_0}, \tfrac{1}{b_4}, \tfrac{1}{b_3}, \tfrac{1}{b_2}, \tfrac{b_0(1+g)}{b_2f}),$$

$$w_0 : (b_0, b_1, b_4, f, g) \mapsto (\tfrac{1}{b_0}, b_0b_1, b_0b_4, \tfrac{f(b_0+b_2g)}{b_0(1+b_2g)}, \tfrac{g}{b_0}),$$

$$w_1 : (b_0, b_1, b_2) \mapsto (b_0b_1, \tfrac{1}{b_1}, b_1b_2), \tag{3.9}$$

$$w_2 : (b_1, b_2, b_3, f, g) \mapsto (b_1b_2, \tfrac{1}{b_2}, b_2b_3, b_2f\tfrac{1+f+g}{1+f+b_2g}, b_2g\tfrac{1+b_2f+b_2g}{1+f+b_2g}),$$

$$w_3 : (b_2, b_3, b_4, f, g) \mapsto (b_2b_3, \tfrac{1}{b_3}, b_3b_4, \tfrac{f}{b_3}, \tfrac{g}{b_3}),$$

$$w_4 : (b_0, b_3, b_4, f, g) \mapsto (b_0b_4, b_3b_4, \tfrac{1}{b_4}, b_4f, \tfrac{g(1+b_4f)}{1+f}),$$

respectively, where the abbreviated variables are invariant with respect to the transformation. It can be shown by direct calculation that these transformations

‡ Actions of these transformations are slightly modified from the original formula to be subtraction-free.

satisfy the fundamental relation of the (extended) affine Weyl group $\widetilde{W}(A_4^{(1)})$:

$$w_i^2 = 1, \ (w_i w_{i\pm 1})^3 = 1, \ (w_i w_j)^2 = 1 \ (j \not\equiv i, i \pm 1), \ \sigma^5 = 1, \ \sigma'^2 = 1,$$
$$\sigma w_i = w_{i+2}\sigma, \ \sigma' w_0 = w_2\sigma', \ \sigma' w_3 = w_4\sigma', \ \sigma' w_1 = w_1\sigma', \ i \in \mathbb{Z}/5\mathbb{Z}. \tag{3.10}$$

We note that $q = 1/(b_0 b_1 b_2 b_3 b_4)$ is invariant with respect to the Weyl group actions. The translation $T_0 = w_4 w_3 w_2 w_1 \sigma^3$ acts on b_i as

$$T_0 : (b_0, b_1, b_2, b_3, b_4) \mapsto (qb_0, b_1, b_2, b_3, b_4/q), \tag{3.11}$$

and the action on f and g is nothing but $\mathrm{dP}(A_4^{(1)})$(1.1) for $t = b_0$ and $s = b_4$. If we define the translations T_i $(i = 1, 2, 3, 4)$ by

$$T_1 = \sigma^3 T_0 \sigma^2, \quad T_2 = \sigma T_0 \sigma^4, \quad T_3 = \sigma^4 T_0 \sigma, \quad T_4 = \sigma^2 T_0 \sigma^3, \tag{3.12}$$

then actions of T_i $(i = 1, 2, 3, 4)$ on the parameters are given by

$$\begin{aligned}
T_1 &: (b_0, b_1, b_2, b_3, b_4) \mapsto (b_0/q, qb_1, b_2, b_3, b_4), \\
T_2 &: (b_0, b_1, b_2, b_3, b_4) \mapsto (b_0, b_1/q, qb_2, b_3, b_4), \\
T_3 &: (b_0, b_1, b_2, b_3, b_4) \mapsto (b_0, b_1, b_2/q, qb_3, b_4), \\
T_4 &: (b_0, b_1, b_2, b_3, b_4) \mapsto (b_0, b_1, b_2, b_3/q, qb_4).
\end{aligned} \tag{3.13}$$

and one can directly verify that $T_i T_j = T_j T_i$ $(i, j = 0, 1, 2, 3, i \neq j)$ and $T_0 T_1 T_2 T_3 T_4 = 1$. Therefore if we regard T_0 as the discrete time evolution, T_i $(i = 1, 2, 3, 4)$ can be regarded as the Bäcklund transformations.

3.2. Determinant formula

Let us apply the Bäcklund transformation T_2 on the Riccati solution obtained in Proposition 2.1. Applying T_2 N times yields the solution for $\mathrm{dP}(A_4^{(1)})[q^{-N}b_1, q^N, b_3]$, which is expressed as rational function in ψ. However the denominator and numerator can be factorized into two factors, respectively, and each factor admits determinant formula. More precisely, we have the following formula, which is the main result of this article:

Theorem 3.1 *Let* $\tau_N(t, b_1, b_3)$ *$(N \in \mathbb{Z})$ be*

$$\tau_N(t, b_1, b_3) = \begin{cases} \det(\psi(t, q^{-j+1}b_1, q^{i-1}b_3))_{i,j=1,\ldots,N} & (N > 0), \\ 1 & (N = 0), \\ \det(\psi(t, q^{j-1}b_1, q^{-i+1}b_3))_{i,j=1,\ldots,M} & (N = -M < 0), \end{cases} \tag{3.14}$$

Then

$$f = \begin{cases} q^{N+1}tb_3(1 - q^{-N}b_1) \dfrac{\tau_N(t, b_1, qb_3) \, \tau_{N+1}(qt, b_1/q, qb_3)}{\tau_N(qt, b_1/q, qb_3) \, \tau_{N+1}(t, b_1, qb_3)} & (N \geq 0), \\[14pt] -\dfrac{\tau_N(t, qb_1, b_3) \, \tau_{N+1}(qt, b_1, b_3)}{\tau_N(qt, b_1, b_3) \, \tau_{N+1}(t, qb_1, b_3)} & (N < 0), \end{cases}$$

$$g = \begin{cases} -\dfrac{\tau_N(t, b_1/q, q^2 b_3) \, \tau_{N+1}(t, b_1, b_3)}{\tau_N(t, b_1/q, qb_3) \, \tau_{N+1}(t, b_1, qb_3)} & (N \geq 0), \\[14pt] qt(b_3 - 1) \dfrac{\tau_N(t, b_1, qb_3) \, \tau_{N+1}(t, qb_1, b_3/q)}{\tau_N(t, b_1, b_3) \, \tau_{N+1}(t, qb_1, b_3)} & (N < 0), \end{cases} \tag{3.15}$$

satisfy $\mathrm{dP}(A_4^{(1)})[q^{-N}b_1, q^N, b_3]$.

