Hypergeometric solutions to the q-Painlevé equation of {type A^(1)}_4

Hamamoto, Taro
Graduate School of Mathematics, Kyushu University

Kajiwara, Kenji
Graduate School of Mathematics, Kyushu University

Witte, Nicholas S.

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T. Hamamoto & K. Kajiwara

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Hypergeometric solutions to the q-Painlevé equation of type $A_4^{(1)}$

Taro Hamamoto and Kenji Kajiwara

Graduate School of Mathematics, Kyushu University, 6-10-1 Hakozaki, Fukuoka 812-8581, Japan

Abstract. We consider the q-Painlevé equation of type $A_4^{(1)}$ (a version of q-Painlevé V equation) and construct a family of solutions expressible in terms of certain basic hypergeometric series. We also present the determinant formula for the solutions.

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1. Introduction

In this article we consider the q-difference equation

$$\begin{cases} \overline{g}g = \dfrac{qt}{b_2}\,\dfrac{(f+b_3)(f+1)}{f+\dfrac{1}{s}}, \\[2ex] \overline{f}f = \dfrac{1}{s}\,\dfrac{\left(\overline{g}+\dfrac{1}{b_2}\right)\left(\overline{g}+\dfrac{1}{b_1b_2}\right)}{\overline{g}+\dfrac{qt}{b_2}}, \end{cases} \tag{1.1}$$

$$\overline{b}_i = b_i\ (i=1,2,3),\quad \overline{t}=qt,\quad s=\frac{1}{qb_1b_2b_3t},$$

where q is a constant and $\overline{}$ denotes the discrete time evolution. (1.1) can be also expressed as

$$\begin{cases} \overline{y}y = \dfrac{\left(x+\dfrac{a_1}{z}\right)\left(x+\dfrac{1}{a_1z}\right)}{1+a_3x}, \\[2ex] x\underline{x} = \dfrac{\left(y+\dfrac{a_2}{\rho}\right)\left(y+\dfrac{1}{a_2\rho}\right)}{1+\dfrac{y}{a_3}}, \end{cases} \tag{1.2}$$

$$\overline{a}_i = a_i\ (i=1,2,3),\quad \overline{z}=qz,\quad z=q^{\frac{1}{2}}\rho,$$

where the variables are related as

$$b_1=a_2^2,\quad b_2=\frac{1}{q^{\frac{1}{2}}a_1a_2a_3^2},\quad b_3=a_1^2,\quad t=\frac{a_3}{q^{\frac{1}{2}}a_2}z,\quad f=a_1zx,\quad g=a_1a_3^2zy. \tag{1.3}$$

(1.2) was first derived and identified as one of the discrete Painlevé equations with a continuous limit to the Painlevé V equation in [21]. Sakai has classified (1.1) as the discrete dynamical system on the rational surface of type $A_4^{(1)}$ which admits the symmetry of affine Weyl group of type $A_4^{(1)}$ [31]. Geometrical structure of the τ functions on the A_4 weight lattice has been investigated in [28] as well as various Bäcklund transformations. In this article, we denote (1.1) (or (1.2)) as $\mathrm{dP}(A_4^{(1)})$ following the notation that was adopted in [23]. We also write (1.1) as $\mathrm{dP}(A_4^{(1)})[b_1, b_2, b_3]$ when it is necessary to specify the values of parameters explicitly.

It is well-known that the Painlevé and discrete Painlevé equations admit two classes of particular solutions; hypergeometric solutions and algebraic solutions. In particular, the determinant formula for the hypergeometric solutions play an important role in applications, for example, to the area related to matrix integration, such as random matrix theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 25, 32]. The simplest hypergeometric solution to $\mathrm{dP}(A_4^{(1)})$ has been obtained in [15, 16, 29]. The purpose of this article is to construct hypergeometric solutions to $\mathrm{dP}(A_4^{(1)})$ (1.1) and present the determinant formula. In section 2, we construct the simplest hypergeometric solution through the Riccati equation which is reduced from (1.1) by imposing a condition on the parameters. By applying a Bäcklund transformation we construct complex hypergeometric solutions and present the determinant formula in section 3. We give the proof in section 4.

2. Riccati solution

We first recall the definition of the basic hypergeometric series[11]

$$ {}_r\phi_s\left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n (q; q)_n} \left[(-1)^n q^{\frac{n(n-1)}{2}}\right]^{1+s-r} z^n, \tag{2.1} $$

where

$$ (a_1, \ldots, a_r; q)_n = \prod_{i=1}^{r} (a_i; q)_n, \quad (a; q)_n = \prod_{k=1}^{n} (1 - a q^{k-1}). \tag{2.2} $$

The simplest solution that is expressible in terms of the basic hypergeometric function is constructed by looking for the special case in which $\mathrm{dP}(A_4^{(1)})$ (1.1) is reduced to the Riccati equation. In fact, imposing the condition on (1.1)

$$ b_2 = 1 \tag{2.3} $$

then it admits a specialization $1 + f + g = 0$ to yield the Riccati equation

$$ \overline{g} = -qt \frac{g + 1 - b_3}{g + 1 - qtb_1 b_3}\ , \ \overline{f} = -(1 + \overline{g}) = \frac{qtb_3(1 - b_1) + (qt - 1)f}{qtb_1 b_3 + f}. \tag{2.4} $$

Linearizing the Riccati equation (2.4) by the standard technique, one obtains the following solution (see also [29, 15, 16]):

Proposition 2.1 *Let $\psi = \psi(t, b_1, b_3)$ a function satisfying*

$$ \psi(qt, b_1, b_3) = b_1 \psi(t, b_1, b_3) + (1 - b_1)\ \psi(qt, b_1/q, b_3), \tag{2.5} $$

$$ b_3 \psi(qt, b_1, b_3) = \psi(t, b_1, b_3) + (b_3 - 1)\ \psi(t, b_1, q b_3), \tag{2.6} $$

$$ qtb_1 b_3 \psi(qt, b_1, b_3) = (qtb_1 - 1)\ \psi(t, b_1, b_3) + \psi(t, q b_1, b_3), \tag{2.7} $$

$$ qt \psi(qt, b_1, b_3) = (qtb_1 - 1)\ \psi(t, b_1, b_3) + \psi(qt, b_1, b_3/q). \tag{2.8} $$

Then

$$f = qtb_3(1-b_1)\frac{\psi(qt,b_1/q,qb_3)}{\psi(t,b_1,qb_3)},\ g = -\frac{\psi(t,b_1,b_3)}{\psi(t,b_1,qb_3)}, \tag{2.9}$$

gives a solution of $dP(A_4^{(1)})$ (1.1) with $b_2 = 1$.

