

Arithmetic of the splitting field of Alexander polynomial

Komatsu, Toru
Faculty of Mathematics, Kyushu University

<https://hdl.handle.net/2324/3390>

出版情報 : MHF Preprint Series. 2006-17, 2006-04-03. 九州大学大学院数理学研究院
バージョン :
権利関係 :



MHF Preprint Series

Kyushu University
21st Century COE Program
Development of Dynamic Mathematics with
High Functionality

Arithmetic of the splitting field of Alexander polynomial

T. Komatsu

MHF 2006-17

(Received April 3, 2006)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Arithmetic of the splitting field of Alexander polynomial

Toru KOMATSU

§ 1. Introduction

In this paper we study the arithmetic of the minimal splitting field of the Alexander polynomial of a knot and present two kinds of infinite families of knots, one is a family of knots which satisfying Heilbronn conjecture (Conjecture 1.3) and the other is a family of counterexamples to the conjecture.

For a knot K in \mathbb{R}^3 let us denote by $\Delta_K(t)$ the Alexander polynomial of K . In general, $\Delta_K(t)$ is defined up to $\pm t^k$. In this paper we assume that $\Delta_K(t)$ is normalized so that $\Delta_K(t)$ is a polynomial in $\mathbb{Z}[t]$ whose order at $t = 0$ is equal to 0. It is a difficult problem to determine the period of a knot K in terms of the Alexander polynomial $\Delta_K(t)$. However, some necessary conditions for a knot to have some period are known. For a positive integer $m \in \mathbb{Z}$, $m \geq 1$ let ζ_m be a primitive m -th root of unity in $\overline{\mathbb{Q}}$. Let F_f be the minimal splitting field of a polynomial $f \in \mathbb{Q}[t]$ over \mathbb{Q} .

Proposition 1.1 (Trotter [7]). *Let K be a fibered knot with $\text{disc}_t \Delta_K(t) \neq 0$. If K has period m , then $\zeta_m \in F_{\Delta_K}$.*

Let p be a prime number. For a positive integer $\lambda \in \mathbb{Z}$, $\lambda \geq 1$ and a polynomial $\mu \in \mathbb{Z}[t]$ let us denote by $M_p(\lambda, \mu)$ the set consisting of polynomials $f \in \mathbb{Z}[t]$ such that $f \equiv \pm t^k(1 + t + \cdots + t^{\lambda-1})^{p-1} \mu^p \pmod{p}$ for some integer $k \in \mathbb{Z}$.

Proposition 1.2 (Murasugi [4]). *If K is a knot with prime period p , then $\Delta_K(t) \in M_p(\lambda, \mu)$ for a positive integer $\lambda \in \mathbb{Z}$, $\lambda \geq 1$ and a polynomial $\mu \in \mathbb{Z}[t]$.*

Let $H(F)$ be the Hilbert class field of a finite number field F .

Conjecture 1.3 (Heilbronn (cf. [5])). *For a (fibered) knot K , if $\Delta_K(t) \in M_p(\lambda, 1)$ for a positive integer $\lambda \geq 1$, then $\zeta_p \in H(F_{\Delta_K})$.*

We have already obtained some counterexamples to the Heilbronn's conjecture (e.g., Example 2.3). Morishita gave a question which revises the Heilbronn's conjecture. For a polynomial $f \in \mathbb{Q}[t]$ we say that f satisfies the condition (U) if $\zeta_p \in H(F_f)$.

Question 1.4 (Morishita (cf. [5])). *What is a condition so that the Alexander polynomial $\Delta_K(t)$ with degree $p - 1$ of a knot K satisfies the condition (U) ?*

In this paper we show the following theorem by constructing the explicit Alexander polynomials of knots.

Theorem 1.5 (Corollary 3.4). *For every odd prime number $p > 3$ (resp. $p = 3$) there exist infinitely many knots K whose Alexander polynomials $\Delta_K(t)$ are monic (resp. non-monic) of degree $p - 1$ and satisfy the two conditions $\Delta_K(t) \in M_p(2, 1)$ and (U).*

Theorem 1.6 (Corollary 3.8). *For every odd prime number $p > 3$ (resp. $p = 3$) there exist infinitely many knots K whose Alexander polynomials $\Delta_K(t)$ are monic (resp. non-monic) of degree $p - 1$ and satisfy the conditions $\Delta_K(t) \in M_p(2, 1)$ but fail the condition (U).*

REMARK 1.7. The Alexander polynomial of a fibered knot is monic. For the Alexander polynomial $\Delta_K(t)$ of a knot K , if $\Delta_K(t)$ is monic, then there exists a fibered knot K' so that $\Delta_{K'}(t) = \Delta_K(t)$ (Burde [1]).