We introduce a notation for simplicity

$$\tau_N(t, q^m b_1, q^n b_3) = \tau_N^{m,n}(t). \tag{3.16}$$

Theorem 3.1 for $N \geq 0$ is a direct consequence of the following Proposition:

Proposition 3.2 *For $N \geq 0$, $\tau_N^{m,n}(t)$ satisfy the following bilinear difference equations.*

$$(1 - q^{-N} b_1)\tau_N^{0,1}(t) \; \tau_{N+1}^{-1,1}(qt) + q^{-N} b_1 \tau_N^{-1,1}(qt) \; \tau_{N+1}^{0,1}(t)$$
$$- q^{-N} \tau_N^{-1,1}(t) \; \tau_{N+1}^{0,1}(qt) = 0, \tag{3.17}$$

$$qt b_3 (1 - q^{-N} b_1)\tau_N^{0,1}(t) \; \tau_{N+1}^{-1,1}(qt) + q^{-N} \tau_N^{-1,1}(qt) \; \tau_{N+1}^{0,1}(t)$$
$$- \tau_N^{-1,2}(qt) \; \tau_{N+1}^{0,0}(t) = 0, \tag{3.18}$$

$$qt(1 - q^{-N} b_1)\tau_N^{0,1}(t) \; \tau_{N+1}^{-1,1}(qt) + q^{-N} \tau_N^{-1,1}(qt) \; \tau_{N+1}^{0,1}(t)$$
$$- q^{-N} \tau_N^{-1,2}(t) \; \tau_{N+1}^{0,0}(qt) = 0, \tag{3.19}$$

$$q^{-N} \tau_N^{-1,1}(t)\tau_{N+1}^{0,1}(t) - q^{-N} b_1 \tau_N^{-1,2}(t)\tau_{N+1}^{0,0}(t)$$
$$- (1 - q^{-N} b_1)(1 - q^{-N} tb_1)\tau_N^{0,1}(t)\tau_{N+1}^{-1,1}(t) = 0, \tag{3.20}$$

$$q^{-N} \tau_N^{-1,1}(t) \; \tau_{N+1}^{0,1}(t) - \tau_N^{-1,2}(t) \; \tau_{N+1}^{0,0}(t)$$
$$+ q^{N+1} tb_3 (1 - q^{-N} b_1) \; \tau_N^{0,1}(t/q) \; \tau_{N+1}^{-1,1}(qt) = 0, \tag{3.21}$$

$$q^{-N} t \; \tau_N^{-1,1}(t) \; \tau_{N+1}^{0,1}(t) - \tau_N^{-1,2}(t) \; \tau_{N+1}^{0,0}(t)$$
$$+ (1 - q^{-N} tb_1) \; \tau_N^{-1,1}(qt) \; \tau_{N+1}^{0,1}(t/q) = 0. \tag{3.22}$$

In fact, Theorem 3.1 for $N \geq 0$ can be derived from Proposition 3.2 as follows. We have from (3.17) by using (3.15)

$$f + \frac{1}{s} = qt b_3 \frac{\tau_N^{-1,1}(t) \; \tau_{N+1}^{0,1}(qt)}{\tau_N^{-1,1}(qt) \; \tau_{N+1}^{0,1}(t)}. \tag{3.23}$$

We also have from (3.18) and (3.19)

$$f + 1 = q^N \frac{\tau_N^{-1,2}(qt) \; \tau_{N+1}^{0,0}(t)}{\tau_N^{-1,1}(qt) \; \tau_{N+1}^{0,1}(t)}, \tag{3.24}$$

$$f + b_3 = b_3 \frac{\tau_N^{-1,2}(t) \; \tau_{N+1}^{0,0}(qt)}{\tau_N^{-1,1}(qt) \; \tau_{N+1}^{0,1}(t)}, \tag{3.25}$$

respectively. Therefore we obtain

$$\frac{(f+1)(f+b_3)}{f + \frac{1}{s}} = \frac{q^{N-1}}{t} \frac{\tau_N^{-1,2}(qt)\tau_{N+1}^{0,0}(qt)\tau_{N+1}^{0,0}(t)\tau_N^{-1,2}(t)}{\tau_N^{-1,1}(qt)\tau_{N+1}^{0,1}(qt)\tau_{N+1}^{0,1}(t)\tau_N^{-1,1}(t)} = \frac{q^{N-1}}{t} \bar{g}g, \tag{3.26}$$

which is the first equation of (1.1). Similarly, from $(3.20)_{t \to qt}$, $(3.21)_{t \to qt}$ and $(3.22)_{t \to qt}$ we get

$$1 + b_1 \bar{g} = q^N (1 - q^{-N} b_1)(1 - q^{-N+1} tb_1)\frac{\tau_N^{0,1}(qt)\tau_{N+1}^{-1,1}(qt)}{\tau_N^{-1,1}(qt)\tau_{N+1}^{0,1}(qt)}, \tag{3.27}$$

$$1 + q^N \bar{g} = -q^{2N+2} tb_3 (1 - q^{-N} b_1)\frac{\tau_N^{0,1}(t)\tau_{N+1}^{-1,1}(q^2 t)}{\tau_N^{-1,1}(qt)\tau_{N+1}^{0,1}(qt)}, \tag{3.28}$$

$$qt + q^N \bar{g} = -q^N (1 - q^{-N+1} tb_1)\frac{\tau_N^{-1,1}(q^2 t)\tau_{N+1}^{0,1}(t)}{\tau_N^{-1,1}(qt) \; \tau_{N+1}^{0,1}(qt)}, \tag{3.29}$$

respectively. Then we have

$$qtb_3 \frac{(1 + b_1\overline{g})(1 + q^N\overline{g})}{qt + q^N\overline{g}} = \overline{f}f, \tag{3.30}$$

which is the second equation of (1.1). Therefore we have verified that Theorem 3.1 for $N \geq 0$ follows from the bilinear equations (3.17) - (3.22) in Proposition 3.2. We omit the proof for the case of $N < 0$ since it can be proved in similar manner.

4. Proof of Proposition 3.2

The bilinear equations (3.17) - (3.22) can be reduced to the Plücker relations which are quadratic identities among the determinants whose columns are properly shifted. This can be done by constructing "difference formulae" that relate the "shifted" determinants with $\tau_N^{m,n}(t)$ by using the contiguous relations of ψ. This technique has been developed in [26, 27] and applied to various discrete Painlevé equations[12, 13, 14, 17, 18, 19, 20, 24, 25, 30]. In this section, we prove the bilinear equation (3.17) as an example. Since other bilinear equations (3.18)-(3.22) can be proved in similar manner, we leave the details in the appendix.

