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# Resolvent estimates <br> for the linearized compressible Navier-Stokes equation in an infinite layer 

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# Resolvent estimates for the linearized compressible Navier-Stokes equation in an infinite layer 

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#### Abstract

The resolvent problem of the linearized compressible Navier-Stokes equation around a given constant state is considered in an infinite layer $\mathbf{R}^{n-1} \times(0, a), n \geq 2$, under the no slip boundary condition for the momentum. It is proved that the linearized operator is sectorial in $W^{1, p} \times L^{p}$ for $1<p<\infty$. The $L^{p}$ estimates for the resolvent are established for all $1 \leq p \leq \infty$. The estimates for the high frequency part of the resolvent are also derived, which lead to the exponential decay of the corresponding part of the semigroup.


## 1. Introduction

This paper studies the following resolvent problem

$$
\begin{equation*}
(\lambda+L) u=f \tag{1.1}
\end{equation*}
$$

in an infinite layer $\Omega=\mathbf{R}^{n-1} \times(0, a), n \geq 2$, where $\lambda \in \mathbf{C}$ is a parameter, $f=f(x)$ is a given function with values in $\mathbf{R}^{n+1}, u=\binom{\phi}{m}$ is the unknown function with $\phi=\phi(x) \in \mathbf{R}$ and $m={ }^{T}\left(m^{1}(x), \cdots, m^{n}(x)\right) \in \mathbf{R}^{n}$, and $L$ is an operator defined by

$$
L=\left(\begin{array}{cc}
0 & \gamma \operatorname{div} \\
\gamma \nabla & -\nu \Delta I_{n}-\widetilde{\nu} \nabla \operatorname{div}
\end{array}\right)
$$

with positive constants $\nu$ and $\gamma$ and a nonnegative constant $\widetilde{\nu}$. Here $x=$ $\binom{x^{\prime}}{x_{n}} \in \Omega$ with $x^{\prime} \in \mathbf{R}^{n-1}, x_{n} \in(0, a)$; the superscript ${ }^{T}$. stands for the transposition; $I_{n}$ is the $n \times n$ identity matrix; and div, $\nabla$ and $\Delta$ are the
usual divergence, gradient and Laplacian with respect to $x$. We consider (1.1) under the boundary condition

$$
\begin{equation*}
\left.m\right|_{\partial \Omega}=0 . \tag{1.2}
\end{equation*}
$$

Problem (1.1)-(1.2) is the resolvent problem associated with the linearization of the compressible Navier-Stokes equation around a motionless state with a positive constant density, where $\phi$ and $m$ are the Laplace transform of the perturbation of the density and the momentum, respectively.

In this paper we establish the $L^{p}$ estimates for the solution of (1.1)-(1.2) for all $1 \leq p \leq \infty$. The estimates show that $-L$ generates an analytic semigroup in $W^{1, p} \times L^{p}$ for $1<p<\infty$. We also establish the estimates for the high frequency part of the resolvent, which lead to the exponential decay of the corresponding part of the semigroup. The low frequency part is investigated in [5]. The precise statement of the main results of this paper will be given in section 2 .

The resolvent problem in an infinite layer was studied in $[1,2,3]$ in the case of the incompressible Stokes equation. They established the $L^{p}$ estimates of the resolvent for $1<p<\infty$, which yields the exponential decay of the Stokes semigroup in $L^{p}$ norms. To analyze the resolvent, they considered the Fourier transform in $x^{\prime} \in \mathbf{R}^{n-1}$ and derived solution formulae for the resolvent problem. The $L^{p}$ estimates were then obtained by applying the Fourier multiplier theorem.

In this paper we will also consider the Fourier transform $\left(\lambda+\widehat{L}_{\xi^{\prime}}\right)^{-1}$ of the resolvent in $x^{\prime} \in \mathbf{R}^{n-1}$, where $\xi^{\prime} \in \mathbf{R}^{n-1}$ denotes the phase space variable. In section 3 we derive an integral representation of $\left(\lambda+\widehat{L}_{\xi^{\prime}}\right)^{-1}$. In the derivation we make use of the invariance of (1.1) under the orthogonal group and Green's formula for (1.1), which naturally leads to a decomposition of the solution into the two parts; one is the solution under the slip boundary condition; and the other is the term arising from the viscous friction stress due to the no slip boundary condition (1.2). (Cf. the solution formula for the half-space problem given in [6, 7].) In section 4 we investigate the resolvent problem based on the integral representation. The $L^{p}$ estimates for $1<p<\infty$ are obtained by applying the Fourier multiplier theorem as in $[1,2,3]$. We obtain the $L^{p}$ estimates for $p=1, \infty$ based on the Riemann-Lebesgue lemma as in the analysis of the Cauchy problem for (1.1) given in [8]. We also establish the estimates for the high frequency part $\left|\xi^{\prime}\right| \gg 1$. In contrast to the incompressible Stokes problem, $\widehat{L}_{\xi^{\prime}}$ has different characters between the cases $\left|\xi^{\prime}\right| \gg 1$ and $\left|\xi^{\prime}\right| \ll 1$. The spectral analysis near the origin is given in [5]. It is shown in [5] that the continuous spectrum reaches the origin $\lambda=0$.

## 2. Main Results

We first introduce some notation which will be used throughout the paper. For a domain $D$ and $1 \leq p \leq \infty$ we denote by $L^{p}(D)$ the usual Lebesgue space on $D$ and its norm is denoted by $\|\cdot\|_{L^{p}(D)}$. Let $\ell$ be a nonnegative integer. The symbol $W^{\ell, p}(D)$ denotes the $\ell$ th order $L^{p}$ Sobolev space on $D$ with norm $\|\cdot\|_{W^{\ell, p}(D)}$. When $p=2$, the space $W^{\ell, 2}(D)$ is denoted by $H^{\ell}(D)$ and its norm is denoted by $\|\cdot\|_{H^{\ell}(D)} . C_{0}^{\ell}(D)$ stands for the set of all $C^{\ell}$ functions which have compact support in $D$. We denote by $W_{0}^{1, p}(D)$ the completion of $C_{0}^{1}(D)$ in $W^{1, p}(D)$. In particular, $W_{0}^{1,2}(D)$ is denoted by $H_{0}^{1}(D)$.

We simply denote by $L^{p}(D)$ (resp., $W^{\ell, p}(D), H^{\ell}(D)$ ) the set of all vector fields $m={ }^{T}\left(m^{1}, \cdots, m^{n}\right)$ on $D$ with $m^{j} \in L^{p}(D)$ (resp., $W^{\ell, p}(D), H^{\ell}(D)$ ), $j=1, \cdots, n$, and its norm is also denoted by $\|\cdot\|_{L^{p}(D)}$ (resp., $\|\cdot\|_{W^{\ell, p}(D)}$, $\left.\|\cdot\|_{H^{\ell}(D)}\right)$. For $u=\binom{\phi}{m}$ with $\phi \in W^{k, p}(D)$ and $m={ }^{T}\left(m^{1}, \cdots, m^{n}\right) \in$ $W^{\ell, q}(D)$, we define $\|u\|_{W^{k, p}(D) \times W^{\ell, q}(D)}$ by $\|u\|_{W^{k, p}(D) \times W^{\ell, q(D)}}=\|\phi\|_{W^{k, p}(D)}+$ $\|m\|_{W^{\ell, q}(D)}$. When $k=\ell$ and $p=q$, we simply write $\|u\|_{W^{k, p}(D) \times W^{k, p}(D)}=$ $\|u\|_{W^{k, p}(D)}$.

In case $D=\Omega$ we abbreviate $L^{p}(\Omega)$ (resp., $\left.W^{\ell, p}(\Omega), H^{\ell}(\Omega)\right)$ as $L^{p}$ (resp., $\left.W^{\ell, p}, H^{\ell}\right)$. In particular, the norm $\|\cdot\|_{L^{p}(\Omega)}=\|\cdot\|_{L^{p}}$ is denoted by $\|\cdot\|_{p}$.

In case $D=(0, a)$ we denote the norm of $L^{p}(0, a)$ by $|\cdot|_{p}$. The inner product of $L^{2}(0, a)$ is denoted by

$$
(f, g)=\int_{0}^{a} f\left(x_{n}\right) \overline{g\left(x_{n}\right)} d x_{n}, \quad f, g \in L^{2}(0, a)
$$

Here $\bar{g}$ denotes the complex conjugate of $g$. We will also denote the bilinear pairing $f$ and $g$ by

$$
((f, g))=\int_{0}^{a} f\left(x_{n}\right) g\left(x_{n}\right) d x_{n}
$$

The norms of $W^{\ell, p}(0, a)$ and $H^{\ell}(0, a)$ are denoted by $|\cdot|_{W^{\ell, p}}$ and $|\cdot|_{H^{\ell}}$, respectively.

We often write $x \in \Omega$ as $x=\binom{x^{\prime}}{x_{n}}, x^{\prime}={ }^{T}\left(x_{1}, \cdots, x_{n-1}\right) \in \mathbf{R}^{n-1}$. Partial derivatives of a function $u$ in $x, x^{\prime}, x_{n}$ and $t$ are denoted by $\partial_{x} u, \partial_{x^{\prime}} u$, $\partial_{x_{n}} u$ and $\partial_{t} u$, respectively. We also write higher order partial derivatives of $u$ in $x$ as $\partial_{x}^{k} u=\left(\partial_{x}^{\alpha} u ;|\alpha|=k\right)$.

We denote the $k \times k$ identity matrix by $I_{k}$. In particular, when $k=n+1$, we simply write $I$ for $I_{n+1}$. We also define $(n+1) \times(n+1)$ diagonal matrices
$Q_{0}, \widetilde{Q}, Q^{\prime}$ and $Q_{n}$ by

$$
\begin{array}{ll}
Q_{0}=\operatorname{diag}(1,0, \cdots, 0,0), & \widetilde{Q}=\operatorname{diag}(0,1 \cdots, 1,1) \\
Q^{\prime}=\operatorname{diag}(0,1, \cdots, 1,0), & Q_{n}=\operatorname{diag}(0,0, \cdots, 0,1)
\end{array}
$$

For $u=\binom{\phi}{m} \in \mathbf{R}^{n+1}$ with $m=\binom{m^{\prime}}{m^{n}} \in \mathbf{R}^{n}$, we have

$$
Q_{0} u=\binom{\phi}{0}, \quad \widetilde{Q} u=\binom{0}{m}, \quad Q^{\prime} u=\left(\begin{array}{c}
0 \\
m^{\prime} \\
0
\end{array}\right), \quad Q_{n} u=\left(\begin{array}{c}
0 \\
0 \\
m^{n}
\end{array}\right)
$$

We next introduce some notation about integral operators. For a function $f=f(z)\left(z \in \mathbf{R}^{k}\right)$, we denote its Fourier transform by $\mathscr{F}_{z} f$ :

$$
\left(\mathscr{F}_{z} f\right)(\zeta)=\int_{\mathbf{R}^{k}} f(z) e^{-i \zeta \cdot z} d z
$$

In particular, when $k=n-1\left(x^{\prime} \in \mathbf{R}^{n-1}\right)$, we denote $\mathscr{F}_{x^{\prime}} f$ by $\widehat{f}$,

$$
\widehat{f}=\left(\mathscr{F}_{x^{\prime}} f\right)\left(\xi^{\prime}\right)=\int_{\mathbf{R}^{n-1}} f\left(x^{\prime}\right) e^{-i \xi^{\prime} \cdot x^{\prime}} d x^{\prime}
$$

The inverse Fourier transform is denoted by $\mathscr{F}_{\zeta}^{-1}$ :

$$
\left(\mathscr{F}_{\zeta}^{-1} f\right)(z)=(2 \pi)^{-k} \int_{\mathbf{R}^{k}} f(\zeta) e^{i \zeta \cdot z} d \zeta .
$$

For a function $K\left(x_{n}, y_{n}\right)$ on $(0, a) \times(0, a)$ we will denote by $K f$ the integral operator $\int_{0}^{a} K\left(x_{n}, y_{n}\right) f\left(y_{n}\right) d y_{n}$. Similarly, for a function $K\left(x^{\prime}, x_{n}, y_{n}\right)$ on $\mathbf{R}^{n-1} \times(0, a) \times(0, a)$ we will denote by $K f$ the integral operator $\int_{\mathbf{R}^{n-1}} \int_{0}^{a} K\left(x^{\prime}-\right.$ $\left.y^{\prime}, x_{n}, y_{n}\right) f\left(y_{n}\right) d y^{\prime} d y_{n}$.

We denote the resolvent set of a closed operator $A$ by $\rho(A)$ and the spectrum of $A$ by $\sigma(A)$. For $\Lambda \in \mathbf{R}$ and $\theta \in\left(\frac{\pi}{2}, \pi\right)$ we will denote the set $\{\lambda \in \mathbf{C} ;|\arg (\lambda-\Lambda)| \leq \theta\}$ by $\Sigma(\Lambda, \theta)$ :

$$
\Sigma(\Lambda, \theta)=\{\lambda \in \mathbf{C} ;|\arg (\lambda-\Lambda)| \leq \theta\}
$$

We now state the main results of this paper.
Theorem 2.1. Let $1<p<\infty$. There exists a number $\theta \in\left(\frac{\pi}{2}, \pi\right)$ such that for any $\eta>0$ problem (1.1)-(1.2) has a unique solution $u \in W^{1, p} \times$
( $\left.W^{2, p} \cap W_{0}^{1, p}\right)$ for any $f \in W^{1, p} \times L^{p}$, provided that $\lambda \in \Sigma(\eta, \theta)$. Furthermore, $u=(\lambda+L)^{-1} f$ satisfies the following estimates uniformly in $\lambda \in \Sigma(\eta, \theta):$

$$
\left\|\partial_{x}^{k}(\lambda+L)^{-1} f\right\|_{p} \leq C\left\{\frac{\left\|Q_{0} f\right\|_{W^{k, p}}}{|\lambda|+1}+\frac{\|\widetilde{Q} f\|_{p}}{(|\lambda|+1)^{1-\frac{k}{2}}}\right\}, \quad k=0,1
$$

and

$$
\left\|\partial_{x}^{2} \widetilde{Q}(\lambda+L)^{-1} f\right\|_{2} \leq C\|f\|_{W^{1, p} \times L^{p}}
$$

Furthermore, if $\left.\widetilde{Q} f\right|_{x_{n}=0, a}=0$, then there holds

$$
\left\|\partial_{x} \widetilde{Q}(\lambda+L)^{-1} f\right\|_{p} \leq \frac{C}{|\lambda|+1}\|f\|_{W^{1, p}}
$$

In the application to the nonlinear problem we will also use the following $H^{s} \rightarrow L^{\infty}$ estimates of the resolvent.

Theorem 2.2. Let $\theta$ be the number given in Theorem 2.1 and let $\eta$ be a positive number. Then the following estimates hold uniformly in $\lambda \in \Sigma(\eta, \theta)$ :

$$
\left\|\partial_{x}^{k} Q_{0}(\lambda+L)^{-1} f\right\|_{\infty} \leq C\left\{\frac{\left\|Q_{0} f\right\|_{H^{\left[\frac{n}{2}\right]+1+k}}}{|\lambda|+1}+\frac{\|\widetilde{Q} f\|_{H^{\left[\frac{n}{2}\right]+k}}}{(|\lambda|+1)^{\frac{3}{4}}}\right\}, \quad k=0,1,
$$

and

$$
\left\|\partial_{x}^{k} \widetilde{Q}(\lambda+L)^{-1} f\right\|_{\infty} \leq C\left\{\frac{\left\|Q_{0} f\right\|_{H^{\left[\frac{n}{2}\right]+k}}}{(|\lambda|+1)^{\frac{3}{4}}}+\frac{\|\widetilde{Q} f\|_{H^{\left[\frac{n}{2}\right]-1+k}}}{(|\lambda|+1)^{\frac{e}{4}}}\right\}, \quad k=0,1
$$

Here $\varepsilon$ is some number satisfying $0<\varepsilon<\frac{1}{3}$.
As for the $L^{p}$ estimates for $p=1, \infty$, we have the following result.
Theorem 2.3. Let $\theta$ be the number given in Theorem 2.1 and let $\eta$ be a positive number. Then the following estimates hold uniformly in $\lambda \in \Sigma(\eta, \theta)$ : for $p=1, \infty$,

$$
\left\|\partial_{x}^{k} Q_{0}(\lambda+L)^{-1} f\right\|_{p} \leq \frac{C}{|\lambda|+1}\|f\|_{W^{k+1, p} \times W^{k, p}}, \quad k=0,1,
$$

and

$$
\left\|\partial_{x}^{k} \widetilde{Q}(\lambda+L)^{-1} f\right\|_{p} \leq C\left\{\frac{\left\|Q_{0} f\right\|_{W^{k, p}}}{|\lambda|+1}+\frac{\|\widetilde{Q} f\|_{p}}{(|\lambda|+1)^{1-\frac{k}{2}}}\right\}, \quad k=0,1 .
$$

Furthermore, if $\left.\widetilde{Q} f\right|_{x_{n}=0, a}=0$, then, there hold, for $p=1, \infty$,

$$
\left\|\partial_{x} \widetilde{Q}(\lambda+L)^{-1} f\right\|_{p} \leq \frac{C}{|\lambda|+1}\|f\|_{W^{1, p}}
$$

We see from Theorem 2.1 that $-L$ generates the analytic semigroup $\mathscr{U}(t)$ in $W^{1, p} \times L^{p}$ for $1<p<\infty$. Based on Theorems 2.1-2.3 we have the following estimates of $\mathscr{U}(t)$ for $0<t \leq 1$.

Corollary 2.4. Let $\ell=0,1$. Then there hold the estimates

$$
\begin{gathered}
\left\|\partial_{x}^{\ell} \mathscr{U}(t) u_{0}\right\|_{p} \leq C t^{-\frac{\ell}{2}}\left\|u_{0}\right\|_{W^{\ell, p} \times L^{p}}, \quad 1<p<\infty \\
\left\|\partial_{x}^{\ell} \mathscr{U}(t) u_{0}\right\|_{\infty} \leq C t^{-(1-\varepsilon)}\left\|u_{0}\right\|_{H^{\left[\frac{n}{2}\right]+1+\ell} \times H^{\left[\frac{n}{2}\right]+\ell}}
\end{gathered}
$$

and

$$
\left\|\partial_{x}^{\ell} \mathscr{U}(t) u_{0}\right\|_{p} \leq C t^{-\frac{\ell}{2}}\left\|u_{0}\right\|_{W^{\ell+1, p} \times W^{\ell, p}}, \quad p=1, \infty
$$

for $0<t \leq 1$ with some constant $0<\varepsilon<1$, provided that $u_{0}$ belongs to the Sobolev spaces indicated on the right-hand side of each inequality above. Furthermore, if $\left.\widetilde{Q} u_{0}\right|_{x_{n}=0, a}=0$, then

$$
\left\|\partial_{x} \mathscr{U}(t) u_{0}\right\|_{1} \leq C\left\|u_{0}\right\|_{W^{2,1} \times W^{1,1}}
$$

holds for $0 \leq t \leq 1$.
To investigate problem (1.1)-(1.2) we consider the Fourier transform in $x^{\prime} \in \mathbf{R}^{n-1}$. Applying the Fourier transform, we have the following boundary value problem for functions $\phi\left(x_{n}\right)$ and $m\left(x_{n}\right)$ on the interval $(0, a)$ :

$$
\begin{equation*}
\lambda u+\widehat{L}_{\xi^{\prime}} u=f, \tag{2.1}
\end{equation*}
$$

where $u=\left(\begin{array}{c}\phi\left(x_{n}\right) \\ m^{\prime}\left(x_{n}\right) \\ m^{n}\left(x_{n}\right)\end{array}\right), f=\left(\begin{array}{c}f^{0}\left(x_{n}\right) \\ f^{\prime}\left(x_{n}\right) \\ f^{n}\left(x_{n}\right)\end{array}\right)$ and $\widehat{L}_{\xi^{\prime}}$ is a closed operator on $H^{1}(0, a) \times L^{2}(0, a)$ defined by $\widehat{L}_{\xi^{\prime}}=\widehat{A}_{\xi^{\prime}}+\widehat{B}_{\xi^{\prime}}$ with domain of definition $D\left(\widehat{L}_{\xi^{\prime}}\right)=H^{1}(0, a) \times\left(H^{2}(0, a) \cap H_{0}^{1}(0, a)\right)$. Here

$$
\widehat{A}_{\xi^{\prime}}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & \nu\left(\left|\xi^{\prime}\right|^{2}-\partial_{x_{n}}^{2}\right) I_{n-1}+\widetilde{\nu} \xi^{\prime} \xi^{\prime} & -i \widetilde{\nu} \xi^{\prime} \partial_{x_{n}} \\
0 & -i \widetilde{\nu}^{T} \xi^{\prime} \partial_{x_{n}} & \nu\left(\left|\xi^{\prime}\right|^{2}-\partial_{x_{n}}^{2}\right)-\widetilde{\nu} \partial_{x_{n}}^{2}
\end{array}\right)
$$

and

$$
\widehat{B}_{\xi^{\prime}}=\left(\begin{array}{ccc}
0 & i \gamma^{T} \xi^{\prime} & \gamma \partial_{x_{n}} \\
i \gamma \xi^{\prime} & 0 & 0 \\
\gamma \partial_{x_{n}} & 0 & 0
\end{array}\right)
$$

In the analysis of the semigroup generated by $-L$, it is convenient to decompose the phase space $\left\{\xi^{\prime} \in \mathbf{R}^{n-1}\right\}$ into the parts $\left|\xi^{\prime}\right| \ll 1$ and $\left|\xi^{\prime}\right| \gg$ 1 , since $\widehat{L}_{\xi^{\prime}}$ has different characters between the cases $\left|\xi^{\prime}\right| \ll 1$ and $\left|\xi^{\prime}\right| \gg 1$. Motivated by this we introduce the following decomposition. Let $r$ be a positive number. We take a function $\chi\left(\xi^{\prime}\right) \in C^{\infty}\left(\mathbf{R}^{n-1}\right)$ satisfying $0 \leq \chi \leq 1$ on $\mathbf{R}^{n-1}, \chi\left(\xi^{\prime}\right)=0$ for $\left|\xi^{\prime}\right| \leq \frac{r}{2}$ and $\chi\left(\xi^{\prime}\right)=1$ for $\left|\xi^{\prime}\right| \geq r$. We define the operator $R^{(1)}(\lambda)$ by

$$
R^{(1)}(\lambda) f=\mathscr{F}_{\xi^{\prime}}^{-1}\left[\chi\left(\xi^{\prime}\right)\left(\lambda+\widehat{L}_{\xi^{\prime}}\right)^{-1} \widehat{f}\right] .
$$

The estimates in Theorems 2.1-2.3 hold for a negative $\eta$ with $(\lambda+L)^{-1}$ replaced by $R^{(1)}(\lambda)$.

Theorem 2.5. Let $r$ be a positive number.
(i) There exist positive numbers $\widetilde{\eta}$ and $\widetilde{\theta}$ with $\widetilde{\theta} \in\left(\frac{\pi}{2}, \pi\right)$ such that $\Sigma(-\widetilde{\eta}, \widetilde{\theta}) \subset$ $\rho\left(-\widehat{L}_{\xi^{\prime}}\right)$ for $\left|\xi^{\prime}\right| \geq r$.
(ii) Let $1<p<\infty$ and define $R^{(1)}(\lambda)$ as above. Then the following estimates hold uniformly in $\lambda \in \Sigma(-\widetilde{\eta}, \widetilde{\theta})$ :

$$
\left\|\partial_{x}^{k} R^{(1)}(\lambda) f\right\|_{p} \leq\left\{\frac{\left\|Q_{0} f\right\|_{W^{k, p}}}{|\lambda|+1}+\frac{\|\widetilde{Q} f\|_{p}}{(|\lambda|+1)^{1-\frac{k}{2}}}\right\}, \quad k=0,1
$$

and

$$
\left\|\partial_{x}^{2} \widetilde{Q} R^{(1)}(\lambda) f\right\|_{2} \leq C\|f\|_{W^{1, p} \times L^{p}}
$$

Theorem 2.6 Let $\widetilde{\eta}$ and $\widetilde{\theta}$ be the numbers as in Theorem 2.5. Then the following estimates hold uniformly in $\lambda \in \Sigma(-\widetilde{\eta}, \widetilde{\theta})$ :

$$
\left\|\partial_{x}^{k} Q_{0} R^{(1)}(\lambda) f\right\|_{\infty} \leq C\left\{\frac{\left\|Q_{0} f\right\|_{H^{\left[\frac{n}{2}\right]+1+k}}}{|\lambda|+1}+\frac{\|\widetilde{Q} f\|_{H^{\left[\frac{n}{2}\right]+k}}}{(|\lambda|+1)^{\frac{3}{4}}}\right\}, \quad k=0,1,
$$

and

$$
\left\|\partial_{x}^{k} \widetilde{Q} R^{(1)}(\lambda) f\right\|_{\infty} \leq C\left\{\frac{\left\|Q_{0} f\right\|_{H^{\left[\frac{n}{2}\right]+k}}}{(|\lambda|+1)^{\frac{3}{4}}}+\frac{\|\widetilde{Q} f\|_{H^{\left[\frac{n}{2}\right]-1+k}}}{(|\lambda|+1)^{\frac{\varepsilon}{4}}}\right\}, \quad k=0,1
$$

Here $\varepsilon$ is some number satisfying $0<\varepsilon<\frac{1}{3}$.
Theorem 2.7. Let $p=1, \infty$. Let $\widetilde{\eta}$ and $\widetilde{\theta}$ be the numbers as in Theorem 2.5. Then the following estimates hold uniformly in $\lambda \in \Sigma(-\widetilde{\eta}, \tilde{\theta})$ :

$$
\left\|\partial_{x}^{k} Q_{0} R^{(1)}(\lambda) f\right\|_{p} \leq \frac{C}{|\lambda|+1}\|f\|_{W^{k+1, p} \times W^{k, p}}, \quad k=0,1,
$$

and

$$
\left\|\partial_{x}^{k} \widetilde{Q} R^{(1)}(\lambda) f\right\|_{p} \leq C\left\{\frac{\left\|Q_{0} f\right\|_{W^{k, p}}}{|\lambda|+1}+\frac{\|\widetilde{Q} f\|_{p}}{(|\lambda|+1)^{1-\frac{k}{2}}}\right\}, \quad k=0,1 .
$$

## 3. An integral representation of the resolvent

In this section we derive an integral representation of the resolvent. For this purpose we take the Fourier transform of (1.1)-(1.2) in $x^{\prime} \in \mathbf{R}^{n-1}$ and consider the boundary value problem (2.1).