Acknowledgement. The author is grateful to Professor Masaaki Morishita for introducing the Heilbronn's conjecture. He is grateful to Professor Kunio Murasugi for his many helpful comments. He is supported by the 21st Century COE Program "Development of Dynamic Mathematics with High Functionality".

§ 2. Known results

For a positive integer n let A_n be the set of the Alexander polynomials $\Delta_K(t)$ of knots K with degree n . We define three conditions (A.1) to (A.3) for a polynomial $f(t) \in \mathbb{Z}[t]$ by

(A.1) the degree n of $f(t)$ is even,

$$(A.2) \ f(t) = t^n f(t^{-1}),$$

$$(A.3) \ f(1) = \pm 1.$$

Proposition 2.1 (Seifert [6]). *For a polynomial $f(t) \in \mathbb{Z}[t]$ with degree n , it holds that $f(t) \in A_n$ if and only if $f(t)$ satisfies all of the three conditions (A.1) to (A.3).*

Let p be a prime number. For a positive integer $\lambda \in \mathbb{Z}$, $\lambda \geq 1$ and a polynomial $\mu \in \mathbb{Z}[t]$ let $M_p(\lambda, \mu)$ be the set as in Introduction, that is, the set consisting of polynomials $f \in \mathbb{Z}[t]$ such that $f \equiv \pm t^k(1 + t + \cdots + t^{\lambda-1})^{p-1} \mu^p \pmod{p}$ for some integer $k \in \mathbb{Z}$. If $f(t) \in A_n \cap M_p(\lambda, 1)$, then $t^k(1 + t + \cdots + t^{\lambda-1})^{p-1} \equiv f(t) = t^n f(t^{-1}) \equiv t^{n-k}(1 + t^{-1} + \cdots + t^{-(\lambda-1)})^{p-1} \pmod{p}$. By considering the degrees, one has that $k + (\lambda - 1)(p - 1) = n - k$, which shows $n = (\lambda - 1)(p - 1) + 2k$. Note that $k \geq 0$ for $f(t) \in \mathbb{Z}[t]$. Let us calculate the Hilbert class field $H(F_{\Delta_K})$ of the Alexander polynomials $\Delta_K(t)$ of some knots K and determine whether or not the Heilbronn's conjecture holds.

EXAMPLE 2.2 (cf. [5]). For $\Delta_K(t) = 4 - 7t + 4t^2 \in A_2 \cap M_3(2, 1)$, we have $F_{\Delta_K} = \mathbb{Q}(\sqrt{-15})$. One can see that $H(F_{\Delta_K}) = F_{\Delta_K}(\zeta_3) \ni \zeta_3$. Thus Heilbronn's conjecture is true for the case $\Delta_K(t) = 4 - 7t + 4t^2$.

EXAMPLE 2.3 (cf. [5]). When $\Delta_K(t) = 1 - 6t + 11t^2 - 6t^3 + t^4 \in A_4 \cap M_5(2, 1)$, we have $F_{\Delta_K} = \mathbb{Q}(\sqrt{5})$ since $\Delta_K(t) = (1 - 3t + t^2)^2$. Note that $H(\mathbb{Q}(\sqrt{5})) = \mathbb{Q}(\sqrt{5})$. This means that $\zeta_5 \notin H(F_{\Delta_K})$, that is, Heilbronn's conjecture is not true for the case $\Delta_K(t) = 1 - 6t + 11t^2 - 6t^3 + t^4$.

We define four conditions (U), (U₁), (U₂) and (U₃) for a finite Galois extension F of \mathbb{Q} such that

$$(U) \ \zeta_p \in H(F),$$

$$(U_1) \ F(\zeta_p)/F \text{ is unramified,}$$

$$(U_2) \ F(\zeta_p)/F \text{ is unramified at all the prime ideals } \mathfrak{p} \text{ of } F \text{ above } p,$$

$$(U_3) \text{ for every prime ideal } \mathfrak{p} \text{ of } F \text{ above } p, \text{ the ramification index of } \mathfrak{p} \text{ in the extension } F/\mathbb{Q} \text{ is a multiple of } p - 1.$$