It should be remarked that several basic hypergeometric functions satisfy the contiguous relations (2.5)-(2.8)[22]. For example, we have

$$\text{(i)}\ \ \psi(t,b_1,b_3) = {}_2\phi_1\left[\begin{matrix}1/b_1, b_3\\ 0\end{matrix}\ ;\ q, qtb_1\right], \tag{2.10}$$

(ii)

$$\psi(t,b_1,b_3) = \frac{(qt,1/t,b_3;q)_\infty}{(qtb_1,b_1b_3;q)_\infty}\ {}_2\phi_1\left[\begin{matrix}q/b_3, 0\\ q/b_1b_3\end{matrix}\ ;\ q, 1/tb_1\right] \tag{2.11}$$

$$= \frac{(qt,1/t,b_3;q)_\infty}{(qtb_1,b_1b_3;q)_\infty}\ {}_1\phi_1\left[\begin{matrix}b_3/q\\ b_1b_3/q\end{matrix}\ ;\ 1/q, 1/qt\right], \tag{2.12}$$

$$\text{(iii)}\ \ \psi(t,b_1,b_3) = \frac{(b_3t, q/b_3t;q)_\infty}{(qtb_1,qb_1,q/b_3;q)_\infty}\ {}_2\phi_1\left[\begin{matrix}b_3/q, 1/qb_1\\ 0\end{matrix}\ ;\ 1/q, tb_1\right]. \tag{2.13}$$

In order to prove Proposition 2.1 we use the following Lemma:

Lemma 2.2 *$\psi(t,b_1,b_3)$ satisfy the contiguous relations*

$$qtb_3\psi(qt,b_1,qb_3) = \psi(t,b_1,b_3) - (1-qtb_1b_3)\ \psi(t,b_1,qb_3), \tag{2.14}$$

$$tb_3\psi(qt,b_1,b_3) = \psi(t/q,qb_1,b_3) - (1-t)\ \psi(t,b_1,b_3). \tag{2.15}$$

Proof of Lemma 2.2. Eliminating $\psi(t,b_1,b_3)$ from (2.5) and (2.7) we have

$$(qtb_1b_3 - qt + \frac{1}{b_1})\ \psi(qt,b_1,b_3) + (qtb_1-1)(\frac{1}{b_1}-1)\psi(qt,b_1/q,b_3) = \psi(t,qb_1,b_3). \tag{2.16}$$

Similarly, eliminating $\psi(qt,b_1/q,b_3)$ from (2.5) and $(2.7)_{b_1\to b_1/q}$ we obtain

$$tb_1b_3\psi(qt,b_1,b_3) - ({tb_1}^2b_3 + 1 - b_1)\ \psi(t,b_1,b_3) = (1-b_1)(tb_1-1)\psi(t,b_1/q,b_3). \tag{2.17}$$

Then eliminating $\psi(qt,b_1/q,b_3)$ from (2.16) and $(2.17)_{t\to qt}$ we get

$$(1-qt)\ \psi(qt,b_1,b_3) + qtb_3\ \psi(q^2t,b_1,b_3) = \psi(t,qb_1,b_3),$$

which is nothing but $(2.15)_{t\to qt}$. Similarly, (2.14) can be derived by eliminating $\psi(qt,b_1,b_3)$ from (2.6) and $(2.8)_{b_3\to qb_3}$. $\square$

Proposition 2.1 follows immediately from Lemma 2.2. In fact, dividing (2.6) by (2.14) we have

$$\frac{1}{qt}\frac{\psi(qt,b_1,b_3)}{\psi(qt,b_1,qb_3)} = -\frac{\overline{g}}{qt} = \frac{\psi(t,b_1,b_3) + (b_3-1)\ \psi(t,b_1,qb_3)}{\psi(t,b_1,b_3) - (1-qtb_1b_3)\ \psi(t,b_1,qb_3)} = \frac{-g + (b_3-1)}{-g - (1-qtb_1b_3)},$$

which is the first equation of (2.4). The second equation of (2.4) can be derived in similar manner by dividing $(2.15)_{t\to qt, b_1\to b_1/q, b_3\to qb_3}$ by $(2.5)_{b_3\to qb_3}$. $\square$

3. Determinant formula and bilinear equations

3.1. *Bäcklund transformations*

Sakai constructed the following transformations for the homogeneous variables x, y, z of $\mathbb{P}^2$ and the parameters b_i $(i = 0, 1, 2, 3, 4)$ on the $A_4^{(1)}$ type (*Mul.5*) surface[31]‡:

$$\sigma : \left(\begin{matrix} b_1\ b_2\ b_3 \\ b_4\ b_0 \end{matrix} \ ;\ x : y : z \right) \mapsto$$

$$\left(\begin{matrix} b_3\ b_4\ b_0 \\ b_1\ b_2 \end{matrix} \ ;\ b_4 xy(z+x) : b_2 z(x+y+z)(x+b_4 y+z) : x(x+z)^2 \right), \tag{3.1}$$

$$\sigma' : \left(\begin{matrix} b_1\ b_2\ b_3 \\ b_4\ b_0 \end{matrix} \ ;\ x : y : z \right) \mapsto$$

$$\left(\begin{matrix} \frac{1}{b_1}\ \frac{1}{b_0}\ \frac{1}{b_4} \\ \frac{1}{b_3}\ \frac{1}{b_2} \end{matrix} \ ;\ b_2 z(x+z)(x+y+z) : y((z+x)(b_0 x+b_2 z)+b_2 yz) : b_0 x(x+z)^2 \right) \tag{3.2}$$

$$w_3 : \left(\begin{matrix} b_1\ b_2\ b_3 \\ b_4\ b_0 \end{matrix} \ ;\ x : y : z \right) \mapsto$$

$$\left(\begin{matrix} b_1\ b_2 b_3\ \frac{1}{b_3} \\ b_3 b_4\ b_0 \end{matrix} \ ;\ b_3 x(b_3 x+y+b_3 z) : y(b_3 x+y+z) : b_3 z(x+y+z) \right), \tag{3.3}$$

$$w_1 : \left(\begin{matrix} b_1\ b_2\ b_3 \\ b_4\ b_0 \end{matrix} \ ;\ x : y : z \right) \mapsto \left(\begin{matrix} \frac{1}{b_1}\ b_1 b_2\ b_3 \\ b_4\ b_1 b_0 \end{matrix} \ ;\ x : y : z \right), \tag{3.4}$$

$$w_2 : \left(\begin{matrix} b_1\ b_2\ b_3 \\ b_4\ b_0 \end{matrix} \ ;\ x : y : z \right) \mapsto \left(\begin{matrix} b_1 b_2\ \frac{1}{b_2}\ b_2 b_3 \\ b_4\ b_0 \end{matrix} \ ;\ x : b_2 y : b_2 z \right), \tag{3.5}$$

$$w_4 : \left(\begin{matrix} b_1\ b_2\ b_3 \\ b_4\ b_0 \end{matrix} \ ;\ x : y : z \right) \mapsto \left(\begin{matrix} b_1\ b_2\ b_3 b_4 \\ \frac{1}{b_4}\ b_4 b_0 \end{matrix} \ ;\ x : b_4 y : z \right), \tag{3.6}$$

$$w_0 = \sigma^2 \circ w_1 \circ \sigma^3. \tag{3.7}$$

Introducing the variables f and g by

$$f = \frac{y}{z+x}, \quad g = \frac{z(x+y+z)}{x(z+x)}, \tag{3.8}$$

then (3.1) - (3.7) can be rewritten as

$$\begin{aligned}
&\sigma : (b_0, b_1, b_2, b_3, b_4, f, g) \mapsto (b_2, b_3, b_4, b_0, b_1, b_2 g, \tfrac{1+b_2 g}{b_4 f}),\\
&\sigma' : (b_0, b_1, b_2, b_3, b_4, g) \mapsto (\tfrac{1}{b_1}, \tfrac{1}{b_0}, \tfrac{1}{b_4}, \tfrac{1}{b_3}, \tfrac{1}{b_2}, \tfrac{b_0(1+g)}{b_2 f}),\\
&w_0 : (b_0, b_1, b_4, f, g) \mapsto (\tfrac{1}{b_0}, b_0 b_1, b_0 b_4, \tfrac{f(b_0+b_2 g)}{b_0(1+b_2 g)}, \tfrac{g}{b_0}),\\
&w_1 : (b_0, b_1, b_2) \mapsto (b_0 b_1, \tfrac{1}{b_1}, b_1 b_2),\\
&w_2 : (b_1, b_2, b_3, f, g) \mapsto (b_1 b_2, \tfrac{1}{b_2}, b_2 b_3, b_2 f \tfrac{1+f+g}{1+f+b_2 g}, b_2 g \tfrac{1+b_2 f+b_2 g}{1+f+b_2 g}),\\
&w_3 : (b_2, b_3, b_4, f, g) \mapsto (b_2 b_3, \tfrac{1}{b_3}, b_3 b_4, \tfrac{f}{b_3}, \tfrac{g}{b_3}),\\
&w_4 : (b_0, b_3, b_4, f, g) \mapsto (b_0 b_4, b_3 b_4, \tfrac{1}{b_4}, b_4 f, \tfrac{g(1+b_4 f)}{1+f}),
\end{aligned} \tag{3.9}$$

respectively, where the abbreviated variables are invariant with respect to the transformation. It can be shown by direct calculation that these transformations