We begin with the following observation on the resolvent of $-\widehat{L}_{\xi^{\prime}}$. If $\lambda \neq 0$, then, by the first row of equation (2.1), $\phi$ is written as

$$
\begin{equation*}
\phi=\frac{1}{\lambda}\left\{f^{0}-i \gamma \xi^{\prime} \cdot m^{\prime}-\gamma \partial_{x_{n}} m^{n}\right\} . \tag{3.1}
\end{equation*}
$$

Substituting (3.1) into the second and third rows of (2.1), we obtain

$$
\begin{equation*}
\mathcal{L}\left(\lambda, \xi^{\prime}\right) m=F \tag{3.2}
\end{equation*}
$$

where $\mathcal{L}\left(\lambda, \xi^{\prime}\right)=\mathcal{A}\left(\lambda, \xi^{\prime}\right)+\mathcal{B}\left(\lambda, \xi^{\prime}\right)$ with domain of definition $D\left(\widetilde{\mathcal{L}}\left(\lambda, \xi^{\prime}\right)\right)=$ $H^{2}(0, a) \cap H_{0}^{1}(0, a)$ and $F=\lambda\binom{f^{\prime}}{f^{n}}-\gamma\binom{i \xi^{\prime}}{\partial_{x_{n}}} f^{0}$. Here

$$
\begin{aligned}
\mathcal{A}\left(\lambda, \xi^{\prime}\right)= & \left(\begin{array}{cc}
\left\{\lambda^{2}+\nu \lambda\left(\left|\xi^{\prime}\right|^{2}-\partial_{x_{n}}^{2}\right)\right\} I_{n-1} & 0 \\
0 & \lambda^{2}+\nu \lambda\left(\left|\xi^{\prime}\right|^{2}-\partial_{x_{n}}^{2}\right)
\end{array}\right) \\
& \mathcal{B}\left(\lambda, \xi^{\prime}\right)=\left(\tilde{\nu} \lambda+\gamma^{2}\right)\left(\begin{array}{cc}
\xi^{T} \xi^{\prime} & -i \xi^{\prime} \partial_{x_{n}} \\
-i^{T} \xi^{\prime} \partial_{x_{n}} & -\partial_{x_{n}}^{2}
\end{array}\right)
\end{aligned}
$$

We thus deduce that (2.1) is equivalent to (3.1) and (3.2) if $\lambda \neq 0$. We also write (3.2) as

$$
\begin{equation*}
\widetilde{\mathcal{L}}\left(\lambda, \xi^{\prime}\right) m=\widetilde{F} \tag{3.3}
\end{equation*}
$$

where

$$
\widetilde{\mathcal{L}}\left(\lambda, \xi^{\prime}\right)=-\partial_{x_{n}}^{2} I_{n}+\widetilde{\mathcal{A}}\left(\lambda, \xi^{\prime}\right)+\widetilde{\mathcal{B}}\left(\lambda, \xi^{\prime}\right) \partial_{x_{n}}
$$

with domain of definition $D\left(\widetilde{\mathcal{L}}\left(\lambda, \xi^{\prime}\right)\right)=H^{2}(0, a) \cap H_{0}^{1}(0, a)$ and $\widetilde{F}=\binom{\frac{1}{\nu \lambda} F^{\prime}}{\frac{1}{\nu_{1} \lambda+\gamma^{2}} F^{n}}$.
Here

$$
\widetilde{\mathcal{A}}\left(\lambda, \xi^{\prime}\right)=\left(\begin{array}{cc}
\frac{1}{\nu}\left(\lambda+\nu\left|\xi^{\prime}\right|^{2}\right) I_{n-1}+\frac{1}{\nu \lambda}\left(\widetilde{\nu} \lambda+\gamma^{2}\right) \xi^{\prime} \xi^{\prime} & 0 \\
0 & \frac{\lambda\left(\lambda+\left.\nu\left|\xi^{\prime}\right|\right|^{2}\right)}{\nu_{1} \lambda+\gamma^{2}}
\end{array}\right)
$$

and

$$
\widetilde{\mathcal{B}}\left(\lambda, \xi^{\prime}\right)=\left(\begin{array}{cc}
0 & -i \frac{\widetilde{\nu} \lambda+\gamma^{2}}{\nu \lambda} \xi^{\prime} \\
-i \frac{\widetilde{\nu} \lambda+\gamma^{2}}{\nu_{1} \lambda+\gamma^{2}} T & \xi^{\prime}
\end{array}\right] .
$$

Here and in what follows we write $\nu_{1}$ for $\nu+\widetilde{\nu}$ :

$$
\nu_{1}=\nu+\widetilde{\nu}
$$

Lemma 3.1. Assume that $\lambda \neq 0$ and $\nu_{1} \lambda+\gamma^{2} \neq 0$. Then $\lambda \in \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$ if and only if $\operatorname{Ker} \mathcal{L}\left(\lambda, \xi^{\prime}\right)=\{0\}$.

Proof. Suppose that $\lambda \in \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$. Let $m \in H^{2}(0, a) \cap H_{0}^{1}(0, a)$ satisfy $\mathcal{L}\left(\lambda, \xi^{\prime}\right) m=0$, namely, $\widetilde{\mathcal{L}}\left(\lambda, \xi^{\prime}\right) m=0$. We define $\phi$ as in (3.1) with $f^{0}=0$ and set $u=\binom{\phi}{m}$. It then follows that $\left(\lambda+\widehat{L}_{\xi^{\prime}}\right) u=0$. Since $\lambda \in \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$, we see that $u=0$, in particular, $m=0$. This shows that $\operatorname{Ker} \mathcal{L}\left(\lambda, \xi^{\prime}\right)=\{0\}$. Conversely, suppose that $\operatorname{Ker} \mathcal{L}\left(\lambda, \xi^{\prime}\right)=\{0\}$, namely, $\operatorname{Ker} \widetilde{\mathcal{L}}\left(\lambda, \xi^{\prime}\right)=\{0\}$. We define $\widetilde{F}$ as in (3.3). Since (2.1) is equivalent to (3.1) and (3.3), it suffices to show the unique existence of the solution $u=\binom{\phi}{m}$ of (3.1) and (3.3) satisfying $|u|_{H^{1} \times H^{2}} \leq C|f|_{H^{1} \times L^{2}}$ with some $C>0$. Since $\operatorname{Ker} \widetilde{\mathcal{L}}\left(\lambda, \xi^{\prime}\right)=\{0\}$, the standard theory of elliptic equations shows that there exists a unique solution $m \in H^{2}(0, a) \cap H_{0}^{1}(0, a)$ of (3.3) and $m$ satisfies

$$
|m|_{H^{2}} \leq C\left(\lambda, \xi^{\prime}\right)|\widetilde{F}|_{2} \leq C\left(\lambda, \xi^{\prime}\right)|f|_{H^{1} \times L^{2}}
$$

It then follows from (3.1) that $\phi \in H^{1}(0, a)$ and $|\phi|_{H^{1}} \leq C\left(\lambda, \xi^{\prime}\right)|f|_{H^{1} \times L^{2}}$. We thus conclude that $\lambda \in \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$. This completes the proof.

We next give a fundamental set of solutions of the ordinary differential equation (3.2) with $F=0$. For this purpose we introduce some notation. We set

$$
\lambda_{1,0}=-\nu\left|\xi^{\prime}\right|^{2}
$$

and

$$
\lambda_{ \pm, 0}=-\frac{\nu_{1}}{2}\left|\xi^{\prime}\right|^{2} \pm \frac{1}{2} \sqrt{\nu_{1}^{2}\left|\xi^{\prime}\right|^{4}-4 \gamma^{2}\left|\xi^{\prime}\right|^{2}}
$$

and define

$$
\mu_{1}=\mu_{1}\left(\lambda,\left|\xi^{\prime}\right|^{2}\right)=\sqrt{\frac{\lambda+\nu\left|\xi^{\prime}\right|^{2}}{\nu}}
$$

and

$$
\mu_{2}=\mu_{2}\left(\lambda,\left|\xi^{\prime}\right|^{2}\right)=\sqrt{\frac{\lambda^{2}+\nu_{1}\left|\xi^{\prime}\right|^{2} \lambda+\gamma^{2}\left|\xi^{\prime}\right|^{2}}{\nu_{1} \lambda+\gamma^{2}}}
$$

Remark 3.2. We observe that $\mu_{1}=\sqrt{\frac{\lambda-\lambda_{1,0}}{\nu}}$ and $\mu_{2}=\sqrt{\frac{\left(\lambda-\lambda_{+, 0}\right)\left(\lambda-\lambda_{-, 0}\right)}{\left(\nu_{1} \lambda+\gamma^{2}\right)}}$. Furthermore, $\lambda_{-, 0}=\overline{\lambda_{+, 0}}$ with $\operatorname{Im} \lambda_{+, 0}=\gamma\left|\xi^{\prime}\right| \sqrt{1-\frac{\nu_{1}^{2}}{4 \gamma^{2}}\left|\xi^{\prime}\right|^{2}}$ when $\left|\xi^{\prime}\right|<2 \gamma / \nu_{1}$ and $\lambda_{ \pm, 0} \in \mathbf{R}$ when $\left|\xi^{\prime}\right|>2 \gamma / \nu_{1}$, and it holds that

$$
\lambda_{ \pm, 0}=-\frac{\nu_{1}}{2}\left|\xi^{\prime}\right|^{2} \pm i\left|\xi^{\prime}\right|+O\left(\left|\xi^{\prime}\right|^{3}\right)
$$

as $\left|\xi^{\prime}\right| \rightarrow 0$, and

$$
\lambda_{+, 0}=-\frac{\gamma^{2}}{\nu_{1}}+O\left(\left|\xi^{\prime}\right|^{-2}\right), \quad \lambda_{-, 0}=-\nu_{1}\left|\xi^{\prime}\right|^{2}+O(1)
$$

as $\left|\xi^{\prime}\right| \rightarrow \infty$.
Proposition 3.3. Assume that $\lambda \neq 0, \widetilde{\nu} \lambda+\gamma^{2} \neq 0, \nu_{1} \lambda+\gamma^{2} \neq 0, \lambda \neq \lambda_{1,0}$ and $\lambda \neq \lambda_{ \pm, 0}$. Then the following functions $v_{1}, \cdots, v_{2 n}$ form a fundamental set of solutions of the ordinary differential equation in (3.2) with $F=0$ :

$$
\begin{gathered}
v_{j}\left(x_{n}\right)=\binom{\boldsymbol{e}_{j}^{\prime} \cosh \mu_{1} x_{n}}{-i \frac{\xi_{j}}{\mu_{1}} \sinh \mu_{1} x_{n}}, \quad v_{n}\left(x_{n}\right)=\binom{i \frac{\xi^{\prime}}{\mu_{2}} \sinh \mu_{2} x_{n}}{\cosh \mu_{2} x_{n}}, \\
v_{n+j}\left(x_{n}\right)=\binom{\boldsymbol{e}_{j}^{\prime} \mu_{1} \sinh \mu_{1} x_{n}}{-i \xi_{j} \cosh \mu_{1} x_{n}}, \quad v_{2 n}\left(x_{n}\right)=\binom{i \xi^{\prime} \cosh \mu_{2} x_{n}}{\mu_{2} \sinh \mu_{2} x_{n}},
\end{gathered}
$$

where $j=1, \cdots, n-1$ and $\boldsymbol{e}_{j}^{\prime}={ }^{T}(0, \cdots, \stackrel{j}{1}, \cdots, 0) \in \mathbf{R}^{n-1}$.
Remark. We note that $v_{1}, \cdots, v_{2 n}$ are analytic in $\lambda$ and $\left|\xi^{\prime}\right|^{2}$.
Proof. Setting $w={ }^{T}\left(w^{1}, \cdots, w^{2 n}\right)$ with $w^{j}=m^{j}, w^{n+j}=\partial_{x_{n}} m^{j}, j=$ $1, \cdots, n$, we see that the ordinary differential equation in (3.2) with $F=0$ is equivalent to

$$
\begin{equation*}
\frac{d w}{d x_{n}}=\mathcal{M}\left(\lambda, \xi^{\prime}\right) w \tag{3.4}
\end{equation*}
$$

where

$$
\mathcal{M}\left(\lambda, \xi^{\prime}\right)=\left(\begin{array}{cc}
0 & I_{n} \\
\widetilde{\mathcal{A}}\left(\lambda, \xi^{\prime}\right) & \widetilde{\mathcal{B}}\left(\lambda, \xi^{\prime}\right)
\end{array}\right)
$$

with $\widetilde{\mathcal{A}}\left(\lambda, \xi^{\prime}\right)$ and $\widetilde{\mathcal{B}}\left(\lambda, \xi^{\prime}\right)$ defined in (3.3).
To obtain a fundamental set of solutions of (3.4) we first consider the characteristic equation of $\mathcal{M}\left(\lambda, \xi^{\prime}\right)$. Let $T^{\prime}$ be an $(n-1) \times(n-1)$ orthogonal matrix and set

$$
\mathcal{T}=\left(\begin{array}{cccc}
T^{\prime} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & T^{\prime} & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

It then follows that $\mathcal{M}\left(\lambda, \xi^{\prime}\right)=\mathcal{T}^{-1} \mathcal{M}\left(\lambda, T^{\prime} \xi^{\prime}\right) \mathcal{T}$.
We take $T^{\prime}$ in such a way that $T^{\prime} \xi^{\prime}=\left|\xi^{\prime}\right| \boldsymbol{e}_{n-1}^{\prime}$. With this $T^{\prime}$ we find that

$$
\begin{aligned}
\operatorname{det}\left(\mu I_{2 n}-\mathcal{M}\left(\lambda, \xi^{\prime}\right)\right) & =\operatorname{det}\left(\mu I_{2 n}-\mathcal{M}\left(\lambda, T^{\prime} \xi^{\prime}\right)\right) \\
& =\operatorname{det}\left(\mu^{2} I_{n}-\mu \widetilde{\mathcal{B}}\left(\lambda, T^{\prime} \xi^{\prime}\right)-\widetilde{\mathcal{A}}\left(\lambda, T^{\prime} \xi^{\prime}\right)\right) \\
& =\left(\mu^{2}-\mu_{1}^{2}\right)^{n-1}\left(\mu^{2}-\mu_{2}^{2}\right)
\end{aligned}
$$

Therefore, the eigenvalues of $\mathcal{M}\left(\lambda, \xi^{\prime}\right)$ are given by $\pm \mu_{1}$ and $\pm \mu_{2}$. Note that $\mu_{1} \neq \pm \mu_{2}$ since $\lambda \neq 0$ and $\widetilde{\nu} \lambda+\gamma^{2} \neq 0$. Furthermore, $\mu_{j} \neq 0, \mathrm{j}=1,2$, since $\lambda \neq \lambda_{1,0}$ and $\lambda \neq \lambda_{ \pm, 0}$.

We next look for eigenvectors associated with $\pm \mu_{j}$. To obtain eigenvectors for $\mu_{1}$ we consider the problem

$$
\mathcal{M}\left(\lambda, \xi^{\prime}\right)\binom{X}{Y}=\mu_{1}\binom{X}{Y}
$$

where $X, Y \in \mathbf{C}^{n}$. This is equivalent to

$$
\left(\widetilde{\mathcal{A}}\left(\lambda, \xi^{\prime}\right)+\mu_{1} \widetilde{\mathcal{B}}\left(\lambda, \xi^{\prime}\right)-\mu_{1}^{2} I_{n}\right) X=0 .
$$

We write $X=\binom{X^{\prime}}{X_{n}}$. Since $\widetilde{\nu} \lambda+\gamma^{2} \neq 0$, we have $\xi^{\prime} \cdot X^{\prime}-i \mu_{1} X_{n}=0$. This implies that $X_{j,+}=\binom{\mu_{1} \boldsymbol{e}_{j}^{\prime}}{-i \xi_{j}}, j=1, \cdots, n-1$, are eigenvectors for $\mu_{1}$. Similarly, one can see that $X_{j,-}=\binom{-\mu_{1} \boldsymbol{e}_{j}^{\prime}}{-i \xi_{j}}, j=1, \cdots, n-1$, are eigenvectors for $-\mu_{1}$.

We next look for eigenvector for $\mu_{2}$. As in the case of $\mu_{1}$, it suffices to find a nontrivial solution $X \in \mathbf{C}^{n}$ of

$$
\left(\widetilde{\mathcal{A}}\left(\lambda, \xi^{\prime}\right)+\mu_{2} \widetilde{\mathcal{B}}\left(\lambda, \xi^{\prime}\right)-\mu_{2}^{2} I_{n}\right) X=0
$$

Since $\widetilde{\nu} \lambda+\gamma^{2} \neq 0$, it follows that

$$
\left\{\begin{array}{l}
\frac{\lambda}{\nu_{1} \lambda+\gamma^{2}} X_{j}+\frac{1}{\lambda}\left(\xi^{\prime} \cdot X^{\prime}\right) \xi_{j}-i \mu_{2} \frac{\xi_{j}}{\lambda} X_{n}=0, \quad j=1, \cdots, n-1 \\
-i \mu_{2}\left(\xi^{\prime} \cdot X^{\prime}\right)-\left|\xi^{\prime}\right|^{2} X_{n}=0
\end{array}\right.
$$

This implies that $X_{n,+}=\binom{\xi^{\prime}}{-i \mu_{2}}$ is an eigenvector for $\mu_{2}$. Similarly, we can find that $X_{n,-}=\binom{\xi^{\prime}}{i \mu_{2}}$ is an eigenvector for $-\mu_{2}$.

Since $\mu_{1} \neq \pm \mu_{2}$ and $\mu_{j} \neq 0, j=1,2$, we have a fundamental set of solutions of the ordinary differential equation in (3.2) with $F=0$ : $u_{j, \pm}=$ $X_{j, \pm} e^{ \pm \mu_{1} x_{n}}(j=1, \cdots, n-1), u_{n, \pm}=X_{n, \pm} e^{ \pm \mu_{2} x_{n}}$. One can now obtain the basis $v_{1}, \cdots, v_{2 n}$ by setting $v_{j}=\frac{1}{2 \mu_{1}}\left(u_{j,+}-u_{j,-}\right), v_{n+j}=\frac{1}{2}\left(u_{j,+}+u_{j,-}\right)$ $(j=1, \cdots, n-1), v_{n}=\frac{i}{2 \mu_{2}}\left(u_{n,+}-u_{n,-}\right)$ and $v_{2 n}=\frac{i}{2}\left(u_{n,+}+u_{n,-}\right)$. This completes the proof.

We next give a characterization of the resolvent set $\rho\left(-\widehat{L}_{\xi^{\prime}}\right)$. We define complex valued functions $b_{j}(j=1,2,3)$ by

$$
\begin{aligned}
& b_{1}\left(\lambda, \xi^{\prime}, x_{n}\right)=b_{1}\left(\lambda,\left|\xi^{\prime}\right|^{2}, x_{n}\right)=\cosh \mu_{1} x_{n}-\cosh \mu_{2} x_{n}, \\
& b_{2}\left(\lambda, \xi^{\prime}, x_{n}\right)=b_{2}\left(\lambda,\left|\xi^{\prime}\right|^{2}, x_{n}\right)=\mu_{1} \sinh \mu_{1} x_{n}-\frac{\left|\xi^{\prime}\right|^{2}}{\mu_{2}} \sinh \mu_{2} x_{n}, \\
& b_{3}\left(\lambda, \xi^{\prime}, x_{n}\right)=b_{3}\left(\lambda,\left|\xi^{\prime}\right|^{2}, x_{n}\right)=\mu_{2} \sinh \mu_{2} x_{n}-\frac{\left|\xi^{\prime}\right|^{2}}{\mu_{1}} \sinh \mu_{1} x_{n},
\end{aligned}
$$

with $\mu_{j}=\mu_{j}\left(\lambda, \xi^{\prime}\right), j=1,2$. We set

$$
D\left(\lambda, \xi^{\prime}\right)=D\left(\lambda,|\xi|^{2}\right)=b_{3}\left(\lambda, \xi^{\prime}, a\right) b_{2}\left(\lambda, \xi^{\prime}, a\right)+\left|\xi^{\prime}\right|^{2} b_{1}\left(\lambda, \xi^{\prime}, a\right)^{2} .
$$

In the following we will frequently abbreviate $b_{j}\left(\lambda, \xi^{\prime}, x_{n}\right)$ to $b_{j}\left(x_{n}\right)$. Note that $b_{j}(j=1,2,3)$ are analytic in $\lambda$ and $\left|\xi^{\prime}\right|^{2}$, and hence, so is $D$. We also set

$$
\lambda_{1, k}=-\nu\left|\xi^{(k)}\right|^{2}
$$

for $\xi^{\prime} \in \mathbf{R}^{n-1}$ and $k=1,2, \cdots$, where

$$
\left|\xi^{(k)}\right|^{2}=\left|\xi^{\prime}\right|^{2}+a_{k}^{2}, \quad a_{k}=\frac{k \pi}{a} .
$$

Lemma 3.4. Assume that $\lambda \neq 0, \widetilde{\nu} \lambda+\gamma^{2} \neq 0, \nu_{1} \lambda+\gamma^{2} \neq 0, \lambda \neq \lambda_{ \pm, 0}$ and $\lambda \neq \lambda_{1, k}$ for any $k=0,1,2, \cdots$. Then $\lambda \in \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$ if and only if $D\left(\lambda, \xi^{\prime}\right) \neq 0$.

Proof. By Proposition 3.3 the solution of the equation in (3.2) with $F=0$ is written as

$$
m=c_{1} v_{1}+\cdots+c_{2 n} v_{2 n}
$$

for some $c_{j} \in \mathbf{C}, j=1, \cdots, 2 n$. This $m$ satisfies the boundary condition $\left.m\right|_{x_{n}=0, a}=0$ if and only if

$$
A\left(\begin{array}{c}
c_{1}  \tag{3.5}\\
\vdots \\
c_{2 n}
\end{array}\right)=0 \text { with } A=\left(\begin{array}{cc}
A_{1}(0) & A_{2}(0) \\
A_{1}(a) & A_{2}(a)
\end{array}\right)
$$

where $A_{1}\left(x_{n}\right)=\left(v_{1}\left(x_{n}\right), \cdots, v_{n}\left(x_{n}\right)\right)$ and $A_{2}\left(x_{n}\right)=\left(v_{n+1}\left(x_{n}\right), \cdots, v_{2 n}\left(x_{n}\right)\right)$. Note that $A_{1}(0)=I_{n}$. In view of Lemma 3.1, $\lambda \in \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$ if and only if (3.5) has only the trivial solution $c_{j}=0, j=1, \cdots, 2 n$, namely, $\operatorname{det} A \neq 0$.

Let us compute $\operatorname{det} A$. Since $A_{1}(0)=I_{n}$, by a well known formula, we have

$$
\operatorname{det} A=\operatorname{det}\left(A_{2}(a)-A_{1}(a) A_{2}(0)\right)
$$

A direct calculation gives

$$
\begin{aligned}
& A_{2}(a)-A_{1}(a) A_{2}(0) \\
& =\left(\begin{array}{cc}
\mu_{1} \sinh \mu_{1} a I_{n-1}-\frac{\xi^{\prime} T \xi^{\prime}}{\mu_{2}} \sinh \mu_{2} a & -i \xi^{\prime} b_{1}(a) \\
-i^{T} \xi^{\prime} b_{1}(a) & b_{3}(a)
\end{array}\right)
\end{aligned}
$$

Let $T^{\prime}$ be the $(n-1) \times(n-1)$ orthogonal matrix given in the proof of Proposition 3.3 such as $T^{\prime} \xi^{\prime}=\left|\xi^{\prime}\right| \boldsymbol{e}_{n-1}^{\prime}$ and set

$$
T=\left(\begin{array}{cc}
T^{\prime} & 0 \\
0 & 1
\end{array}\right)
$$

Then

$$
\begin{aligned}
& T\left(A_{2}(a)-A_{1}(a) A_{2}(0)\right) T^{-1} \\
& \quad=\left(\begin{array}{ccc}
\mu_{1} \sinh \mu_{1} a I_{n-2} & 0 & 0 \\
0 & b_{2}(a) & -i\left|\xi^{\prime}\right| b_{1}(a) \\
0 & -i\left|\xi^{\prime}\right| b_{1}(a) & b_{3}(a)
\end{array}\right)
\end{aligned}
$$

It follows that

$$
\begin{align*}
\operatorname{det} A & =\operatorname{det}\left(T\left(A_{2}(a)-A_{1}(a) A_{2}(0)\right) T^{-1}\right) \\
& =\left\{\mu_{1} \sinh \mu_{1} a\right\}^{n-2} D\left(\lambda, \xi^{\prime}\right) \tag{3.6}
\end{align*}
$$

Therefore, we see that $\lambda \in \rho\left(-\widehat{L}_{\xi}^{\prime}\right)$ if and only if $D\left(\lambda, \xi^{\prime}\right) \neq 0$. This completes the proof.

To obtain a solution formula for (2.1) we next consider the Fourier series expansion of the solution $u=\binom{\phi}{m}$ of (2.1). We expand $\phi$ and $m$ into the Fourier series as

$$
\phi=\frac{1}{2} \phi_{0}+\sum_{k=1}^{\infty} \phi_{k} \cos a_{k} x_{n}, \quad m^{\prime}=\frac{1}{2} m_{0}^{\prime}+\sum_{k=1}^{\infty} m_{k}^{\prime} \cos a_{k} x_{n}
$$

and

$$
m^{n}=\frac{1}{2} m_{0}^{n}+\sum_{k=1}^{\infty} m_{k}^{n} \sin a_{k} x_{n}
$$

where $a_{k}=k \pi / a$; and $m_{0}^{n}=0$,

$$
\phi_{k}=\frac{2}{a} \int_{0}^{a} \phi\left(x_{n}\right) \cos a_{k} x_{n} d x_{n}, \quad m_{k}^{\prime}=\frac{2}{a} \int_{0}^{a} m^{\prime}\left(x_{n}\right) \cos a_{k} x_{n} d x_{n}
$$

and

$$
m_{k}^{n}=\frac{2}{a} \int_{0}^{a} m^{n}\left(x_{n}\right) \sin a_{k} x_{n} d x_{n}
$$

Similarly, we expand $f=\left(\begin{array}{c}f^{0} \\ f^{\prime} \\ f^{n}\end{array}\right)$ as

$$
f^{0}=\frac{1}{2} f_{0}^{0}+\sum_{k=1}^{\infty} f_{k}^{0} \cos a_{k} x_{n}, \quad f^{\prime}=\frac{1}{2} f_{0}^{\prime}+\sum_{k=1}^{\infty} f_{k}^{\prime} \cos a_{k} x_{n}
$$

and

$$
f^{n}=\frac{1}{2} f_{0}^{n}+\sum_{k=1}^{\infty} f_{k}^{n} \sin a_{k} x_{n}
$$

where the Fourier coefficients are defined as above.
Similarly to $\lambda_{1, k}$, we also introduce $\lambda_{ \pm, k}$ by

$$
\lambda_{ \pm, k}=-\frac{\nu_{1}}{2}\left|\xi^{(k)}\right|^{2} \pm \frac{1}{2} \sqrt{\nu_{1}^{2}\left|\xi^{(k)}\right|^{4}-4 \gamma^{2}\left|\xi^{(k)}\right|^{2}}
$$

for $\xi^{\prime} \in \mathbf{R}^{n-1}$ and $k=1,2, \cdots$. Here, as above, $\left|\xi^{(k)}\right|^{2}=\left|\xi^{\prime}\right|^{2}+a_{k}^{2}$.
Remark 3.5. We note that $\lambda_{ \pm, k}$ have properties similar to those of $\lambda_{ \pm, 0}$, namely, $\lambda_{ \pm, k}$ are the two roots of $\lambda^{2}+\nu_{1}\left|\xi^{(k)}\right|^{2} \lambda+\gamma^{2}\left|\xi^{(k)}\right|^{2}=0 ; \lambda_{-, k}=\overline{\lambda_{+, k}}$
with $\operatorname{Im} \lambda_{+, k}=\gamma\left|\xi^{(k)}\right| \sqrt{1-\frac{\nu_{1}^{2}}{4 \gamma^{2}}\left|\xi^{(k)}\right|^{2}}$ when $\left|\xi^{(k)}\right|<2 \gamma / \nu_{1}$ and $\lambda_{ \pm, k} \in \mathbf{R}$ when $\left|\xi^{(k)}\right|>2 \gamma / \nu_{1}$; and

$$
\lambda_{+, k}=-\frac{\gamma^{2}}{\nu_{1}}+O\left(\left|\xi^{(k)}\right|^{-2}\right), \quad \lambda_{-, k}=-\nu_{1}\left|\xi^{(k)}\right|^{2}+O(1)
$$

as $\left|\xi^{(k)}\right| \rightarrow \infty$. In contrast to the case $k=0$, we find that there exists a positive number $\eta_{0}$ such that

$$
\lambda_{1, k}, \quad \lambda_{ \pm, k} \notin\left\{\lambda ; \operatorname{Re} \lambda \geq-\eta_{0}\right\}
$$

for all $k \geq 1$ and $\xi^{\prime} \in \mathbf{R}^{n-1}$.
The Fourier expansions described above are based on the reflection symmetry of equation (1.1), but they do not fit in the boundary condition (1.2). This leads to a decomposition of the Fourier coefficients $\phi_{k}, m_{k}^{\prime}$ and $m_{k}^{n}$ into two parts, one of which involves the boundary values of the $x_{n}$ derivative of $m^{\prime}$ at $x_{n}=0, a$.

Proposition 3.6. Let $\xi^{\prime} \neq 0$. Assume that $\lambda \neq 0, \widetilde{\nu} \lambda+\gamma^{2} \neq 0, \nu_{1} \lambda+\gamma^{2} \neq 0$, $\lambda \neq \lambda_{1, k}$ and $\lambda \neq \lambda_{ \pm, k}$ for any $k=0,1,2, \cdots$. Then the Fourier coefficients $u_{k}=\binom{\phi_{k}}{m_{k}}, k=0,1,2, \cdots$, are given by

$$
u_{k}=\widehat{L}_{k}\left(\lambda, \xi^{\prime}\right)^{-1} f_{k}+\widehat{L}_{k}\left(\lambda, \xi^{\prime}\right)^{-1} Y_{k}
$$

Here

$$
\begin{aligned}
\widehat{L}_{k}\left(\lambda, \xi^{\prime}\right)^{-1}= & \frac{1}{\lambda-\lambda_{1, k}}\left(\begin{array}{cc}
0 & 0 \\
0 & P_{0, k}
\end{array}\right) \\
& +\frac{1}{\left(\lambda-\lambda_{+, k}\right)\left(\lambda-\lambda_{-, k}\right)}\left(\begin{array}{ccc}
\lambda+\nu_{1}\left|\xi^{(k)}\right|^{2} & -i \gamma^{T} \xi^{\prime} & -\gamma a_{k} \\
-i \gamma \xi^{\prime} & \lambda \frac{\xi^{\prime} \xi^{\prime}}{\left|\xi^{(k)}\right|^{2}} & -i \lambda \frac{a_{k} \xi^{\prime}}{\left|\xi^{(k)}\right|^{2}} \\
\gamma a_{k} & i \lambda \frac{a_{k}^{T} \xi^{\prime}}{\left|\xi^{(k)}\right|^{2}} & \lambda \frac{a_{k}^{2}}{\left|\xi^{(k)}\right|^{2}}
\end{array}\right)
\end{aligned}
$$

where $P_{0, k}$ is an $n \times n$ matrix defined by

$$
P_{0, k}=I_{n}-\left(\begin{array}{cc}
\frac{\xi^{\prime} T \xi^{\prime}}{\left|\xi^{(k)}\right|^{2}} & -i \frac{a_{k} \xi^{\prime}}{\left|\xi^{(k)}\right|^{2}} \\
i \frac{a_{k}^{T} \xi^{\prime}}{\left|\xi^{(k)}\right|^{2}} & \frac{a_{k}^{2}}{\left|\xi^{(k)}\right|^{2}}
\end{array}\right)
$$

and $Y_{k}$ is given by

$$
Y_{k}=\left(\begin{array}{c}
0 \\
\frac{2 \nu}{a}\left\{(-1)^{k} \partial_{x_{n}} m^{\prime}(a)-\partial_{x_{n}} m^{\prime}(0)\right\} \\
0
\end{array}\right) .
$$

Proof. Since

$$
\begin{aligned}
\frac{2}{a} \int_{0}^{a} \partial_{x_{n}} w\left(x_{n}\right) \cos a_{k} x_{n} d x_{n}= & \frac{2}{a}\left\{(-1)^{k} w(a)-w(0)\right\} \\
& +a_{k} \frac{2}{a} \int_{0}^{a} w\left(x_{n}\right) \sin a_{k} x_{n} d x_{n}
\end{aligned}
$$

and

$$
\frac{2}{a} \int_{0}^{a} \partial_{x_{n}} w\left(x_{n}\right) \sin a_{k} x_{n} d x_{n}=-a_{k} \frac{2}{a} \int_{0}^{a} w\left(x_{n}\right) \cos a_{k} x_{n} d x_{n}
$$

it is not so difficult to obtain the desired expression of $u_{k}$ from (2.1). We omit the details.