The ramification theory of algebraic number theory implies

Lemma 2.4. *Assume that p is odd. If F is totally real, then $F(\zeta_p)/F$ is ramified at all infinite places of F , in particular, $\zeta_p \notin H(F)$. When F is totally imaginary, the four conditions (U), (U₁), (U₂) and (U₃) are equivalent to each other.*

Morishita and Taguchi showed a sufficient condition to hold (U). For a polynomial $g(t) = \sum_{i=0}^n a_i t^i \in \mathbb{Z}[t]$ we call $g(t)$ a p -Eisenstein polynomial if the coefficients a_i satisfy that $p \nmid a_n$, $p \mid a_i$ for $0 \leq i \leq n-1$ and $p^2 \nmid a_0$. In this paper we do not assume that a p -Eisenstein polynomial is monic.

Proposition 2.5 (Morishita–Taguchi (cf. [5])). *If $g(t) \in \mathbb{Z}[t]$ is a p -Eisenstein polynomial of degree $p-1$, then $\zeta_p \in H(F_g)$.*

EXAMPLE 2.6 (cf. [5]). If $g(t) = t^4 + 5t^3 - 40t^2 + 70t - 35$, then $g(t+1) = t^4 + 9t^3 - 19t^2 + 9t + 1$, which is the Alexander polynomial of some knot. One can see that F_g is a \mathcal{D}_4 -extension of \mathbb{Q} where \mathcal{D}_4 is the dihedral group of degree 4 with order 8. Note that F_g has no subfields which are cyclic quartic fields since \mathcal{D}_4 has no normal subgroups N such that \mathcal{D}_4/N are isomorphic to the cyclic group \mathcal{C}_4 of order 4. Thus we have $\zeta_5 \notin F_{\Delta_K}$. Proposition 1.1 implies that K does not have period 5. On the other hand, Proposition 2.5 yields $\zeta_5 \in H(F_{\Delta_K})$.

For the use of Proposition 2.5 one needs a criterion whether or not a p -Eisenstein polynomial $g(t)$ can become the Alexander polynomial $\Delta_K(t)$ of a knot K by some translation $t \mapsto at+b$ for integers a and $b \in \mathbb{Z}$ with $a \neq 0$, that is, $g(at+b) = \Delta_K(t)$. However, such a criterion is not easy in general. By using a characterization of the Alexander polynomials we present a family of the Alexander polynomials which become p -Eisenstein polynomials in the next section.

§ 3. Construction of the Alexander polynomials

For a positive and even integer $n = 2r$ let B_n be the set of polynomials $f(t)$ in $\mathbb{Z}[t]$ such that $f(t) = t^n f(t^{-1})$. Note that $A_n = \{f(t) \in B_n \mid \deg_t f(t) = n, f(1) = \pm 1\}$.

Lemma 3.1 (cf. Crowell–Fox [3]). *The set B_n is a \mathbb{Z} -module with free basis $\{(t^2 - 2t + 1)^i t^{r-i} \mid 0 \leq i \leq r\}$, that is, $B_n = \{\sum_{0 \leq i \leq r} b_i (t^2 - 2t + 1)^i t^{r-i} \mid b_i \in \mathbb{Z}\}$.*

Let p be a prime number and $\lambda \geq 1$ a positive integer such that $n \geq (\lambda - 1)(p - 1)$. We define a polynomial $\alpha_\lambda(t) = \alpha_{n,p,\lambda}(t)$ by

$$\alpha_\lambda(t) = t^k(1 + t + \cdots + t^{\lambda-1})^{p-1} - (\lambda^{p-1} - 1)t^r$$

where $k = r - (\lambda - 1)(p - 1)/2$. For integers $c_i \in \mathbb{Z}$ with $1 \leq i \leq r$ let us denote $\alpha_\lambda(t) + p \sum_{1 \leq i \leq r} c_i(t^2 - 2t + 1)^i t^{r-i}$ by $f_\lambda(\mathbf{c}, t)$ where $\mathbf{c} = (c_1, c_2, \dots, c_r)$. We put $S_r(\mathbb{Z}) = \mathbb{Z}^r$ and $S_r^\times(\mathbb{Z}) = \{\mathbf{c} = (c_1, c_2, \dots, c_r) \in S_r(\mathbb{Z}) | c_r \neq 0\}$.