We first introduce the following notation:

$$\tau_N^{m,n}(t) = \begin{vmatrix} \psi(t, q^m b_1, q^n b_3) & \psi(t, q^{m-1} b_1, q^n b_3) & \cdots & \psi(t, q^{m-N+1} b_1, q^n b_3) \\ \psi(t, q^m b_1, q^{n+1} b_3) & \psi(t, q^{m-1} b_1, q^{n+1} b_3) & \cdots & \psi(t, q^{m-N+1} b_1, q^{n+1} b_3) \\ \vdots & \vdots & \cdots & \vdots \\ \psi(t, q^m b_1, q^{n+N-1} b_3) & \psi(t, q^{m-1} b_1, q^{n+N-1} b_3) & \cdots & \psi(t, q^{m-N+1} b_1, q^{n+N-1} b_3) \end{vmatrix}$$

$$= \begin{vmatrix} \Psi_{m,n}(t) & \Psi_{m-1,n}(t) & \cdots & \Psi_{m-N+1,n}(t) \end{vmatrix}, \tag{4.1}$$

where $\Psi_{m,n}(t)$ denotes a column vector

$$\Psi_{m,n}(t) = \begin{pmatrix} \psi(t, q^m b_1, q^n b_3) \\ \psi(t, q^m b_1, q^{n+1} b_3) \\ \vdots \\ \psi(t, q^m b_1, q^{n+N-1} b_3) \end{pmatrix}. \tag{4.2}$$

Here we note that the height of the column vector is N, but we use the same symbol for the column vector with different height. Then we have the following difference formula:

Lemma 4.1

$$\begin{vmatrix} \Psi_{m,n}(t) & \Psi_{m,n}(t/q) & \cdots & \Psi_{m-N+2,n}(t/q) \end{vmatrix}$$

$$= \frac{\prod_{k=0}^{N-2}(q^{m-k}b_1 - 1)}{q^{\frac{(N-1)(2m-N+2)}{2}} b_1^{N-1}} \tau_N^{m,n}(t), \tag{4.3}$$

$$\begin{vmatrix} \Psi_{m,n}(t/q) & \Psi_{m-1,n}(t) & \Psi_{m-1,n}(t/q) & \cdots & \Psi_{m-N+2,n}(t/q) \end{vmatrix}$$

$$= \frac{\prod_{k=1}^{N-2}(q^{m-k}b_1 - 1)}{q^{\frac{(N-1)(2m-N+2)}{2}} b_1^{N-1}} \tau_N^{m,n}(t). \tag{4.4}$$

Proof. Using the contiguous relation (2.5) on the N-th column of the determinant in (4.1), we have

$$\tau_N^{m,n}(t) = \left| \ \Psi_{m,n}(t) \quad \Psi_{m-1,n}(t) \quad \cdots \quad \Psi_{m-N+2,n}(t) \quad \Psi_{m-N+1,n}(t) \ \right|$$

$$= \left| \ \Psi_{m,n}(t) \quad \cdots \quad \Psi_{m-N+2,n}(t) \quad \frac{-\Psi_{m-N+2,n}(t)+q^{m-N+2}b_1\Psi_{m-N+2,n}(t/q)}{q^{m-N+2}b_1-1} \ \right|$$

$$= \frac{q^{m-N+2}b_1}{q^{m-N+2}b_1-1} \left| \ \Psi_{m,n}(t) \quad \cdots \quad \Psi_{m-N+2,n}(t) \quad \Psi_{m-N+2,n}(t/q) \ \right|.$$

Applying this procedure from the N-th column to the second column we obtain

$$\tau_N^{m,n}(t) = \frac{q^{\frac{(N-1)(2m-N+2)}{2}}b_1^{N-1}}{\displaystyle\prod_{k=0}^{N-2}(q^{m-k}b_1-1)} \left| \ \Psi_{m,n}(t) \quad \Psi_{m,n}(t/q) \quad \cdots \quad \Psi_{m-N+2,n}(t/q) \ \right|, \quad (4.5)$$

which is nothing but (4.3). At the stage where the above procedure has been applied up to the third column, we have by using (2.5) on the first column

$$\tau_N^{m,n}(t)$$

$$= \frac{q^{\frac{(N-2)(2m-N+1)}{2}}b_1^{N-2}}{\displaystyle\prod_{k=1}^{N-2}(q^{m-k}b_1-1)} \left| \ \Psi_{m,n}(t) \quad \Psi_{m-1,n}(t) \quad \Psi_{m-1,n}(t/q) \quad \cdots \quad \Psi_{m-N+2,n}(t/q) \ \right|$$

$$= \frac{q^{\frac{(N-2)(2m-N+1)}{2}}b_1^{N-2}}{\displaystyle\prod_{k=1}^{N-2}(q^{m-k}b_1-1)}$$

$$\times \left| \ q^m b_1 \Psi_{m,n}(t/q) - (q^m b_1 - 1)\Psi_{m-1,n}(t) \quad \Psi_{m-1,n}(t) \quad \Psi_{m-1,n}(t/q) \quad \cdots \quad \Psi_{m-N+2,n}(t/q) \ \right|$$

$$= \frac{q^{\frac{(N-1)(2m-N+2)}{2}}b_1^{N-1}}{\displaystyle\prod_{k=1}^{N-2}(q^{m-k}b_1-1)}$$

$$\times \left| \ \Psi_{m,n}(t/q) \quad \Psi_{m-1,n}(t) \quad \Psi_{m-1,n}(t/q) \quad \cdots \quad \Psi_{m-N+2,n}(t/q) \ \right|, \quad (4.6)$$

which is (4.4). This completes the proof. $\square$

Now consider the Plücker relation

$$0 = \left| \ \Psi_{m+1,n}(t/q) \quad \Psi_{m,n}(t) \quad \Psi_{m,n}(t/q) \quad \cdots \quad \Psi_{m-N+3,n}(t/q) \ \right|$$

$$\times \left| \ \Psi_{m,n}(t/q) \quad \cdots \quad \Psi_{m-N+3,n}(t/q) \quad \Psi_{m-N+2,n}(t/q) \quad \phi \ \right|$$

$$- \left| \ \Psi_{m+1,n}(t/q) \quad \Psi_{m,n}(t/q) \quad \cdots \quad \Psi_{m-N+3,n}(t/q) \quad \Psi_{m-N+2,n}(t/q) \ \right|$$

$$\times \left| \ \Psi_{m,n}(t) \quad \Psi_{m,n}(t/q) \quad \cdots \quad \Psi_{m-N+3,n}(t/q) \quad \phi \ \right|$$