We next compute the term $Y_{k}$ which involves the boundary values of the $x_{n}$ derivative of $m^{\prime}$. To do so, we make use of Green's formula for (3.2). We define $(n-1) \times(n-1)$ matrices

$$
P_{1,0}^{\prime}=\frac{\xi^{\prime T} \xi^{\prime}}{\left|\xi^{\prime}\right|^{2}}, \quad P_{0,0}^{\prime}=I_{n-1}-P_{1,0}^{\prime}
$$

Proposition 3.7. Let the assumption of Proposition 3.6 be satisfied. Then $Y_{k}$ has the form

$$
Y_{k}=\frac{2}{a} \int_{0}^{a}\left\{(-1)^{k} B\left(\lambda, \xi^{\prime}, y_{n}\right)+\check{B}\left(\lambda, \xi^{\prime}, a-y_{n}\right)\right\} f\left(y_{n}\right) d y_{n},
$$

where $B\left(\lambda, \xi^{\prime}, y_{n}\right)$ and $\check{B}\left(\lambda, \xi^{\prime}, y_{n}\right)$ are $(n+1) \times(n+1)$ matrices of the form

$$
B\left(\lambda, \xi^{\prime}, y_{n}\right)=\left(\begin{array}{ccc}
0 & 0 & 0 \\
\boldsymbol{b}_{0}\left(\lambda, \xi^{\prime}, y_{n}\right) & B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right) & \boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right) \\
0 & 0 & 0
\end{array}\right)
$$

and

$$
\check{B}\left(\lambda, \xi^{\prime}, y_{n}\right)=B\left(\lambda, \xi^{\prime}, y_{n}\right) \operatorname{diag}(1, \cdots, 1,-1)
$$

Here $\boldsymbol{b}_{0}\left(\lambda, \xi^{\prime}, y_{n}\right)$ and $\boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)$ are $(n-1)$-vectors of the form

$$
\boldsymbol{b}_{0}\left(\lambda, \xi^{\prime}, y_{n}\right)=i \xi^{\prime} \beta_{0}\left(\lambda, \xi^{\prime}, y_{n}\right)
$$

with

$$
\beta_{0}\left(\lambda, \xi^{\prime}, y_{n}\right)=\frac{\gamma \lambda}{\nu_{1} \lambda+\gamma^{2}} \frac{1}{D\left(\lambda, \xi^{\prime}\right)}\left\{b_{3}(a) \frac{1}{\mu_{2}} \sinh \mu_{2} y_{n}+b_{1}(a) \cosh \mu_{2} y_{n}\right\}
$$

and

$$
\boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)=-\frac{i \xi^{\prime}}{D\left(\lambda, \xi^{\prime}\right)}\left\{b_{3}(a) b_{1}\left(y_{n}\right)-b_{1}(a) b_{3}\left(y_{n}\right)\right\}
$$

and $B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)$ is an $(n-1) \times(n-1)$ matrix defined by

$$
B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)=-\frac{\sinh \mu_{1} y_{n}}{\sinh \mu_{1} a} P_{0,0}^{\prime}-\beta_{1}\left(\lambda, \xi^{\prime}, y_{n}\right) P_{1,0}^{\prime}
$$

with

$$
\beta_{1}\left(\lambda, \xi^{\prime}, y_{n}\right)=\frac{1}{D\left(\lambda, \xi^{\prime}\right)}\left\{b_{3}(a) b_{2}\left(y_{n}\right)+\left|\xi^{\prime}\right|^{2} b_{1}(a) b_{1}\left(y_{n}\right)\right\} .
$$

Proof. In the proof we denote $Y_{k}=\binom{0}{\tilde{Y}_{k}}$. Since $\lambda \neq 0,(2.1)$ is equivalent to (3.1) and (3.2). Consider the following equation

$$
\begin{equation*}
\widetilde{M}\left(\lambda, \xi^{\prime}\right) m=0, \tag{3.7}
\end{equation*}
$$

where

$$
\begin{aligned}
& \widetilde{M}\left(\lambda, \xi^{\prime}\right) \\
= & \left(\begin{array}{cc}
\left\{\lambda^{2}+\nu \lambda\left(\left|\xi^{\prime}\right|^{2}-\partial_{x_{n}}^{2}\right)\right\} I_{n-1}+\left(\tilde{\nu} \lambda+\gamma^{2}\right) \xi^{\prime T} \xi^{\prime} & i\left(\tilde{\nu} \lambda+\gamma^{2}\right) \xi^{\prime} \partial_{x_{n}} \\
i\left(\tilde{\nu} \lambda+\gamma^{2}\right)^{T} \xi^{\prime} \partial_{x_{n}} & \lambda^{2}+\nu \lambda\left|\xi^{\prime}\right|^{2}-\left\{\nu_{1} \lambda+\gamma^{2}\right\} \partial_{x_{n}}^{2}
\end{array}\right) .
\end{aligned}
$$

As in the proof of Proposition 3.3, one can see that the following functions $\widetilde{v}_{1}, \cdots, \widetilde{v}_{2 n}$ form a fundamental set of solutions of (3.7):

$$
\begin{gathered}
\widetilde{v}_{j}\left(x_{n}\right)=\binom{\boldsymbol{e}_{j}^{\prime} \cosh \mu_{1} x_{n}}{i \frac{\xi_{j}}{\mu_{1}} \sinh \mu_{1} x_{n}}, \quad \widetilde{v}_{n}\left(x_{n}\right)=\binom{-i \frac{\xi^{\prime}}{\mu_{2}} \sinh \mu_{2} x_{n}}{\cosh \mu_{2} x_{n}} \\
\widetilde{v}_{n+j}\left(x_{n}\right)=\binom{\boldsymbol{e}_{j}^{\prime} \mu_{1} \sinh \mu_{1} x_{n}}{i \xi_{j} \cosh \mu_{1} x_{n}}, \quad \widetilde{v}_{2 n}\left(x_{n}\right)=\binom{-i \xi^{\prime} \cosh \mu_{2} x_{n}}{\mu_{2} \sinh \mu_{2} x_{n}},
\end{gathered}
$$

where $j=1, \cdots, n-1$. We note that $\widetilde{v}_{1}, \cdots, \widetilde{v}_{2 n}$ are analytic in $\lambda$ and $\left|\xi^{\prime}\right|^{2}$. By Green's formula for (3.2), we see that

$$
\begin{aligned}
\left(\left(F, v_{j}\right)\right)= & \left(\left(m, \widetilde{M}\left(\lambda, \xi^{\prime}\right) \widetilde{v}_{j}\right)\right)-\lambda \nu\left[\partial_{x_{n}} m^{\prime} \cdot \widetilde{v}_{j}^{\prime}\right]_{x_{n}=0}^{x_{n}=a} \\
& -\left\{\nu_{1} \lambda+\gamma^{2}\right\}\left[\partial_{x_{n}} m^{n} \widetilde{v}_{j}^{n}\right]_{x_{n}=0}^{x_{n}=a} \\
= & -\partial_{x_{n}} V(a) \cdot \widetilde{v}_{j}(a)+\partial_{x_{n}} V(0) \cdot \widetilde{v}_{j}(0),
\end{aligned}
$$

where

$$
V\left(x_{n}\right)=\binom{\nu \lambda m^{\prime}\left(x_{n}\right)}{\left\{\nu_{1} \lambda+\gamma^{2}\right\} m^{n}\left(x_{n}\right)}
$$

We thus obtain

$$
\tilde{A}\binom{\partial_{x_{n}} V(0)}{\partial_{x_{n}} V(a)}=\binom{\widetilde{F}^{(1)}}{\widetilde{F}^{(2)}}
$$

with

$$
\begin{gathered}
\widetilde{A}=\left(\begin{array}{cc}
A^{(1)}(0) & -A^{(1)}(a) \\
A^{(2)}(0) & -A^{(2)}(a)
\end{array}\right), \quad F^{(j)}=\left(\left(A^{(j)}, F\right)\right) \quad(j=1,2), \\
A^{(1)}\left(x_{n}\right)=\left(\begin{array}{c}
T \widetilde{v}_{1}\left(x_{n}\right) \\
\vdots \\
T \widetilde{v}_{n}\left(x_{n}\right)
\end{array}\right), \quad A^{(2)}\left(x_{n}\right)=\left(\begin{array}{c}
T \widetilde{v}_{n+1}\left(x_{n}\right) \\
\vdots \\
T \widetilde{v}_{2 n}\left(x_{n}\right)
\end{array}\right)
\end{gathered}
$$

For a moment we assume that $\widetilde{A}$ is invertible. We write the inverse of $\widetilde{A}$ as

$$
\tilde{A}^{-1}=\left(\begin{array}{cc}
S_{1} & S_{2} \\
S_{3} & S_{4}
\end{array}\right)
$$

Then we have

$$
\binom{\partial_{x_{n}} V(0)}{\partial_{x_{n}} V(a)}=\int_{0}^{a}\binom{S_{1} A^{(1)}\left(y_{n}\right)+S_{2} A^{(2)}\left(y_{n}\right)}{S_{3} A^{(1)}\left(y_{n}\right)+S_{4} A^{(2)}\left(y_{n}\right)} F\left(y_{n}\right) d y_{n},
$$

and, hence,

$$
\begin{aligned}
\widetilde{Y}_{k}= & \frac{2}{a \lambda} \widetilde{Q}^{\prime}\left\{(-1)^{k} \partial_{x_{n}} V(a)-\partial_{x_{n}} V(0)\right\} \\
= & \frac{2}{a \lambda} \int_{0}^{a} \widetilde{Q}^{\prime}\left\{(-1)^{k}\left(S_{3} A^{(1)}\left(y_{n}\right)+S_{4} A^{(2)}\left(y_{n}\right)\right)\right. \\
& \left.\quad-\left(S_{1} A^{(1)}\left(y_{n}\right)+S_{2} A^{(2)}\left(y_{n}\right)\right)\right\} F\left(y_{n}\right) d y_{n}
\end{aligned}
$$

Here and in what follows we denote the $n \times n$ matrix $\operatorname{diag}(1, \cdots, 1,0)$ by $\widetilde{Q}^{\prime}$ :

$$
\widetilde{Q}^{\prime}=\operatorname{diag}(1, \cdots, 1,0)
$$

By a direct calculation we have

$$
A^{(1)}\left(x_{n}\right)=\left(\begin{array}{cc}
\cosh \mu_{1} x_{n} I_{n-1} & i \frac{\xi^{\prime}}{\mu_{1}} \sinh \mu_{1} x_{n} \\
-i \frac{T \xi^{\prime}}{\mu_{2}} \sinh \mu_{2} x_{n} & \cosh \mu_{2} x_{n}
\end{array}\right)
$$

and

$$
A^{(2)}\left(x_{n}\right)=\left(\begin{array}{cc}
\mu_{1} \sinh \mu_{1} x_{n} I_{n-1} & i \xi^{\prime} \cosh \mu_{1} x_{n} \\
-i^{T} \xi^{\prime} \cosh \mu_{2} x_{n} & \mu_{2} \sinh \mu_{2} x_{n}
\end{array}\right)
$$

We thus obtain

$$
\widetilde{A}=\left(\begin{array}{cc}
I_{n} & -A^{(1)}(a) \\
A^{(2)}(0) & -A^{(2)}(a)
\end{array}\right)
$$

We define $A\left(y_{n}\right)$ by

$$
\begin{aligned}
A\left(y_{n}\right) & =A^{(2)}(0) A^{(1)}\left(y_{n}\right)-A^{(2)}\left(y_{n}\right) \\
& =\left(\begin{array}{cc}
-\mu_{1} \sinh \mu_{1} y_{n} I_{n-1}+\frac{\left|\xi^{\prime}\right|^{2}}{\mu_{2}} \sinh \mu_{2} y_{n} P_{1,0}^{\prime} & -i b_{1}\left(y_{n}\right) \xi^{\prime} \\
-i b_{1}\left(y_{n}\right)^{T} \xi^{\prime} & -b_{3}\left(y_{n}\right)
\end{array}\right)
\end{aligned}
$$

One can see that if $A(a)$ is invertible, then so is $\tilde{A}$, and $\tilde{A}^{-1}$ is given by

$$
\tilde{A}^{-1}=\left(\begin{array}{cc}
I_{n}-A^{(1)}(a) A(a)^{-1} A^{(2)}(0) & A^{(1)}(a) A(a)^{-1} \\
-A(a)^{-1} A^{(2)}(0) & A(a)^{-1}
\end{array}\right)
$$

We now verify that $A(a)$ is invertible under the assumption of Proposition 3.7. We denote the $n \times n$ diagonal matrix $\operatorname{diag}(1, \cdots, 1,-1)$ by $\check{I}_{n}$. We observe that $A^{(j)}\left(x_{n}\right)=^{T}\left[\check{I}_{n} A_{j}\left(x_{n}\right) \check{I}_{n}\right], j=1,2$, where $A_{j}\left(x_{n}\right)$ are the matrices given in the proof of Lemma 3.4. Therefore, we have

$$
\widetilde{A}=^{T}\left[\operatorname{diag}\left(\check{I}_{n},-\check{I}_{n}\right) A \operatorname{diag}\left(\check{I}_{n}, \check{I}_{n}\right)\right] .
$$

It then follows from (3.6) that

$$
\operatorname{det} \widetilde{A}=(-1)^{n} \operatorname{det} A=(-1)^{n}\left\{\mu_{1} \sinh \mu_{1} a\right\}^{n-2} D\left(\lambda, \xi^{\prime}\right)
$$

and hence, by Lemma 3.4, $A(a)$ is invertible under the assumption of Proposition 3.7.

We thus obtain

$$
\begin{equation*}
\widetilde{Y}_{k}=\frac{2}{a \lambda} \int_{0}^{a}\left\{(-1)^{k} \widetilde{B}^{(1)}\left(\lambda, \xi^{\prime}, y_{n}\right)+\widetilde{B}^{(2)}\left(\lambda, \xi^{\prime}, y_{n}\right)\right\} F\left(y_{n}\right) d y_{n}, \tag{3.8}
\end{equation*}
$$

where $\widetilde{B}^{(1)}\left(\lambda, \xi, y_{n}\right)=-\widetilde{Q}^{\prime} A(a)^{-1} A\left(y_{n}\right)$ and $\widetilde{B}^{(2)}\left(\lambda, \xi^{\prime}, y_{n}\right)=\widetilde{Q}^{\prime}\left(A^{(1)}\left(y_{n}\right)-\right.$ $\left.A^{(1)}(a) A(a)^{-1} A\left(y_{n}\right)\right)$.

We next show that

$$
\begin{equation*}
-A(a)^{-1} A\left(y_{n}\right)=K^{(1)}\left(\lambda, \xi^{\prime}, y_{n}\right)+K^{(2)}\left(\lambda, \xi^{\prime}, y_{n}\right), \tag{3.9}
\end{equation*}
$$

where

$$
K^{(1)}\left(\lambda, \xi^{\prime}, y_{n}\right)=-\frac{\sinh \mu_{1} y_{n}}{\sinh \mu_{1} a}\left(\begin{array}{cc}
P_{0,0}^{\prime} & 0 \\
0 & 0
\end{array}\right)
$$

and

$$
\begin{aligned}
& K^{(2)}\left(\lambda, \xi^{\prime}, y_{n}\right) \\
= & -\frac{1}{D\left(\lambda, \xi^{\prime}\right)}\left(\begin{array}{cc}
\left\{b_{3}(a) b_{2}\left(y_{n}\right)+\left|\xi^{\prime}\right|^{2} b_{1}(a) b_{1}\left(y_{n}\right) P_{1,0}^{\prime}\right. & i\left\{b_{3}(a) b_{1}\left(y_{n}\right)-b_{1}(a) b_{3}\left(y_{n}\right)\right\} \xi^{\prime} \\
i\left\{b_{2}(a) b_{1}\left(y_{n}\right)-b_{1}(a) b_{2}\left(y_{n}\right)\right\}^{T} \xi^{\prime} & b_{2}(a) b_{3}\left(y_{n}\right)+\left|\xi^{\prime}\right|^{2} b_{1}(a) b_{1}\left(y_{n}\right)
\end{array}\right) .
\end{aligned}
$$

Let us prove (3.9). Noting that $P_{1,0}^{\prime}$ is an orthogonal projection onto the subspace spanned by $\xi^{\prime}$, namely, $P_{1,0} \xi^{\prime}=\xi^{\prime},{ }^{T} \xi^{\prime} P_{1,0}^{\prime}={ }^{T} \xi^{\prime}$ and $P_{1,0}^{\prime}{ }^{2}=P_{1,0}^{\prime}$, we see that

$$
A(a)^{-1}=-\frac{1}{\mu_{1} \sinh \mu_{1} a}\left(\begin{array}{cc}
P_{0,0}^{\prime} & 0 \\
0 & 0
\end{array}\right)+\frac{1}{D\left(\lambda, \xi^{\prime}\right)}\left(\begin{array}{cc}
-b_{3}(a) P_{1,0}^{\prime} & i b_{1}(a) \xi^{\prime} \\
i b_{1}(a)^{T} \xi^{\prime} & -b_{2}(a)
\end{array}\right)
$$

since $\mu_{1} \sinh \mu_{1} a \neq 0$ and $D \neq 0$. We now obtain (3.9) by a direct computation.

We next prove $\widetilde{B}^{(2)}\left(\lambda, \xi^{\prime}, a-y_{n}\right)=\widetilde{B}^{(1)}\left(\lambda, \xi^{\prime}, y_{n}\right) \check{I}_{n}$. To do so, we write $\widetilde{B}^{(j)}\left(\lambda, \xi^{\prime}, y_{n}\right)$ as

$$
\widetilde{B}^{(j)}\left(\lambda, \xi^{\prime}, y_{n}\right)=\left(\begin{array}{cc}
B^{\prime(j)}\left(\lambda, \xi^{\prime}, y_{n}\right) & \boldsymbol{b}_{n}^{(j)}\left(\lambda, \xi^{\prime}, y_{n}\right)  \tag{3.10}\\
0 & 0
\end{array}\right)
$$

for $j=1,2$. Then we need to show

$$
\begin{equation*}
\left.{B^{\prime(1)}}^{\prime} \lambda, \xi^{\prime}, y_{n}\right)=B^{\prime(2)}\left(\lambda, \xi^{\prime}, a-y_{n}\right) \tag{3.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{b}_{n}^{(1)}\left(\lambda, \xi^{\prime}, y_{n}\right)=-\boldsymbol{b}_{n}^{(2)}\left(\lambda, \xi^{\prime}, a-y_{n}\right) . \tag{3.12}
\end{equation*}
$$

To prove (3.11) and (3.12) we make use of the following reflection symmetry of problem (3.2). For $m\left(x_{n}\right)=\binom{m^{\prime}\left(x_{n}\right)}{m^{n}\left(x_{n}\right)}$ we define $\check{m}\left(x_{n}\right)$ by

$$
\check{m}\left(x_{n}\right)=\binom{m^{\prime}\left(a-x_{n}\right)}{-m^{n}\left(a-x_{n}\right)} .
$$

Let $m\left(x_{n}\right)$ be a solution of problem (3.2) with $F\left(x_{n}\right)=\lambda\binom{f^{\prime}\left(x_{n}\right)}{f^{n}\left(x_{n}\right)}-$ $\gamma\binom{i \xi^{\prime}}{\partial_{x_{n}}} f^{0}\left(x_{n}\right)$. Then $\check{m}\left(x_{n}\right)$ is a solution of problem (3.2) with $F\left(x_{n}\right)$ replaced by $\check{F}\left(x_{n}\right)=\binom{F^{\prime}\left(a-x_{n}\right)}{-F^{n}\left(a-x_{n}\right)}$. Since $\partial_{x_{n}}(\check{m})^{\prime}(a)=-\partial_{x_{n}} m^{\prime}(0)$ and $\partial_{x_{n}}(\check{m})^{\prime}(0)=-\partial_{x_{n}} m^{\prime}(a)$, we have
(3.13) $(-1)^{k} \partial_{x_{n}}(\check{m})^{\prime}(a)-\partial_{x_{n}}(\check{m})^{\prime}(0)=(-1)^{k}\left\{(-1)^{k} \partial_{x_{n}} m^{\prime}(a)-\partial_{x_{n}} m^{\prime}(0)\right\}$.

We next define $B_{k}^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)$ and $\boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, y_{n}\right)$ by

$$
\begin{equation*}
B_{k}^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)=(-1)^{k} B^{\prime(1)}\left(\lambda, \xi^{\prime}, y_{n}\right)+B^{\prime(2)}\left(\lambda, \xi^{\prime}, y_{n}\right) \tag{3.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, y_{n}\right)=(-1)^{k} \boldsymbol{b}_{n}^{(1)}\left(\lambda, \xi^{\prime}, y_{n}\right)+\boldsymbol{b}_{n}^{(2)}\left(\lambda, \xi^{\prime}, y_{n}\right) . \tag{3.15}
\end{equation*}
$$

Since $\boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, 0\right)=\boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, a\right)=0$, integrating by parts, we see from (3.8) that

$$
\begin{align*}
& \frac{2}{a} \lambda \nu\left\{(-1)^{k} \partial_{x_{n}}(\check{m})^{\prime}(a)-\partial_{x_{n}}(\check{m})^{\prime}(0)\right\} \\
& =\frac{2}{a} \int_{0}^{a}\left[B_{k}^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)\left\{\lambda f^{\prime}\left(a-y_{n}\right)-i \gamma \xi^{\prime} f^{0}\left(a-y_{n}\right)\right\}\right. \\
& \quad-\lambda \boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, y_{n}\right) f^{n}\left(a-y_{n}\right) \\
& \left.\quad+\gamma \partial_{y_{n}} \boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, y_{n}\right) f^{0}\left(a-y_{n}\right)\right] d y_{n}  \tag{3.16}\\
& =\frac{2}{a} \int_{0}^{a}\left[B_{k}^{\prime}\left(\lambda, \xi^{\prime}, a-y_{n}\right)\left\{\lambda f^{\prime}\left(y_{n}\right)-i \gamma \xi^{\prime} f^{0}\left(y_{n}\right)\right\}\right. \\
& \quad-\lambda \boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, a-y_{n}\right) f^{n}\left(y_{n}\right) \\
& \left.\quad+\gamma \partial_{y_{n}} \boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, a-y_{n}\right) f^{0}\left(y_{n}\right)\right] d y_{n}
\end{align*}
$$

and

$$
\begin{align*}
& \frac{2}{a} \lambda \nu\left\{(-1)^{k} \partial_{x_{n}} m^{\prime}(a)-\partial_{x_{n}} m^{\prime}(0)\right\} \\
& =\frac{2}{a} \int_{0}^{a}\left[B_{k}^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)\left\{\lambda f^{\prime}\left(y_{n}\right)-i \gamma \xi^{\prime} f^{0}\left(y_{n}\right)\right\}\right.  \tag{3.17}\\
& \left.\quad+\lambda \boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, y_{n}\right) f^{n}\left(y_{n}\right)+\gamma \partial_{y_{n}} \boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, y_{n}\right) f^{0}\left(y_{n}\right)\right] d y_{n} .
\end{align*}
$$

Combining (3.13), (3.16) and (3.17) we obtain $B_{k}^{\prime}\left(\lambda, \xi^{\prime}, a-y_{n}\right)=(-1)^{k} B_{k}^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)$ and $\boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, a-y_{n}\right)=(-1)^{k+1} \boldsymbol{b}_{n, k}\left(\lambda, \xi^{\prime}, y_{n}\right)$. This, together with (3.14) and (3.15), implies that

$$
\begin{align*}
& (-1)^{k} B^{\prime(1)}\left(\lambda, \xi^{\prime}, a-y_{n}\right)+B^{\prime(2)}\left(\lambda, \xi^{\prime}, a-y_{n}\right) \\
& =B^{\prime(1)}\left(\lambda, \xi^{\prime}, y_{n}\right)+(-1)^{k} B^{\prime(2)}\left(\lambda, \xi^{\prime}, y_{n}\right) \tag{3.18}
\end{align*}
$$

and

$$
\begin{gather*}
(-1)^{k} \boldsymbol{b}_{n}^{(1)}\left(\lambda, \xi^{\prime}, a-y_{n}\right)+\boldsymbol{b}_{n}^{(2)}\left(\lambda, \xi^{\prime}, a-y_{n}\right) \\
=-\boldsymbol{b}_{n}^{(1)}\left(\lambda, \xi^{\prime}, y_{n}\right)-(-1)^{k} \boldsymbol{b}_{n}^{(2)}\left(\lambda, \xi^{\prime}, y_{n}\right) . \tag{3.19}
\end{gather*}
$$

It follows from (3.18), with $k$ being odd and even respectively, that
which implies that $B^{(2)}\left(\lambda, \xi^{\prime}, a-y_{n}\right)=B^{\prime(1)}\left(\lambda, \xi^{\prime}, y_{n}\right)$. This shows (3.11). Similarly, one can prove (3.12) by using (3.19).