Lemma 3.2. *The intersection set $A_n \cap M_p(\lambda, 1)$ consists of the polynomials $\pm f_\lambda(\mathbf{c}, t)$, $\mathbf{c} = (c_1, c_2, \dots, c_r)$ where $\mathbf{c} \in S_r(\mathbb{Z})$ if $n = (\lambda - 1)(p - 1)$ and $\mathbf{c} \in S_r^\times(\mathbb{Z})$ otherwise.*

Proof. Let \mathbf{c} be an element in $S_r(\mathbb{Z})$ if $n = (\lambda - 1)(p - 1)$ and in $S_r^\times(\mathbb{Z})$ otherwise. It follows from the definition that $\deg_t f_\lambda(\mathbf{c}, t) \leq n$. If $n = (\lambda - 1)(p - 1)$, then the coefficient of t^{p-1} in $f_\lambda(\mathbf{c}, t)$ is equal to $1 + pc_r$. Since $c_r \in \mathbb{Z}$, one has that $1 + pc_r \neq 0$ and $\deg_t f_\lambda(\mathbf{c}, t) = n$. When $n > (\lambda - 1)(p - 1)$, the coefficient of t^{p-1} in $f_\lambda(\mathbf{c}, t)$ is equal to pc_r . Thus it holds that $\deg_t f_\lambda(\mathbf{c}, t) = n$ for $c_r \neq 0$. It is easy to see that $\pm f_\lambda(\mathbf{c}, t)$ satisfies the conditions (A.2) and (A.3). Hence the polynomial $\pm f_\lambda(\mathbf{c}, t)$ belongs to $A_n \cap M_p(\lambda, 1)$. Let $f(t)$ be a polynomial in $A_n \cap M_p(\lambda, 1)$. For a non-negative integer $k \in \mathbb{Z}$ with $k = r - (\lambda - 1)(p - 1)/2$, one has $\pm t^k(1 + t + \cdots + t^{\lambda-1})^{p-1} \in B_n$. Since B_n is a \mathbb{Z} -module, we have $f(t) \mp t^k(1 + t + \cdots + t^{\lambda-1})^{p-1} \in B_n$ and $(f(t) \mp t^k(1 + t + \cdots + t^{\lambda-1})^{p-1})/p \in B_n$ if $f(1) = \pm 1$, respectively. Lemma 3.1 implies that $f(t) = \pm t^k(1 + t + \cdots + t^{\lambda-1})^{p-1} + p \sum_{0 \leq i \leq r} c_i(t^2 - 2t + 1)^i t^{r-i}$ where $c_i \in \mathbb{Z}$ for $0 \leq i \leq r$. It follows from $f(1) = \pm 1$ that $\pm \lambda^{p-1} + pc_0 = \pm 1$ and $c_0 = \mp(\lambda^{p-1} - 1)/p \in \mathbb{Z}$. Hence $f(t)$ is of the form $\pm f_\lambda(\mathbf{c}, t)$. \square

For considering Question 1.4 we have $n = p - 1 \geq 2$, which means that $\lambda = 1$ or 2. Here one has that $\alpha_1(t) = t^{(p-1)/2}$ and $\alpha_2(t) = (1 + t)^{p-1} - (2^{p-1} - 1)t^{(p-1)/2}$. Let us define a number $\varepsilon_p \in \mathbb{Q}$ by $\varepsilon_p = -(2^{p-1} - 1)/(4p)$. Note that $v_p(\varepsilon_p) \geq 0$.

Theorem 3.3. *Let $h(X) = \sum_{j=0}^{(p-3)/2} s_j X^j \in \mathbb{Z}[X]$ be a polynomial of degree less than $(p - 1)/2$ such that $s_0 \not\equiv \varepsilon_p \pmod{p}$ and $s_0 > \varepsilon_p$. Then*

$$f(t) = \alpha_2(t) + pt^{(p-1)/2}(t + t^{-1} - 2)h(t + t^{-1} + 2)$$

is a polynomial in $A_{p-1} \cap M_p(2, 1)$ satisfying $\zeta_p \in H(F_f)$. The polynomial $f(t)$ is monic if and only if $s_{(p-3)/2} = 0$.