$$+ \left| \ \Psi_{m+1,n}(t/q) \quad \Psi_{m,n}(t/q) \quad \cdots \quad \Psi_{m-N+3,n}(t/q) \quad \phi \ \right|$$

$$\times \left| \ \Psi_{m,n}(t) \quad \Psi_{m,n}(t/q) \quad \cdots \quad \Psi_{m-N+3,n}(t/q) \quad \Psi_{m-N+2,n}(t/q) \ \right|, \quad (4.7)$$

where ϕ is the column vector

$$\phi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \quad (4.8)$$

Applying Lemma 4.1 to (4.7) we have

$$\tau_N^{m+1,n}(t)\,\tau_{N-1}^{m,n}(t/q) - q^{m+1}b_1\,\tau_N^{m+1,n}(t/q)\,\tau_{N-1}^{m,n}(t)$$
$$+ q^{N-1}(1 - q^{m-N+2}b_1)\,\tau_{N-1}^{m+1,n}(t/q)\,\tau_N^{m,n}(t) = 0.$$

Putting $N \to N+1$, $t \to qt$, $m = -1$ and $n = 1$ we obtain (3.17). $\square$

Acknowledgments

Appendix: Proof of bilinear equations

In this appendix we prove the bilinear equations (3.18)-(3.22). We first note that $\tau_N^{m,n}(t)$ admits various determinantal expressions, which play an important role in proving the bilinear equations. Taking the transpose of the right hand side of (4.1), we have

$$\tau_N^{m,n}(t) = \left|\ \widetilde{\Psi}_{m,n}(t) \quad \widetilde{\Psi}_{m,n+1}(t) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-1}(t)\ \right|, \tag{A.1}$$

where

$$\widetilde{\Psi}_{m,n}(t) = \begin{pmatrix} \psi(t, q^m b_1, q^n b_3) \\ \psi(t, q^{m-1} b_1, q^n b_3) \\ \vdots \\ \psi(t, q^{m-N+1} b_1, q^n b_3) \end{pmatrix}. \tag{A.2}$$

It is also possible to express $\tau_N^{m,n}(t)$ by the determinants with different structure of shifts.

Lemma A.1 $\tau_N^{m,n}(t)$ *can be expressed as follows:*

$$\tau_N^{m,n}(t) = \prod_{k=1}^{N-1}\left(\frac{q^{n+k-1}b_3}{q^{n+k-1}b_3 - 1}\right)^{N-k}$$
$$\times \left|\ \widetilde{\Psi}_{m,n}(t) \quad \widetilde{\Psi}_{m,n}(qt) \quad \cdots \quad \widetilde{\Psi}_{m,n}(q^{N-1}t)\ \right| \tag{A.3}$$
$$= \prod_{k=1}^{N-1}\left(\frac{q^{m-k+1}b_1}{q^{m-k+1}b_1 - 1}\right)^{N-k}$$
$$\times \left|\ \check{\Psi}_{m,n}(t) \quad \check{\Psi}_{m,n+1}(t) \quad \cdots \quad \check{\Psi}_{m,n+N-1}(t)\ \right| \tag{A.4}$$
$$= (-1)^{\frac{N(N-1)}{2}}\prod_{k=1}^{N-1}(q^{n+k-1}b_3)^{N-k}\ \prod_{k=1}^{N-1}\left(\frac{q^{m-k+1}b_1}{q^{m-k+1}b_1 - 1}\right)^{N-k}$$
$$\times \left|\ \widehat{\Psi}_{m,n}(t) \quad \widehat{\Psi}_{m,n+1}(q^{-1}t) \quad \cdots \quad \widehat{\Psi}_{m,n+N-1}(q^{-N+1}t)\ \right|, \tag{A.5}$$

where the column vectors are given by

$$\check{\Psi}_{m,n}(t) = \begin{pmatrix} \psi(t, q^m b_1, q^n b_3) \\ \psi(q^{-1}t, q^m b_1, q^n b_3) \\ \vdots \\ \psi(q^{-N+1}t, q^m b_1, q^n b_3) \end{pmatrix}, \quad \widehat{\Psi}_{m,n}(t) = \begin{pmatrix} \psi(t, q^m b_1, q^n b_3) \\ \psi(qt, q^m b_1, q^n b_3) \\ \vdots \\ \psi(q^{N-1}t, q^m b_1, q^n b_3) \end{pmatrix}, \quad (A.6)$$

respectively.

Proof. We prove (A.3). Using the contiguous relation (2.6) to the N-th column of the right hand side of (A.1), we get

$$\tau_N^{m,n}(t) = \left| \begin{array}{ccccc} \widetilde{\Psi}_{m,n}(t) & \widetilde{\Psi}_{m,n+1}(t) & \cdots & \widetilde{\Psi}_{m,n+N-2}(t) & \frac{q^{n+N-2}b_3 \widetilde{\Psi}_{m,n+N-2}(qt) - \widetilde{\Psi}_{m,n+N-2}(t)}{q^{n+N-2}b_3 - 1} \end{array} \right|$$

$$= \frac{q^{n+N-2}b_3}{q^{n+N-2}b_3 - 1} \left| \begin{array}{ccccc} \widetilde{\Psi}_{m,n}(t) & \widetilde{\Psi}_{m,n+1}(t) & \cdots & \widetilde{\Psi}_{m,n+N-2}(t) & \widetilde{\Psi}_{m,n+N-2}(qt) \end{array} \right|.$$

Applying this procedure up to the second column, we have

$$\tau_N^{m,n}(t) = \prod_{k=1}^{N-1} \frac{q^{n+k-1}b_3}{q^{n+k-1}b_3 - 1} \left| \begin{array}{ccccc} \widetilde{\Psi}_{m,n}(t) & \widetilde{\Psi}_{m,n}(qt) & \cdots & \widetilde{\Psi}_{m,n+N-3}(qt) & \widetilde{\Psi}_{m,n+N-2}(qt) \end{array} \right|.$$