‡ Actions of these transformations are slightly modified from the original formula to be subtraction-free.

satisfy the fundamental relation of the (extended) affine Weyl group $\widetilde{W}(A_4^{(1)})$:

$$\begin{aligned} &w_i^2 = 1,\ (w_iw_{i\pm1})^3 = 1,\ (w_iw_j)^2 = 1\ (j \not\equiv i, i\pm 1),\ \sigma^5 = 1,\ \sigma'^2 = 1, \\ &\sigma w_i = w_{i+2}\sigma,\ \sigma' w_0 = w_2\sigma',\ \sigma' w_3 = w_4\sigma',\ \sigma' w_1 = w_1\sigma',\ i \in \mathbb{Z}/5\mathbb{Z}. \end{aligned} \tag{3.10}$$

We note that $q = 1/(b_0b_1b_2b_3b_4)$ is invariant with respect to the Weyl group actions. The translation $T_0 = w_4w_3w_2w_1\sigma^3$ acts on b_i as

$$T_0 : (b_0, b_1, b_2, b_3, b_4) \mapsto (qb_0, b_1, b_2, b_3, b_4/q), \tag{3.11}$$

and the action on f and g is nothing but $\mathrm{dP}(A_4^{(1)})$(1.1) for $t = b_0$ and $s = b_4$. If we define the translations T_i $(i = 1, 2, 3, 4)$ by

$$T_1 = \sigma^3T_0\sigma^2, \quad T_2 = \sigma T_0\sigma^4, \quad T_3 = \sigma^4T_0\sigma, \quad T_4 = \sigma^2T_0\sigma^3, \tag{3.12}$$

then actions of T_i $(i = 1, 2, 3, 4)$ on the parameters are given by

$$\begin{aligned} T_1 &: (b_0, b_1, b_2, b_3, b_4) \mapsto (b_0/q, qb_1, b_2, b_3, b_4), \\ T_2 &: (b_0, b_1, b_2, b_3, b_4) \mapsto (b_0, b_1/q, qb_2, b_3, b_4), \\ T_3 &: (b_0, b_1, b_2, b_3, b_4) \mapsto (b_0, b_1, b_2/q, qb_3, b_4), \\ T_4 &: (b_0, b_1, b_2, b_3, b_4) \mapsto (b_0, b_1, b_2, b_3/q, qb_4). \end{aligned} \tag{3.13}$$

and one can directly verify that $T_iT_j = T_jT_i$ $(i, j = 0, 1, 2, 3,\ i \neq j)$ and $T_0T_1T_2T_3T_4 = 1$. Therefore if we regard T_0 as the discrete time evolution, T_i $(i = 1, 2, 3, 4)$ can be regarded as the Bäcklund transformations.

3.2. Determinant formula

Let us apply the Bäcklund transformation T_2 on the Riccati solution obtained in Proposition 2.1. Applying T_2 N times yields the solution for $\mathrm{dP}(A_4^{(1)})[q^{-N}b_1, q^N, b_3]$, which is expressed as rational function in ψ. However the denominator and numerator can be factorized into two factors, respectively, and each factor admits determinant formula. More precisely, we have the following formula, which is the main result of this article:

Theorem 3.1 *Let $\tau_N(t, b_1, b_3)$ $(N \in \mathbb{Z})$ be*

$$\tau_N(t, b_1, b_3) = \begin{cases} \det(\psi(t, q^{-j+1}b_1, q^{i-1}b_3))_{i,j=1,\ldots,N} & (N > 0), \\ 1 & (N = 0), \\ \det(\psi(t, q^{j-1}b_1, q^{-i+1}b_3))_{i,j=1,\ldots,M} & (N = -M < 0), \end{cases} \tag{3.14}$$

Then

$$\begin{aligned} f &= \begin{cases} q^{N+1}tb_3(1 - q^{-N}b_1)\ \dfrac{\tau_N(t, b_1, qb_3)\ \tau_{N+1}(qt, b_1/q, qb_3)}{\tau_N(qt, b_1/q, qb_3)\ \tau_{N+1}(t, b_1, qb_3)} & (N \geq 0), \\ -\dfrac{\tau_N(t, qb_1, b_3)\ \tau_{N+1}(qt, b_1, b_3)}{\tau_N(qt, b_1, b_3)\ \tau_{N+1}(t, qb_1, b_3)} & (N < 0), \end{cases} \\ g &= \begin{cases} -\dfrac{\tau_N(t, b_1/q, q^2b_3)\ \tau_{N+1}(t, b_1, b_3)}{\tau_N(t, b_1/q, qb_3)\ \tau_{N+1}(t, b_1, qb_3)} & (N \geq 0), \\ qt(b_3 - 1)\dfrac{\tau_N(t, b_1, qb_3)\ \tau_{N+1}(t, qb_1, b_3/q)}{\tau_N(t, b_1, b_3)\ \tau_{N+1}(t, qb_1, b_3)} & (N < 0), \end{cases} \end{aligned} \tag{3.15}$$

satisfy $dP(A_4^{(1)})[q^{-N}b_1, q^N, b_3]$.

We introduce a notation for simplicity

$$\tau_N(t,q^m b_1,q^n b_3)=\tau_N^{m,n}(t). \tag{3.16}$$

Theorem 3.1 for $N\geq 0$ is a direct consequence of the following Proposition:

Proposition 3.2 *For* $N \geq 0$, $\tau_N^{m,n}(t)$ *satisfy the following bilinear difference equations.*

$$\begin{aligned}&(1-q^{-N}b_1)\tau_N^{0,1}(t)\ \tau_{N+1}^{-1,1}(qt)+q^{-N}b_1\tau_N^{-1,1}(qt)\ \tau_{N+1}^{0,1}(t)\\ &\qquad -q^{-N}\tau_N^{-1,1}(t)\ \tau_{N+1}^{0,1}(qt)=0,\end{aligned} \tag{3.17}$$

$$\begin{aligned}&qtb_3(1-q^{-N}b_1)\tau_N^{0,1}(t)\ \tau_{N+1}^{-1,1}(qt)+q^{-N}\tau_N^{-1,1}(qt)\ \tau_{N+1}^{0,1}(t)\\ &\qquad -\tau_N^{-1,2}(qt)\ \tau_{N+1}^{0,0}(t)=0,\end{aligned} \tag{3.18}$$