We finally set $B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)=B^{\prime(1)}\left(\lambda, \xi^{\prime}, y_{n}\right), \boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)=\boldsymbol{b}_{n}^{(1)}\left(\lambda, \xi^{\prime}, y_{n}\right)$ and $\boldsymbol{b}_{0}\left(\lambda, \xi^{\prime}, y_{n}\right)=-\frac{\gamma}{\lambda}\left\{i B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right) \xi^{\prime}-\partial_{y_{n}} \boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)\right\}$. Then, by a direct calculation, we see that

$$
\boldsymbol{b}_{0}=\frac{i \gamma \lambda \xi^{\prime}}{\nu_{1} \lambda+\gamma^{2}} \frac{1}{D\left(\lambda, \xi^{\prime}\right)}\left\{b_{3}(a) \frac{1}{\mu_{2}} \sinh \mu_{2} y_{n}+b_{1}(a) \cosh \mu_{2} y_{n}\right\}
$$

and the desired expression of $Y_{k}$ is obtained. This completes the proof.
We now give an integral representation of $\left(\lambda+\widehat{L}_{\xi^{\prime}}\right)^{-1} f$. To do so, we introduce some functions. We define $g_{\mu_{j}}^{D}\left(x_{n}, y_{n}\right)(j=1,2)$ by

$$
g_{\mu_{j}}^{D}\left(x_{n}, y_{n}\right)=\frac{1}{\mu_{j} \sinh \mu_{j} a} \sinh \mu_{j}\left(a-x_{n}\right) \sinh \mu_{j} y_{n}, \quad y_{n} \leq x_{n}
$$

and $x_{n}, y_{n}$ exchanged for $x_{n} \leq y_{n}$. Similarly, we define $g_{\mu_{j}}^{N}\left(x_{n}, y_{n}\right)$ by

$$
g_{\mu_{j}}^{N}\left(x_{n}, y_{n}\right)=\frac{1}{\mu_{j} \sinh \mu_{j} a} \cosh \mu_{j}\left(a-x_{n}\right) \cosh \mu_{j} y_{n}, \quad y_{n} \leq x_{n}
$$

and $x_{n}, y_{n}$ exchanged for $x_{n} \leq y_{n}$. We set

$$
g_{\mu_{1}, \mu_{2}}^{M}\left(x_{n}, y_{n}\right)=g_{\mu_{1}}^{M}\left(x_{n}, y_{n}\right)-g_{\mu_{2}}^{M}\left(x_{n}, y_{n}\right), \quad M=D, N .
$$

Note that $g_{\mu_{j}}^{D}$ and $g_{\mu_{j}}^{N}$ are the Green functions of the equation $\mu_{j}^{2} v-\partial_{x_{n}}^{2} v=$ 0 under the Dirichlet and Neumann boundary conditions at $\left\{x_{n}=0, a\right\}$, respectively. We also define $h_{\mu_{j}}\left(x_{n}\right)$ and $h_{\mu_{1}, \mu_{2}}\left(x_{n}\right)$ by

$$
h_{\mu_{j}}\left(x_{n}\right)=\frac{1}{\mu_{j} \sinh \mu_{j} a} \cosh \mu_{j} x_{n}
$$

and

$$
h_{\mu_{1}, \mu_{2}}\left(x_{n}\right)=h_{\mu_{1}}\left(x_{n}\right)-h_{\mu_{2}}\left(x_{n}\right) .
$$

We will denote the Dirac measure with point mass at $x_{n}$ by $\delta_{x_{n}}$, namely,

$$
\delta_{x_{n}} f=\int_{0}^{a} \delta\left(x_{n}-y_{n}\right) f\left(y_{n}\right) d y_{n}=f\left(x_{n}\right)
$$

Theorem 3.8. Let $\lambda$ satisfy $\lambda \neq 0, \widetilde{\nu} \lambda+\gamma^{2} \neq 0, \nu_{1} \lambda+\gamma^{2} \neq 0$. Assume that $\lambda \neq \lambda_{1, k}$ and $\lambda \neq \lambda_{ \pm, k}$ for any $k=0,1,2, \cdots$. Assume also that $\lambda \in \rho\left(-\widehat{L}_{\xi}^{\prime}\right)$. Then the solution $\left(\lambda+\widehat{L}_{\xi^{\prime}}\right)^{-1} f$ of (2.1) is written as

$$
\left(\lambda+\widehat{L}_{\xi^{\prime}}\right)^{-1} f=\widehat{G}\left(\lambda, \xi^{\prime}\right) f+\widehat{K}\left(\lambda, \xi^{\prime}\right) f
$$

with integral operators $\widehat{G}\left(\lambda, \xi^{\prime}\right)$ and $\widehat{K}\left(\lambda, \xi^{\prime}\right)$ defined by

$$
\left(\widehat{G}\left(\lambda, \xi^{\prime}\right) f\right)\left(x_{n}\right)=\int_{0}^{a} \widehat{G}\left(\lambda, \xi^{\prime}, x_{n}, y_{n}\right) f\left(y_{n}\right) d y_{n}
$$

and

$$
\left(\widehat{K}\left(\lambda, \xi^{\prime}\right) f\right)\left(x_{n}\right)=\int_{0}^{a} \widehat{K}\left(\lambda, \xi^{\prime}, x_{n}, y_{n}\right) f\left(y_{n}\right) d y_{n}
$$

Here $\widehat{G}\left(\lambda, \xi^{\prime}, x_{n}, y_{n}\right)$ is an $(n+1) \times(n+1)$ matrix of the form

$$
\begin{aligned}
& \widehat{G}\left(\lambda, \xi^{\prime}, x_{n}, y_{n}\right) \\
& =\left(\frac{\nu_{1}}{d(\lambda)} \delta\left(x_{n}-y_{n}\right) Q_{0}\right. \\
& \quad+\frac{\gamma}{d(\lambda)}\left(\begin{array}{ccc}
\frac{\gamma \lambda}{d(\lambda)} g_{\mu_{2}}^{N}\left(x_{n}, y_{n}\right) & -i^{T} \xi^{\prime} g_{\mu_{2}}^{N}\left(x_{n}, y_{n}\right) & -\partial_{x_{n}} g_{\mu_{2}}^{D}\left(x_{n}, y_{n}\right) \\
-i \xi^{\prime} g_{\mu_{2}}^{N}\left(x_{n}, y_{n}\right) & 0 & 0 \\
-\partial_{x_{n}} g_{\mu_{2}}^{N}\left(x_{n}, y_{n}\right) & 0 & 0
\end{array}\right) \\
& \quad+\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & \frac{1}{\nu} g_{\mu_{1}}^{N}\left(x_{n}, y_{n}\right) I_{n-1} & 0 \\
0 & 0 & \frac{1}{\nu} g_{\mu_{1}}^{D}\left(x_{n}, y_{n}\right)
\end{array}\right) \\
& \quad+\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & \frac{\xi^{\prime} T \xi^{\prime}}{\lambda} g_{\mu_{1}, \mu_{2}}^{N}\left(x_{n}, y_{n}\right) & -\frac{i \xi^{\prime}}{\lambda} \partial_{x_{n}} g_{\mu_{1}, \mu_{2}}^{D}\left(x_{n}, y_{n}\right) \\
0 & -\frac{i \xi^{\prime}}{\lambda} \partial_{x_{n}} g_{\mu_{1}, \mu_{2}}^{N}\left(x_{n}, y_{n}\right) & -\frac{1}{\lambda} \partial_{x_{n}}^{2} g_{\mu_{1}, \mu_{2}}^{D}\left(x_{n}, y_{n}\right)
\end{array}\right)
\end{aligned}
$$

where $d(\lambda)=\nu_{1} \lambda+\gamma^{2}$ and $\mu_{j}=\mu_{j}\left(\lambda, \xi^{\prime}\right), j=1,2 ;$ and

$$
\widehat{K}\left(\lambda, \xi^{\prime}, x_{n}, y_{n}\right)=\widehat{H}\left(\lambda, \xi^{\prime}, x_{n}, y_{n}\right)+\check{H}\left(\lambda, \xi^{\prime}, a-x_{n}, a-y_{n}\right),
$$

where

$$
\check{H}\left(\lambda, \xi^{\prime}, a-x_{n}, a-y_{n}\right)=\widehat{H}\left(\lambda, \xi^{\prime}, a-x_{n}, a-y_{n}\right) \operatorname{diag}\left(I_{n},-1\right)
$$

and

$$
\begin{aligned}
H & \left(\lambda, \xi^{\prime}, x_{n}, y_{n}\right) \\
= & \left(\begin{array}{ccc}
0 & 0 & 0 \\
\frac{i \xi^{\prime}}{\nu} h_{\mu_{1}}\left(x_{n}\right) \beta_{0}\left(y_{n}\right) & \frac{1}{\nu} h_{\mu_{1}}\left(x_{n}\right) B^{\prime}\left(y_{n}\right) & \frac{1}{\nu} h_{\mu_{1}}\left(x_{n}\right) \boldsymbol{b}_{n}\left(y_{n}\right) \\
0 & 0 & 0
\end{array}\right) \\
& +\left(\begin{array}{ccc}
\frac{\gamma\left|\xi^{\prime}\right|^{2}}{d(\lambda)} h_{\mu_{2}}\left(x_{n}\right) \beta_{0}\left(y_{n}\right) & \frac{i \gamma^{T} \xi^{\prime}}{d(\lambda)} h_{\mu_{2}}\left(x_{n}\right) \beta_{1}\left(y_{n}\right) & -\frac{i \gamma^{T} \xi^{\prime}}{d(\lambda)} h_{\mu_{2}}\left(x_{n}\right) \boldsymbol{b}_{n}\left(y_{n}\right) \\
\frac{i\left|\xi^{\prime}\right|^{2} \xi^{\prime}}{\lambda} h_{\mu_{1}, \mu_{2}}\left(x_{n}\right) \beta_{0}\left(y_{n}\right) & 0 & 0 \\
-\frac{\left|\xi^{\prime}\right|^{2}}{\lambda} \partial_{x_{n}} h_{\mu_{1}, \mu_{2}}\left(x_{n}\right) \beta_{0}\left(y_{n}\right) & 0 & 0 \\
0 & 0 & 0 \\
0 & -\frac{\xi^{\prime} T \xi^{\prime}}{\lambda} h_{\mu_{1}, \mu_{2}}\left(x_{n}\right) \beta_{1}\left(y_{n}\right) & \frac{\left|\xi^{\prime}\right|^{2}}{\lambda} h_{\mu_{1}, \mu_{2}}\left(x_{n}\right) \boldsymbol{b}_{n}\left(y_{n}\right) \\
0 & -\frac{i^{\prime} \xi^{\prime}}{\lambda} \partial_{x_{n}} h_{\mu_{1}, \mu_{2}}\left(x_{n}\right) \beta_{1}\left(y_{n}\right) & \frac{i^{T} \xi^{\prime}}{\lambda} \partial_{x_{n}} h_{\mu_{1}, \mu_{2}}\left(x_{n}\right) \boldsymbol{b}_{n}\left(y_{n}\right)
\end{array}\right)
\end{aligned}
$$

with $d(\lambda)=\nu_{1} \lambda+\gamma^{2}, \mu_{j}=\mu_{j}\left(\lambda, \xi^{\prime}\right), j=1,2, \beta_{j}\left(y_{n}\right)=\beta_{j}\left(\lambda, \xi^{\prime}, y_{n}\right), j=0,1$, $B^{\prime}\left(y_{n}\right)=B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)$ and $\boldsymbol{b}_{n}\left(y_{n}\right)=\boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)$.

Proof. By assumption we see that the Fourier series of $\left(\lambda+\widehat{L}_{\xi^{\prime}}\right)^{-1} f$ takes the form as in Proposition 3.6 with $Y_{k}$ given in Proposition 3.7.

The Fourier sine and cosine series of $w^{M}\left(x_{n}\right)=\int_{0}^{a} g_{\mu}^{M}\left(x_{n}, y_{n}\right) v\left(y_{n}\right) d y_{n}$ with $M=D, N$ are given by

$$
w^{D}\left(x_{n}\right)=\sum_{k=1}^{\infty} \frac{1}{\mu^{2}+a_{k}^{2}} v_{s, k} \sin a_{k} x_{n}
$$

and

$$
w^{N}\left(x_{n}\right)=\frac{1}{2} v_{c, 0}+\sum_{k=1}^{\infty} \frac{1}{\mu^{2}+a_{k}^{2}} v_{c, k} \cos a_{k} x_{n},
$$

respectively. Here $v_{s, k}$ and $v_{c, k}$ are the Fourier sine and cosine coefficients of $v$ respectively. Since

$$
\begin{gathered}
\frac{1}{\lambda-\lambda_{1, k}}=\frac{1}{\nu} \frac{1}{\mu_{1}^{2}+a_{k}^{2}}, \\
\frac{1}{\left(\lambda-\lambda_{+, k}\right)\left(\lambda-\lambda_{-, k}\right)}=\frac{1}{\nu_{1} \lambda+\gamma^{2}} \frac{1}{\mu_{2}^{2}+a_{k}^{2}}
\end{gathered}
$$

and

$$
\frac{1}{\lambda-\lambda_{1, k}} \frac{1}{\left|\xi^{(k)}\right|^{2}}-\frac{\lambda}{\left(\lambda-\lambda_{+, k}\right)\left(\lambda-\lambda_{-, k}\right)} \frac{1}{\left|\xi^{(k)}\right|^{2}}=-\frac{1}{\lambda}\left(\frac{1}{\mu_{1}^{2}+a_{k}^{2}}-\frac{1}{\mu_{2}^{2}+a_{k}^{2}}\right),
$$

we see that the Fourier coefficients of $\widehat{G}\left(\lambda, \xi^{\prime}\right) f$ are equal to $\widehat{L}_{k}\left(\lambda, \xi^{\prime}\right)^{-1} f_{k}$.
Also, since the Fourier cosine coefficients of $e^{\mu x_{n}}$ are given by

$$
\frac{2}{a} \frac{\mu\left\{(-1)^{k} e^{\mu a}-1\right\}}{\mu^{2}+a_{k}^{2}}, \quad k=0,1,2, \cdots
$$

we see from Propositions 3.6 and 3.7 that the Fourier coefficients of $\widehat{K}\left(\lambda, \xi^{\prime}\right) f$ are equal to $\widehat{L}_{k}\left(\lambda, \xi^{\prime}\right)^{-1} Y_{k}$. This completes the proof.

## 4. Preliminary estimates

In this section we prepare some estimates for the analysis of the integral kernel given in Theorem 3.8.

We first give an estimate of the resolvent set by the energy method.
Proposition 4.1. (i) There exists a positive constant $c_{1}$ such that the set

$$
\Sigma_{0} \equiv\left\{\lambda ; \operatorname{Re} \lambda+c_{1}|\operatorname{Im} \lambda|^{2}>0\right\}
$$

is in the resolvent set $\rho\left(-\widehat{L}_{\xi^{\prime}}\right)$ for all $\xi^{\prime}$.
(ii) There exists a positive number $\eta_{1}$ such that the set

$$
\left\{\lambda ; \operatorname{Re} \lambda \geq-\eta_{1}\right\} \cap\{\lambda ; \operatorname{Re} \lambda \leq 0, \operatorname{Im} \lambda=0\}^{c}
$$

is in the resolvent set $\rho\left(-\widehat{L}_{\xi^{\prime}}\right)$ for all $\xi^{\prime}$. Here and in what follows, for a set $E$, the symbol $E^{c}$ denotes the complementary set of $E$.

Proof. Let us consider problem (2.1). We note that $\widehat{A}_{\xi^{\prime}}$ is self adjoint in $L^{2}(0, a)$ and $\widehat{B}_{\xi^{\prime}}$ is skew-symmetric and the following relations hold:

$$
\begin{align*}
& \left(\widehat{A}_{\xi^{\prime}} u, u\right)=\nu\left|\xi^{\prime}\right|^{2}|m|_{2}^{2}+\nu\left|\partial_{x_{n}} m\right|_{2}^{2}+\widetilde{\nu}\left|i \xi^{\prime} \cdot m^{\prime}+\partial_{x_{n}} m^{n}\right|_{2}^{2} \\
& \left(\widehat{B}_{\xi^{\prime}} u, u\right)=2 i \gamma \operatorname{Im}\left(i \xi^{\prime} \cdot m^{\prime}+\partial_{x_{n}} m^{n}, \phi\right) \tag{4.1}
\end{align*}
$$

Let $u=\binom{\phi}{m}$ be a solution of (2.1). Taking the $L^{2}$-inner product of (2.1) with $u$ we see from (4.1) that

$$
\begin{align*}
& \lambda|u|_{2}^{2}+\nu\left|\xi^{\prime}\right|^{2}|m|_{2}^{2}+\nu\left|\partial_{x_{n}} m\right|_{2}^{2}+\widetilde{\nu}\left|i \xi^{\prime} \cdot m^{\prime}+\partial_{x_{n}} m^{n}\right|_{2}^{2} \\
& \quad+2 i \gamma \operatorname{Im}\left(i \xi^{\prime} \cdot m^{\prime}+\partial_{x_{n}} m^{n}, \phi\right)=(f, u) \tag{4.2}
\end{align*}
$$

The real part of (4.2) gives

$$
\begin{align*}
& \operatorname{Re} \lambda|u|_{2}^{2}+\nu\left|\xi^{\prime}\right|^{2}|m|_{2}^{2}+\nu\left|\partial_{x_{n}} m\right|_{2}^{2}+\widetilde{\nu}\left|i \xi^{\prime} \cdot m^{\prime}+\partial_{x_{n}} m^{n}\right|_{2}^{2} \\
& \quad=\operatorname{Re}(f, u) \leq \varepsilon|u|_{2}^{2}+\frac{1}{\varepsilon}|f|_{2}^{2} \tag{4.3}
\end{align*}
$$

for any $\varepsilon>0$. On the other hand, the imaginary part of (4.2) gives

$$
\operatorname{Im} \lambda|u|_{2}^{2}=\operatorname{Im}(f, u)-2 \gamma \operatorname{Im}\left(i \xi^{\prime} \cdot m^{\prime}+\partial_{x_{n}} m^{n}, \phi\right)
$$

from which we obtain

$$
\begin{equation*}
|\operatorname{Im} \lambda|^{2}|u|_{2}^{2} \leq 2\left\{|f|_{2}^{2}+\gamma^{2}\left|i \xi^{\prime} \cdot m^{\prime}+\partial_{x_{n}} m^{n}\right|_{2}^{2}\right\} . \tag{4.4}
\end{equation*}
$$

It then follows from (4.3) and (4.4) that

$$
\begin{equation*}
\left(\operatorname{Re} \lambda+c_{1}|\operatorname{Im} \lambda|^{2}-\varepsilon\right)|u|_{2}^{2}+\frac{\nu}{2}\left(\left|\xi^{\prime}\right|^{2}|m|_{2}^{2}+\left|\partial_{x_{n}} m\right|_{2}^{2}\right) \leq C_{\varepsilon}|f|_{2}^{2} \tag{4.5}
\end{equation*}
$$

for any $\varepsilon>0$ with some constant $c_{1}=c_{1}(\nu, \gamma)>0$.
We next estimate $\left|\partial_{x_{n}} \phi\right|_{2}$. Differentiating the first row of (2.1) with respect to $x_{n}$ we have

$$
\begin{equation*}
\lambda \partial_{x_{n}} \phi+\gamma \partial_{x_{n}}^{2} m^{n}=\partial_{x_{n}} f^{0}-i \gamma \xi^{\prime} \cdot \partial_{x_{n}} m^{\prime} \tag{4.6}
\end{equation*}
$$

We also see from the third row of (2.1) that

$$
\begin{equation*}
-\nu_{1} \partial_{x_{n}}^{2} m^{n}+\gamma \partial_{x_{n}} \phi=g \tag{4.7}
\end{equation*}
$$

where $g=f^{n}-\left\{\lambda m^{n}+\nu\left|\xi^{\prime}\right|^{2} m^{n}-i \widetilde{\nu} \xi^{\prime} \cdot \partial_{x_{n}} m^{\prime}\right\}$. By adding (4.7) $\times \frac{\gamma}{\nu_{1}}$ to (4.6) we obtain

$$
\begin{equation*}
\left(\lambda+\frac{\gamma^{2}}{\nu_{1}}\right) \partial_{x_{n}} \phi=\partial_{x_{n}} f^{0}-i \gamma \xi^{\prime} \cdot \partial_{x_{n}} m^{\prime}+\frac{\gamma}{\nu_{1}} g . \tag{4.8}
\end{equation*}
$$

This implies that if $\lambda \neq-\frac{\gamma^{2}}{\nu_{1}}$, then

$$
\begin{equation*}
\left|\partial_{x_{n}} \phi\right|_{2} \leq \frac{C}{\left|\lambda+\frac{\gamma^{2}}{\nu_{1}}\right|}\left\{|f|_{H^{1} \times L^{2}}+|\lambda||m|_{2}+\left|\xi^{\prime}\right|^{2}|m|_{2}+\left|\xi^{\prime}\right|\left|\partial_{x_{n}} m\right|_{2}\right\} . \tag{4.9}
\end{equation*}
$$

We thus deduce from (4.5) and (4.9) that

$$
\begin{equation*}
|u|_{H^{1} \times L^{2}} \leq C|f|_{H^{1} \times L^{2}} \tag{4.10}
\end{equation*}
$$

for some $C>0$, provided that $\lambda \in \Sigma_{0}$. This, together with (2.1), yields

$$
\begin{equation*}
|m|_{H^{2}} \leq \widetilde{C}|f|_{H^{1} \times L^{2}} \tag{4.11}
\end{equation*}
$$

with some constant $\widetilde{C}=\widetilde{C}\left(\eta_{0}, \theta_{0}, \xi^{\prime}\right)>0$. Since $\lambda \neq 0$ and $\nu_{1} \lambda+\gamma^{2} \neq 0$ when $\lambda \in \Sigma_{0}$, it follows from (4.10) and (4.11) that $\operatorname{Ker} \mathcal{L}=\{0\}$ for $\lambda \in \Sigma_{0}$. Lemma 3.1 then implies that $\Sigma_{0} \subset \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$. This completes the proof of (i).

We next prove (ii). In view of Lemma 3.1 it suffices to prove that (3.2) is uniquely solvable for any $F \in L^{2}(0, a)$ if $\lambda \in\left\{\lambda ; \operatorname{Re} \lambda \geq-\eta_{1}\right\} \cap\{\lambda ; \operatorname{Re} \lambda \leq$ $0, \operatorname{Im} \lambda=0\}^{c}$ with some $\eta_{1}>0$.

We take the $L^{2}$-inner product of (3.2) with $m$. Then, integrating by parts, we have
(4.12) $\left(\lambda^{2}+\nu \lambda\left|\xi^{\prime}\right|^{2}\right)|m|_{2}^{2}+\nu \lambda\left|\partial_{x_{n}} m\right|_{2}^{2}+\left(\widetilde{\nu} \lambda+\gamma^{2}\right)\left|i \xi^{\prime} \cdot m^{\prime}+\partial_{x_{n}} m^{n}\right|_{2}^{2}=(F, m)$.

The imaginary part of (4.12) gives

$$
\operatorname{Im} \lambda\left\{\left(2 \operatorname{Re} \lambda+\nu\left|\xi^{\prime}\right|^{2}\right)|m|_{2}^{2}+\nu\left|\partial_{x_{n}} m\right|_{2}^{2}+\widetilde{\nu}\left|i \xi^{\prime} \cdot m^{\prime}+\partial_{x_{n}} m^{n}\right|_{2}^{2}\right\}=\operatorname{Im}(F, m)
$$

It follows that if $\operatorname{Im} \lambda \neq 0$, then

$$
\begin{equation*}
\left(2 \operatorname{Re} \lambda+\nu\left|\xi^{\prime}\right|^{2}\right)|m|_{2}^{2}+\nu\left|\partial_{x_{n}} m\right|_{2}^{2}+\widetilde{\nu}\left|i \xi^{\prime} \cdot m^{\prime}+\partial_{x_{n}} m^{n}\right|_{2}^{2}=\frac{\operatorname{Im}(F, m)}{\operatorname{Im} \lambda} \tag{4.13}
\end{equation*}
$$

By the Poincaré inequality, there exists a positive constant $\eta_{1}$ such that the left-hand side of (4.13) is bounded from below by

$$
\left(2 \operatorname{Re} \lambda+\nu\left|\xi^{\prime}\right|^{2}+4 \eta_{1}\right)|m|_{2}^{2}+\frac{\nu}{2}\left|\partial_{x_{n}} m\right|_{2}^{2}
$$

while the right-hand side of (4.13) is bounded from above by

$$
\eta_{1}|m|_{2}^{2}+\frac{C}{|\operatorname{Im} \lambda|^{2}}|F|_{2}^{2}
$$

We thus conclude that

$$
\left(2 \operatorname{Re} \lambda+\nu\left|\xi^{\prime}\right|^{2}+3 \eta_{1}\right)|m|_{2}^{2}+\frac{\nu}{2}\left|\partial_{x_{n}} m\right|_{2}^{2} \leq \frac{C}{|\operatorname{Im} \lambda|^{2}}|F|_{2}^{2}
$$

in particular, if $\lambda \in \operatorname{Re} \lambda \geq-\eta_{1}-\frac{\left.\nu \xi^{\prime}\right|^{2}}{2}, \operatorname{Im} \lambda \neq 0$ and $F=0$, then $m=0$. This, together with (i), implies that $\left\{\lambda ; \operatorname{Re} \lambda \geq-\eta_{1}-\frac{\nu\left|\xi^{\prime}\right|^{2}}{2}\right\} \cap\{\lambda ; \operatorname{Re} \lambda \leq$ $0, \operatorname{Im} \lambda=0\}^{c} \subset \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$. This completes the proof.

We next investigate $D\left(\lambda, \xi^{\prime}\right)$. We first estimate $\left|D\left(\lambda, \xi^{\prime}\right)\right|$ from below for small $\lambda$.

Proposition 4.2. (i) Let $r_{1}$ and $R_{1}$ be any positive numbers with $r_{1}<R_{1}$. Then there exists a positive number $\Lambda_{1}=\Lambda_{1}\left(r_{1}, R_{1}\right)$ such that the inequality

$$
\left|D\left(\lambda, \xi^{\prime}\right)\right| \geq \frac{a^{4}}{24 \nu^{2}} r_{1}^{2}|\lambda|^{2}
$$

holds for $|\lambda| \leq \Lambda_{1}$ and $r_{1} \leq\left|\xi^{\prime}\right| \leq R_{1}$.
(ii) There are positive numbers $\Lambda_{2}$ and $R_{2}=R_{2}\left(\Lambda_{2}\right)$ such that the inequality

$$
\left|D\left(\lambda, \xi^{\prime}\right)\right| \geq \frac{1}{16 \nu^{2}}\left(\frac{\sinh \left|\xi^{\prime}\right| a}{\left|\xi^{\prime}\right|}\right)^{2}|\lambda|^{2}
$$

holds for $|\lambda| \leq \Lambda_{2}$ and $\left|\xi^{\prime}\right| \geq R_{2}$.
Proof. Assume that $\left|\nu_{1} \lambda+\gamma^{2}\right| \geq \frac{\gamma^{2}}{2 \nu_{1}} \equiv c_{2}$. Let $\Lambda$ and $r$ be positive numbers satisfying

$$
\begin{equation*}
\frac{\Lambda}{\nu r^{2}}<1 \text { and } \frac{\Lambda^{2}}{c_{2} r^{2}}<1 \tag{4.14}
\end{equation*}
$$

Then for $|\lambda| \leq \Lambda$ and $\left|\xi^{\prime}\right| \geq r$, we have

$$
\left|\frac{\lambda}{\nu\left|\xi^{\prime}\right|^{2}}\right| \leq \frac{\Lambda}{\nu r^{2}}<1 \quad \text { and } \quad\left|\frac{\lambda^{2}}{\nu_{1} \lambda+\gamma^{2}} \frac{1}{\left|\xi^{\prime}\right|^{2}}\right| \leq \frac{\Lambda^{2}}{c_{2} r^{2}}<1,
$$

and hence,

$$
\mu_{1}=\left|\xi^{\prime}\right|+\frac{\lambda}{2 \nu} \frac{1}{\left|\xi^{\prime}\right|}-\frac{\lambda^{2}}{8 \nu^{2}} \frac{1}{\left|\xi^{\prime}\right|^{3}}+O\left(|\lambda|^{3}\left|\xi^{\prime}\right|^{-5}\right)
$$

and

$$
\mu_{2}=\left|\xi^{\prime}\right|+\mu_{2}^{(1)} \frac{1}{\left|\xi^{\prime}\right|}+\mu_{2}^{(2)} \frac{1}{\left|\xi^{\prime}\right|^{3}}+O\left(|\lambda|^{6}\left|\xi^{\prime}\right|^{-5}\right)
$$

Here

$$
\mu_{2}^{(1)}=\frac{\lambda^{2}}{2\left(\nu_{1} \lambda+\gamma^{2}\right)} \text { and } \mu_{2}^{(2)}=-\frac{\lambda^{4}}{8\left(\nu_{1} \lambda+\gamma^{2}\right)^{2}} .
$$

It then follows that

$$
\begin{aligned}
& \sinh \mu_{j} a= \sinh \left|\xi^{\prime}\right| a+\frac{e_{j}^{(1)}}{\left|\xi^{\prime}\right|} \cosh \left|\xi^{\prime}\right| a+\frac{e_{j}^{(2)}}{\left|\xi^{\prime}\right|^{2}} \sinh \left|\xi^{\prime}\right| a+\frac{e_{j}^{(3)}}{\left|\xi^{\prime}\right|^{3}} \cosh \left|\xi^{\prime}\right| a \\
&+O\left(|\lambda|^{3}\left|\xi^{\prime}\right|^{-4} e^{\left|\xi^{\prime}\right| a}\right)
\end{aligned}
$$

and

$$
\begin{gathered}
\cosh \mu_{j} a=\cosh \left|\xi^{\prime}\right| a+\frac{e_{j}^{(1)}}{\left|\xi^{\prime}\right|} \sinh \left|\xi^{\prime}\right| a+\frac{e_{j}^{(2)}}{\left|\xi^{\prime}\right|^{2}} \cosh \left|\xi^{\prime}\right| a+\frac{e_{j}^{(3)}}{\left|\xi^{\prime}\right|^{3}} \sinh \left|\xi^{\prime}\right| a \\
+O\left(|\lambda|^{3}\left|\xi^{\prime}\right|^{-4} e^{\left|\xi^{\prime}\right| a}\right)
\end{gathered}
$$

where $j=1,2$,

$$
e_{1}^{(1)}=\frac{\lambda a}{2 \nu}, \quad e_{2}^{(1)}=\frac{\lambda^{2} a}{2\left(\nu_{1} \lambda+\gamma^{2}\right)}, \quad e_{1}^{(2)}=\frac{\lambda^{2} a^{2}}{8 \nu^{2}}, \quad e_{2}^{(2)}=\frac{\lambda^{4} a^{2}}{8\left(\nu_{1} \lambda+\gamma^{2}\right)^{2}}
$$

and

$$
e_{1}^{(3)}=O\left(|\lambda|^{2}\right), \quad e_{2}^{(3)}=O\left(|\lambda|^{4}\right)
$$