Continuing this procedure we obtain

$$\tau_N^{m,n}(t) = \left(\prod_{k=1}^{N-1} \frac{q^{n+k-1}b_3}{q^{n+k-1}b_3 - 1} \right) \times \left(\prod_{k=1}^{N-2} \frac{q^{n+k-1}b_3}{q^{n+k-1}b_3 - 1} \right) \times \cdots \times \left(\frac{q^n b_3}{q^n b_3 - 1} \right)$$

$$\times \left| \begin{array}{ccccc} \widetilde{\Psi}_{m,n}(t) & \widetilde{\Psi}_{m,n}(qt) & \cdots & \widetilde{\Psi}_{m,n}(q^{N-2}t) & \widetilde{\Psi}_{m,n}(q^{N-1}t) \end{array} \right|$$

$$= \prod_{k=1}^{N-1} \left(\frac{q^{n+k-1}b_3}{q^{n+k-1}b_3 - 1} \right)^{N-k}$$

$$\times \left| \begin{array}{ccccc} \widetilde{\Psi}_{m,n}(t) & \widetilde{\Psi}_{m,n}(qt) & \cdots & \widetilde{\Psi}_{m,n}(q^{N-2}t) & \widetilde{\Psi}_{m,n}(q^{N-1}t) \end{array} \right|,$$

which is (A.3). As to (A.4) and (A.5) we omit the details and only describe the method, since one can prove them by the similar calculations. In order to prove (A.4) we use the contiguous relation (2.5) on (4.1) repeatedly. For (A.5) we use (2.5) on (A.3) to express $\tau_N^{m,n}(t)$ by the determinant in which t is shifted in both horizontal and vertical directions. Finally we use (2.6) on this determinant to derive (A.5). □

Now the bilinear equations (3.18)-(3.22) can be proved by the same procedure as that in section 4. Therefore we do not repeat the procedure, but give the list of data which are necessary for proof of each bilinear equation.

(3.18)

(i) Expression of $\tau_N^{m,n}$: (A.1)

(ii) Difference formula:

$$\left| \begin{array}{cccc} \overline{\widetilde{\Psi}_{m,n}(t)} & \widetilde{\Psi}_{m-1,n+1}(qt) & \cdots & \widetilde{\Psi}_{m-1,n+N-1}(qt) \end{array} \right|$$

$$= \frac{1}{q^{\frac{(N-1)(2n+N)}{2}} (tb_3)^{N-1} \prod_{k=0}^{N-1} (q^{m-k}b_1 - 1)} \tau_N^{m,n}(t), \qquad (A.7)$$

$$\left| \begin{array}{cccccc} \widetilde{\Psi}_{m-1,n+1}(qt) & \overline{\widetilde{\Psi}_{m,n+1}(t)} & \widetilde{\Psi}_{m-1,n+2}(qt) & \cdots & \widetilde{\Psi}_{m-1,n+N-1}(qt) \end{array} \right|$$

$$= -\frac{1}{q^{\frac{(N-1)(2n+N)}{2}}(tb_3)^{N-1}\prod\limits_{k=0}^{N-1}(q^{m-k}b_1-1)} \, \tau_N^{m,n}(t), \qquad (A.8)$$

where

$$\overline{\widetilde{\Psi}_{m,n}(t)} = \begin{pmatrix} \frac{1}{q^m b_1 - 1}\psi(t, q^m b_1, q^n b_3) \\ \vdots \\ \frac{1}{q^{m-N+1}b_1 - 1}\psi(t, q^{m-N+1}b_1, q^n b_3) \end{pmatrix}. \qquad (A.9)$$

(iii) Contiguous relation to be used for derivation of difference formula:

$$\psi(t, b_1, qb_3) = \psi(t, b_1, b_3) + qtb_3(b_1 - 1)\,\psi(qt, b_1/q, qb_3). \quad (A.10)$$

(A.10) can be derived by eliminating $\psi(qt, b_1, qb_3)$ from $(2.5)_{b_3 \to qb_3}$ and (2.14).

(iv) Plücker relation:

$$0 = \left| \begin{array}{ccccc} \widetilde{\Psi}_{m-1,n}(qt) & \overline{\widetilde{\Psi}_{m,n}(t)} & \widetilde{\Psi}_{m-1,n+1}(qt) & \cdots & \widetilde{\Psi}_{m-1,n+N-2}(qt) \end{array} \right|$$

$$\times \left| \begin{array}{ccccc} \widetilde{\Psi}_{m-1,n+1}(qt) & \widetilde{\Psi}_{m-1,n+2}(qt) & \cdots & \widetilde{\Psi}_{m-1,n+N-1}(qt) & \phi' \end{array} \right|$$

$$- \left| \begin{array}{cccc} \widetilde{\Psi}_{m-1,n}(qt) & \widetilde{\Psi}_{m-1,n+1}(qt) & \cdots & \widetilde{\Psi}_{m-1,n+N-1}(qt) \end{array} \right|$$

$$\times \left| \begin{array}{ccccc} \overline{\widetilde{\Psi}_{m,n}(t)} & \widetilde{\Psi}_{m-1,n+1}(qt) & \cdots & \widetilde{\Psi}_{m-1,n+N-2}(qt) & \phi' \end{array} \right|$$

$$+ \left| \begin{array}{ccccc} \widetilde{\Psi}_{m-1,n}(qt) & \widetilde{\Psi}_{m-1,n+1}(qt) & \cdots & \widetilde{\Psi}_{m-1,n+N-2}(qt) & \phi' \end{array} \right|$$

$$\times \left| \begin{array}{cccc} \overline{\widetilde{\Psi}_{m,n}(t)} & \widetilde{\Psi}_{m-1,n+1}(qt) & \cdots & \widetilde{\Psi}_{m-1,n+N-1}(qt) \end{array} \right|, \qquad (A.11)$$

where

$$\phi' = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \qquad (A.12)$$

(3.19):

(i) Expression of $\tau_N^{m,n}$: (A.1)

(ii) Difference formula:

$$\left| \begin{array}{cccc} \underline{\widetilde{\Psi}_{m,n}(t)} & \widetilde{\Psi}_{m,n+1}(t/q) & \cdots & \widetilde{\Psi}_{m,n+N-1}(t/q) \end{array} \right|$$

$$= \frac{t^{N-1}}{\prod\limits_{k=0}^{N-1}(q^{m-k}tb_1 - 1)} \, \tau_N^{m,n}(t), \qquad (A.13)$$

$$\left| \begin{array}{cccc} \widetilde{\Psi}_{m,n+1}(t/q) & \underline{\widetilde{\Psi}_{m,n+1}(t)} & \widetilde{\Psi}_{m,n+2}(t/q) & \cdots & \widetilde{\Psi}_{m,n+N-1}(t/q) \end{array} \right|$$

$$= -\frac{t^{N-2}}{\prod\limits_{k=0}^{N-1}(q^{m-k}tb_1 - 1)} \, \tau_N^{m,n}(t), \qquad (A.14)$$

where

$$\underline{\widetilde{\Psi}_{m,n}(t)} = \begin{pmatrix} \frac{1}{q^m t b_1 - 1} \psi(t, q^m b_1, q^n b_1) \\ \vdots \\ \frac{1}{q^{m-N+1} t b_1 - 1} \psi(t, q^{m-N+1} b_1, q^n b_3) \end{pmatrix}. \qquad (A.15)$$