$$\begin{aligned}&qt(1-q^{-N}b_1)\tau_N^{0,1}(t)\ \tau_{N+1}^{-1,1}(qt)+q^{-N}\tau_N^{-1,1}(qt)\ \tau_{N+1}^{0,1}(t)\\ &\qquad -q^{-N}\tau_N^{-1,2}(t)\ \tau_{N+1}^{0,0}(qt)=0,\end{aligned} \tag{3.19}$$

$$\begin{aligned}&q^{-N}\tau_N^{-1,1}(t)\tau_{N+1}^{0,1}(t)-q^{-N}b_1\tau_N^{-1,2}(t)\tau_{N+1}^{0,0}(t)\\ &\qquad -(1-q^{-N}b_1)(1-q^{-N}tb_1)\tau_N^{0,1}(t)\tau_{N+1}^{-1,1}(t)=0,\end{aligned} \tag{3.20}$$

$$\begin{aligned}&q^{-N}\tau_N^{-1,1}(t)\ \tau_{N+1}^{0,1}(t)-\tau_N^{-1,2}(t)\ \tau_{N+1}^{0,0}(t)\\ &\qquad +q^{N+1}tb_3(1-q^{-N}b_1)\ \tau_N^{0,1}(t/q)\ \tau_{N+1}^{-1,1}(qt)=0,\end{aligned} \tag{3.21}$$

$$\begin{aligned}&q^{-N}t\ \tau_N^{-1,1}(t)\ \tau_{N+1}^{0,1}(t)-\tau_N^{-1,2}(t)\ \tau_{N+1}^{0,0}(t)\\ &\qquad +(1-q^{-N}tb_1)\ \tau_N^{-1,1}(qt)\ \tau_{N+1}^{0,1}(t/q)=0.\end{aligned} \tag{3.22}$$

In fact, Theorem 3.1 for $N\geq 0$ can be derived from Proposition 3.2 as follows. We have from (3.17) by using (3.15)

$$f+\frac{1}{s}=qtb_3\frac{\tau_N^{-1,1}(t)\ \tau_{N+1}^{0,1}(qt)}{\tau_N^{-1,1}(qt)\ \tau_{N+1}^{0,1}(t)}. \tag{3.23}$$

We also have from (3.18) and (3.19)

$$f+1=q^N\frac{\tau_N^{-1,2}(qt)\ \tau_{N+1}^{0,0}(t)}{\tau_N^{-1,1}(qt)\ \tau_{N+1}^{0,1}(t)}, \tag{3.24}$$

$$f+b_3=b_3\frac{\tau_N^{-1,2}(t)\ \tau_{N+1}^{0,0}(qt)}{\tau_N^{-1,1}(qt)\ \tau_{N+1}^{0,1}(t)}, \tag{3.25}$$

respectively. Therefore we obtain

$$\frac{(f+1)(f+b_3)}{f+\frac{1}{s}}=\frac{q^{N-1}}{t}\ \frac{\tau_N^{-1,2}(qt)\tau_{N+1}^{0,0}(qt)\tau_{N+1}^{0,0}(t)\tau_N^{-1,2}(t)}{\tau_N^{-1,1}(qt)\tau_{N+1}^{0,1}(qt)\tau_{N+1}^{0,1}(t)\tau_N^{-1,1}(t)}=\frac{q^{N-1}}{t}\ \overline{g}g, \tag{3.26}$$

which is the first equation of (1.1). Similarly, from $(3.20)_{t\to qt}$, $(3.21)_{t\to qt}$ and $(3.22)_{t\to qt}$ we get

$$1+b_1\overline{g}=q^N(1-q^{-N}b_1)(1-q^{-N+1}tb_1)\frac{\tau_N^{0,1}(qt)\tau_{N+1}^{-1,1}(qt)}{\tau_N^{-1,1}(qt)\tau_{N+1}^{0,1}(qt)}, \tag{3.27}$$

$$1+q^N\overline{g}=-q^{2N+2}tb_3(1-q^{-N}b_1)\frac{\tau_N^{0,1}(t)\tau_{N+1}^{-1,1}(q^2t)}{\tau_N^{-1,1}(qt)\tau_{N+1}^{0,1}(qt)}, \tag{3.28}$$

$$qt+q^N\overline{g}=-q^N(1-q^{-N+1}tb_1)\frac{\tau_N^{-1,1}(q^2t)\tau_{N+1}^{0,1}(t)}{\tau_N^{-1,1}(qt)\ \tau_{N+1}^{0,1}(qt)}, \tag{3.29}$$

respectively. Then we have

$$qtb_3\frac{(1+b_1\overline{g})(1+q^N\overline{g})}{qt+q^N\overline{g}}=\overline{f}f, \tag{3.30}$$

which is the second equation of (1.1). Therefore we have verified that Theorem 3.1 for $N\geq 0$ follows from the bilinear equations (3.17) - (3.22) in Proposition 3.2. We omit the proof for the case of $N<0$ since it can be proved in similar manner.

4. Proof of Proposition 3.2

The bilinear equations (3.17) - (3.22) can be reduced to the Plücker relations which are quadratic identities among the determinants whose columns are properly shifted. This can be done by constructing "difference formulae" that relate the "shifted" determinants with $\tau_N^{m,n}(t)$ by using the contiguous relations of ψ. This technique has been developed in [26, 27] and applied to various discrete Painlevé equations[12, 13, 14, 17, 18, 19, 20, 24, 25, 30]. In this section, we prove the bilinear equation (3.17) as an example. Since other bilinear equations (3.18)-(3.22) can be proved in similar manner, we leave the details in the appendix.

We first introduce the following notation:

$$\begin{aligned}\tau_N^{m,n}(t)&=\begin{vmatrix}\psi(t,q^mb_1,q^nb_3) & \psi(t,q^{m-1}b_1,q^nb_3) & \cdots & \psi(t,q^{m-N+1}b_1,q^nb_3)\\ \psi(t,q^mb_1,q^{n+1}b_3) & \psi(t,q^{m-1}b_1,q^{n+1}b_3) & \cdots & \psi(t,q^{m-N+1}b_1,q^{n+1}b_3)\\ \vdots & \vdots & \cdots & \vdots\\ \psi(t,q^mb_1,q^{n+N-1}b_3) & \psi(t,q^{m-1}b_1,q^{n+N-1}b_3) & \cdots & \psi(t,q^{m-N+1}b_1,q^{n+N-1}b_3)\end{vmatrix}\\ &=\begin{vmatrix}\Psi_{m,n}(t) & \Psi_{m-1,n}(t) & \cdots & \Psi_{m-N+1,n}(t)\end{vmatrix},\end{aligned} \tag{4.1}$$

where $\Psi_{m,n}(t)$ denotes a column vector

$$\Psi_{m,n}(t)=\begin{pmatrix}\psi(t,q^mb_1,q^nb_3)\\ \psi(t,q^mb_1,q^{n+1}b_3)\\ \vdots\\ \psi(t,q^mb_1,q^{n+N-1}b_3)\end{pmatrix}. \tag{4.2}$$

Here we note that the height of the column vector is N, but we use the same symbol for the column vector with different height. Then we have the following difference formula:

Lemma 4.1

$$\begin{aligned}&\begin{vmatrix}\Psi_{m,n}(t) & \Psi_{m,n}(t/q) & \cdots & \Psi_{m-N+2,n}(t/q)\end{vmatrix}\\ &\qquad=\frac{\prod_{k=0}^{N-2}(q^{m-k}b_1-1)}{q^{\frac{(N-1)(2m-N+2)}{2}}b_1^{N-1}}\ \tau_N^{m,n}(t),\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\begin{vmatrix}\Psi_{m,n}(t/q) & \Psi_{m-1,n}(t) & \Psi_{m-1,n}(t/q) & \cdots & \Psi_{m-N+2,n}(t/q)\end{vmatrix}\\ &\qquad=\frac{\prod_{k=1}^{N-2}(q^{m-k}b_1-1)}{q^{\frac{(N-1)(2m-N+2)}{2}}b_1^{N-1}}\ \tau_N^{m,n}(t).\end{aligned} \tag{4.4}$$

Proof. Using the contiguous relation (2.5) on the N-th column of the determinant in (4.1), we have

$$\begin{aligned}\tau_N^{m,n}(t) &= \left|\ \Psi_{m,n}(t)\quad \Psi_{m-1,n}(t)\quad \cdots\quad \Psi_{m-N+2,n}(t)\quad \Psi_{m-N+1,n}(t)\ \right| \\ &= \left|\ \Psi_{m,n}(t)\quad \cdots\quad \Psi_{m-N+2,n}(t)\quad \frac{-\Psi_{m-N+2,n}(t)+q^{m-N+2}b_1\Psi_{m-N+2,n}(t/q)}{q^{m-N+2}b_1-1}\ \right| \\ &= \frac{q^{m-N+2}b_1}{q^{m-N+2}b_1-1}\left|\ \Psi_{m,n}(t)\quad \cdots\quad \Psi_{m-N+2,n}(t)\quad \Psi_{m-N+2,n}(t/q)\ \right|.\end{aligned}$$

Applying this procedure from the N-th column to the second column we obtain

$$\tau_N^{m,n}(t) = \frac{q^{\frac{(N-1)(2m-N+2)}{2}}b_1^{N-1}}{\prod_{k=0}^{N-2}(q^{m-k}b_1-1)}\left|\ \Psi_{m,n}(t)\quad \Psi_{m,n}(t/q)\quad \cdots\quad \Psi_{m-N+2,n}(t/q)\ \right|, \tag{4.5}$$

which is nothing but (4.3). At the stage where the above procedure has been applied up to the third column, we have by using (2.5) on the first column

$$\begin{aligned}&\tau_N^{m,n}(t) \\ &= \frac{q^{\frac{(N-2)(2m-N+1)}{2}}b_1^{N-2}}{\prod_{k=1}^{N-2}(q^{m-k}b_1-1)}\left|\ \Psi_{m,n}(t)\quad \Psi_{m-1,n}(t)\quad \Psi_{m-1,n}(t/q)\quad \cdots\quad \Psi_{m-N+2,n}(t/q)\ \right| \\ &= \frac{q^{\frac{(N-2)(2m-N+1)}{2}}b_1^{N-2}}{\prod_{k=1}^{N-2}(q^{m-k}b_1-1)} \\ &\times \left|\ q^m b_1\Psi_{m,n}(t/q)-(q^m b_1-1)\Psi_{m-1,n}(t)\quad \Psi_{m-1,n}(t)\quad \Psi_{m-1,n}(t/q)\quad \cdots\quad \Psi_{m-N+2,n}(t/q)\ \right| \\ &= \frac{q^{\frac{(N-1)(2m-N+2)}{2}}b_1^{N-1}}{\prod_{k=1}^{N-2}(q^{m-k}b_1-1)} \\ &\times \left|\ \Psi_{m,n}(t/q)\quad \Psi_{m-1,n}(t)\quad \Psi_{m-1,n}(t/q)\quad \cdots\quad \Psi_{m-N+2,n}(t/q)\ \right|,\end{aligned} \tag{4.6}$$

which is (4.4). This completes the proof. □

Now consider the Plücker relation

$$\begin{aligned}0 &= \left|\ \Psi_{m+1,n}(t/q)\quad \Psi_{m,n}(t)\quad \Psi_{m,n}(t/q)\quad \cdots\quad \Psi_{m-N+3,n}(t/q)\ \right| \\ &\quad\times\left|\ \Psi_{m,n}(t/q)\quad \cdots\quad \Psi_{m-N+3,n}(t/q)\quad \Psi_{m-N+2,n}(t/q)\quad \phi\ \right| \\ &\quad- \left|\ \Psi_{m+1,n}(t/q)\quad \Psi_{m,n}(t/q)\quad \cdots\quad \Psi_{m-N+3,n}(t/q)\quad \Psi_{m-N+2,n}(t/q)\ \right| \\ &\quad\times\left|\ \Psi_{m,n}(t)\quad \Psi_{m,n}(t/q)\quad \cdots\quad \Psi_{m-N+3,n}(t/q)\quad \phi\ \right| \\ &\quad+ \left|\ \Psi_{m+1,n}(t/q)\quad \Psi_{m,n}(t/q)\quad \cdots\quad \Psi_{m-N+3,n}(t/q)\quad \phi\ \right| \\ &\quad\times\left|\ \Psi_{m,n}(t)\quad \Psi_{m,n}(t/q)\quad \cdots\quad \Psi_{m-N+3,n}(t/q)\quad \Psi_{m-N+2,n}(t/q)\ \right|,\end{aligned} \tag{4.7}$$

where ϕ is the column vector

$$\phi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{4.8}$$

Applying Lemma 4.1 to (4.7) we have

$$\tau_N^{m+1,n}(t)\ \tau_{N-1}^{m,n}(t/q) - q^{m+1}b_1\ \tau_N^{m+1,n}(t/q)\ \tau_{N-1}^{m,n}(t) \\ + q^{N-1}(1-q^{m-N+2}b_1)\ \tau_{N-1}^{m+1,n}(t/q)\ \tau_N^{m,n}(t) = 0.$$

Putting $N \to N+1$, $t \to qt$, $m=-1$ and $n=1$ we obtain (3.17). □

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Appendix: Proof of bilinear equations

In this appendix we prove the bilinear equations (3.18)-(3.22). We first note that $\tau_N^{m,n}(t)$ admits various determinantal expressions, which play an important role in proving the bilinear equations. Taking the transpose of the right hand side of (4.1), we have

$$\tau_N^{m,n}(t) = \left| \ \widetilde{\Psi}_{m,n}(t) \quad \widetilde{\Psi}_{m,n+1}(t) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-1}(t)\ \right|, \tag{A.1}$$

where

$$\widetilde{\Psi}_{m,n}(t) = \begin{pmatrix} \psi(t, q^m b_1, q^n b_3) \\ \psi(t, q^{m-1} b_1, q^n b_3) \\ \vdots \\ \psi(t, q^{m-N+1} b_1, q^n b_3) \end{pmatrix}. \tag{A.2}$$

It is also possible to express $\tau_N^{m,n}(t)$ by the determinants with different structure of shifts.