We thus obtain

$$
\begin{equation*}
\frac{1}{\lambda^{2}} D\left(\lambda, \xi^{\prime}\right)=-\frac{a^{2}}{4 \nu^{2}}+d_{0}(\lambda)+\left(\frac{1}{4 \nu^{2}}+d_{1}(\lambda)\right)\left(\frac{\sinh \left|\xi^{\prime}\right| a}{\left|\xi^{\prime}\right|}\right)^{2}+d_{2}\left(\lambda, \xi^{\prime}\right) \tag{4.15}
\end{equation*}
$$

where $d_{0}(\lambda), d_{1}(\lambda)$ and $d_{2}\left(\lambda, \xi^{\prime}\right)$ are some functions satisfying

$$
\begin{equation*}
\left|d_{0}(\lambda)\right| \leq C|\lambda|, \quad\left|d_{1}(\lambda)\right| \leq C|\lambda|, \quad\left|d_{2}\left(\lambda, \xi^{\prime}\right)\right| \leq C \frac{|\lambda|}{\left|\xi^{\prime}\right|^{3}} e^{2\left|\xi^{\prime}\right| a} \tag{4.16}
\end{equation*}
$$

uniformly in $\lambda$ and $\xi^{\prime}$ with $|\lambda| \leq \Lambda$ and $\left|\xi^{\prime}\right| \geq r$ for some $C=C\left(r_{1}\right)>0$.
Let $r_{1}>0$ and $R_{1}>0$ be fixed. Since

$$
\left(\frac{\sinh \left|\xi^{\prime}\right| a}{\left|\xi^{\prime}\right|}\right)^{2}-a^{2}=\left(\frac{\sinh \left|\xi^{\prime}\right| a}{\left|\xi^{\prime}\right|}+a\right)\left(\frac{\sinh \left|\xi^{\prime}\right| a}{\left|\xi^{\prime}\right|}-a\right) \geq \frac{a^{4}}{3}\left|\xi^{\prime}\right|^{2}
$$

for $\left|\xi^{\prime}\right|>0$, we see from (4.15) and (4.16) that if $r_{1} \leq\left|\xi^{\prime}\right| \leq R_{1}$, then

$$
\left|\frac{1}{\lambda^{2}} D\left(\lambda, \xi^{\prime}\right)\right| \geq \frac{a^{4}}{12 \nu^{2}}\left|\xi^{\prime}\right|^{2}-C|\lambda|\left\{1+\frac{e^{2 R_{1} a}}{\left|\xi^{\prime}\right|^{3}}\right\},
$$

provided that $|\lambda| \leq \Lambda_{1}$ for some $\Lambda_{1}$ satisfying (4.14) with $\Lambda=\Lambda_{1}$ and $r=r_{1}$. Here $C=C\left(r_{1}\right)>0$. Therefore, there exists a constant $\Lambda_{1}=\Lambda\left(r_{1}, R_{1}\right)>0$ such that

$$
\left|D\left(\lambda, \xi^{\prime}\right)\right| \geq \frac{a^{4}}{24 \nu^{2}} r_{1}^{2}|\lambda|^{2}
$$

for $\lambda$ and $\xi^{\prime}$ with $|\lambda| \leq \Lambda_{1}$ and $r_{1} \leq\left|\xi^{\prime}\right| \leq R_{1}$. This proves (i).
We next prove (ii). From (4.16) we see that there exists a positive number $\Lambda_{2}$ such that

$$
\left|\frac{1}{4 \nu^{2}}+d_{1}(\lambda)\right| \geq \frac{1}{8 \nu^{2}}
$$

for $|\lambda| \leq \Lambda_{2}$. Furthermore, there exists a positive number $R_{2}=R_{2}\left(\Lambda_{2}\right)$ such that (4.14) is satisfied with $\Lambda=\Lambda_{2}$ and $r=R_{2}$, and the inequalities

$$
\left|-\frac{a^{2}}{4 \nu^{2}}+d_{0}(\lambda)\right| \leq \frac{1}{32 \nu^{2}}\left(\frac{\sinh \left|\xi^{\prime}\right| a}{\left|\xi^{\prime}\right|}\right)^{2}, \quad\left|d_{2}\left(\lambda, \xi^{\prime}\right)\right| \leq \frac{1}{32 \nu^{2}}\left(\frac{\sinh \left|\xi^{\prime}\right| a}{\left|\xi^{\prime}\right|}\right)^{2}
$$

hold for all $|\lambda| \leq \Lambda_{2}$ and $\left|\xi^{\prime}\right| \geq R_{2}$. It then follows from (4.15) that

$$
\left|\frac{1}{\lambda^{2}} D\left(\lambda, \xi^{\prime}\right)\right| \geq \frac{1}{16 \nu^{2}}\left(\frac{\sinh \left|\xi^{\prime}\right| a}{\left|\xi^{\prime}\right|}\right)^{2}
$$

for all $|\lambda| \leq \Lambda_{2}$ and $\left|\xi^{\prime}\right| \geq R_{2}$. This completes the proof.
By Propositions 4.1 and 4.2 we have the following consequence on $\rho\left(-\widehat{L}_{\xi^{\prime}}\right)$.
Lemma 4.3. (i) There exists a number $\theta_{1} \in\left(\frac{\pi}{2}, \pi\right)$ such that $\Sigma\left(\eta, \theta_{1}\right) \subset$ $\rho\left(-\widehat{L}_{\xi^{\prime}}\right)$ for any $\eta>0$ and $\xi^{\prime} \in \mathbf{R}^{n-1}$. Furthermore, the integral representation of $\left(\lambda+\widehat{L}_{\xi^{\prime}}\right)^{-1}$ in Theorem 3.8 holds for $\lambda \in \Sigma\left(\eta, \theta_{1}\right)$ for any $\eta>0$ and $\xi^{\prime} \in \mathbf{R}^{n-1}$.
(ii) For any $r>0$ there exist positive numbers $\eta_{2}$ and $\theta_{2}$ with $\theta_{2} \in\left(\frac{\pi}{2}, \pi\right)$ such that $\Sigma\left(-\eta_{2}, \theta_{2}\right) \subset \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$ for $\left|\xi^{\prime}\right| \geq r$. Furthermore, the integral representation of $\left(\lambda+\widehat{L}_{\xi^{\prime}}\right)^{-1}$ in Theorem 3.8 holds for $\lambda \in \Sigma\left(-\eta_{2}, \theta_{2}\right)$ and $\left|\xi^{\prime}\right| \geq r$.

Proof. The first assertion of (i) is an easy consequence of Proposition 4.1. By Remarks 3.2 and 3.5 , changing $\theta_{1}$ suitably if necessary, we see that $\lambda_{1, k}, \lambda_{ \pm, k} \notin$ $\Sigma\left(\eta, \theta_{1}\right)$ for any $k=0,1,2, \cdots$. Therefore, the integral representation in Theorem 3.8 holds for $\lambda \in \Sigma\left(\eta, \theta_{1}\right)$.

Similarly, we see from Lemma 3.4 and Propositions 4.1 and 4.2 that for any $r>0$ there exist positive numbers $\eta_{2}$ and $\theta_{2}$ with $\theta_{2} \in\left(\frac{\pi}{2}, \pi\right)$ such that $\Sigma\left(-\eta_{2}, \theta_{2}\right)-\{0\} \subset \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$ for $\left|\xi^{\prime}\right| \geq r$. Furthermore, by Remarks 3.2 and 3.5 , changing $\eta_{2}$ and $\theta_{2}$ suitably if necessary, we deduce that $\lambda_{1, k}, \lambda_{ \pm, k} \notin$ $\Sigma\left(-\eta_{2}, \theta_{2}\right)$ for any $k=0,1,2, \cdots$, and the integral representation in Theorem 3.8 holds for $\lambda \in \Sigma\left(-\eta_{2}, \theta_{2}\right)-\{0\}$ and $\left|\xi^{\prime}\right| \geq r$.

Let us prove $0 \in \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$. Assume that $\xi^{\prime}$ satisfies $\left|\xi^{\prime}\right| \geq r$. We note that $\left(\lambda+\widehat{L}_{\xi^{\prime}}\right)^{-1}$ is analytic in $\Sigma\left(-\eta_{2}, \theta_{2}\right)-\{0\}$. Furthermore, $\widehat{G}\left(\lambda, \xi^{\prime}\right)$ and $\widehat{K}\left(\lambda, \xi^{\prime}\right)$ are also analytic in $\Sigma\left(-\eta_{2}, \theta_{2}\right)-\{0\}$. Therefore, it suffices to prove that $\widehat{G}\left(\lambda, \xi^{\prime}\right)$ and $\widehat{K}\left(\lambda, \xi^{\prime}\right)$ are bounded in $\Sigma\left(-\eta_{2}, \theta_{2}\right)-\{0\}$.

Since

$$
\begin{equation*}
\mu_{1}-\mu_{2}=\frac{\lambda}{\mu_{1}+\mu_{2}}\left\{\frac{1}{\nu}+\frac{\lambda}{\nu_{1} \lambda+\gamma^{2}}\right\}, \tag{4.17}
\end{equation*}
$$

we see that $\widetilde{Q} \widehat{G}\left(\lambda, \xi^{\prime}\right) \widetilde{Q}$ is bounded in $\Sigma\left(-\eta_{2}, \theta_{2}\right)-\{0\}$. It is easy to see that the other components of $\widehat{G}\left(\lambda, \xi^{\prime}\right)$ are bounded in $\Sigma\left(-\eta_{2}, \theta_{2}\right)-\{0\}$, and hence, $\widehat{G}\left(\lambda, \xi^{\prime}\right)$ is bounded in $\Sigma\left(-\eta_{2}, \theta_{2}\right)-\{0\}$.

As for $\widehat{K}\left(\lambda, \xi^{\prime}\right)$, we easily see that $b_{j}\left(\lambda, \xi^{\prime}, y_{n}\right)=O(\lambda)$ as $\lambda \rightarrow 0$ for $j=1,2,3$. This, together with Proposition 4.2, implies that $\beta_{j}\left(\lambda, \xi^{\prime}, y_{n}\right)=$ $O(1)(j=0,1), B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)=O(1)$ and $\boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)=O(1)$ as $\lambda \rightarrow 0$. Furthermore, using (4.17), we also see that $h_{\mu_{1}, \mu_{2}}\left(x_{n}\right)=O(\lambda)$ as $\lambda \rightarrow 0$. It then follows that $\widehat{K}\left(\lambda, \xi^{\prime}\right)$ is bounded in $\Sigma\left(-\eta_{2}, \theta_{2}\right)-\{0\}$ and assertion (ii) is proved. This completes the proof.

We next derive estimates for $\left|D\left(\lambda, \xi^{\prime}\right)\right|$ from below when $|\lambda|+\left|\xi^{\prime}\right|^{2}$ is large. In the following we specify branches of $\mu_{j}\left(\lambda, \xi^{\prime}\right), j=1,2$, as a function of $\lambda$. As for $\mu_{1}\left(\lambda, \xi^{\prime}\right)$ we take the principal branch of the square root of $\frac{\lambda+\nu\left|\xi^{\prime}\right|^{2}}{\nu}$, i.e., for $\lambda \notin\left(-\infty,-\nu\left|\xi^{\prime}\right|^{2}\right]$ we denote by $\mu_{1}\left(\lambda, \xi^{\prime}\right)$ the square root of $\frac{\lambda+\nu\left|\xi^{\prime}\right|^{2}}{\nu}$ with $\operatorname{Re} \mu_{1}\left(\lambda, \xi^{\prime}\right)>0$. As for $\mu_{2}\left(\lambda, \xi^{\prime}\right)$ we take the branch in the following way. When $\left|\xi^{\prime}\right|<\frac{2 \gamma}{\nu_{1}}$, we use the branch specified by the requirement

$$
\arg \left(\lambda-\lambda_{ \pm, 0}\right)=\mp \frac{\pi}{2} \text { at } \lambda=\operatorname{Re} \lambda_{+, 0} \text { and } \arg \left(\lambda+\frac{\gamma^{2}}{\nu_{1}}\right)=0 \text { at } \lambda=0
$$

and take the branch cut

$$
\left\{\lambda ; \operatorname{Re} \lambda \leq-\frac{\gamma^{2}}{\nu_{1}}, \operatorname{Im} \lambda=0\right\} \cup\left\{\lambda \in \Gamma_{\gamma, \nu_{1}} ; \operatorname{Re} \lambda \leq \operatorname{Re} \lambda_{ \pm, 0}\right\} .
$$

Here $\Gamma_{\gamma, \nu_{1}}$ is the circle defined by $\Gamma_{\gamma, \nu_{1}}=\left\{\lambda ;\left|\lambda+\frac{\gamma^{2}}{\nu_{1}}\right|=\frac{\gamma^{2}}{\nu_{1}}\right\}$. When $\left|\xi^{\prime}\right| \geq \frac{2 \gamma}{\nu_{1}}$, we use the branch specified by $\arg \left(\lambda-\lambda_{ \pm, 0}\right)=\arg \left(\lambda+\frac{\gamma^{2}}{\nu_{1}}\right)=0$ at $\lambda=0$ and take the branch cut

$$
\left\{\lambda ; \operatorname{Re} \lambda \leq-\frac{\gamma^{2}}{\nu_{1}}, \operatorname{Im} \lambda=0\right\}
$$

As shown in [6], it holds $\operatorname{Re} \mu_{2}\left(\lambda, \xi^{\prime}\right)>0$ for $\lambda$ outside the branch cut.
We introduce a function $D_{1}\left(\lambda, \xi^{\prime}\right)$ defined by

$$
D_{1}\left(\lambda, \xi^{\prime}\right)=e^{-\left(\mu_{1}+\mu_{2}\right) a} D\left(\lambda, \xi^{\prime}\right)
$$

To investigate the resolvent $(\lambda+L)^{-1}=\mathscr{F}^{-1}\left[\left(\lambda+L_{\xi}^{\prime}\right)^{-1}\right]$ for large $\lambda+\left|\xi^{\prime}\right|^{2}$, it is convenient to consider $D_{1}\left(\lambda, \xi^{\prime}\right)$ rather than $D\left(\lambda, \xi^{\prime}\right)$.

Lemma 4.4. (i) There are positive numbers $\eta_{3}$ and $\theta_{3}$ with $\theta_{3} \in\left(\frac{\pi}{2}, \pi\right)$ such that the inequality

$$
\left|D_{1}\left(\lambda, \xi^{\prime}\right)\right| \geq C \frac{|\lambda|^{2}}{|\lambda|+1+\left|\xi^{\prime}\right|^{2}}
$$

holds uniformly in $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right) \cap\{\lambda ;|\lambda| \geq \delta\}$ and $\left|\xi^{\prime}\right| \geq R_{3}$ for any $\delta>0$ with some constant $R_{3}=R_{3}(\delta)>0$.
(ii) There are positive numbers $\eta_{4}$ and $\theta_{4}$ with $\theta_{4} \in\left(\frac{\pi}{2}, \pi\right)$ such that the inequality

$$
\left|D_{1}\left(\lambda, \xi^{\prime}\right)\right| \geq C \frac{|\lambda|^{2}}{|\lambda|+1+\left|\xi^{\prime}\right|^{2}}
$$

holds uniformly in $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$ and $\xi^{\prime} \in \mathbf{R}^{n-1}$.
Proof. We first observe that there are positive numbers $\eta_{3}, \theta_{3} \in\left(\frac{\pi}{2}, \pi\right)$ and $\widetilde{R}_{3}$ such that

$$
\begin{equation*}
\operatorname{Re} \mu_{j} \geq C\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{\frac{1}{2}} \quad(j=1,2) \tag{4.18}
\end{equation*}
$$

for $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\left|\xi^{\prime}\right| \geq \widetilde{R}_{3}$. Furthermore, there exist positive numbers $\tilde{\eta}_{4}$ and $\theta_{4} \in\left(\frac{\pi}{2}, \pi\right)$ such that (4.18) also holds for $\lambda \in \Sigma\left(\widetilde{\eta}_{4}, \theta_{4}\right)$ and $\xi^{\prime} \in \mathbf{R}^{n-1}$. In fact, it is easy to show the inequalities (4.18) for $j=1$ with appropriate $\eta_{3}$, $\tilde{\eta}_{4}, \theta_{3}, \theta_{4}$ and $\widetilde{R}_{3}$. As for $j=2$ one can find (4.18) by using the observation in Remark 3.2. It is also possible to see that

$$
\begin{equation*}
\left|\mu_{j}\right| \leq C\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{\frac{1}{2}} \tag{4.19}
\end{equation*}
$$

for $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\xi^{\prime}$ with $\left|\xi^{\prime}\right| \geq \widetilde{R}_{3}$, and for $\lambda \in \Sigma\left(\widetilde{\eta}_{4}, \theta_{4}\right)$ and $\xi^{\prime} \in \mathbf{R}^{n-1}$.
Consider next the quadratic equation $\omega^{2}+(2 \nu+\widetilde{\nu})\left|\xi^{\prime}\right|^{2} \omega+\gamma^{2}\left|\xi^{\prime}\right|^{2}=0$ for $\omega$. This equation has two roots

$$
\omega_{ \pm}=-\frac{1}{2}(2 \nu+\widetilde{\nu})\left|\xi^{\prime}\right|^{2} \pm \frac{1}{2} \sqrt{(2 \nu+\widetilde{\nu})^{2}\left|\xi^{\prime}\right|^{4}-4 \gamma^{2}\left|\xi^{\prime}\right|^{2}}
$$

Therefore, as in Remark 3.2, we see that

$$
\omega_{ \pm}=-\frac{1}{2}(2 \nu+\widetilde{\nu})\left|\xi^{\prime}\right|^{2} \pm i \gamma\left|\xi^{\prime}\right|+O\left(\left|\xi^{\prime}\right|^{3}\right) \text { as }\left|\xi^{\prime}\right| \rightarrow 0
$$

and $\omega_{ \pm} \in \mathbf{R}$ for $\left|\xi^{\prime}\right| \geq \frac{2 \gamma}{2 \nu+\nu}$, and

$$
\omega_{+}=-\frac{\gamma^{2}}{2 \nu+\widetilde{\nu}}+O\left(\left|\xi^{\prime}\right|^{-2}\right), \quad \omega_{-}=-(2 \nu+\widetilde{\nu})\left|\xi^{\prime}\right|^{2}+O(1) \text { as }\left|\xi^{\prime}\right| \rightarrow \infty
$$

We thus deduce, by suitably changing $\theta_{3}, \theta_{4}, \widetilde{R}_{3}, \eta_{3}$ and $\widetilde{\eta}_{4}$ if necessary, that

$$
\begin{align*}
\left.\left|\lambda^{2}+(2 \nu+\widetilde{\nu})\right| \xi^{\prime}\right|^{2} \lambda+\gamma^{2}\left|\xi^{\prime}\right|^{2} \mid & =\left|\left(\lambda-\omega_{+}\right)\left(\lambda-\omega_{-}\right)\right|  \tag{4.20}\\
& \geq C\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)(|\lambda|+1)
\end{align*}
$$

for $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\xi^{\prime}$ with $\left|\xi^{\prime}\right| \geq \widetilde{R}_{3}$, and for $\lambda \in \Sigma\left(\widetilde{\eta}_{4}, \theta_{4}\right)$ and $\xi^{\prime} \in \mathbf{R}^{n-1}$.
We now estimate $D_{1}\left(\lambda, \xi^{\prime}\right)$. We write $D_{1}\left(\lambda, \xi^{\prime}\right)$ as

$$
\begin{equation*}
D_{1}\left(\lambda, \xi^{\prime}\right)=\frac{1}{4}\left(\frac{\left|\xi^{\prime}\right|^{4}}{\mu_{1} \mu_{2}}-2\left|\xi^{\prime}\right|^{2}+\mu_{1} \mu_{2}\right)+\widetilde{D}_{1}\left(\lambda, \xi^{\prime}\right) \tag{4.21}
\end{equation*}
$$

where

$$
\begin{align*}
\widetilde{D}_{1}\left(\lambda, \xi^{\prime}\right)= & \frac{1}{4}\left\{8\left|\xi^{\prime}\right|^{2} e^{-\left(\mu_{1}+\mu_{2}\right) a}+\left(\frac{\left|\xi^{\prime}\right|^{4}}{\mu_{1} \mu_{2}}-2\left|\xi^{\prime}\right|^{2}+\mu_{1} \mu_{2}\right) e^{-2\left(\mu_{1}+\mu_{2}\right) a}\right.  \tag{4.22}\\
& \left.-\left(\frac{\left|\xi^{\prime}\right|^{4}}{\mu_{1} \mu_{2}}+2\left|\xi^{\prime}\right|^{2}+\mu_{1} \mu_{2}\right)\left(e^{-2 \mu_{1} a}+e^{-2 \mu_{2} a}\right)\right\} .
\end{align*}
$$

Since

$$
\left.\left|\frac{\left|\xi^{\prime}\right|^{4}}{\mu_{1} \mu_{2}}-2\right| \xi^{\prime}\right|^{2}+\left.\mu_{1} \mu_{2}\left|=\left|\frac{1}{\mu_{1} \mu_{2}}\right|\right| \frac{\left(\mu_{1} \mu_{2}\right)^{2}-\left|\xi^{\prime}\right|^{4}}{\mu_{1} \mu_{2}+\left|\xi^{\prime}\right|^{2}}\right|^{2}
$$

and

$$
\left(\mu_{1} \mu_{2}\right)^{2}-\left|\xi^{\prime}\right|^{4}=\frac{\lambda\left\{\lambda^{2}+(2 \nu+\widetilde{\nu})\left|\xi^{\prime}\right|^{2} \lambda+\gamma^{2}\left|\xi^{\prime}\right|^{2}\right\}}{\nu_{1} \lambda+\gamma^{2}},
$$

we see from (4.19) and (4.20) that

$$
\begin{equation*}
\left.\left.\frac{1}{4}\left|\frac{\left|\xi^{\prime}\right|^{4}}{\mu_{1} \mu_{2}}-2\right| \xi^{\prime}\right|^{2}+\mu_{1} \mu_{2} \right\rvert\, \geq C_{0} \frac{|\lambda|^{2}}{|\lambda|+1+\left|\xi^{\prime}\right|^{2}} \tag{4.23}
\end{equation*}
$$

uniformly for $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\xi^{\prime}$ with $\left|\xi^{\prime}\right| \geq \widetilde{R}_{3}$, and for $\lambda \in \Sigma\left(\widetilde{\eta}_{4}, \theta_{4}\right)$ and $\xi^{\prime} \in \mathbf{R}^{n-1}$.

We also see from (4.18) and (4.19) that

$$
\left|e^{-\mu_{j} a}\right| \leq C_{k}\left(\operatorname{Re} \mu_{j} a\right)^{-k} \leq C_{k}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{k}{2}}
$$

for any $k$. This, together with (4.22), implies that for any $\delta>0$ there exists a positive number $R_{3}=R_{3}(\delta)$ with $R_{3} \geq \widetilde{R}_{3}$ such that if $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right) \cap$ $\{\lambda ;|\lambda| \geq \delta\}$ and $\left|\xi^{\prime}\right| \geq R_{3}$, then

$$
\left|\widetilde{D}_{1}\left(\lambda, \xi^{\prime}\right)\right| \leq \frac{C|\lambda|^{2}}{\delta^{2}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{2}} \leq \frac{C_{0}}{2} \frac{|\lambda|^{2}}{|\lambda|+1+\left|\xi^{\prime}\right|^{2}}
$$

Combining this with (4.23), we have

$$
\left|D_{1}\left(\lambda, \xi^{\prime}\right)\right| \geq \frac{C_{0}}{2} \frac{|\lambda|^{2}}{|\lambda|+1+\left|\xi^{\prime}\right|^{2}}
$$

for $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right) \cap\{\lambda ;|\lambda| \geq \delta\}$ and $\xi^{\prime}$ with $\left|\xi^{\prime}\right| \geq R_{3}$. This proves (i).
We can also find a positive number $\eta_{4}$ with $\eta_{4} \geq \widetilde{\eta}_{4}$ such that

$$
\left|\widetilde{D}_{1}\left(\lambda, \xi^{\prime}\right)\right| \leq \frac{C|\lambda|^{2}}{\eta_{4}^{2}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{2}} \leq \frac{C_{0}}{2} \frac{|\lambda|^{2}}{|\lambda|+1+\left|\xi^{\prime}\right|^{2}}
$$

for all $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$ and $\xi^{\prime} \in \mathbf{R}^{n-1}$, and hence,

$$
\left|D_{1}\left(\lambda, \xi^{\prime}\right)\right| \geq \frac{C_{0}}{2} \frac{|\lambda|^{2}}{|\lambda|+1+\left|\xi^{\prime}\right|^{2}}
$$

for all such $\lambda$ and $\xi^{\prime}$. This completes the proof.
To estimate each component of the integral kernel in Theorem 3.8, we will use the following lemma.

Lemma 4.5. Let $\alpha^{\prime} \in \mathbf{Z}^{n-1}$ be any multi-index with $\left|\alpha^{\prime}\right| \leq n$ and let $\theta_{4}$ be the number given in Lemma 4.4. Then there exists a positive number $\eta_{4}$ such that the following estimates hold uniformly in $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$ and $\xi^{\prime} \in \mathbf{R}^{n-1}$ :

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}} \mu_{j}\right| \leq C\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{\frac{1}{2}-\frac{\left|\alpha^{\prime}\right|}{2}} \quad(j=1,2) \tag{i}
\end{equation*}
$$

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\mu_{1}-\mu_{2}\right)\right| \leq C|\lambda|\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{1}{2}-\frac{\left|\alpha^{\prime}\right|}{2}} \tag{ii}
\end{equation*}
$$

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\mu_{1} \mu_{2}-\left|\xi^{\prime}\right|^{2}\right)\right| \leq C|\lambda|\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{\left|\alpha^{\prime}\right|}{2}} \tag{iii}
\end{equation*}
$$

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}} e^{-\mu_{j} x_{n}}\right| \leq C\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{\left|\alpha^{\prime}\right|}{2}} e^{-\frac{1}{2} \operatorname{Re} \mu_{j} x_{n}} \quad(j=1,2) \tag{iv}
\end{equation*}
$$

for all $0 \leq x_{n} \leq a$,

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\frac{1}{D_{1}\left(\lambda, \xi^{\prime}\right)}\right)\right| \leq C|\lambda|^{-2}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{1-\frac{\left|\alpha^{\prime}\right|}{2}} \tag{v}
\end{equation*}
$$

$$
\begin{align*}
& \left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(e^{-\mu_{1} x_{n}}-e^{-\mu_{2} x_{n}}\right)\right|  \tag{vi}\\
& \quad \leq C|\lambda|\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-1-\frac{\left|\alpha^{\prime}\right|}{2}}\left\{e^{-\frac{1}{8} \operatorname{Re} \mu_{1} x_{n}}+e^{-\frac{1}{8} \operatorname{Re} \mu_{2} x_{n}}\right\}
\end{align*}
$$

for all $0 \leq x_{n} \leq a$.
The inequalities (i)-(vi) also hold uniformly in $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\left|\xi^{\prime}\right| \geq$ $R_{3}$ for some $R_{3}>0$, where $\eta_{3}$ and $\theta_{3}$ are the numbers given in Lemma 4.4.

Proof. Let $\eta_{4}$ and $\theta_{4}$ are the numbers given in Lemma 4.4 (ii). Then a direct calculation gives the inequality in (i). The inequality in (ii) follows from (i), (4.17) and (4.18).