Proof. It is easy to check that $f(t) \in A_{p-1} \cap M_p(2, 1)$. In fact, it holds that $f(t) = f_2(\mathbf{c}, t)$ for an element $\mathbf{c} = (c_1, c_2, \dots, c_{(p-1)/2}) \in S_{(p-1)/2}(\mathbb{Z})$ satisfying $\sum_{i=1}^{(p-1)/2} c_i(t + t^{-1} - 2)^{i-1} = \sum_{j=0}^{(p-3)/2} s_j(t + t^{-1} + 2)^j$. One has that $f(t - 1) \equiv t^{p-1} \pmod{p}$ and $f(-1) = (-1)^{(p+1)/2}(2^{p-1} - 1 + 4ps_0)$. The condition $s_0 \not\equiv \varepsilon_p \pmod{p}$ is equivalent to $v_p(f(-1)) = 1$. Thus $f(t - 1)$ is a p -Eisenstein polynomial of degree $p - 1$. Now put $X = t + t^{-1} + 2$. Then we have $f(t)/t^{(p-1)/2} = X^{(p-1)/2} - (2^{p-1} - 1) + p(X - 4)h(X)$, which is denoted by $\tilde{f}(X)$. Here it is seen that $\tilde{f}(0) = -(2^{p-1} - 1) - 4ps_0$ and $\tilde{f}(4) = 1$. It follows from $s_0 > \varepsilon_p$ that $\tilde{f}(0) < 0$. This shows that $\tilde{f}(X) = 0$ has a real solution x with $0 < x < 4$. For a complex number $z \in \mathbb{C}$ with $z + z^{-1} \in \mathbb{R}$, the condition $0 < z + z^{-1} + 2 < 4$ holds if and only if z is not real. This means that $f(t)$ has a non-real zero in \mathbb{C} . Thus F_f is not totally real but totally imaginary. Hence Lemma 2.4 verifies that $\zeta_p \in H(F_f)$. \square

Corollary 3.4 (Theorem 1.5). *For every odd prime number $p > 3$ (resp. $p = 3$) there exist infinitely many knots K whose Alexander polynomials $\Delta_K(t)$ are monic (resp. non-monic) and satisfy the two conditions $\Delta_K(t) \in A_{p-1} \cap M_p(2, 1)$ and (U).*

Proof. For example, one may take an arbitrary integer $s \in \mathbb{Z}$ satisfying $s \not\equiv \varepsilon_p \pmod{p}$ and $s > \varepsilon_p$ for the polynomial $h(X)$ in Theorem 3.3. \square

EXAMPLE 3.5 (Examples 2.2 and 2.6). For the case that $p = 3$ and $h(X) = 1$ in Theorem 3.3, one has $f(X) = 4t^2 - 7t + 4$. Since $\varepsilon_3 = -1/4 \equiv 2 \pmod{3}$, we have $\zeta_3 \in H(F_f)$. If $p = 5$ and $h(X) = 1$, then $f(t) = t^4 + 9t^3 - 19t^2 + 9t + 1$ and $\zeta_5 \in H(F_f)$ for $\varepsilon_5 = -3/4 \equiv 3 \pmod{5}$.

Theorem 3.6. *Assume $p \geq 5$. Let $h(X) = \sum_{i=0}^{(p-3)/2} s_i X^i \in \mathbb{Z}[X]$ be a polynomial of degree less than $(p - 1)/2$. If $s_0 \equiv \varepsilon_p \pmod{p}$ and $s_1 \not\equiv \varepsilon_p/4 \pmod{p}$, then $f(t) = \alpha_2(t) + pt^{(p-1)/2}(t + t^{-1} - 2)h(t + t^{-1} + 2)$ is a polynomial in $A_{p-1} \cap M_p(2, 1)$ such that $\zeta_p \notin H(F_f)$. The polynomial $f(t)$ is monic if and only if $s_{(p-3)/2} = 0$.*

Proof. In the same way as in the proof of Theorem 3.3 one sees $f(t) \in A_{p-1} \cap M_p(2, 1)$. For $0 \leq j \leq p - 1$ let $\gamma_j \in \mathbb{Z}$ be integers such that $f(t - 1) = \sum_{j=0}^{p-1} \gamma_j t^j$.

It is calculated that

$$\begin{aligned}\gamma_0 &= (-1)^{(p+1)/2} 4p(s_0 - \varepsilon_p), \\ \gamma_1 &= (-1)^{(p-1)/2} 2p(p-1)(s_0 - \varepsilon_p), \\ \gamma_2 &= (-1)^{(p-1)/2} 4p\left(s_1 - \frac{p^2 - 4p + 5}{8}s_0 - \frac{(2^{p-1} - 1)(p-1)(p-3)}{32p}\right).\end{aligned}$$