Lemma A.1 $\tau_N^{m,n}(t)$ *can be expressed as follows:*

$$\begin{aligned}
\tau_N^{m,n}(t) &= \prod_{k=1}^{N-1}\left(\frac{q^{n+k-1}b_3}{q^{n+k-1}b_3-1}\right)^{N-k} \\
&\quad \times \left| \ \widetilde{\Psi}_{m,n}(t) \quad \widetilde{\Psi}_{m,n}(qt) \quad \cdots \quad \widetilde{\Psi}_{m,n}(q^{N-1}t)\ \right| && \text{(A.3)}\\
&= \prod_{k=1}^{N-1}\left(\frac{q^{m-k+1}b_1}{q^{m-k+1}b_1-1}\right)^{N-k} \\
&\quad \times \left| \ \check{\Psi}_{m,n}(t) \quad \check{\Psi}_{m,n+1}(t) \quad \cdots \quad \check{\Psi}_{m,n+N-1}(t)\ \right| && \text{(A.4)}\\
&= (-1)^{\frac{N(N-1)}{2}} \prod_{k=1}^{N-1}(q^{n+k-1}b_3)^{N-k} \prod_{k=1}^{N-1}\left(\frac{q^{m-k+1}b_1}{q^{m-k+1}b_1-1}\right)^{N-k} \\
&\quad \times \left| \ \widehat{\Psi}_{m,n}(t) \quad \widehat{\Psi}_{m,n+1}(q^{-1}t) \quad \cdots \quad \widehat{\Psi}_{m,n+N-1}(q^{-N+1}t)\ \right|, && \text{(A.5)}
\end{aligned}$$

where the column vectors are given by

$$\check{\Psi}_{m,n}(t)=\begin{pmatrix}\psi(t,q^mb_1,q^nb_3)\\ \psi(q^{-1}t,q^mb_1,q^nb_3)\\ \vdots\\ \psi(q^{-N+1}t,q^mb_1,q^nb_3)\end{pmatrix},\ \widehat{\Psi}_{m,n}(t)=\begin{pmatrix}\psi(t,q^mb_1,q^nb_3)\\ \psi(qt,q^mb_1,q^nb_3)\\ \vdots\\ \psi(q^{N-1}t,q^mb_1,q^nb_3)\end{pmatrix},\tag{A.6}$$

respectively.

Proof. We prove (A.3). Using the contiguous relation (2.6) to the N-th column of the right hand side of (A.1), we get

$$\begin{aligned}\tau_N^{m,n}(t)&=\left|\ \widetilde{\Psi}_{m,n}(t)\quad \widetilde{\Psi}_{m,n+1}(t)\quad\cdots\quad \widetilde{\Psi}_{m,n+N-2}(t)\quad \frac{q^{n+N-2}b_3\widetilde{\Psi}_{m,n+N-2}(qt)-\widetilde{\Psi}_{m,n+N-2}(t)}{q^{n+N-2}b_3-1}\ \right|\\ &=\frac{q^{n+N-2}b_3}{q^{n+N-2}b_3-1}\left|\ \widetilde{\Psi}_{m,n}(t)\quad \widetilde{\Psi}_{m,n+1}(t)\quad\cdots\quad \widetilde{\Psi}_{m,n+N-2}(t)\quad \widetilde{\Psi}_{m,n+N-2}(qt)\ \right|.\end{aligned}$$

Applying this procedure up to the second column, we have

$$\tau_N^{m,n}(t)=\prod_{k=1}^{N-1}\frac{q^{n+k-1}b_3}{q^{n+k-1}b_3-1}\left|\ \widetilde{\Psi}_{m,n}(t)\quad \widetilde{\Psi}_{m,n}(qt)\quad\cdots\quad \widetilde{\Psi}_{m,n+N-3}(qt)\quad \widetilde{\Psi}_{m,n+N-2}(qt)\ \right|.$$

Continuing this procedure we obtain

$$\begin{aligned}\tau_N^{m,n}(t)&=\left(\prod_{k=1}^{N-1}\frac{q^{n+k-1}b_3}{q^{n+k-1}b_3-1}\right)\times\left(\prod_{k=1}^{N-2}\frac{q^{n+k-1}b_3}{q^{n+k-1}b_3-1}\right)\times\cdots\times\left(\frac{q^nb_3}{q^nb_3-1}\right)\\ &\qquad\times\left|\ \widetilde{\Psi}_{m,n}(t)\quad \widetilde{\Psi}_{m,n}(qt)\quad\cdots\quad \widetilde{\Psi}_{m,n}(q^{N-2}t)\quad \widetilde{\Psi}_{m,n}(q^{N-1}t)\ \right|\\ &=\prod_{k=1}^{N-1}\left(\frac{q^{n+k-1}b_3}{q^{n+k-1}b_3-1}\right)^{N-k}\\ &\qquad\times\left|\ \widetilde{\Psi}_{m,n}(t)\quad \widetilde{\Psi}_{m,n}(qt)\quad\cdots\quad \widetilde{\Psi}_{m,n}(q^{N-2}t)\quad \widetilde{\Psi}_{m,n}(q^{N-1}t)\ \right|,\end{aligned}$$

which is (A.3). As to (A.4) and (A.5) we omit the details and only describe the method, since one can prove them by the similar calculations. In order to prove (A.4) we use the contiguous relation (2.5) on (4.1) repeatedly. For (A.5) we use (2.5) on (A.3) to express $\tau_N^{m,n}(t)$ by the determinant in which t is shifted in both horizontal and vertical directions. Finally we use (2.6) on this determinant to derive (A.5). □

Now the bilinear equations (3.18)-(3.22) can be proved by the same procedure as that in section 4. Therefore we do not repeat the procedure, but give the list of data which are necessary for proof of each bilinear equation.

(3.18)

(i) Expression of $\tau_N^{m,n}$: (A.1)

(ii) Difference formula:

$$\begin{aligned}&\left|\ \overline{\widetilde{\Psi}_{m,n}(t)}\quad \widetilde{\Psi}_{m-1,n+1}(qt)\quad\cdots\quad \widetilde{\Psi}_{m-1,n+N-1}(qt)\ \right|\\ &\qquad=\frac{1}{q^{\frac{(N-1)(2n+N)}{2}}(tb_3)^{N-1}\prod_{k=0}^{N-1}(q^{m-k}b_1-1)}\ \tau_N^{m,n}(t),\end{aligned}\tag{A.7}$$

$$\begin{aligned}
&\left| \widetilde{\Psi}_{m-1,n+1}(qt) \quad \overline{\widetilde{\Psi}_{m,n+1}(t)} \quad \widetilde{\Psi}_{m-1,n+2}(qt) \quad \cdots \quad \widetilde{\Psi}_{m-1,n+N-1}(qt) \right| \\
&\qquad = -\frac{1}{q^{\frac{(N-1)(2n+N)}{2}}(tb_3)^{N-1}\prod_{k=0}^{N-1}(q^{m-k}b_1-1)}\ \tau_N^{m,n}(t),
\end{aligned} \tag{A.8}$$

where

$$\overline{\widetilde{\Psi}_{m,n}(t)} = \begin{pmatrix} \frac{1}{q^m b_1-1}\psi(t,q^m b_1,q^n b_3) \\ \vdots \\ \frac{1}{q^{m-N+1}b_1-1}\psi(t,q^{m-N+1}b_1,q^n b_3) \end{pmatrix}. \tag{A.9}$$

(iii) Contiguous relation to be used for derivation of difference formula:

$$\psi(t,b_1,qb_3) = \psi(t,b_1,b_3) + qtb_3(b_1-1)\ \psi(qt,b_1/q,qb_3). \tag{A.10}$$

(A.10) can be derived by eliminating $\psi(qt,b_1,qb_3)$ from $(2.5)_{b_3\to qb_3}$ and (2.14).