Let us prove (iii). Suppose first that $|\lambda| \leq C_{0}\left|\xi^{\prime}\right|^{2}$, where $C_{0}$ is a positive constant to be determined later. We write $\mu_{1} \mu_{2}-\left|\xi^{\prime}\right|^{2}$ as

$$
\begin{equation*}
\mu_{1} \mu_{2}-\left|\xi^{\prime}\right|^{2}=\frac{\left(\mu_{1} \mu_{2}\right)^{2}-\left|\xi^{\prime}\right|^{4}}{\mu_{1} \mu_{2}+\left|\xi^{\prime}\right|^{2}} \tag{4.24}
\end{equation*}
$$

Since

$$
\left(\mu_{1} \mu_{2}\right)^{2}-\left|\xi^{\prime}\right|^{4}=\frac{\lambda^{3}}{\nu\left(\nu_{1} \lambda+\gamma^{2}\right)}+\lambda\left(\frac{1}{\nu}+\frac{\lambda}{\nu_{1} \lambda+\gamma^{2}}\right)\left|\xi^{\prime}\right|^{2}
$$

we see that

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\left(\mu_{1} \mu_{2}\right)^{2}-\left|\xi^{\prime}\right|^{4}\right)\right| \leq C|\lambda|\left(|\lambda|+\left|\xi^{\prime}\right|^{2}\right)^{1-\frac{\left|\alpha^{\prime}\right|}{2}} \tag{4.25}
\end{equation*}
$$

On the other hand, since $\left|\frac{\lambda^{2}}{\nu_{1} \lambda+\gamma^{2}}\right| \leq C|\lambda|$ for $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$, taking $C_{0}$ sufficiently small, we find that

$$
\left|\frac{\lambda}{\nu}\right| \leq \frac{\left|\xi^{\prime}\right|^{2}}{2}, \quad\left|\frac{\lambda^{2}}{\nu_{1} \lambda+\gamma^{2}}\right| \leq \frac{\left|\xi^{\prime}\right|^{2}}{2}
$$

for $\lambda$ and $\xi^{\prime}$ with $|\lambda| \leq C_{0}\left|\xi^{\prime}\right|^{2}$. Applying the mean value theorem to $\mu_{1}=$ $\left|\xi^{\prime}\right| \sqrt{1+\frac{\lambda}{\nu\left|\xi^{\prime}\right|^{2}}}$ and $\mu_{2}=\left|\xi^{\prime}\right| \sqrt{1+\frac{\lambda^{2}}{\left(\nu_{1} \lambda+\gamma^{2}\right)\left|\xi^{\prime}\right|^{2}}}$, we obtain

$$
\mu_{1} \mu_{2}+\left|\xi^{\prime}\right|^{2}=2\left|\xi^{\prime}\right|^{2}+\frac{\lambda}{\nu} q_{1}\left(\lambda, \xi^{\prime}\right)+\frac{\lambda^{2}}{\nu_{1} \lambda+\gamma^{2}} q_{2}\left(\lambda, \xi^{\prime}\right)+\frac{\lambda^{3}}{\nu\left(\nu_{1} \lambda+\gamma^{2}\right)\left|\xi^{\prime}\right|^{2}} q_{3}\left(\lambda, \xi^{\prime}\right),
$$

where $q_{j}\left(\lambda, \xi^{\prime}\right), j=1,2,3$, are some functions satisfying

$$
\left|q_{j}\left(\lambda, \xi^{\prime}\right)\right| \leq C, \quad j=1,2,3,
$$

for $\lambda$ and $\xi^{\prime}$ with $|\lambda| \leq C_{0}\left|\xi^{\prime}\right|^{2}$. Therefore, taking $C_{0}$ smaller if necessary, we see that

$$
\begin{equation*}
\left|\mu_{1} \mu_{2}+\left|\xi^{\prime}\right|^{2}\right| \geq 2\left|\xi^{\prime}\right|^{2}-C|\lambda| \geq\left|\xi^{\prime}\right|^{2} \geq C\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right) \tag{4.26}
\end{equation*}
$$

uniformly for $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$ and $\xi^{\prime}$ with $|\lambda| \leq C_{0}\left|\xi^{\prime}\right|^{2}$. It then follows from (i) and (4.24)-(4.26) that

$$
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\mu_{1} \mu_{2}-\left|\xi^{\prime}\right|^{2}\right)\right| \leq C|\lambda|\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{\left|\alpha^{\prime}\right|}{2}}
$$

uniformly for $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$ and $\xi^{\prime}$ with $|\lambda| \leq C_{0}\left|\xi^{\prime}\right|^{2}$. As for the case $|\lambda| \geq$ $C_{0}\left|\xi^{\prime}\right|^{2}$, we easily see that

$$
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\mu_{1} \mu_{2}-\left|\xi^{\prime}\right|^{2}\right)\right| \leq C\left(|\lambda|+\left|\xi^{\prime}\right|^{2}\right)^{1-\frac{\left|\alpha^{\prime}\right|}{2}} \leq C|\lambda|\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{\left|\alpha^{\prime}\right|}{2}}
$$

This proves (iii).
We next consider (iv). The case $\alpha^{\prime}=0$ is trivial. Let $\left|\alpha^{\prime}\right| \geq 1$. Then, by (i) and (4.18), we have

$$
\begin{aligned}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}} e^{-\mu_{j} x_{n}}\right| & \leq C \sum_{\ell=1}^{\left|\alpha^{\prime}\right|} e^{-\operatorname{Re} \mu_{j} x_{n}} \prod_{\alpha_{1}+\cdots+\alpha_{\ell}=\alpha^{\prime}}\left|\partial_{\xi^{\prime}}^{\alpha_{1}} \mu_{j} x_{n}\right| \cdots\left|\partial_{\xi^{\prime}}^{\alpha_{\ell}} \mu_{j} x_{n}\right| \\
& \leq C \sum_{\ell=1}^{\left|\alpha^{\prime}\right|} x_{n}^{\ell} e^{-\operatorname{Re} \mu_{j} x_{n}}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{\frac{\ell}{2}-\frac{\left|\alpha^{\prime}\right|}{2}} \\
& \leq C \sum_{\ell=1}^{\left|\alpha^{\prime}\right|}\left(\operatorname{Re} \mu_{j}\right)^{-\ell} e^{-\frac{1}{2} \operatorname{Re} \mu_{j} x_{n}}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{\frac{\ell}{2}-\frac{\left|\alpha^{\prime}\right|}{2}} \\
& \leq C\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{\left|\alpha^{\prime}\right|}{2}} e^{-\frac{1}{2} \operatorname{Re} \mu_{j} x_{n}} .
\end{aligned}
$$

This shows (iv).
As for (v), since $\frac{\left|\xi^{\prime}\right|^{4}}{\mu_{1} \mu_{2}}-2\left|\xi^{\prime}\right|^{2}+\mu_{1} \mu_{2}=\frac{1}{\mu_{1} \mu_{2}}\left(\mu_{1} \mu_{2}-\left|\xi^{\prime}\right|^{2}\right)^{2}$, we see from (i) and (iii) that

$$
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\frac{\left|\xi^{\prime}\right|^{4}}{\mu_{1} \mu_{2}}-2\left|\xi^{\prime}\right|^{2}+\mu_{1} \mu_{2}\right)\right| \leq C|\lambda|^{2}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-1-\frac{\left|\alpha^{\prime}\right|}{2}}
$$

This, together with (4.21) and (4.22), implies that

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}} D_{1}\left(\lambda, \xi^{\prime}\right)\right| \leq C|\lambda|^{2}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-1-\frac{\left|\alpha^{\prime}\right|}{2}} \tag{4.27}
\end{equation*}
$$

Combining Lemma 4.4 (ii) and (4.27), we obtain the desired inequality in (v).

We finally prove (vi). Since

$$
e^{-\mu_{1} x_{n}}-e^{-\mu_{2} x_{n}}=\left(\mu_{2}-\mu_{1}\right) x_{n} \int_{0}^{1} e^{-\left\{\theta \mu_{1}+(1-\theta) \mu_{2}\right\} x_{n}} d \theta
$$

we see from (ii) and (iv) that

$$
\begin{aligned}
& \left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(e^{-\mu_{1} x_{n}}-e^{-\mu_{2} x_{n}}\right)\right| \\
& \quad \leq C x_{n}|\lambda|\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{1}{2}-\frac{\left|\alpha^{\prime}\right|}{2}} \int_{0}^{1} e^{-\frac{1}{2} \operatorname{Re}\left\{\theta \mu_{1}+(1-\theta) \mu_{2}\right\} x_{n}} d \theta \\
& \quad \leq C x_{n}|\lambda|\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{1}{2}-\frac{\left|\alpha^{\prime}\right|}{2}}\left(e^{-\frac{1}{4} \operatorname{Re} \mu_{1} x_{n}}+e^{-\frac{1}{4} \operatorname{Re} \mu_{2} x_{n}}\right) \\
& \quad \leq C|\lambda|\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-1-\frac{\left|\alpha^{\prime}\right|}{2}}\left(e^{-\frac{1}{8} \operatorname{Re} \mu_{1} x_{n}}+e^{-\frac{1}{8} \operatorname{Re} \mu_{2} x_{n}}\right) .
\end{aligned}
$$

The desired inequalities are thus proved for $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$.
Let us consider the case $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$. In the same way as above, one can show the inequalities (i)-(iv) and (vi) for $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\left|\xi^{\prime}\right| \geq R_{3}$, where $R_{3}$ is the positive number given in Lemma 4.4 (i). As for (v), we take $\delta=\Lambda_{2}$ in Lemma 4.4 (i) with $\Lambda_{2}$ being the number given in Proposition 4.2 (ii). Similarly to the case $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$, by using Lemma 4.4 (i), we can also prove the inequality (v) for $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\left|\xi^{\prime}\right| \geq R_{3}$, provided that $|\lambda| \geq \delta=\Lambda_{2}$. In case $|\lambda| \leq \Lambda_{2}$, since $e^{-\left(\mu_{1}+\mu_{2}\right) a} e^{2\left|\xi^{\prime}\right| a}=e^{\bar{O}\left(\left|\xi^{\prime}\right|^{-1}\right)}$, we see from Proposition 4.2 (ii) that

$$
\left|D_{1}\left(\lambda, \xi^{\prime}\right)\right| \geq C \frac{|\lambda|^{2}}{\left|\xi^{\prime}\right|^{2}}
$$

for large $\left|\xi^{\prime}\right|$. Furthermore, similarly to the case $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$, one can obtain (4.27) with a more detailed computation for $\widetilde{D}_{1}\left(\lambda, \xi^{\prime}\right)$ given in (4.22)
for $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and large $\left|\xi^{\prime}\right|$. The desired inequality (v) then follows by changing $R_{3}$ suitably large if necessary. This completes the proof.

We next derive the estimates for the integral kernel of $\widehat{G}\left(\lambda, \xi^{\prime}\right)$. We set

$$
\left\{\begin{array}{l}
g_{\mu_{j}}^{(1)}\left(x_{n}, y_{n}\right)=\frac{1}{2 \mu_{j}} e^{-\mu_{j}\left|x_{n}-y_{n}\right|}  \tag{4.28}\\
g_{\mu_{j}}^{(2)}\left(x_{n}, y_{n}\right)=\frac{1}{2 \mu_{j}}\left(e^{-\mu_{j}\left(x_{n}+y_{n}\right)}+e^{-\mu_{j}\left\{\left(a-x_{n}\right)+\left(a-y_{n}\right)\right\}}\right) \\
g_{\mu_{j}}^{(3)}\left(x_{n}, y_{n}\right)=\frac{1}{2 \mu_{j}} \frac{e^{-2 \mu_{j} a}}{1-e^{-2 \mu_{j}}}\left(e^{-\mu_{j}\left|x_{n}-y_{n}\right|}+e^{\mu_{j}\left|x_{n}-y_{n}\right|}\right) \\
g_{\mu_{j}}^{(4)}\left(x_{n}, y_{n}\right)=\frac{1}{2 \mu_{j}} \frac{e^{-2 \mu_{j} a}}{1-e^{-2 \mu_{j}}}\left(e^{-\mu_{j}\left(x_{n}+y_{n}\right)}+e^{-\mu_{j}\left\{\left(a-x_{n}\right)+\left(a-y_{n}\right)\right\}}\right),
\end{array}\right.
$$

where $\mu_{j}=\mu_{j}\left(\lambda, \xi^{\prime}\right), j=1,2$. In what follows we will denote $|\lambda|+1+\left|\xi^{\prime}\right|^{2}$ by $\sigma\left(\lambda, \xi^{\prime}\right)$ :

$$
\sigma\left(\lambda, \xi^{\prime}\right)=|\lambda|+1+\left|\xi^{\prime}\right|^{2}
$$

Lemma 4.6. Let $g_{\mu_{j}}^{(k)}\left(x_{n}, y_{n}\right), j=1,2, k=1, \cdots, 4$, be defined in (4.28). Then

$$
g_{\mu_{j}}^{D}\left(x_{n}, y_{n}\right)=\sum_{k=1}^{4}(-1)^{k+1} g_{\mu_{j}}^{(k)}\left(x_{n}, y_{n}\right), \quad g_{\mu_{j}}^{N}\left(x_{n}, y_{n}\right)=\sum_{k=1}^{4} g_{\mu_{j}}^{(k)}\left(x_{n}, y_{n}\right)
$$

and the following estimates hold for any multi-index $\alpha^{\prime} \in \mathbf{Z}^{n-1}$ with $\left|\alpha^{\prime}\right| \leq$ $\left[\frac{n-1}{2}\right]+1$ and any nonnegative integer $\ell$ uniformly in $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right), \xi^{\prime} \in \mathbf{R}^{n-1}$ and $x_{n}, y_{n} \in[0, a]$ :

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left[\left(\partial_{x_{n}}^{\ell} g_{\mu_{j}}^{(k)}\right)\left(x_{n}, y_{n}\right)\right]\right| \leq C_{\ell} \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{\ell}{2}-\frac{1}{2}-\frac{\left|\alpha^{\prime}\right|}{2}} E^{(k)}\left(x_{n}, y_{n}\right) \tag{i}
\end{equation*}
$$

for $k=1,2$ and $j=1,2$, and

$$
\begin{align*}
& \left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left[\left(\partial_{x_{n}}^{\ell} g_{\mu_{1}}^{(k)}\right)\left(x_{n}, y_{n}\right)-\left(\partial_{x_{n}}^{\ell} g_{\mu_{2}}^{(k)}\right)\left(x_{n}, y_{n}\right)\right]\right| \\
& \quad \leq C|\lambda| \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{\ell}{2}-\frac{3}{2}-\frac{\left|\alpha^{\prime}\right|}{2}} E^{(k)}\left(x_{n}, y_{n}\right) \tag{ii}
\end{align*}
$$

for $k=1,2$. Here $E^{(k)}\left(x_{n}, y_{n}\right), k=1,2$, are the functions defined by

$$
\begin{aligned}
& E^{(1)}\left(x_{n}, y_{n}\right)=e^{-c \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{1}{2}}\left|x_{n}-y_{n}\right|}, \\
& E^{(2)}\left(x_{n}, y_{n}\right)=e^{-c \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{1}{2}}\left(x_{n}+y_{n}\right)}+e^{-c \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{1}{2}}\left\{\left(a-x_{n}\right)+\left(a-y_{n}\right)\right\}}
\end{aligned}
$$

with some constant $c>0$ independent of $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right), \xi^{\prime} \in \mathbf{R}^{n-1}$ and $x_{n}, y_{n} \in[0, a]$. Furthermore, for any nonnegative integer $q$, there hold the estimates

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left[\left(\partial_{x_{n}}^{\ell} g_{\mu_{j}}^{(k)}\right)\left(x_{n}, y_{n}\right)\right]\right| \leq C_{\ell, q} \sigma\left(\lambda, \xi^{\prime}\right)^{-\frac{q+\left|\alpha^{\prime}\right|}{2}}, \quad k=3,4, j=1,2 \tag{iii}
\end{equation*}
$$

The inequalities (i)-(iii) also hold uniformly in $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\left|\xi^{\prime}\right| \geq$ $R_{3}$ for some $R_{3}>0$, where $\eta_{3}$ and $\theta_{3}$ are the numbers given in Lemma 4.4.

Proof. A direct computation shows that $g_{\mu_{j}}^{M}\left(x_{n}, y_{n}\right), M=D, N$, are written as above. One can prove inequalities (i) and (iii) by a direct application of Lemma 4.5.

As for (ii), we write $g_{\mu_{1}}^{(1)}\left(x_{n}, y_{n}\right)-g_{\mu_{2}}^{(1)}\left(x_{n}, y_{n}\right)$ as

$$
\begin{aligned}
g_{\mu_{1}}^{(1)}\left(x_{n}, y_{n}\right)-g_{\mu_{2}}^{(1)}\left(x_{n}, y_{n}\right)= & \frac{1}{2}\left(\frac{1}{\mu_{1}}-\frac{1}{\mu_{2}}\right) e^{-\mu_{1}\left|x_{n}-y_{n}\right|} \\
& +\frac{1}{2 \mu_{2}}\left(e^{-\mu_{1}\left|x_{n}-y_{n}\right|}-e^{-\mu_{2}\left|x_{n}-y_{n}\right|}\right) .
\end{aligned}
$$

Since $\partial_{x_{n}}^{\ell} e^{-\mu_{j}\left|x_{n}-y_{n}\right|}=\left(-\mu_{j} \operatorname{sgn}\left(x_{n}-y_{n}\right)\right)^{\ell} e^{-\mu_{j}\left|x_{n}-y_{n}\right|}$, we can obtain inequality (ii) for $k=1$. The case $k=2$ can be proved similarly. This completes the proof.

In the same manner we can estimate $h_{\mu_{j}}\left(x_{n}\right)$.
Lemma 4.7. Let $\alpha^{\prime} \in \mathbf{Z}^{n-1}$ be a multi-index with $\left|\alpha^{\prime}\right| \leq\left[\frac{n-1}{2}\right]+1$ and let $\ell$ be a nonnegative integer. Then the following estimates hold uniformly in $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right), \xi^{\prime}$ and $x_{n} \in[0, a]:$

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}} \partial_{x_{n}}^{\ell} h_{\mu_{j}}\left(x_{n}\right)\right| \leq C_{\ell} \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{\ell}{2}-\frac{1}{2}-\frac{\left|\alpha^{\prime}\right|}{2}} e^{-c \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{1}{2}}\left(a-x_{n}\right)} \tag{i}
\end{equation*}
$$

for $j=1,2$, and

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}} \partial_{x_{n}}^{\ell} h_{\mu_{1}, \mu_{2}}\left(x_{n}\right)\right| \leq C_{\ell} \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{\ell}{2}-\frac{3}{2}-\frac{\left|\alpha^{\prime}\right|}{2}} e^{-c \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{1}{2}}\left(a-x_{n}\right)} \tag{ii}
\end{equation*}
$$

The inequalities (i) and (ii) also hold uniformly in $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\left|\xi^{\prime}\right| \geq R_{3}$ for some $R_{3}>0$, where $\eta_{3}$ and $\theta_{3}$ are the numbers given in Lemma 4.4.

Proof. The proof is similar to that of Lemma 4.6. We omit it.
We next estimate $\beta_{j}\left(\lambda, \xi^{\prime}, y_{n}\right), B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)$ and $\boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)$.

Lemma 4.8. Let $\alpha^{\prime} \in \mathbf{Z}^{n-1}$ be a multi-index with $\left|\alpha^{\prime}\right| \leq\left[\frac{n-1}{2}\right]+1$. Then the following estimates hold uniformly in $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right), \xi^{\prime} \in \mathbf{R}^{n-1}$ and $y_{n} \in[0, a]$ :

$$
\begin{equation*}
\left.\mid \partial_{\xi^{\prime}}^{\alpha^{\prime}} \beta_{0}\left(\lambda, \xi^{\prime}, y_{n}\right)\right) \left\lvert\, \leq C_{\ell}(|\lambda|+1)^{-1} \sigma\left(\lambda, \xi^{\prime}\right)^{-\frac{\left|\alpha^{\prime}\right|}{2}} e^{-c \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{1}{2}}\left(a-y_{n}\right)}\right. \tag{i}
\end{equation*}
$$

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}} \beta_{1}\left(\lambda, \xi^{\prime}, y_{n}\right)\right| \leq C \sigma\left(\lambda, \xi^{\prime}\right)^{-\frac{\left|\alpha^{\prime}\right|}{2}} e^{-c \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{1}{2}}\left(a-y_{n}\right)} \tag{ii}
\end{equation*}
$$

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}} B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)\right| \leq C \sigma\left(\lambda, \xi^{\prime}\right)^{-\frac{\left|\alpha^{\prime}\right|}{2}} e^{-c \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{1}{2}}\left(a-y_{n}\right)} \tag{iii}
\end{equation*}
$$

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}} \boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)\right| \leq C \sigma\left(\lambda, \xi^{\prime}\right)^{-\frac{\left|\alpha^{\prime}\right|}{2}} e^{-c \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{1}{2}}\left(a-y_{n}\right)} \tag{iv}
\end{equation*}
$$

The inequalities (i)-(iv) also hold uniformly in $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\left|\xi^{\prime}\right| \geq$ $R_{3}$ for some $R_{3}>0$, where $\eta_{3}$ and $\theta_{3}$ are the numbers given in Lemma 4.4.

Proof. A direct calculation shows that

$$
\begin{align*}
\beta_{0}\left(\lambda, \xi^{\prime}, y_{n}\right)= & \frac{\gamma \lambda}{\nu_{1} \lambda+\gamma^{2}} \frac{1}{4 D_{1}\left(\lambda, \xi^{\prime}\right)} \frac{\mu_{1} \mu_{2}-\left|\xi^{\prime}\right|^{2}}{\mu_{1} \mu_{2}} e^{-\mu_{2}\left(a-y_{n}\right)} \\
& +\frac{1}{4 D_{1}\left(\lambda, \xi^{\prime}\right)} \widetilde{\beta}_{0}\left(\lambda, \xi^{\prime}, y_{n}\right) \tag{4.29}
\end{align*}
$$

where $\widetilde{\beta}_{0}\left(\lambda, \xi^{\prime}, y_{n}\right)$ is written as the sum of terms of the form

$$
e^{-\mu_{k} a-\mu_{1} z_{n}-\mu_{2} w_{n}} \times\left\{\text { polynomial in } \mu_{1}, \mu_{2}, \frac{1}{\mu_{1}}, \frac{1}{\mu_{2}} \text { and } \xi^{\prime}\right\}, k=1,2,
$$

with $z_{n}$ and $w_{n}$ being linear functions of $y_{n}$ that satisfy $z_{n} \geq 0, w_{n} \geq 0$ and $\operatorname{Re}\left(\mu_{k} a+\mu_{1} z_{n}+\mu_{2} w_{n}\right) \geq \operatorname{Re} \mu_{2}\left(a-y_{n}\right)$ for $y_{n} \in[0, a]$. The desired inequality in (i) is obtained by applying Lemma 4.5 to (4.29). Similarly, by a direct computation, we see that $\beta_{1}\left(\lambda, \xi^{\prime}, y_{n}\right)$ and $\boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)$ can be written as

$$
\begin{aligned}
\beta_{1}\left(\lambda, \xi^{\prime}, y_{n}\right)= & \frac{\mu_{1} \mu_{2}-\left|\xi^{\prime}\right|^{2}}{4 D_{1}\left(\lambda, \xi^{\prime}\right)}\left\{\frac{\mu_{1} \mu_{2}-\left|\xi^{\prime}\right|^{2}}{\mu_{1} \mu_{2}} e^{-\mu_{2}\left(a-y_{n}\right)}-\left(e^{-\mu_{1}\left(a-y_{n}\right)}-e^{-\mu_{2}\left(a-y_{n}\right)}\right)\right\} \\
& +\widetilde{\beta}_{1}\left(\lambda, \xi^{\prime}, y_{n}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)= & -\frac{i \xi^{\prime}}{4 D_{1}\left(\lambda, \xi^{\prime}\right)} \frac{\mu_{1} \mu_{2}-\left|\xi^{\prime}\right|^{2}}{\mu_{1}}\left(e^{-\mu_{1}\left(a-y_{n}\right)}-e^{-\mu_{2}\left(a-y_{n}\right)}\right) \\
& +i \xi^{\prime} \widetilde{\beta}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)
\end{aligned}
$$

where $\widetilde{\beta}_{j}\left(\lambda, \xi^{\prime}, y_{n}\right), j=1, n$ ，are written as the sum of terms of the same form as that for $\widetilde{\beta}_{0}\left(\lambda, \xi^{\prime}, y_{n}\right)$ with $z_{n}$ and $w_{n}$ satisfying $z_{n} \geq 0, w_{n} \geq 0$ and $\operatorname{Re}\left(\mu_{k} a+\mu_{1} z_{n}+\mu_{2} w_{n}\right) \geq \operatorname{Re} \mu_{\ell}\left(a-y_{n}\right)$ for $y_{n} \in[0, a], k=1,2$ and $\ell=1$ or 2．The desired inequalities in（ii）and（iv）now follow from Lemma 4．5．

We next consider inequality（iii）．We write $B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)$ as

$$
B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)=-\frac{\sinh \mu_{1} y_{n}}{\sinh \mu_{1} a} I_{n-1}-\beta\left(\lambda, \xi^{\prime}, y_{n}\right) P_{1,0}^{\prime}
$$

where

$$
\beta\left(\lambda, \xi^{\prime}, y_{n}\right)=\frac{\sinh \mu_{1} y_{n}}{\sinh \mu_{1} a}-\beta_{1}\left(\lambda, \xi^{\prime}, y_{n}\right) .
$$

A direct computation shows that $\beta\left(\lambda, \xi^{\prime}, y_{n}\right)$ is written as
where $\widetilde{\beta}\left(\lambda, \xi^{\prime}, y_{n}\right)$ is written as the sum of the same form as that for $\widetilde{\beta}_{1}\left(\lambda, \xi^{\prime}, y_{n}\right)$ ． Consequently we have

$$
\begin{aligned}
B^{\prime}(\lambda, & \left.\xi^{\prime}, y_{n}\right) \\
= & \frac{1}{\mu_{1}} e^{-\mu_{1}\left(a-y_{n}\right)} I_{n-1}+\frac{\xi^{\prime} \xi^{\prime}}{4 D_{1}\left(\lambda, \xi^{\prime}\right)} \frac{\mu_{1} \mu_{2}-\left|\xi^{\prime}\right|^{2}}{\mu_{1} \mu_{2}}\left(e^{-\mu_{1}\left(a-y_{n}\right)}-e^{-\mu_{2}\left(a-y_{n}\right)}\right) \\
& \quad+\frac{1}{\mu_{1}} \frac{e^{-\mu_{1} a}}{1-e^{-2 \mu_{1} a}}\left(e^{-\mu_{1}\left(2 a-y_{n}\right)}-e^{-\mu_{1} y_{n}}\right) I_{n-1}+\frac{\xi^{\prime} T \xi^{\prime}}{4 D_{1}\left(\lambda, \xi^{\prime}\right)\left(1-e^{-2 \mu_{1} a} a\right.} \widetilde{\beta}\left(\lambda, \xi^{\prime}, y_{n}\right) .
\end{aligned}
$$

The desired inequality in（iii）now follows from Lemma 4．5．
Similarly to above，one can prove the desired inequalities for $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\left|\xi^{\prime}\right| \geq R_{3}$ with $|\lambda| \geq \delta$ ，where $\delta$ is any positive number．

Let us consider the case $|\lambda| \leq \delta$ ．In this case a direct computation shows that

$$
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}} \tilde{\beta}_{j}\left(\lambda, \xi^{\prime}, y_{n}\right)\right| \leq C|\lambda|^{2} e^{-c\left|\xi^{\prime}\right| a}, \quad j=0,1, n
$$

for small $|\lambda|$ and large $\left|\xi^{\prime}\right|$ ．Combining this with Lemma 4．5，we can obtain the desired inequalities for $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\left|\xi^{\prime}\right| \geq R_{3}$ with some large $R_{3}>0$ ．This completes the proof．

## 5．Proof of the main results

In this section we prove the main results stated in section 2.
To prove Theorem 2.1 we will apply the following Fourier multiplier the－ orem．

Lemma 5.1. (Fourier multiplier theorem) Let $1<p<\infty$ and let $s$ be an integer satisfying $s \geq\left[\frac{k}{2}\right]+1$. Suppose that $\Psi(\zeta) \in C^{s}\left(\mathbf{R}^{k}-\{0\}\right) \cap L^{\infty}\left(\mathbf{R}^{k}\right)$ and that there exists a positive constant $C_{0}$ such that

$$
|\zeta|^{|\alpha|}\left|\partial_{\zeta}^{\alpha} \Psi(\zeta)\right| \leq C_{0}
$$

for all $\zeta \in \mathbf{R}^{k}-\{0\}$ and $|\alpha| \leq s$. Then the operator $\mathscr{F}_{\zeta}^{-1}[\Psi(\zeta)(\mathscr{F} f)(\zeta)]$ is extended to a bounded linear operator on $L^{p}\left(\mathbf{R}^{k}\right)$ and there holds the estimate

$$
\left\|\mathscr{F} \zeta_{\zeta}^{-1}[\Psi(\zeta)(\mathscr{F} f)(\zeta)]\right\|_{L^{p}\left(\mathbf{R}^{k}\right)} \leq C C_{0}\|f\|_{L^{p}\left(\mathbf{R}^{k}\right)}
$$

See, e.g., [4] for the proof.
We will also use the following lemma concerning integral operator.
Lemma 5.2. Let $\Phi\left(x_{n}, y_{n}\right)$ be a measurable function on $(0, a) \times(0, a)$.
(i) Suppose that there exists a positive constant $M$ such that $\int_{0}^{a}\left|\Phi\left(x_{n}, y_{n}\right)\right| d y_{n} \leq$ $M$ for a.e. $x_{n}$ and $\int_{0}^{a}\left|\Phi\left(x_{n}, y_{n}\right)\right| d x_{n} \leq M$ for a.e. $y_{n}$. Let $1 \leq p \leq \infty$. Then it holds that

$$
\left|\int_{0}^{a} \Phi\left(\cdot, y_{n}\right) f\left(y_{n}\right) d y_{n}\right|_{p} \leq M|f|_{p}
$$

(ii) Suppose that there exists a positive constant $M$ such that $\left(\int_{0}^{a}\left|\Phi\left(x_{n}, y_{n}\right)\right|^{2} d y_{n}\right)^{\frac{1}{2}} \leq$ $M$ for a.e. $x_{n}$. Then it holds that

$$
\left|\int_{0}^{a} \Phi\left(\cdot, y_{n}\right) f\left(y_{n}\right) d y_{n}\right|_{\infty} \leq M|f|_{2} .
$$

The proof of the above lemma is well known. We omit it.
To obtain the estimates for derivatives of the resolvent we still need some consideration. We proceed as in [3].