(iii) Contiguous relation to be used for derivation of difference formula: (2.8)

(iv) Plücker relation:

$$0 = \left| \; \widetilde{\Psi}_{m,n}(t/q) \quad \underline{\widetilde{\Psi}_{m,n}(t)} \quad \widetilde{\Psi}_{m,n+1}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-2}(t/q) \; \right|$$

$$\times \left| \; \widetilde{\Psi}_{m,n+1}(t/q) \quad \widetilde{\Psi}_{m,n+2}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-1}(t/q) \quad \phi' \; \right|$$

$$- \left| \; \widetilde{\Psi}_{m,n}(t/q) \quad \widetilde{\Psi}_{m,n+1}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-1}(t/q) \; \right|$$

$$\times \left| \; \underline{\widetilde{\Psi}_{m,n}(t)} \quad \widetilde{\Psi}_{m,n+1}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-2}(t/q) \quad \phi' \; \right|$$

$$+ \left| \; \widetilde{\Psi}_{m,n}(t/q) \quad \widetilde{\Psi}_{m,n+1}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-2}(t/q) \quad \phi' \; \right|$$

$$\times \left| \; \underline{\widetilde{\Psi}_{m,n}(t)} \quad \widetilde{\Psi}_{m,n+1}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-1}(t/q) \; \right|. \qquad (A.16)$$

(v) Derivation of (3.19): applying the difference formula to the Plücker relation we have

$$qt \, \tau_N^{-1,1}(t) \, \tau_{N+1}^{0,1}(qt) + (1 - qt b_1) \, \tau_N^{-1,1}(qt) \, \tau_{N+1}^{0,1}(t) - \tau_N^{-1,2}(t) \, \tau_{N+1}^{0,0}(qt) = 0. \quad (A.17)$$

We obtain (3.19) by eliminating the term $\tau_N^{-1,1}(t) \, \tau_{N+1}^{0,1}(qt)$ from (3.17) and (A.17).

(3.20):

(i) Expression of $\tau_N^{m,n}$: (4.1)

(ii) Difference formula:

$$\left| \; \Psi_{m,n}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+2,n-1}(t) \; \right|$$

$$= \frac{\displaystyle\prod_{k=0}^{N-2} (q^{m-k} b_1 - 1) \, (1 - q^{m-k} t b_1)}{q^{\frac{(N-1)(2m-N+2)}{2}} b_1^{N-1}} \, \tau_N^{m,n}(t), \qquad (A.18)$$

$$\left| \; \Psi_{m,n-1}(t) \quad \Psi_{m-1,n}(t) \quad \Psi_{m-1,n-1}(t) \quad \cdots \quad \Psi_{m-N+2,n-1}(t) \; \right|$$

$$= \frac{\displaystyle\prod_{k=1}^{N-2} (q^{m-k} b_1 - 1) \, (1 - q^{m-k} t b_1)}{q^{\frac{(N-1)(2m-N+2)}{2}} b_1^{N-1}} \, \tau_N^{m,n}(t). \qquad (A.19)$$

(iii) Contiguous relation to be used for derivation of difference formula:

$$\psi(t, b_1/q, b_3) = \frac{\psi(t, b_1, b_3) - b_1 \, \psi(t, b_1, b_3/q)}{(1 - b_1)(1 - t b_1)}. \qquad (A.20)$$

(A.20) can be derived by eliminating $\psi(t/q, b_1, b_3)$ from $(2.5)_{t \to t/q}$ and $(2.8)_{t \to t/q}$.

(iv) Plücker relation:

$$
\begin{aligned}
0 = & \ \big| \ \Psi_{m+1,n-1}(t) \quad \Psi_{m,n}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \ \big| \\
& \times \big| \ \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \quad \Psi_{m-N+2,n-1}(t) \quad \phi' \ \big| \\
& - \big| \ \Psi_{m+1,n-1}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \quad \Psi_{m-N+2,n-1}(t) \ \big| \\
& \times \big| \ \Psi_{m,n}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \quad \phi' \ \big| \\
& + \big| \ \Psi_{m+1,n-1}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \quad \phi' \ \big| \\
& \times \big| \ \Psi_{m,n}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \quad \Psi_{m-N+2,n-1}(t) \ \big|. \quad \text{(A.21)}
\end{aligned}
$$

(3.21):

(i) Expression of $\tau_N^{m,n}$: (A.4)

(ii) Difference formula:

$$
\Big| \ \overline{\overline{\check{\Psi}_{m,n}(t)}} \quad \check{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \check{\Psi}_{m-1,n+N-1}(qt) \ \Big|
$$
$$
= (q^n t b_3 (q^m b_1 - 1))^{1-N} \prod_{k=1}^{N-1} \left(\frac{q^{m-k+1} b_1}{q^{m-k+1} b_1 - 1} \right)^{k-N} \tau_N^{m,n}(t), \quad \text{(A.22)}
$$

$$
\Big| \ \check{\Psi}_{m-1,n+1}(qt) \quad \overline{\overline{\check{\Psi}_{m,n+1}(t)}} \quad \check{\Psi}_{m-1,n+2}(qt) \quad \cdots \quad \check{\Psi}_{m-1,n+N-1}(qt) \ \Big|
$$
$$
= - (q^n t b_3 (q^m b_1 - 1))^{1-N} \prod_{k=1}^{N-1} \left(\frac{q^{m-k+1} b_1}{q^{m-k+1} b_1 - 1} \right)^{k-N} \tau_N^{m,n}(t), \quad \text{(A.23)}
$$

where

$$
\overline{\check{\Psi}_{m,n}(q^l t)} = \begin{pmatrix} \psi(q^l t, q^m b_1, q^n b_3) \\ q\psi(q^{l-1} t, q^m b_1, q^n b_3) \\ \vdots \\ q^{N-1}\psi(q^{l-N+1} t, q^m b_1, q^n b_3) \end{pmatrix}. \quad \text{(A.24)}
$$

(iii) Contiguous relation to be used for derivation of difference formula:

$$
\psi(t, b_1, q b_3) = \psi(t, b_1, b_3) + q t b_3 (b_1 - 1)\, \psi(q t, b_1/q, q b_3). \quad \text{(A.25)}
$$

(A.25) can be derived by eliminating $\psi(qt, b_1, qb_3)$ from $(2.5)_{b_3 \to qb_3}$ and (2.14).