(iv) Plücker relation:

$$\begin{aligned}
0 = &\left| \widetilde{\Psi}_{m-1,n}(qt) \quad \overline{\widetilde{\Psi}_{m,n}(t)} \quad \widetilde{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \widetilde{\Psi}_{m-1,n+N-2}(qt) \right| \\
&\times \left| \widetilde{\Psi}_{m-1,n+1}(qt) \quad \widetilde{\Psi}_{m-1,n+2}(qt) \quad \cdots \quad \widetilde{\Psi}_{m-1,n+N-1}(qt) \quad \phi' \right| \\
- &\left| \widetilde{\Psi}_{m-1,n}(qt) \quad \widetilde{\Psi}_{m-1,n+1}(qt) \quad \ldots \quad \widetilde{\Psi}_{m-1,n+N-1}(qt) \right| \\
&\times \left| \overline{\widetilde{\Psi}_{m,n}(t)} \quad \widetilde{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \widetilde{\Psi}_{m-1,n+N-2}(qt) \quad \phi' \right| \\
+ &\left| \widetilde{\Psi}_{m-1,n}(qt) \quad \widetilde{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \widetilde{\Psi}_{m-1,n+N-2}(qt) \quad \phi' \right| \\
&\times \left| \overline{\widetilde{\Psi}_{m,n}(t)} \quad \widetilde{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \widetilde{\Psi}_{m-1,n+N-1}(qt) \right|,
\end{aligned} \tag{A.11}$$

where

$$\phi' = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{A.12}$$

(3.19):

(i) Expression of $\tau_N^{m,n}$: (A.1)

(ii) Difference formula:

$$\begin{aligned}
&\left| \underline{\widetilde{\Psi}_{m,n}(t)} \quad \widetilde{\Psi}_{m,n+1}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-1}(t/q) \right| \\
&\qquad = \frac{t^{N-1}}{\prod_{k=0}^{N-1}(q^{m-k}tb_1-1)}\ \tau_N^{m,n}(t),
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
&\left| \widetilde{\Psi}_{m,n+1}(t/q) \quad \underline{\widetilde{\Psi}_{m,n+1}(t)} \quad \widetilde{\Psi}_{m,n+2}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-1}(t/q) \right| \\
&\qquad = -\frac{t^{N-2}}{\prod_{k=0}^{N-1}(q^{m-k}tb_1-1)}\ \tau_N^{m,n}(t),
\end{aligned} \tag{A.14}$$

where

$$\underline{\widetilde{\Psi}_{m,n}(t)} = \begin{pmatrix} \frac{1}{q^m tb_1-1}\psi(t, q^m b_1, q^n b_1) \\ \vdots \\ \frac{1}{q^{m-N+1}tb_1-1}\psi(t, q^{m-N+1}b_1, q^n b_3) \end{pmatrix}. \tag{A.15}$$

(iii) Contiguous relation to be used for derivation of difference formula: (2.8)

(iv) Plücker relation:

$$\begin{aligned} 0 = &\left| \widetilde{\Psi}_{m,n}(t/q) \quad \underline{\widetilde{\Psi}_{m,n}(t)} \quad \widetilde{\Psi}_{m,n+1}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-2}(t/q) \right| \\ &\times \left| \widetilde{\Psi}_{m,n+1}(t/q) \quad \widetilde{\Psi}_{m,n+2}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-1}(t/q) \quad \phi' \right| \\ -&\left| \widetilde{\Psi}_{m,n}(t/q) \quad \widetilde{\Psi}_{m,n+1}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-1}(t/q) \right| \\ &\times \left| \underline{\widetilde{\Psi}_{m,n}(t)} \quad \widetilde{\Psi}_{m,n+1}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-2}(t/q) \quad \phi' \right| \\ +&\left| \widetilde{\Psi}_{m,n}(t/q) \quad \widetilde{\Psi}_{m,n+1}(t/q) \quad \ldots \quad \widetilde{\Psi}_{m,n+N-2}(t/q) \quad \phi' \right| \\ &\times \left| \underline{\widetilde{\Psi}_{m,n}(t)} \quad \widetilde{\Psi}_{m,n+1}(t/q) \quad \cdots \quad \widetilde{\Psi}_{m,n+N-1}(t/q) \right|. \end{aligned} \tag{A.16}$$

(v) Derivation of (3.19): applying the difference formula to the Plücker relation we have

$$qt\ \tau_N^{-1,1}(t)\ \tau_{N+1}^{0,1}(qt) + (1-qtb_1)\ \tau_N^{-1,1}(qt)\ \tau_{N+1}^{0,1}(t) - \tau_N^{-1,2}(t)\ \tau_{N+1}^{0,0}(qt) = 0. \tag{A.17}$$

We obtain (3.19) by eliminating the term $\tau_N^{-1,1}(t)\ \tau_{N+1}^{0,1}(qt)$ from (3.17) and (A.17).

(3.20):

(i) Expression of $\tau_N^{m,n}$: (4.1)

(ii) Difference formula:

$$\left| \Psi_{m,n}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+2,n-1}(t) \right| = \frac{\prod_{k=0}^{N-2} (q^{m-k}b_1 - 1)\ (1 - q^{m-k}tb_1)}{q^{\frac{(N-1)(2m-N+2)}{2}} b_1^{N-1}}\ \tau_N^{m,n}(t), \tag{A.18}$$

$$\left| \Psi_{m,n-1}(t) \quad \Psi_{m-1,n}(t) \quad \Psi_{m-1,n-1}(t) \quad \cdots \quad \Psi_{m-N+2,n-1}(t) \right| = \frac{\prod_{k=1}^{N-2} (q^{m-k}b_1 - 1)\ (1 - q^{m-k}tb_1)}{q^{\frac{(N-1)(2m-N+2)}{2}} b_1^{N-1}}\ \tau_N^{m,n}(t). \tag{A.19}$$

(iii) Contiguous relation to be used for derivation of difference formula:

$$\psi(t, b_1/q, b_3) = \frac{\psi(t, b_1, b_3) - b_1\ \psi(t, b_1, b_3/q)}{(1-b_1)(1-tb_1)}. \tag{A.20}$$

(A.20) can be derived by eliminating $\psi(t/q, b_1, b_3)$ from $(2.5)_{t\to t/q}$ and $(2.8)_{t\to t/q}$.

(iv) Plücker relation:

$$\begin{aligned}
0 =& \left| \Psi_{m+1,n-1}(t) \quad \Psi_{m,n}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \right| \\
&\times \left| \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \quad \Psi_{m-N+2,n-1}(t) \quad \phi' \right| \\
&- \left| \Psi_{m+1,n-1}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \quad \Psi_{m-N+2,n-1}(t) \right| \\
&\times \left| \Psi_{m,n}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \quad \phi' \right| \\
&+ \left| \Psi_{m+1,n-1}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \quad \phi' \right| \\
&\times \left| \Psi_{m,n}(t) \quad \Psi_{m,n-1}(t) \quad \cdots \quad \Psi_{m-N+3,n-1}(t) \quad \Psi_{m-N+2,n-1}(t) \right|. \qquad (A.21)
\end{aligned}$$

(3.21):

(i) Expression of $\tau_N^{m,n}$: (A.4)