By the condition $s_0 \equiv \varepsilon_p \pmod{p}$ we have that $v_p(\gamma_0) = v_p(\gamma_1) \geq 2$ and $\gamma_2 \equiv (-1)^{(p-1)/2} 4p(s_1 - \varepsilon_p/4) \pmod{p^2}$. If $s_1 \not\equiv \varepsilon_p/4 \pmod{p}$, then $v_p(\gamma_2) = 1$. Since $f(t-1) \equiv t^{p-1} \pmod{p}$, it holds that $v_p(\gamma_j) \geq 1$ for $3 \leq j \leq p-2$. Using the Newton polygon method (cf. [2]), one can show that the ramification index of every prime ideal of F_f above p is equal to $p-3$. Hence we have $\zeta_p \notin H(F_f)$ for Lemma 2.4. \square

Lemma 3.7. *Let us put $f(t) = (3s+1)t^2 - (6s+1)t + 3s+1$ for an integer $s \in \mathbb{Z}$. Then $f(t)$ is a polynomial in $A_2 \cap M_3(2, 1)$. The condition $\zeta_3 \in H(F_f)$ holds if and only if $v_3(s+1/4)$ is even and s is greater than $-1/4$.*

Proof. The discriminant of the polynomial $f(t)$ is equal to $-12(s+1/4)$, that is, $F_f = \mathbb{Q}(\sqrt{-3(s+1/4)})$. The ramification index of 3 in F_f/\mathbb{Q} is divisible by 2 if and only if $v_3(s+1/4)$ is even. The field F_f is totally imaginary if and only if $s > -1/4$. Hence the assertion holds. \square

Theorem 3.6 and Lemma 3.7 imply

Corollary 3.8 (Theorem 1.6). *For every odd prime number $p > 3$ (resp. $p = 3$) there exist infinitely many knots K whose Alexander polynomials $\Delta_K(t)$ are monic (resp. non-monic) and satisfy the conditions $\Delta_K(t) \in A_{p-1} \cap M_p(2, 1)$ but fail the condition (U).*

Proof. When $p \geq 5$, for the polynomial $h(X)$ in Theorem 3.6 one may take a polynomial $s_1 t + s_0$ where s_1 and $s_0 \in \mathbb{Z}$ are integers such that $s_0 \equiv \varepsilon_p \pmod{p}$ and $s_1 \not\equiv \varepsilon_p/4 \pmod{p}$. In fact, $\varepsilon_5 = -3/4 \not\equiv 0 \pmod{5}$, which means that we can have $s_{(p-3)/2} = s_1 = 0$ for the case $p = 5$. For $p = 3$ let us set $s = -(3u^2 + 3u + 1)$ in Lemma 3.7 where $u \in \mathbb{Z}$ is an integer. Then one has $f(t) = -((3u+1)t - (3u+2))((3u+2)t - (3u+1))$, which implies that $F_f = \mathbb{Q}$. \square

EXAMPLE 3.9 (Example 2.3). When $p = 5$ and $h(X) = -2$, one has $f(t) = t^4 - 6t^3 + 11t^2 - 6t + 1$. Since $\varepsilon_5 = -3/4 \equiv 3 \pmod{5}$, the polynomial $h(X) = s_1X + s_0 = -2$ satisfies that $s_0 \equiv \varepsilon_5 \pmod{5}$ and $s_1 \not\equiv \varepsilon_5/4 \equiv 2 \pmod{5}$. Theorem 3.6 shows that $\zeta_5 \notin H(F_{\Delta_K})$.

EXAMPLE 3.10. For an integer $s \in \mathbb{Z}$ let us put $f_s(t) = t^4 + (5s + 4)t^3 - (10s + 9)t^2 + (5s + 4)t + 1$, which is obtained as $p = 5$ and $h(X) = s$ in Theorems 3.3 and 3.6. It is calculated that the discriminant $\delta(s)$ of the polynomial $f_s(t)$ is equal to $-5^5(s + 2)^2(5s + 6)^2(4s + 3)$. Here $f_{-3}(t)$ (resp. $f_{-1}(t)$) have four real (resp. four non-real) zeros. It holds that $f_{-2}(t) = (t - (3 + \sqrt{5})/2)^2(t - (3 - \sqrt{5})/2)^2$. The polynomial $f_0(t)$ has two real and two non-real zeros. Thus F_{f_s} is totally real (resp. totally imaginary) provided $s = -3, -2$ (resp. $s = -1, 0$). Note that there exist no zeros of $\delta(s)$ in the areas $s \leq -3$ or $s \geq 0$. Hence F_{f_s} is totally real (resp. totally imaginary) when $s \leq -2$ (resp. $s \geq -1$). Theorems 3.3 and 3.6 with the argument above imply that Heilbronn's conjecture is true for $f_s(t)$ if and only if $s \geq -1$ and $s \not\equiv 3 \pmod{5}$.