Lemma 5.3. There holds
$\mathscr{F}_{\xi^{\prime}}^{-1}\left[\int_{0}^{a} \frac{1}{2 \mu_{j}} e^{-\mu_{j}\left|x_{n}-y_{n}\right|} \widehat{f}\left(\xi^{\prime}, y_{n}\right) d y_{n}\right]\left(x^{\prime}, x_{n}\right)=\mathscr{F}_{\xi}^{-1}\left[\frac{1}{\mu_{j}^{2}+\xi_{n}^{2}} \mathscr{F} E f(\xi)\right]\left(x^{\prime}, x_{n}\right)$
for $j=1,2$, where $\xi=\left(\xi^{\prime}, \xi_{n}\right)$ and $E f\left(x^{\prime}, x_{n}\right)$ is the zero-extension of $f\left(x^{\prime}, x_{n}\right)$ to $\mathbf{R}^{n}$.

Proof. Since $\mathscr{F}_{x_{n}}\left[\frac{1}{2 \mu_{j}} e^{-\mu_{j}\left|x_{n}\right|}\right]=\frac{1}{\mu_{j}^{2}+\xi_{n}^{2}}$, we obtain the desired relation. This completes the proof.

We will also use the following lemma ([3, Lemma 2.6]).
Lemma 5.4. ([3, Lemma 2.6]) Let $1<p<\infty$ and define the operator $T$ by

$$
T f\left(x_{n}\right)=\int_{0}^{a} \frac{f\left(y_{n}\right)}{x_{n}+y_{n}} d y_{n}
$$

for $x_{n} \in(0, a)$ and $f \in L^{p}(0, a)$. Then there exists a positive constant $C=$ $C(p)$ such that

$$
|T f|_{p} \leq C|f|_{p}
$$

This lemma follows from the fact that $T f$ can be written in a form of the Hilbert transformation on $\mathbf{R}$. See [3, Lemma 2.6].

We are now in a position to Prove Theorem 2.1.
Proof of Theorem 2.1. Let $\eta$ be any positive number. We will prove Theorem 2.1 with $\theta$ satisfying $\frac{\pi}{2}<\theta<\min \left\{\theta_{1}, \theta_{3}, \theta_{4}\right\}$, where $\theta_{j}(j=1,3,4)$ are the numbers given in Lemmas 4.3 and 4.4. We note that $\Sigma(\eta, \theta) \subset$ $\Sigma\left(\eta, \theta_{1}\right) \cap \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and that $\overline{\Sigma(\eta, \theta)-\Sigma\left(\eta_{4}, \theta_{4}\right)}$ is a compact set. Here $\eta_{3}$ and $\eta_{4}$ are the numbers given in Lemma 4.4.

We see from Theorem 3.8 that there exists a solution $u$ of problem (1.1)(1.2) which takes the form $u=R(\lambda) f$, where $\widehat{R}\left(\lambda, \xi^{\prime}\right)=\left(\lambda+\widehat{L_{\xi^{\prime}}}\right)^{-1}=$ $\widehat{G}\left(\lambda, \xi^{\prime}\right)+\widehat{K}\left(\lambda, \xi^{\prime}\right)$ with $\widehat{G}\left(\lambda, \xi^{\prime}\right)$ and $\widehat{K}\left(\lambda, \xi^{\prime}\right)$ being the integral operators given in Theorem 3.8. We first prove that the estimates in Theorem 2.1 hold for $u=R(\lambda) f$. We then prove the uniqueness of solutions of problem (1.1)-(1.2).

Let us estimate $u=R(\lambda) f$. We first consider the case $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$. We note that one can see from the form of $\widehat{R}\left(\lambda, \xi^{\prime}\right)$ that $k$-th order derivatives of $Q_{0} R(\lambda) f$ are estimated as $(k+1)$-th order ones of $\widetilde{Q} R(\lambda) f$. So we here give the proof of the estimates for $\widetilde{Q} R(\lambda) f$ only.

Hereafter we set $G(\lambda)=\mathscr{F}_{\xi^{\prime}}^{-1}\left[\widehat{G}\left(\lambda, \xi^{\prime}\right)\right]$ and $K(\lambda)=\mathscr{F}_{\xi^{\prime}}^{-1}\left[\widehat{K}\left(\lambda, \xi^{\prime}\right)\right]$. We also write $G(\lambda)=G^{(1)}(\lambda)+\cdots+G^{(4)}(\lambda)$, where $G^{(\ell)}(\lambda)$ is the matrix with $g_{\mu_{j}}^{D}$ and $g_{\mu_{j}}^{N}$ in $G(\lambda)$ replaced by $(-1)^{\ell+1} g_{\mu_{j}}^{(\ell)}$ and $g_{\mu_{j}}^{(\ell)}$, respectively. Here $g_{\mu_{j}}^{(\ell)}$ are the functions defined in (4.28).

Let us consider $\widetilde{Q} R(\lambda) \widetilde{Q} f$. We first estimate $\partial_{x}^{k} \widetilde{Q} G(\lambda) \widetilde{Q} f$ for $k \leq 1$. Since $\partial_{x_{n}}^{2}\left(g_{\mu_{1}, \mu_{2}}^{M} \widetilde{Q} \hat{f}\right)=\left(\partial_{x_{n}}^{2} g_{\mu_{1}}^{M}-\partial_{x_{n}}^{2} g_{\mu_{2}}^{M}\right)[\widetilde{Q} \hat{f}]$, we see from Lemma 4.6 that, for

$$
\left|\beta^{\prime}\right|+\ell \leq 1 \text { and } j=1,2,
$$

$$
\begin{aligned}
& \left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left[\left(i \xi^{\prime}\right)^{\beta^{\prime}} \partial_{x_{n}}^{\ell} \widetilde{Q} \widehat{G}^{(j)}\left(\lambda, \xi^{\prime} x_{n}, y_{n}\right) \widetilde{Q}\right]\right| \\
& \quad \leq C \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{\left|\beta^{\prime}\right|+\ell}{2}-\frac{1}{2}-\frac{\left|\alpha^{\prime}\right|}{2}}\left\{E^{(1)}\left(x_{n}, y_{n}\right)+E^{(2)}\left(x_{n}, y_{n}\right)\right\} \\
& \quad \leq C(|\lambda|+1)^{\frac{\left|\beta^{\prime}\right|+\ell}{2}-\frac{1}{2}} \mathcal{E}\left(\lambda, x_{n}, y_{n}\right)\left|\xi^{\prime}\right|^{-\left|\alpha^{\prime}\right|},
\end{aligned}
$$

where
$\mathcal{E}\left(\lambda, x_{n}, y_{n}\right)=e^{-c(|\lambda|+1)^{\frac{1}{2}}\left|x_{n}-y_{n}\right|}+e^{-c(|\lambda|+1)^{\frac{1}{2}}\left(x_{n}+y_{n}\right)}+e^{-c(|\lambda|+1)^{\frac{1}{2}}\left\{\left(a-x_{n}\right)+\left(a-y_{n}\right)\right\}}$.
Since

$$
\sup _{0 \leq x_{n} \leq a} \int_{0}^{a} \mathcal{E}\left(\lambda, x_{n}, y_{n}\right) d y_{n}+\sup _{0 \leq y_{n} \leq a} \int_{0}^{a} \mathcal{E}\left(\lambda, x_{n}, y_{n}\right) d x_{n} \leq C(|\lambda|+1)^{-\frac{1}{2}},
$$

we see from Lemmas 5.1 and 5.2 that

$$
\left\|\partial_{x}^{k} \widetilde{Q} G^{(j)}(\lambda) \widetilde{Q} f\right\|_{p} \leq \frac{C}{(|\lambda|+1)^{1-\frac{k}{2}}}\|\widetilde{Q} f\|_{p}, \quad j=1,2
$$

for $k \leq 1$. Similarly, one can estimate $\partial_{x}^{k} \widetilde{Q} G^{(j)}(\lambda) \widetilde{Q} f(j=3,4)$ and $\partial_{x}^{k} \widetilde{Q} K(\lambda) \widetilde{Q} f$ for $k \leq 1$.

We next consider $\partial_{x}^{2} \widetilde{Q} R(\lambda) \widetilde{Q} f$. We first estimate $\partial_{x}^{2} \widetilde{Q} G^{(1)}(\lambda) \widetilde{Q} f$. By Lemma 5.3, we have

$$
\mathscr{F}_{\xi^{\prime}}^{-1}\left[g_{\mu_{j}}^{(1)} \widetilde{Q} \widehat{f}\right]\left(x^{\prime}, x_{n}\right)=\mathscr{F}_{\xi}^{-1}\left[\frac{1}{\mu_{j}^{2}+\xi_{n}^{2}} \mathscr{F}_{x}[\widetilde{Q} E f]\right]\left(x^{\prime}, x_{n}\right), \quad j=1,2,
$$

and, in particular,

$$
\mathscr{F}_{\xi^{\prime}}^{-1}\left[g_{\mu_{1}, \mu_{2}}^{(1)} \widetilde{Q} \widehat{f}\right]\left(x^{\prime}, x_{n}\right)=\mathscr{F}_{\xi}^{-1}\left[M(\lambda, \xi) \mathscr{F}_{x}[\widetilde{Q} E f]\right]\left(x^{\prime}, x_{n}\right),
$$

where $E f$ denotes the zero extension of $f$ to $\mathbf{R}^{n}$ and

$$
M(\lambda, \xi)=\frac{1}{\mu_{1}^{2}+\xi_{n}^{2}}-\frac{1}{\mu_{2}^{2}+\xi_{n}^{2}}
$$

with $\mu_{j}=\mu_{j}\left(\lambda, \xi^{\prime}\right), j=1,2$. Since $\mu_{1}^{2}+\xi_{n}^{2}=\frac{\lambda+\nu|\xi|^{2}}{\nu}, \mu_{2}^{2}+\xi_{n}^{2}=\frac{\lambda^{2}+\left.\nu_{1}|\xi|\right|^{2} \lambda+\gamma^{2}|\xi|^{2}}{\nu_{1} \lambda+\gamma^{2}}$ with $|\xi|^{2}=\left|\xi^{\prime}\right|^{2}+\xi_{n}^{2}$, and $\nu_{1}=\nu+\widetilde{\nu}$, we have

$$
M(\lambda, \xi)=-\frac{\lambda\left(\widetilde{\nu} \lambda+\gamma^{2}\right)}{\left(\lambda+\nu|\xi|^{2}\right)\left(\lambda^{2}+\nu_{1}|\xi|^{2} \lambda+\gamma^{2}|\xi|^{2}\right)}
$$

Therefore, in view of the observation in Remarks 3.2 and 3.5, we obtain $\left|\partial_{\xi}^{\alpha}\left(\frac{\xi^{\beta}}{\mu_{j}^{2}+\xi_{n}^{2}}\right)\right| \leq C|\xi|^{-|\alpha|}$ for $|\beta|=2$ and $\left|\partial_{\xi}^{\alpha}\left(\frac{\xi^{\beta}}{\lambda} M(\lambda, \xi)\right)\right| \leq C|\xi|^{-|\alpha|}$ for $|\beta|=$ 4 uniformly in $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$. It then follows from Lemma 5.1 that

$$
\left\|\partial_{x}^{2} \widetilde{Q} G^{(1)}(\lambda) \widetilde{Q} f\right\|_{p} \leq C\|\widetilde{Q} f\|_{p}
$$

We next consider $\partial_{x}^{2} \widetilde{Q} G^{(2)}(\lambda) \widetilde{Q} f$. For $\left|\beta^{\prime}\right|+\ell=2$, we see from Lemma 4.6 that

$$
\begin{aligned}
& \left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left[\left(i \xi^{\prime}\right)^{\beta^{\prime}} \partial_{x_{n}}^{\ell} \widetilde{Q} \widehat{G}^{(2)}\left(\lambda, \xi^{\prime}, x_{n}, y_{n}\right) \widetilde{Q}\right]\right| \\
& \quad \leq C \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{1}{2}-\frac{\left|\alpha^{\prime}\right|}{2}} E^{(2)}\left(x_{n}, y_{n}\right) \\
& \quad \leq C\left\{\frac{1}{x_{n}+y_{n}}+\frac{1}{\left(a-x_{n}\right)+\left(a-y_{n}\right)}\right\}\left|\xi^{\prime}\right|^{-\left|\alpha^{\prime}\right|} .
\end{aligned}
$$

It then follows from Lemma 5.1 that

$$
\begin{aligned}
& \left\|\partial_{x}^{2} \widetilde{Q} G^{(2)}(\lambda) \widetilde{Q} f\left(\cdot, y_{n}\right)\right\|_{L^{p}\left(\mathbf{R}^{n-1}\right)} \\
& \quad \leq C \int_{0}^{a}\left\{\frac{1}{x_{n}+y_{n}}+\frac{1}{\left(a-x_{n}\right)+\left(a-y_{n}\right)}\right\}\left\|\widetilde{Q} f\left(\cdot, y_{n}\right)\right\|_{L^{p}\left(\mathbf{R}^{n-1}\right)} d y_{n}
\end{aligned}
$$

and hence, by Lemma 5.4, we obtain

$$
\left\|\partial_{x}^{2} \widetilde{Q} G^{(2)}(\lambda) \widetilde{Q} f\right\|_{p} \leq C\|\widetilde{Q} f\|_{p}
$$

In the same way one can also obtain the desired estimates for $\partial_{x}^{2} \widetilde{Q} G^{(j)}(\lambda) \widetilde{Q} f$ $(j=3,4)$ and $\partial_{x}^{2} \widetilde{Q} K(\lambda) \widetilde{Q} f$.

We next consider $\widetilde{Q} R(\lambda) Q_{0} f$. Similarly to above, one can prove $\left\|\partial_{x}^{k} \widetilde{Q} R(\lambda) Q_{0} f\right\|_{p} \leq$ $\frac{C}{|\lambda|+1}\left\|Q_{0} f\right\|_{p}$ for $k \leq 1$. As for $\partial_{x}^{2} \widetilde{Q} R(\lambda) Q_{0} f$, we first note that $\partial_{x_{n}} g_{\mu_{2}}^{N} Q_{0} \widehat{f}=$ $g_{\mu_{2}}^{D}\left[\partial_{x_{n}} Q_{0} \widehat{f}\right]$, which is obtained by integration by parts. It then follows that

$$
\widetilde{Q} \widehat{G}\left(\lambda, \xi^{\prime}\right) Q_{0} \widehat{f}=-\frac{\gamma}{\nu_{1} \lambda+\gamma^{2}}\left(\begin{array}{c}
0 \\
g_{\mu_{2}}^{N}\left[\widetilde{\nabla^{\prime}} \widehat{Q_{0}} f\right] \\
g_{\mu_{2}}^{D}\left[\partial_{x_{n}} \widehat{Q_{0}} f\right]
\end{array}\right)
$$

where $\nabla^{\prime}={ }^{T}\left(\partial_{x_{1}}, \cdots, \partial_{x_{n-1}}\right)$. Therefore, similarly to above, we obtain

$$
\left\|\partial_{x}^{2} \widetilde{Q} G(\lambda) Q_{0} f\right\|_{p} \leq \frac{C}{|\lambda|+1}\left\|Q_{0} f\right\|_{W^{1, p}}
$$

As for $\partial_{x}^{2} \widetilde{Q} K(\lambda) Q_{0} f$, we write

$$
\widetilde{Q} \widehat{H}\left(\lambda, \xi^{\prime}\right) Q_{0} \widehat{f}=\left(\begin{array}{c}
0 \\
\left(\frac{1}{\nu} h_{\mu_{1}} \beta_{0}+\frac{\left|\xi^{\prime}\right|^{2}}{\lambda} h_{\mu_{1}, \mu_{2}}\right)\left[\nabla^{\prime} \widehat{Q}_{0} f\right] \\
\frac{i \xi^{\prime}}{\lambda} \partial_{x_{n}} h_{\mu_{1}, \mu_{2}}\left[\nabla^{\prime} Q_{0} f\right]
\end{array}\right) .
$$

The desired estimate for $\partial_{x}^{2} \widetilde{Q} K(\lambda) Q_{0} f$ now follows similarly to above.
As mentioned before, one can prove the estimates for $\partial_{x}^{k} Q_{0} R(\lambda) f$ with $k \leq 1$ similarly to the estimates for $\partial_{x}^{k+1} \widetilde{Q} R(\lambda) f$. We thus omit the details.

Let us prove the last inequality in Theorem 2.1. We assume that $\left.\widetilde{Q} f\right|_{x_{n}=0, a}=$ 0 . It suffices to prove

$$
\left\|\partial_{x_{n}} \widetilde{Q} R(\lambda) \widetilde{Q} f\right\|_{p} \leq \frac{C}{|\lambda|+1}\|\widetilde{Q} f\|_{W^{1, p}}
$$

Since $\left.\widetilde{Q} f\right|_{x_{n}=0, a}=0$, by integration by parts, we also have

$$
\partial_{x_{n}} g_{\mu_{j}}^{D} \widetilde{Q} \widehat{f}=g_{\mu_{j}}^{N}\left[\partial_{x_{n}} \widetilde{Q} \widehat{f}\right], \quad j=1,2 .
$$

Therefore, we see from Lemmas 4.6, 5.1 and 5.2 that

$$
\left\|\partial_{x_{n}} \widetilde{Q} G(\lambda) \widetilde{Q} f\right\|_{p} \leq \frac{C}{|\lambda|+1}\left\|\partial_{x_{n}} \widetilde{Q} f\right\|_{p}
$$

As for $\widetilde{Q} K(\lambda) \widetilde{Q} f$, we note that an application $\partial_{x_{n}}$ to $h_{\mu_{j}}\left(x_{n}\right)$ yields one of the factors $\mu_{j}, j=1,2$. But, since $B^{\prime}\left(\lambda, \xi^{\prime}, y_{n}\right)$ and $\boldsymbol{b}_{n}\left(\lambda, \xi^{\prime}, y_{n}\right)$ are written as linear combinations of $e^{ \pm \mu_{j} y_{n}}(j=1,2)$, we see that if $\left.\widetilde{Q} f\right|_{x_{n}=0, a}=0$, then

$$
\int_{0}^{a} e^{ \pm \mu_{j} y_{n}} \widetilde{Q} \widehat{f}\left(y_{n}\right) d y_{n}=\mp \frac{1}{\mu_{j}} \int_{0}^{a} e^{ \pm \mu_{j} y_{n}} \partial_{y_{n}} \widetilde{Q} \widehat{f}\left(y_{n}\right) d y_{n}, \quad j=1,2 .
$$

Therefore, we can gain one of the factors $\mu_{j}^{-1}, j=1,2$. It then follows that

$$
\left\|\partial_{x_{n}} \widetilde{Q} K(\lambda) \widetilde{Q} f\right\|_{p} \leq \frac{C}{|\lambda|+1}\left\|\partial_{x_{n}} \widetilde{Q} f\right\|_{p}
$$

We have thus obtained the desired estimates when $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$.
We next consider the case $\lambda \in \Sigma(\eta, \theta)-\Sigma\left(\eta_{4}, \theta_{4}\right)$. We fix a positive number $R_{3}$ in such a way that Lemmas 4.5-4.8 hold for $\lambda \in \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and $\left|\xi^{\prime}\right| \geq R_{3}$.

We decompose $R(\lambda) f$ in the following way. Let $\widetilde{\chi}\left(\xi^{\prime}\right)$ be a $C^{\infty}$ function on $\mathbf{R}^{n-1}$ satisfying $0 \leq \widetilde{\chi} \leq 1$ on $\mathbf{R}^{n-1}, \tilde{\chi}\left(\xi^{\prime}\right)=0$ for $\left|\xi^{\prime}\right| \leq R_{3}$ and $\widetilde{\chi}\left(\xi^{\prime}\right)=1$ for $\left|\xi^{\prime}\right| \geq 2 R_{3}$. We write $R(\lambda) f$ as

$$
R(\lambda) f=\widetilde{R}^{(0)}(\lambda) f+\widetilde{R}^{(\infty)}(\lambda) f
$$

where

$$
\widetilde{R}^{(0)}(\lambda) f=\mathscr{F}_{\xi^{\prime}}^{-1}\left[\left(1-\widetilde{\chi}\left(\xi^{\prime}\right)\right) \widehat{R}\left(\lambda, \xi^{\prime}\right) \widehat{f}\right]
$$

and

$$
\widetilde{R}^{(\infty)}(\lambda) f=\mathscr{F}_{\xi^{\prime}}^{-1}\left[\widetilde{\chi}\left(\xi^{\prime}\right) \widehat{R}\left(\lambda, \xi^{\prime}\right) \hat{f}\right] .
$$

Similarly to the case of $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$, we can obtain the desired estimates for $\widetilde{R}^{(\infty)}(\lambda) f$ since $\Sigma(\eta, \theta) \subset \Sigma\left(-\eta_{3}, \theta_{3}\right)$. The desired estimates also hold for $\widetilde{R}^{(0)}(\lambda) f$. In fact, by Lemmas 3.4 and 4.3 , we see that $D\left(\lambda, \xi^{\prime}\right) \neq 0$ on the compact set $\overline{\Sigma(\eta, \theta)-\Sigma\left(\eta_{4}, \theta_{4}\right)} \times\left\{\xi^{\prime} ;\left|\xi^{\prime}\right| \leq 2 R_{3}\right\}$. Therefore, $\widehat{R}\left(\lambda, \xi^{\prime}, x_{n}, y_{n}\right)$ is analytic, and so we can obtain the desired estimates for $\widetilde{R}^{(0)}(\lambda) f$. We thus conclude that $u=R(\lambda) f$ satisfies the estimates in Theorem 2.1.

It remains to prove the uniqueness of solutions of problem (1.1)-(1.2). To prove the uniqueness, we consider the adjoint problem

$$
\begin{equation*}
\left(\bar{\lambda}+L^{*}\right) w=g,\left.\quad \widetilde{Q} w\right|_{x_{n}=0, a}=0, \tag{5.1}
\end{equation*}
$$

where

$$
L=\left(\begin{array}{cc}
0 & -\gamma \operatorname{div} \\
-\gamma \nabla & -\nu \Delta I_{n}-\widetilde{\nu} \nabla \operatorname{div}
\end{array}\right)
$$

Similarly to the case of problem (1.1)-(1.2), we consider the Fourier transform of (5.1) in $x^{\prime} \in \mathbf{R}^{n-1}$ :

$$
\left(\bar{\lambda}+\widehat{L}_{\xi^{\prime}}^{*}\right) \widehat{w}=\widehat{g},\left.\quad \widetilde{Q} \widehat{w}\right|_{x_{n}=0, a}=0
$$

It is easily verified that $\left(\bar{\lambda}+\widehat{L}_{\xi^{\prime}}^{*}\right)^{-1}$ has an integral representation $\widehat{R}^{*}\left(\bar{\lambda}, \xi^{\prime}\right)$ corresponding to Theorem 3.8, which is written in terms of the components of $\widehat{R}\left(\bar{\lambda}, \xi^{\prime}\right)$. Therefore, $R^{*}(\bar{\lambda}) g=\mathscr{F}_{\xi^{\prime}}^{-1}\left[\widehat{R}^{*}\left(\bar{\lambda}, \xi^{\prime}\right) \widehat{g}\right]$, which is a solution of (5.1), has the same estimates as those for $R(\lambda) f$. We thus conclude that for any $\lambda \in \Sigma(\eta, \theta)$ and $g \in W^{1, p} \times L^{p}$ problem (5.1) has a solution $w \in$ $W^{1, p} \times\left(W^{2, p} \cap W_{0}^{1, p}\right)$.

Assume now that $u \in W^{1, p} \times\left(W^{2, p} \cap W_{0}^{1, p}\right)$ and $(\lambda+L) u=0$. Let $g \in C_{0}^{\infty}(\bar{\Omega}) \times C_{0}^{\infty}(\Omega)$ and let $w$ be a solution of (5.1) satisfying $w \in W^{1, p^{\prime}} \times$ ( $\left.W^{2, p^{\prime}} \cap W_{0}^{1, p^{\prime}}\right)$ with $1 / p^{\prime}=1-1 / p$. It then follows that

$$
0=((\lambda+L) u, w)=\left(u,\left(\bar{\lambda}+L^{*}\right) w\right)=(u, g)
$$

This implies that $u=0$, and the uniqueness of problem (1.1)-(1.2) holds for $\lambda \in \Sigma(\eta, \theta)$. This completes the proof.

We next give a proof of Theorem 2.2.
Proof of Theorem 2.2. We here prove the second inequality in Theorem 2.2 for $\partial_{x_{n}} Q_{n} G(\lambda) f$ only, since the other cases can be proved similarly by applying Lemmas 4.6-4.8 and 5.2.

We first consider the case $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$. Let $E^{(j)}\left(x_{n}, y_{n}\right), j=1,2$, be the functions defined in Lemma 4.6 and set $E\left(x_{n}, y_{n}\right)=E^{(1)}\left(x_{n}, y_{n}\right)+$ $E^{(2)}\left(x_{n}, y_{n}\right)$. Then we have

$$
\sup _{0 \leq x_{n} \leq a}\left(\int_{0}^{a}\left|E\left(x_{n}, y_{n}\right)\right|^{2} d y_{n}\right)^{\frac{1}{2}} \leq C \sigma\left(\lambda, \xi^{\prime}\right)^{-\frac{1}{4}} .
$$

Since $\partial_{x_{n}}^{2}\left(g_{\mu_{2}}^{N} Q_{0} \hat{f}\right)=\left(\partial_{x_{n}}^{2} g_{\mu_{2}}^{N}\right)\left[Q_{0} \hat{f}\right]-Q_{0} \widehat{f}$, we see from Lemmas 4.6 and 5.2 that

$$
\begin{aligned}
\left\|\partial_{x_{n}} Q_{n} G(\lambda) Q_{0} f\right\|_{\infty} & \leq \frac{C}{|\lambda|+1}\left\{\left\|Q_{0} f\right\|_{\infty}+\left\|\left(\partial_{x_{n}}^{2} g_{\mu_{2}}^{N}\right)\left[Q_{0} \widehat{f}\right]\right\|_{L_{\xi^{\prime}}^{1} L_{x_{n}}^{\infty}}\right\} \\
& \leq \frac{C}{|\lambda|+1}\left\{\left\|Q_{0} f\right\|_{\infty}+\left\|\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{\frac{1}{4}}\left|Q_{0} \widehat{f}\left(\xi^{\prime}\right)\right|_{2}\right\|_{L_{\xi^{\prime}}^{1}}\right\} \\
& \leq C\left\{\frac{1}{|\lambda|+1}\left\|Q_{0} f\right\|_{\infty}+\frac{1}{(|\lambda|+1)^{\frac{3}{4}}}\left\|Q_{0} f\right\|_{H^{\left[\frac{n}{2}\right]+1}}\right\} \\
& \leq \frac{C}{(|\lambda|+1)^{\frac{3}{4}}}\left\|Q_{0} f\right\|_{H^{\left[\frac{n}{2}\right]+1}}
\end{aligned}
$$

We next consider $\partial_{x_{n}} Q_{n} G(\lambda) Q^{\prime} f$. Similarly to above, we see from Lemmas 4.6 and 5.2 that, for $0<\varepsilon<1$,

$$
\begin{aligned}
\left\|\partial_{x_{n}} Q_{n} G(\lambda) Q^{\prime} f\right\|_{\infty} & \leq C\left\|\partial_{x_{n}} Q_{n} \widehat{G}\left(\lambda, \xi^{\prime}\right) Q^{\prime} \widehat{f}\right\|_{L_{\xi^{\prime}}^{1} L_{x_{n}}^{\infty}} \\
& \leq C\left\|\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{3}{4}}\left|\xi^{\prime}\right|\left|Q^{\prime} \widehat{f}\left(\xi^{\prime}\right)\right|_{2}\right\|_{L_{\xi^{\prime}}^{1}} \\
& \leq C\left\|(|\lambda|+1)^{-\frac{\varepsilon}{4}}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{1}{4}(1-\varepsilon)}\left|Q^{\prime} \widehat{f}\left(\xi^{\prime}\right)\right|_{2}\right\|_{L_{\xi^{\prime}}^{1}} \\
& \leq \frac{C}{(|\lambda|+1)^{\frac{5}{4}}}\left\|Q^{\prime} \widehat{f}\right\|_{H^{s}}
\end{aligned}
$$

with $s>\frac{n}{2}-\varepsilon$. Therefore, we can take $s=\left[\frac{n}{2}\right]$. Similarly, we can obtain

$$
\left\|\partial_{x_{n}} Q_{n} G(\lambda) Q_{n} f\right\|_{\infty} \leq \frac{C}{(|\lambda|+1)^{\frac{\varepsilon}{4}}}\left\|f^{\prime}\left(\xi^{\prime}\right)\right\|_{H^{\left[\frac{n}{2}\right]}} .
$$

We next consider the case $\lambda \in \Sigma(\eta, \theta)-\Sigma\left(\eta_{4}, \theta_{4}\right)$. We decompose $(\lambda+$ $L)^{-1} f$ into $(\lambda+L)^{-1} f=\widetilde{R}^{(0)}(\lambda) f+\widetilde{R}^{(\infty)}(\lambda) f$ as in the proof of Theorem 2.1. It then follows that $\widetilde{R}^{(\infty)}(\lambda) f$ can be estimated as above. One can also see that $\widetilde{R}^{(0)}(\lambda) f$ has the desired estimates since $D\left(\lambda, \xi^{\prime}\right) \neq 0$ on the compact set $\overline{\Sigma(\eta, \theta)-\Sigma\left(\eta_{4}, \theta_{4}\right)} \times\left\{\xi^{\prime} ;\left|\xi^{\prime}\right| \leq 2 R_{3}\right\}$. This completes the proof.