(ii) Difference formula:

$$\begin{aligned}
&\left| \overline{\check{\Psi}_{m,n}(t)} \quad \check{\Psi}_{m-1,n+1}(qt) \quad \cdots \quad \check{\Psi}_{m-1,n+N-1}(qt) \right| \\
&\qquad = (q^n t b_3 (q^m b_1 - 1))^{1-N} \prod_{k=1}^{N-1} \left(\frac{q^{m-k+1} b_1}{q^{m-k+1} b_1 - 1} \right)^{k-N} \tau_N^{m,n}(t), \qquad (A.22)
\end{aligned}$$

$$\begin{aligned}
&\left| \check{\Psi}_{m-1,n+1}(qt) \quad \overline{\check{\Psi}_{m,n+1}(t)} \quad \check{\Psi}_{m-1,n+2}(qt) \quad \cdots \quad \check{\Psi}_{m-1,n+N-1}(qt) \right| \\
&\qquad = -(q^n t b_3 (q^m b_1 - 1))^{1-N} \prod_{k=1}^{N-1} \left(\frac{q^{m-k+1} b_1}{q^{m-k+1} b_1 - 1} \right)^{k-N} \tau_N^{m,n}(t), \qquad (A.23)
\end{aligned}$$

where

$$\overline{\check{\Psi}_{m,n}(q^l t)} = \begin{pmatrix} \psi(q^l t, q^m b_1, q^n b_3) \\ q\psi(q^{l-1} t, q^m b_1, q^n b_3) \\ \vdots \\ q^{N-1}\psi(q^{l-N+1} t, q^m b_1, q^n b_3) \end{pmatrix}. \qquad (A.24)$$

(iii) Contiguous relation to be used for derivation of difference formula:

$$\psi(t, b_1, qb_3) = \psi(t, b_1, b_3) + qtb_3(b_1 - 1)\ \psi(qt, b_1/q, qb_3). \qquad (A.25)$$

(A.25) can be derived by eliminating $\psi(qt, b_1, qb_3)$ from $(2.5)_{b_3 \to qb_3}$ and (2.14).

(iv) Plücker relation:

$$\begin{aligned}
0 =& \left| \check{\Psi}_{m-1,n}(qt) \quad \overline{\check{\Psi}_{m,n}(t)} \quad \check{\Psi}_{m-1,n+1}(qt) \quad \dots \quad \check{\Psi}_{m-1,n+N-2}(qt) \right| \\
&\times \left| \check{\Psi}_{m-1,n+1}(qt) \quad \dots \quad \check{\Psi}_{m-1,n+N-2}(qt) \quad \check{\Psi}_{m-1,n+N-1}(qt) \quad \phi' \right| \\
&- \left| \check{\Psi}_{m-1,n}(qt) \quad \check{\Psi}_{m-1,n+1}(qt) \quad \dots \quad \check{\Psi}_{m-1,n+N-2}(qt) \quad \check{\Psi}_{m-1,n+N-1}(qt) \right| \\
&\times \left| \overline{\check{\Psi}_{m,n}(t)} \quad \check{\Psi}_{m-1,n+1}(qt) \quad \dots \quad \check{\Psi}_{m-1,n+N-2}(qt) \quad \phi' \right| \\
&+ \left| \check{\Psi}_{m-1,n}(qt) \quad \check{\Psi}_{m-1,n+1}(qt) \quad \dots \quad \check{\Psi}_{m-1,n+N-2}(qt) \quad \phi' \right| \\
&\times \left| \overline{\check{\Psi}_{m,n}(t)} \quad \check{\Psi}_{m-1,n+1}(qt) \quad \dots \quad \check{\Psi}_{m-1,n+N-1}(qt) \right|. \qquad (A.26)
\end{aligned}$$

(3.22):

(i) Expression of $\tau_N^{m,n}$: (A.5)

(ii) Difference formula:

$$\begin{aligned}
&\left| \ \overline{\widehat{\Psi}_{m,n}(t)} \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-1}(q^{-N+2}t) \ \right| \\
&= (-1)^{\frac{N(N-1)}{2}} \left(qt(q^m b_1 - 1)\right)^{1-N} \prod_{k=1}^{N-1} \left(q^{n+k-1} b_3\right)^{k-N} \\
&\qquad \times \prod_{k=1}^{N-1} \left(\frac{q^{m-k+1} b_1}{q^{m-k+1} b_1 - 1} \right)^{k-N} \ \tau_N^{m,n}(t),
\end{aligned} \tag{A.27}$$

$$\begin{aligned}
&\left| \ \widehat{\Psi}_{m-1,n+1}(t) \quad \overline{\widehat{\Psi}_{m,n+1}(t/q)} \quad \widehat{\Psi}_{m-1,n+2}(t/q) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-1}(q^{-N+2}t) \ \right| \\
&= -(-1)^{\frac{N(N-1)}{2}} \Big(qt(q^m b_1 - 1)\Big)^{1-N} \prod_{k=1}^{N-1} \left(q^{n+k-1} b_3\right)^{k-N} \\
&\qquad \times \prod_{k=1}^{N-1} \left(\frac{q^{m-k+1} b_1}{q^{m-k+1} b_1 - 1} \right)^{k-N} \ \tau_N^{m,n}(t),
\end{aligned} \tag{A.28}$$

where

$$\overline{\widehat{\Psi}_{m,n}(q^l t)} = \begin{pmatrix} \psi(q^l t, q^m b_1, q^n b_3) \\ q^{-1}\psi(q^{l+1}t, q^m b_1, q^n b_3) \\ \vdots \\ q^{-N+1}\psi(q^{l+N-1}t, q^m b_1, q^n b_3) \end{pmatrix}. \tag{A.29}$$

(iii) Contiguous relation to be used for derivation of difference formula:

$$\psi(t, b_1, b_3) = \psi(qt, b_1, b_3/q) + qt(b_1 - 1)\ \psi(qt, b_1/q, b_3). \tag{A.30}$$

(A.30) can be derived by eliminating $\psi(qt, b_1, b_3)$ from (2.5) and (2.8).

(iv) Plücker relation:

$$\begin{aligned}
0 = &\left| \ \widehat{\Psi}_{m-1,n}(qt) \quad \overline{\widehat{\Psi}_{m,n}(t)} \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-2}(q^{-N+3}t) \ \right| \\
&\times \left| \ \widehat{\Psi}_{m-1,n+1}(t) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-2}(q^{-N+3}t) \quad \widehat{\Psi}_{m-1,n+N-1}(q^{-N+2}t) \quad \phi' \ \right| \\
- &\left| \ \widehat{\Psi}_{m-1,n}(qt) \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \cdots \quad \widehat{\Psi}_{m-1,n+N-2}(q^{-N+3}t) \quad \widehat{\Psi}_{m-1,n+N-1}(q^{-N+2}t) \ \right| \\
&\times \left| \ \overline{\widehat{\Psi}_{m,n}(t)} \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \ldots \quad \widehat{\Psi}_{m-1,n+N-2}(q^{-N+3}t) \quad \phi' \ \right| \\
+ &\left| \ \widehat{\Psi}_{m-1,n}(qt) \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \ldots \quad \widehat{\Psi}_{m-1,n+N-2}(q^{-N+3}t) \quad \phi' \ \right| \\
&\times \left| \ \overline{\widehat{\Psi}_{m,n}(t)} \quad \widehat{\Psi}_{m-1,n+1}(t) \quad \ldots \quad \widehat{\Psi}_{m-1,n+N-1}(q^{-N+2}t) \ \right|.
\end{aligned} \tag{A.31}$$

(v) Derivation of (3.22): applying the above difference formula to the Plücker relation we have

$$\tau_N^{-1,1}(qt)\ \tau_{N+1}^{0,1}(t/q) + t(1 - q^{-N}b_1)\ \tau_N^{0,1}(t)\ \tau_{N+1}^{-1,1}(t) - \tau_N^{-1,2}(t)\ \tau_{N+1}^{0,0}(t) = 0. \tag{A.32}$$

We obtain (3.22) by eliminating the term $\tau_N^{0,1}(t)\ \tau_{N+1}^{-1,1}(t)$ from (3.20) and (A.32).

This completes the proof of Proposition 3.2. □

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