(iv) Plücker relation:

$$
\begin{aligned}
0 = & \ \big| \ \check{\Psi}_{m-1,n}(qt) \quad \overline{\overline{\check{\Psi}_{m,n}(t)}} \quad \check{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \check{\Psi}_{m-1,n+N-2}(qt) \ \big| \\
& \times \big| \ \check{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \check{\Psi}_{m-1,n+N-2}(qt) \quad \check{\Psi}_{m-1,n+N-1}(qt) \quad \phi' \ \big| \\
& - \big| \ \check{\Psi}_{m-1,n}(qt) \quad \check{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \check{\Psi}_{m-1,n+N-2}(qt) \quad \check{\Psi}_{m-1,n+N-1}(qt) \ \big| \\
& \times \big| \ \overline{\overline{\check{\Psi}_{m,n}(t)}} \quad \check{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \check{\Psi}_{m-1,n+N-2}(qt) \quad \phi' \ \big| \\
& + \big| \ \check{\Psi}_{m-1,n}(qt) \quad \check{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \check{\Psi}_{m-1,n+N-2}(qt) \quad \phi' \ \big| \\
& \times \big| \ \overline{\overline{\check{\Psi}_{m,n}(t)}} \quad \check{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \check{\Psi}_{m-1,n+N-1}(qt) \ \big|. \quad \text{(A.26)}
\end{aligned}
$$

(3.22):

(i) Expression of $\tau_N^{m,n}$: (A.5)

(ii) Difference formula:

$$
\left| \; \overline{\widehat{\Psi}_{m,n}(t)} \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-1}(q^{-N+2}t) \; \right|
$$

$$
= (-1)^{\frac{N(N-1)}{2}} \, (qt(q^m b_1 - 1))^{1-N} \prod_{k=1}^{N-1} \left(q^{n+k-1} b_3 \right)^{k-N}
$$

$$
\times \prod_{k=1}^{N-1} \left(\frac{q^{m-k+1} b_1}{q^{m-k+1} b_1 - 1} \right)^{k-N} \tau_N^{m,n}(t), \tag{A.27}
$$

$$
\left| \; \widehat{\Psi}_{m-1,n+1}(t) \quad \overline{\widehat{\Psi}_{m,n+1}(t/q)} \quad \widehat{\Psi}_{m-1,n+2}(t/q) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-1}(q^{-N+2}t) \; \right|
$$

$$
= -(-1)^{\frac{N(N-1)}{2}} \left(qt(q^m b_1 - 1) \right)^{1-N} \prod_{k=1}^{N-1} \left(q^{n+k-1} b_3 \right)^{k-N}
$$

$$
\times \prod_{k=1}^{N-1} \left(\frac{q^{m-k+1} b_1}{q^{m-k+1} b_1 - 1} \right)^{k-N} \tau_N^{m,n}(t), \tag{A.28}
$$

where

$$
\overline{\widehat{\Psi}_{m,n}(q^l t)} = \begin{pmatrix} \psi(q^l t, q^m b_1, q^n b_3) \\ q^{-1} \psi(q^{l+1} t, q^m b_1, q^n b_3) \\ \vdots \\ q^{-N+1} \psi(q^{l+N-1} t, q^m b_1, q^n b_3) \end{pmatrix}. \tag{A.29}
$$

(iii) Contiguous relation to be used for derivation of difference formula:

$$
\psi(t, b_1, b_3) = \psi(qt, b_1, b_3/q) + qt(b_1 - 1)\, \psi(qt, b_1/q, b_3). \tag{A.30}
$$

(A.30) can be derived by eliminating $\psi(qt, b_1, b_3)$ from (2.5) and (2.8).

(iv) Plücker relation:

$$
0 = \left| \; \widehat{\Psi}_{m-1,n}(qt) \quad \overline{\widehat{\Psi}_{m,n}(t)} \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-2}(q^{-N+3}t) \; \right|
$$

$$
\times \left| \; \widehat{\Psi}_{m-1,n+1}(t) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-2}(q^{-N+3}t) \quad \widehat{\Psi}_{m-1,n+N-1}(q^{-N+2}t) \quad \phi' \; \right|
$$

$$
- \left| \; \widehat{\Psi}_{m-1,n}(qt) \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-2}(q^{-N+3}t) \quad \widehat{\Psi}_{m-1,n+N-1}(q^{-N+2}t) \; \right|
$$

$$
\times \left| \; \overline{\widehat{\Psi}_{m,n}(t)} \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-2}(q^{-N+3}t) \quad \phi' \; \right|
$$

$$
+ \left| \; \widehat{\Psi}_{m-1,n}(qt) \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-2}(q^{-N+3}t) \quad \phi' \; \right|
$$

$$
\times \left| \; \overline{\widehat{\Psi}_{m,n}(t)} \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-1}(q^{-N+2}t) \; \right|. \tag{A.31}
$$

(v) Derivation of (3.22): applying the above difference formula to the Plücker relation we have

$$
\tau_N^{-1,1}(qt)\, \tau_{N+1}^{0,1}(t/q) + t(1 - q^{-N} b_1)\, \tau_N^{0,1}(t)\, \tau_{N+1}^{-1,1}(t) - \tau_N^{-1,2}(t)\, \tau_{N+1}^{0,0}(t) = 0. \tag{A.32}
$$

We obtain (3.22) by eliminating the term $\tau_N^{0,1}(t)\, \tau_{N+1}^{-1,1}(t)$ from (3.20) and (A.32).

This completes the proof of Proposition 3.2. $\square$

References

[1] Adler M and van Moerbeke P 2003 Recursion relations for unitary integrals, combinatorics and the Toeplitz lattice *Comm. Math. Phys.* **237** 397–440.

[2] Borodin A and Boyarchenko D 2003 Distribution of the first particle in discrete orthogonal polynomial ensembles *Comm. Math. Phys.* **234** 287–338.

[3] Borodin A 2003 Discrete gap probabilities and discrete Painlevé equations *Duke Math. J.* **117** 489–542.

[4] Chen Y and Feigin M V 2006 Painlevé IV and degenerate Gaussian unitary ensembles *J. Phys. A: Math. Gen.* **39** 12381-12393.

[5] Forrester P J 2003 Growth models, random matrices and Painlevé transcendents *Nonlinearity* **16** R27–R49.

[6] Forrester P J. and Witte N S 2006 Bi-orthogonal polynomials on the unit circle, regular semi-classical weights and integrable systems *Constr. Approx.* **24** 201–237.

[7] Forrester P J and Witte N S 2005 Discrete Painlevé equations for a class of P_{VI} τ-functions given as U(N) averages *Nonlinearity* **18** 2061–2088.