References

- [1] G. Burde, *Alexanderpolynome Neuwirthscher Knoten*, Topology **5** (1966), 321–330.
- [2] H. Cohen, A course in computational algebraic number theory, Grad. Texts in Math. **138**, 1993.
- [3] R.H. Crowell, R.H. Fox, Introduction to knot theory, Grad. Texts in Math. **57**, 1977.
- [4] K. Murasugi, *On periodic knots*, Comment. Math. Helv. **46** (1971), 162–174.
- [5] K. Murasugi, *Classical knot invariants and elementary number theory*, preprint.
- [6] H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann. **110** (1935), no. 1, 571–592.
- [7] H.F. Trotter, *Periodic automorphisms of groups and knots*, Duke Math. J. **28** (1961), 553–557.

(Toru KOMATSU) FACULTY OF MATHEMATICS, KYUSHU UNIVERSITY, 6-10-1 HAKOZAKI HIGASHIKU, FUKUOKA, 812-8581 JAPAN

E-mail address: trkomatu@math.kyushu-u.ac.jp

List of MHF Preprint Series, Kyushu University

21st Century COE Program

Development of Dynamic Mathematics with High Functionality

- MHF2003-1 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Yoshitaka WATANABE
A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems
- MHF2003-2 Masahisa TABATA & Daisuke TAGAMI
Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients
- MHF2003-3 Tomohiro ANDO, Sadanori KONISHI & Seiya IMOTO
Adaptive learning machines for nonlinear classification and Bayesian information criteria
- MHF2003-4 Kazuhiro YOKOYAMA
On systems of algebraic equations with parametric exponents
- MHF2003-5 Masao ISHIKAWA & Masato WAKAYAMA
Applications of Minor Summation Formulas III, Plücker relations, Lattice paths and Pfaffian identities
- MHF2003-6 Atsushi SUZUKI & Masahisa TABATA
Finite element matrices in congruent subdomains and their effective use for large-scale computations
- MHF2003-7 Setsuo TANIGUCHI
Stochastic oscillatory integrals - asymptotic and exact expressions for quadratic phase functions -
- MHF2003-8 Shoki MIYAMOTO & Atsushi YOSHIKAWA
Computable sequences in the Sobolev spaces
- MHF2003-9 Toru FUJII & Takashi YANAGAWA
Wavelet based estimate for non-linear and non-stationary auto-regressive model
- MHF2003-10 Atsushi YOSHIKAWA
Maple and wave-front tracking — an experiment
- MHF2003-11 Masanobu KANEKO
On the local factor of the zeta function of quadratic orders
- MHF2003-12 Hidefumi KAWASAKI
Conjugate-set game for a nonlinear programming problem

- MHF2004-1 Koji YONEMOTO & Takashi YANAGAWA
Estimating the Lyapunov exponent from chaotic time series with dynamic noise
- MHF2004-2 Rui YAMAGUCHI, Eiko TSUCHIYA & Tomoyuki HIGUCHI
State space modeling approach to decompose daily sales of a restaurant into time-dependent multi-factors
- MHF2004-3 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Cubic pencils and Painlevé Hamiltonians
- MHF2004-4 Atsushi KAWAGUCHI, Koji YONEMOTO & Takashi YANAGAWA
Estimating the correlation dimension from a chaotic system with dynamic noise
- MHF2004-5 Atsushi KAWAGUCHI, Kentarou KITAMURA, Koji YONEMOTO, Takashi YANAGAWA & Kiyofumi YUMOTO
Detection of auroral breakups using the correlation dimension
- MHF2004-6 Ryo IKOTA, Masayasu MIMURA & Tatsuyuki NAKAKI
A methodology for numerical simulations to a singular limit
- MHF2004-7 Ryo IKOTA & Eiji YANAGIDA
Stability of stationary interfaces of binary-tree type
- MHF2004-8 Yuko ARAKI, Sadanori KONISHI & Seiya IMOTO
Functional discriminant analysis for gene expression data via radial basis expansion
- MHF2004-9 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Hypergeometric solutions to the q -Painlevé equations
- MHF2004-10 Raimundas VIDŪNAS
Expressions for values of the gamma function
- MHF2004-11 Raimundas VIDŪNAS
Transformations of Gauss hypergeometric functions
- MHF2004-12 Koji NAKAGAWA & Masakazu SUZUKI
Mathematical knowledge browser
- MHF2004-13 Ken-ichi MARUNO, Wen-Xiu MA & Masayuki OIKAWA
Generalized Casorati determinant and Positon-Negaton-Type solutions of the Toda lattice equation
- MHF2004-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCO
Generating function associated with the determinant formula for the solutions of the Painlevé II equation