We next prove the $L^{p}$ estimates of the resolvent for $p=1, \infty$. For this purpose we prepare the following lemma.

Lemma 5.5. Let $\ell=0,1$ and let $\widehat{\Phi}\left(\xi^{\prime}, x_{n}\right)$ be a function satisfying

$$
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}} \widehat{\Phi}\left(\xi^{\prime}, \cdot\right)\right|_{1} \leq C \sigma\left(\lambda, \xi^{\prime}\right)^{\frac{\ell}{2}-1-\frac{\left|\alpha^{\prime}\right|}{2}}
$$

for all $\alpha^{\prime}$ with $\left|\alpha^{\prime}\right| \leq n$. Then $\Phi\left(x^{\prime}, x_{n}\right)=\left(\mathscr{F}_{\xi^{\prime}}^{-1} \widehat{\Phi}\right)\left(x^{\prime}, x_{n}\right)$ satisfies

$$
\|\Phi\|_{1} \leq C(|\lambda|+1)^{\frac{\ell}{2}-1}
$$

Proof. The proof is based on the Riemann-Lebesgue lemma as in the estimates of solutions to the Cauchy problem given in [8].

Since $2-\ell+n>n-1$, by integration by parts, we see that

$$
\begin{aligned}
\left|\Phi\left(x^{\prime}, \cdot\right)\right|_{1} & =\left|(2 \pi)^{n-1} \int_{\mathbf{R}^{n-1}} \widehat{\Phi}\left(\xi^{\prime}, x_{n}\right)\left(\sum_{j=1}^{n-1} \frac{x_{j}}{i\left|x^{\prime}\right|^{2}} \partial_{\xi_{j}}\right)^{n} e^{i x^{\prime} \cdot \xi^{\prime}} d \xi^{\prime}\right|_{L_{x_{n}}^{1}} \\
& \leq C\left|x^{\prime}\right|^{-n} \int_{\mathbf{R}^{n-1}} \sum_{\left|\alpha^{\prime}\right|=n}\left|\partial_{\xi_{j}}^{\alpha^{\prime}} \widehat{\Phi}\left(\xi^{\prime}, x_{n}\right)\right|_{L_{x_{n}}^{1}} d \xi^{\prime} \\
& \leq C\left|x^{\prime}\right|^{-n} \int_{\mathbf{R}^{n-1}}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{\frac{\ell}{2}-1-\frac{n}{2}} d \xi^{\prime} \\
& \leq C\left|x^{\prime}\right|^{-n}(|\lambda|+1)^{-\frac{3}{2}+\frac{\ell}{2}}
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\left\|\left|\Phi\left(x^{\prime}, \cdot\right)\right|_{1}\right\|_{L^{1}\left(\left|x^{\prime}\right| \geq(|\lambda|+1)^{-\frac{1}{2}}\right)} & \leq C(|\lambda|+1)^{-\frac{3}{2}+\frac{\ell}{2}}\left\|\left|x^{\prime}\right|^{-n}\right\|_{L^{1}\left(\left\lvert\, x^{\prime} \geq(|\lambda|+1)^{-\frac{1}{2}}\right.\right)} \\
& \leq C(|\lambda|+1)^{-1+\frac{\ell}{2}}
\end{aligned}
$$

We next show that

$$
\left\|\left|\Phi\left(x^{\prime}, \cdot\right)\right|_{1}\right\|_{L^{1}\left(\left|x^{\prime}\right| \leq(|\lambda|+1)^{-\frac{1}{2}}\right)} \leq C(|\lambda|+1)^{-1+\frac{\ell}{2}} .
$$

In case $\ell=0$, similarly to above, we have

$$
\begin{aligned}
\left|\Phi\left(x^{\prime}, \cdot\right)\right|_{1} & =\left|(2 \pi)^{n-1} \int_{\mathbf{R}^{n-1}} \widehat{\Phi}\left(\xi^{\prime}, x_{n}\right)\left(\sum_{j=1}^{n-1} \frac{x_{j}}{i\left|x^{\prime}\right|^{2}} \partial_{\xi_{j}}\right)^{n-2} e^{i x^{\prime} \cdot \xi^{\prime}} d \xi^{\prime}\right|_{L_{x_{n}}^{1}} \\
& \leq C\left|x^{\prime}\right|^{-(n-2)} \int_{\mathbf{R}^{n-1}} \sum_{\left|\alpha^{\prime}\right|=n-2}\left|\partial_{\xi_{j}}^{\alpha_{j}} \widehat{\Phi}\left(\xi^{\prime}, x_{n}\right)\right|_{L_{x_{n}}^{1}} d \xi^{\prime} \\
& \leq C\left|x^{\prime}\right|^{-(n-2)} \int_{\mathbf{R}^{n-1}}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{\frac{\ell}{2}-1-\frac{n-2}{2}} d \xi^{\prime} \\
& \leq C\left|x^{\prime}\right|^{-(n-2)}(|\lambda|+1)^{-\frac{1}{2}}
\end{aligned}
$$

This implies

$$
\begin{aligned}
\left\|\left|\Phi\left(x^{\prime}, \cdot\right)\right|_{1}\right\|_{L^{1}\left(\left|x^{\prime}\right| \leq(|\lambda|+1)^{-\frac{1}{2}}\right)} & \leq C(|\lambda|+1)^{-\frac{1}{2}}\left\|\left|x^{\prime}\right|^{-(n-2)}\right\|_{L^{1}\left(\left|x^{\prime}\right| \leq(|\lambda|+1)^{-\frac{1}{2}}\right)} \\
& \leq C(|\lambda|+1)^{-1}
\end{aligned}
$$

The proof of the lemma for $\ell=0$ is thus complete.
Let us consider the case $\ell=1$. We first observe that

$$
\begin{align*}
\int_{\mathbf{R}^{n-1}} \partial_{\xi^{\prime}}^{\alpha^{\prime}} \widehat{\Phi}\left(\xi^{\prime}, x_{n}\right) e^{i x^{\prime} \cdot \xi^{\prime}} d \xi^{\prime} & =-\int_{\mathbf{R}^{n-1}} \partial_{\xi^{\prime}}^{\alpha^{\prime}} \widehat{\Phi}\left(\xi^{\prime}, x_{n}\right) e^{i x^{\prime} \cdot\left(\xi^{\prime}+\frac{x^{\prime}}{\left|x^{\prime}\right|^{2}} \pi\right)} d \xi^{\prime}  \tag{5.2}\\
& =-\int_{\mathbf{R}^{n-1}} \partial_{\xi^{\prime}}^{\alpha^{\prime}} \widehat{\Phi}\left(\xi^{\prime}-z^{\prime}, x_{n}\right) e^{i x^{\prime} \cdot \xi^{\prime}} d \xi^{\prime}
\end{align*}
$$

Here and in what follows we write $z^{\prime}=\frac{x^{\prime}}{\left|x^{\prime}\right|^{2}} \pi$. From (5.2) we see that

$$
\begin{align*}
& \left|\Phi\left(x^{\prime}\right)\right|_{1} \\
& \leq C\left|x^{\prime}\right|^{-(n-2)} \int_{\mathbf{R}^{n-1}} \sum_{\left|\alpha^{\prime}\right|=n-2}\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\widehat{\Phi}\left(\xi^{\prime}-z^{\prime}, x_{n}\right)-\widehat{\Phi}\left(\xi^{\prime}, x_{n}\right)\right)\right|_{L_{x_{n}}^{1}} d \xi^{\prime} \tag{5.3}
\end{align*}
$$

By assumption, we have

$$
\begin{equation*}
\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\widehat{\Phi}\left(\xi^{\prime}-z^{\prime}, x_{n}\right)-\widehat{\Phi}\left(\xi^{\prime}, x_{n}\right)\right)\right|_{L_{x_{n}}^{1}} \leq C \Phi^{(1)}\left(\xi^{\prime}, x^{\prime}\right) \tag{5.4}
\end{equation*}
$$

where

$$
\Phi^{(1)}\left(\xi^{\prime}, x^{\prime}\right)=\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{n-1}{2}}+\left(|\lambda|+1+\left|\xi^{\prime}-z^{\prime}\right|^{2}\right)^{-\frac{n-1}{2}}
$$

We also have

$$
\begin{align*}
& \left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\widehat{\Phi}\left(\xi^{\prime}-z^{\prime}, x_{n}\right)-\widehat{\Phi}\left(\xi^{\prime}, x_{n}\right)\right)\right|_{L_{x_{n}}^{1}} \\
& \quad \leq C\left|z^{\prime}\right| \sum_{\left|\beta^{\prime}\right|=n-1} \int_{0}^{1}\left|\partial_{\xi^{\prime}}^{\beta^{\prime}} \widehat{\Phi}\left(\xi^{\prime}-\theta z^{\prime}, x_{n}\right)\right|_{L_{x_{n}}^{1}} d \theta  \tag{5.5}\\
& \quad \leq C\left|x^{\prime}\right|^{-1} \Phi^{(2)}\left(\xi^{\prime}, x^{\prime}\right)
\end{align*}
$$

where

$$
\Phi^{(2)}\left(\xi^{\prime}, x^{\prime}\right)=\int_{0}^{1}\left(|\lambda|+1+\left|\xi^{\prime}-\theta z^{\prime}\right|^{2}\right)^{-\frac{n}{2}} d \theta
$$

Let $\tau$ be a number satisfying $1-\frac{1}{n}<\tau<1$. It then follows from (5.4) and (5.5) that, for $\left|\alpha^{\prime}\right|=n-2$,

$$
\begin{align*}
& \int_{\mathbf{R}^{n-1}}\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\widehat{\Phi}\left(\xi^{\prime}-z^{\prime}, x_{n}\right)-\widehat{\Phi}\left(\xi^{\prime}, x_{n}\right)\right)\right|_{L_{x_{n}}^{1}} d \xi^{\prime} \\
& \quad \leq C\left|x^{\prime}\right|^{-(1-\tau)} \int_{\mathbf{R}^{n-1}} \widehat{\Phi}^{(1)}\left(\xi^{\prime}, x^{\prime}\right)^{\tau} \widehat{\Phi}^{(2)}\left(\xi^{\prime}, x^{\prime}\right)^{1-\tau} d \xi^{\prime}  \tag{5.6}\\
& \quad=C\left|x^{\prime}\right|^{-(1-\tau)} \int_{\Omega_{1}}+\int_{\Omega_{2}}+\int_{\Omega_{3}} \widehat{\Phi}^{(1)}\left(\xi^{\prime}, x^{\prime}\right)^{\tau} \widehat{\Phi}^{(2)}\left(\xi^{\prime}, x^{\prime}\right)^{1-\tau} d \xi^{\prime}
\end{align*}
$$

where

$$
\begin{aligned}
& \Omega_{1}=\left\{\xi^{\prime} ;\left|\xi^{\prime}\right| \geq 2\left|z^{\prime}\right|\right\}, \\
& \Omega_{2}=\left\{\xi^{\prime} ;\left|\xi^{\prime}\right| \leq 2\left|z^{\prime}\right|\right\} \cap\left\{\xi^{\prime} ;\left|\xi^{\prime}-z^{\prime}\right| \geq 2\left|z^{\prime}\right|\right\}, \\
& \Omega_{3}=\left\{\xi^{\prime} ;\left|\xi^{\prime}\right| \leq 2\left|z^{\prime}\right|\right\} \cap\left\{\xi^{\prime} ;\left|\xi^{\prime}-z^{\prime}\right| \leq 2\left|z^{\prime}\right|\right\} .
\end{aligned}
$$

On $\Omega_{1}$ we have $\left|\xi^{\prime}-z^{\prime}\right| \geq \frac{1}{2}\left|\xi^{\prime}\right|$ and $\left|\xi^{\prime}-\theta z^{\prime}\right| \geq \frac{1}{2}\left|\xi^{\prime}\right|$ for any $\theta \in[0,1]$. It follows that

$$
\begin{align*}
\int_{\Omega_{1}} \widehat{\Phi}^{(1)}\left(\xi^{\prime}, x^{\prime}\right)^{\tau} \widehat{\Phi}^{(2)}\left(\xi^{\prime}, x^{\prime}\right)^{1-\tau} d \xi^{\prime} & \leq C \int_{\mathbf{R}^{n-1}}\left(|\lambda|+1+\left|\xi^{\prime}\right|^{2}\right)^{-\frac{n}{2}+\frac{\tau}{2}} d \xi^{\prime}  \tag{5.7}\\
& \leq C(|\lambda|+1)^{-\frac{1}{2}+\frac{\tau}{2}}
\end{align*}
$$

On $\Omega_{2}$ we have $\left|\xi^{\prime}-z^{\prime}\right| \geq\left|\xi^{\prime}\right|$ and $\left|\xi^{\prime}-\theta z^{\prime}\right| \geq\left|\xi^{\prime}-z^{\prime}\right|-(1-\theta)\left|z^{\prime}\right| \geq \frac{1}{2}\left|\xi^{\prime}\right|$ for any $\theta \in[0,1]$. Therefore, as above, we obtain

$$
\begin{equation*}
\int_{\Omega_{2}} \widehat{\Phi}^{(1)}\left(\xi^{\prime}, x^{\prime}\right)^{\tau} \widehat{\Phi}^{(2)}\left(\xi^{\prime}, x^{\prime}\right)^{1-\tau} d \xi^{\prime} \leq C(|\lambda|+1)^{-\frac{1}{2}+\frac{\tau}{2}} \tag{5.8}
\end{equation*}
$$

As for the integral on $\Omega_{3}$, we have

$$
\begin{align*}
\int_{\Omega_{3}} & \widehat{\Phi}^{(1)}\left(\xi^{\prime}, x^{\prime}\right)^{\tau} \widehat{\Phi}^{(2)}\left(\xi^{\prime}, x^{\prime}\right)^{1-\tau} d \xi^{\prime} \\
\leq & C \int_{\left\{\left|\xi^{\prime}\right| \leq 2\left|z^{\prime}\right|\right\}}\left|\xi^{\prime}\right|^{-(n-1) \tau}(|\lambda|+1)^{-\frac{n}{2}(1-\tau)} d \xi^{\prime}  \tag{5.9}\\
& +C \int_{\left\{\left|\xi^{\prime}-z^{\prime}\right| \leq 2\left|z^{\prime}\right|\right\}}\left|\xi^{\prime}-z^{\prime}\right|^{-(n-1) \tau}(|\lambda|+1)^{-\frac{n}{2}(1-\tau)} d \xi^{\prime} \\
\leq & C(|\lambda|+1)^{-\frac{n}{2}(1-\tau)}\left|x^{\prime}\right|^{-(1-\tau)(n-1)} .
\end{align*}
$$

Consequently we see from (5.6)-(5.9) that

$$
\begin{aligned}
& \int_{\mathbf{R}^{n-1}}\left|\partial_{\xi^{\prime}}^{\alpha^{\prime}}\left(\widehat{\Phi}\left(\xi^{\prime}-z^{\prime}, x_{n}\right)-\widehat{\Phi}\left(\xi^{\prime}, x_{n}\right)\right)\right|_{L_{x_{n}}^{1}} d \xi^{\prime} \\
& \quad \leq C\left\{(|\lambda|+1)^{-\frac{1}{2}+\frac{\tau}{2}}\left|x^{\prime}\right|^{-(1-\tau)}+(|\lambda|+1)^{-\frac{n}{2}(1-\tau)}\left|x^{\prime}\right|^{-(1-\tau) n}\right\} .
\end{aligned}
$$

This, together with (5.3), implies that

$$
\begin{aligned}
\left|\Phi\left(x^{\prime}\right)\right|_{1} \leq & C\left\{(|\lambda|+1)^{-\frac{1}{2}+\frac{\tau}{2}}\left|x^{\prime}\right|^{-(n-1)+\tau}\right. \\
& \left.+(|\lambda|+1)^{-\frac{n}{2}(1-\tau)}\left|x^{\prime}\right|^{-(n-2)-(1-\tau) n}\right\}
\end{aligned}
$$

Since $1-\frac{1}{n}<\tau$, integrating this over $\left\{x^{\prime} ;\left|x^{\prime}\right| \leq(|\lambda|+1)^{-\frac{1}{2}}\right\}$, we obtain

$$
\left\|\left|\Phi\left(x^{\prime}, \cdot\right)\right|_{1}\right\|_{L^{1}\left(\left|x^{\prime}\right| \leq(|\lambda|+1)^{-\frac{1}{2}}\right)} \leq C(|\lambda|+1)^{-\frac{1}{2}}
$$

This completes the proof.
We now prove the $L^{p}$ estimates of the resolvent for $p=1, \infty$.
Proof of Theorem 2.3. We here prove the estimates only for $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$. The other case can be treated by using the decomposition $(\lambda+L)^{-1} f=$ $\widetilde{R}^{(0)}(\lambda) f+\widetilde{R}^{(\infty)}(\lambda) f$ as in the proof of Theorem 2.1.

Let us denote the integral kernels of $(\lambda+L)^{-1}, G(\lambda) f$ and $K(\lambda) f$ by $R\left(\lambda, x^{\prime}-y^{\prime}, x_{n}, y_{n}\right), G\left(\lambda, x^{\prime}-y^{\prime}, x_{n}, y_{n}\right)$ and $K\left(\lambda, x^{\prime}-y^{\prime}, x_{n}, y_{n}\right)$, respectively.

Let $E\left(x_{n}, y_{n}\right)=E^{(1)}\left(x_{n}, y_{n}\right)+E^{(2)}\left(x_{n}, y_{n}\right)$ be defined as in the proof of Theorem 2.2. It holds that

$$
\sup _{0 \leq y_{n} \leq a} \int_{0}^{a} E\left(x_{n}, y_{n}\right) d x_{n}+\sup _{0 \leq x_{n} \leq a} \int_{0}^{a} E\left(x_{n}, y_{n}\right) d y_{n} \leq C \sigma\left(\lambda, \xi^{\prime}\right)^{-\frac{1}{2}} .
$$

Therefore, by Lemmas 4.6-4.8 and 5.5, we have

$$
\begin{aligned}
\left\|\partial_{x}^{k} \widetilde{Q}(\lambda+L)^{-1} \widetilde{Q} f\right\|_{1} & \leq \sup _{0 \leq y_{n} \leq a}\left\|\partial_{x}^{k} \widetilde{Q} R\left(\lambda, \cdot, \cdot, y_{n}\right)\right\|_{1}\|\widetilde{Q} f\|_{1} \\
& \leq \frac{C}{(|\lambda|+1)^{1-\frac{k}{2}}}\|\widetilde{Q} f\|_{1}, \quad k=0,1
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\partial_{x}^{k} \widetilde{Q}(\lambda+L)^{-1} \widetilde{Q} f\right\|_{\infty} & \leq \sup _{0 \leq x_{n} \leq a}\left\|\partial_{x}^{k} \widetilde{Q} R\left(\lambda, \cdot, x_{n}, \cdot\right)\right\|_{1}\|\widetilde{Q} f\|_{\infty} \\
& \leq \frac{C}{(|\lambda|+1)^{1-\frac{k}{2}}}\|\widetilde{Q} f\|_{\infty}, \quad k=0,1
\end{aligned}
$$

Similarly one can estimate $Q_{0}(\lambda+L)^{-1} f$ to obtain the desired estimate.
We next consider $\partial_{x} Q_{0}(\lambda+L)^{-1} \widetilde{Q} f$. We here estimate only $\partial_{x} Q_{0} G(\lambda) Q_{n} f$. Since $\partial_{x^{\prime}}^{\alpha^{\prime}} G(\lambda) f=G(\lambda) \partial_{x^{\prime}}^{\alpha^{\prime}} f$, we see from Lemmas 4.6 and 5.5 that

$$
\left\|\partial_{x^{\prime}}^{\alpha^{\prime}} Q_{0} G(\lambda) Q_{n} f\right\|_{p} \leq \frac{C}{|\lambda|+1}\left\|\partial_{x^{\prime}}^{\alpha^{\prime}} Q_{n} f\right\|_{p}, \quad p=1, \infty,\left|\alpha^{\prime}\right|=1
$$

Since $\partial_{x_{n}}^{2}\left(g_{\mu_{2}}^{(D)} \widehat{f}\right)=\mu_{2}^{2} g_{\mu_{2}}^{(D)} \widehat{f}-\widehat{f}$ and $\mu_{2}^{2}=\frac{\lambda^{2}}{\nu_{1} \lambda+\gamma^{2}}+\left|\xi^{\prime}\right|^{2}$, we have

$$
\begin{aligned}
& \partial_{x_{n}} Q_{0} G(\lambda) Q_{n} f \\
& \quad=-\frac{\gamma}{\nu_{1} \lambda+\gamma^{2}}\left\{\frac{\lambda^{2}}{\nu_{1} \lambda+\gamma^{2}} Q_{0} G(\lambda) Q_{n} f+\nabla^{\prime} Q_{0} G(\lambda)\left[\nabla^{\prime} Q_{n} f\right]-Q_{n} f\right\} .
\end{aligned}
$$

It then follows from Lemmas 4.6 and 5.5 that

$$
\left\|\partial_{x_{n}} Q_{0} G(\lambda) Q_{n} f\right\|_{p} \leq \frac{C}{|\lambda|+1}\left\{\left\|Q_{n} f\right\|_{p}+\left\|\nabla^{\prime} Q_{n} f\right\|_{p}\right\}, \quad p=1, \infty
$$

The remaining part of $\partial_{x} Q_{0}(\lambda+L)^{-1} \widetilde{Q} f$ can be estimated similarly. One can also estimate $\partial_{x} \widetilde{Q}(\lambda+L)^{-1} Q_{0} f$ in a similar manner.

We next estimate $\partial_{x} Q_{0}(\lambda+L)^{-1} Q_{0} f$. Consider first $Q_{0} K(\lambda) Q_{0} f$. We set

$$
\begin{aligned}
& \widetilde{k}\left(\lambda, x^{\prime}, x_{n}, y_{n}\right)=\mathscr{F}^{-1}\left[h_{\mu_{2}}\left(\lambda, \xi^{\prime}, x_{n}\right) \beta_{0}\left(\lambda, \xi^{\prime}, y_{n}\right)\right]\left(x^{\prime}\right), \\
& \widetilde{K}(\lambda) f=\int_{\mathbf{R}^{n-1}} \widetilde{k}\left(x^{\prime}-y^{\prime}, x_{n}, y_{n}\right) Q_{0} f\left(y^{\prime}, y_{n}\right) d y^{\prime} d y_{n} .
\end{aligned}
$$

Then

$$
Q_{0} K(\lambda) Q_{0} f=-\frac{\gamma}{\nu_{1} \lambda+\gamma^{2}} \nabla^{\prime} \widetilde{K}(\lambda)\left[\nabla^{\prime} Q_{0} f\right] .
$$

It then follows from Lemmas 4.7, 4.8 and 5.5 that

$$
\left\|\partial_{x^{\prime}} Q_{0} K(\lambda) Q_{0} f\right\|_{p} \leq \frac{C}{|\lambda|+1}\left\|\partial_{x^{\prime}} \nabla^{\prime} Q_{0} f\right\|_{p}, \quad p=1, \infty
$$

Also, since

$$
Q_{0} K(\lambda) Q_{0} f=-\frac{\gamma}{\nu_{1} \lambda+\gamma^{2}} \widetilde{K}(\lambda)\left[\Delta^{\prime} Q_{0} f\right]
$$

we similarly obtain, by Lemmas 4.7, 4.8 and 5.5,

$$
\left\|\partial_{x_{n}} Q_{0} K(\lambda) Q_{0} f\right\|_{p} \leq \frac{C}{|\lambda|+1}\left\|\Delta^{\prime} Q_{0} f\right\|_{p}
$$

As for $\partial_{x} Q_{0} G(\lambda) Q_{0} f$, one can treat it as in the case of $\partial_{x} \widetilde{Q}(\lambda+L)^{-1} \widetilde{Q} f$.
Let us finally consider $\partial_{x} \widetilde{Q}(\lambda+L)^{-1} f$, assuming that $\left.\widetilde{Q} f\right|_{x_{n}=0, a}=0$. As for $\partial_{x} \widetilde{Q}(\lambda+L)^{-1} \widetilde{Q} f$, we rewrite it as in the proof of the last inequality of Theorem 2.1. One can then estimate it similarly to the case of $\partial_{x}^{k} \widetilde{Q}(\lambda+$ $L)^{-1} \widetilde{Q} f$ with $k \leq 1$ to obtain the desired inequality. The estimate for $\partial_{x} \widetilde{Q}(\lambda+$ $L)^{-1} Q_{0} f$ has been already obtained. This completes the proof.

We finally prove Theorems 2.5-2.7.
Proof of Theorems 2.5-2.7. We prove Theorems 2.5-2.7 for $\tilde{\eta}=\min \left\{\eta_{2}, \eta_{3}\right\}$ and $\frac{\pi}{2}<\widetilde{\theta}<\min \left\{\theta_{2}, \theta_{3}, \theta_{4}\right\}$, where $\eta_{j}(j=2,3)$ and $\theta_{j}(j=2,3,4)$ are the numbers given in Lemmas 4.3 and 4.4. For this $\{\widetilde{\eta}, \widetilde{\theta}\}$ we see from Lemma 4.3 that $\Sigma(-\widetilde{\eta}, \widetilde{\theta}) \subset \rho\left(-\widehat{L}_{\xi^{\prime}}\right)$ for $\left|\xi^{\prime}\right| \geq r$. Furthermore, we deduce that $\Sigma(-\widetilde{\eta}, \widetilde{\theta}) \subset \Sigma\left(-\eta_{2}, \theta_{2}\right) \cap \Sigma\left(-\eta_{3}, \theta_{3}\right)$ and that $\overline{\Sigma(-\widetilde{\eta}, \widetilde{\theta})-\Sigma\left(\eta_{4}, \theta_{4}\right)}$ is a compact set. Here $\eta_{4}$ and $\theta_{4}$ are the numbers given in Lemma 4.4.

In view of the proof of Theorems 2.1-2.3, the desired estimates for $R^{(1)}(\lambda) f$ hold for $\lambda \in \Sigma\left(\eta_{4}, \theta_{4}\right)$.

In case $\lambda \in \Sigma(-\widetilde{\eta}, \widetilde{\theta})-\Sigma\left(\eta_{4}, \theta_{4}\right)$, we decompose $R^{(1)}(\lambda) f$ in the following way. Let $\chi\left(\xi^{\prime}\right)$ and $\widetilde{\chi}\left(\xi^{\prime}\right)$ be the cut-off functions given in the definition of $R^{(1)}(\lambda) f$ and in the proof of Theorem 2.1, respectively. We write $R^{(1)}(\lambda) f$ as

$$
R^{(1)}(\lambda) f=\widetilde{R}^{(2)}(\lambda) f+\widetilde{R}^{(\infty)}(\lambda) f
$$

where

$$
\widetilde{R}^{(2)}(\lambda) f=\mathscr{F}_{\xi^{\prime}}^{-1}\left[\chi\left(\xi^{\prime}\right)\left(1-\widetilde{\chi}\left(\xi^{\prime}\right)\right) \widehat{R}\left(\lambda, \xi^{\prime}\right) \widehat{f}\right]
$$

and

$$
\widetilde{R}^{(\infty)}(\lambda) f=\mathscr{F}_{\xi^{\prime}}^{-1}\left[\chi\left(\xi^{\prime}\right) \widetilde{\chi}\left(\xi^{\prime}\right) \widehat{R}\left(\lambda, \xi^{\prime}\right) \widehat{f}\right]
$$

By Lemma 3.4 and Lemma 4.3 (ii) we see that $D\left(\lambda, \xi^{\prime}\right) \neq 0$ on the compact set $\overline{\Sigma(-\widetilde{\eta}, \tilde{\theta})-\Sigma\left(\eta_{4}, \theta_{4}\right)}$. Therefore, as above, one can see that $R^{(1)}(\lambda) f$ has the desired estimates for $\lambda \in \Sigma(-\widetilde{\eta}, \widetilde{\theta})-\Sigma\left(\eta_{4}, \theta_{4}\right)$ as in the proof of Theorems 2.1-2.3. This completes the proof.

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