[8] Forrester P J and Witte N S 2004 Application of the τ-function theory of Painlevé equations to random matrices: P_{VI}, the JUE, CyUE, cJUE and scaled limits *Nagoya Math. J.* **174** 29–114.

[9] Forrester P J and Witte N S 2004 Discrete Painlevé equations, orthogonal polynomials on the unit circle, and N-recurrences for averages over $U(N)$—$P_{III'}$ and $P_V \tau$-functions *Int. Math. Res. Not.* **2004** 160–183.

[10] Forrester P J and Witte N S 2003 Discrete Painlevé equations and random matrix averages. *Nonlinearity* **16** 1919–1944.

[11] Gasper G and Rahman M 2004 Basic hypergeometric series 2nd ed. *Encyclopedia of Mathematics and its Applications Vol 96* (Cambridge: Cambridge University Press).

[12] Hamamoto T, Kajiwara K and Witte N S 2006 Hypergeometric solutions to the q-Painlevé equation of type $(A_1 + A_1')^{(1)}$ *Int. Math. Res. Not.* **2006** Article ID 84169.

[13] Kajiwara K 2003 On a q-difference Painlevé III equation: II. rational solutions *J. Nonlinear Math. Phys.* **10** 282-303.

[14] Kajiwara K and Kimura K 2003 On a q-difference Painlevé III equation: I. derivation, symmetry and Riccati type solutions *J. Nonlinear Math. Phys.* **10** 86-202.

[15] Kajiwara K, Masuda T, Noumi M, Ohta Y and Yamada Y 2004 Hypergeometric solutions to the q-Painlevé equations *Int. Math. Res. Not.* **2004** 2497-2521.

[16] Kajiwara K, Masuda T, Noumi M, Ohta Y and Yamada Y 2005 Construction of hypergeometric solutions to the q-Painlevé equations *Int. Math. Res. Not.* **2005** 1439-1463.

[17] Kajiwara K, Noumi M and Yamada Y 2001 A study on the fourth q-Painlevé equation *J. Phys. A: Math. Gen.* **34** 8563-8581.

[18] Kajiwara K, Ohta Y and Satsuma J 1995 Casorati determinant solutions for the discrete Painlevé III equation *J. Math. Phys.* **36** 4162-4174.

[19] Kajiwara K, Ohta Y, Satsuma J, Grammaticos B and Ramani A 1994 Casorati determinant solutions for the discrete Painlevé–II equation *J. Phys. A: Math. Gen.* **27** 915–922.

[20] Kajiwara K, Yamamoto K and Ohta Y 1997 Rational solutions for the discrete Painlevé II equation *Phys. Lett. A* **232** 189-199.

[21] Kruskal M D, Tamizhmani K M, Grammaticos B and Ramani A 2000 Asymmetric discrete Painlevé equations *Reg. Chaot. Dyn.* **5** 273-281.

[22] Masuda T *private communication.*

[23] Murata Y, Sakai H and Yoneda J 2002 Riccati solutions of discrete Painlevé equation with Weyl group symmetry of type $E_8^{(1)}$ *J. Math. Phys.* **44** 1396-1414.

[24] Nijhoff F, Satsuma J, Kajiwara K, Grammaticos B and Ramani A 1996 A study of the alternate discrete Painlevé II equation *Inverse Problems* **12** 697-716.

[25] Ohta Y, Kajiwara K and Satsuma J 1996 Bilinear structure and exact solutions of the discrete Painlevé I equation *Symmetries and integrability of difference equations (CRM Proc. Lect. Notes Vol 9)* ed D Levi, L Vinet and P Winternitz (Providence: AMS) 265-268.

[26] Ohta Y, Hirota R, Tsujimoto S and Imai T 1993 Casorati and discrete Gram type determinant representation of solutions to the discrete KP hierarchy *J. Phys. Soc. Jpn.* **62** 1872-1886.

[27] Ohta Y, Kajiwara K, Matsukidaira J and Satsuma J 1993 Casorati determinant solution for the relativistic Toda lattice equation *J. Math. Phys.* **34** 5190-5204.

[28] Ramani A, Grammaticos B and Ohta Y 2001 The q-Painlevé V equation and its geometrical description *J. Phys. A: Math. Gen.* **34** 2505-2513.

[29] Ramani A, Grammaticos B, Tamizhmani T and Tamizhmani K M 2001 Special function solutions

of the discrete Painlevé equations *J. Comput. Math. Appl.* **42** 603-614.

[30] Sakai H 1998 Casorati determinant solutions for the q-difference sixth Painlevé equation *Nonlinearity* **11** 823-833.

[31] Sakai H 2001 Rational surfaces associated with affine root systems and geometry of the Painlevé equations *Commun. Math. Phys.* **220** 165-229.

[32] Tracy C A and Widom H 1999 Random unitary matrices, permutations and Painlevé *Comm. Math. Phys.* **207** 665–685.

List of MHF Preprint Series, Kyushu University

21st Century COE Program
Development of Dynamic Mathematics with High Functionality

MHF2005-1　Hideki KOSAKI
Matrix trace inequalities related to uncertainty principle

MHF2005-2　Masahisa TABATA
Discrepancy between theory and real computation on the stability of some finite element schemes

MHF2005-3　Yuko ARAKI & Sadanori KONISHI
Functional regression modeling via regularized basis expansions and model selection

MHF2005-4　Yuko ARAKI & Sadanori KONISHI
Functional discriminant analysis via regularized basis expansions

MHF2005-5　Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Point configurations, Cremona transformations and the elliptic difference Painlevé equations

MHF2005-6　Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Construction of hypergeometric solutions to the q　Painlevé equations

MHF2005-7　Hiroki MASUDA
Simple estimators for non-linear Markovian trend from sampled data:
I. ergodic cases

MHF2005-8　Hiroki MASUDA & Nakahiro YOSHIDA
Edgeworth expansion for a class of Ornstein-Uhlenbeck-based models

MHF2005-9　Masayuki UCHIDA
Approximate martingale estimating functions under small perturbations of dynamical systems

MHF2005-10　Ryo MATSUZAKI & Masayuki UCHIDA
One-step estimators for diffusion processes with small dispersion parameters from discrete observations

MHF2005-11　Junichi MATSUKUBO, Ryo MATSUZAKI & Masayuki UCHIDA
Estimation for a discretely observed small diffusion process with a linear drift

MHF2005-12　Masayuki UCHIDA & Nakahiro YOSHIDA
AIC for ergodic diffusion processes from discrete observations