- MHF2004-15 Kouji HASHIMOTO, Ryohei ABE, Mitsuhiro T. NAKAO & Yoshitaka WATANABE
Numerical verification methods of solutions for nonlinear singularly perturbed problem
- MHF2004-16 Ken-ichi MARUNO & Gino BIONDINI
Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete versions
- MHF2004-17 Ryuei NISHII & Shinto EGUCHI
Supervised image classification in Markov random field models with Jeffreys divergence
- MHF2004-18 Kouji HASHIMOTO, Kenta KOBAYASHI & Mitsuhiro T. NAKAO
Numerical verification methods of solutions for the free boundary problem
- MHF2004-19 Hiroki MASUDA
Ergodicity and exponential β -mixing bounds for a strong solution of Lévy-driven stochastic differential equations
- MHF2004-20 Setsuo TANIGUCHI
The Brownian sheet and the reflectionless potentials
- MHF2004-21 Ryuei NISHII & Shinto EGUCHI
Supervised image classification based on AdaBoost with contextual weak classifiers
- MHF2004-22 Hideki KOSAKI
On intersections of domains of unbounded positive operators
- MHF2004-23 Masahisa TABATA & Shoichi FUJIMA
Robustness of a characteristic finite element scheme of second order in time increment
- MHF2004-24 Ken-ichi MARUNO, Adrian ANKIEWICZ & Nail AKHMEDIEV
Dissipative solitons of the discrete complex cubic-quintic Ginzburg-Landau equation
- MHF2004-25 Raimundas VIDŪNAS
Degenerate Gauss hypergeometric functions
- MHF2004-26 Ryo IKOTA
The boundedness of propagation speeds of disturbances for reaction-diffusion systems
- MHF2004-27 Ryusuke KON
Convex dominates concave: an exclusion principle in discrete-time Kolmogorov systems

- MHF2004-28 Ryusuke KON
Multiple attractors in host-parasitoid interactions: coexistence and extinction
- MHF2004-29 Kentaro IHARA, Masanobu KANEKO & Don ZAGIER
Derivation and double shuffle relations for multiple zeta values
- MHF2004-30 Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Generalized partitioned quantum cellular automata and quantization of classical CA
- MHF2005-1 Hideki KOSAKI
Matrix trace inequalities related to uncertainty principle
- MHF2005-2 Masahisa TABATA
Discrepancy between theory and real computation on the stability of some finite element schemes
- MHF2005-3 Yuko ARAKI & Sadanori KONISHI
Functional regression modeling via regularized basis expansions and model selection
- MHF2005-4 Yuko ARAKI & Sadanori KONISHI
Functional discriminant analysis via regularized basis expansions
- MHF2005-5 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Point configurations, Cremona transformations and the elliptic difference Painlevé equations
- MHF2005-6 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Construction of hypergeometric solutions to the q -Painlevé equations
- MHF2005-7 Hiroki MASUDA
Simple estimators for non-linear Markovian trend from sampled data:
I. ergodic cases
- MHF2005-8 Hiroki MASUDA & Nakahiro YOSHIDA
Edgeworth expansion for a class of Ornstein-Uhlenbeck-based models
- MHF2005-9 Masayuki UCHIDA
Approximate martingale estimating functions under small perturbations of dynamical systems
- MHF2005-10 Ryo MATSUZAKI & Masayuki UCHIDA
One-step estimators for diffusion processes with small dispersion parameters from discrete observations
- MHF2005-11 Junichi MATSUKUBO, Ryo MATSUZAKI & Masayuki UCHIDA
Estimation for a discretely observed small diffusion process with a linear drift

- MHF2005-12 Masayuki UCHIDA & Nakahiro YOSHIDA
AIC for ergodic diffusion processes from discrete observations
- MHF2005-13 Hiromichi GOTO & Kenji KAJIWARA
Generating function related to the Okamoto polynomials for the Painlevé IV equation
- MHF2005-14 Masato KIMURA & Shin-ichi NAGATA
Precise asymptotic behaviour of the first eigenvalue of Sturm-Liouville problems with large drift
- MHF2005-15 Daisuke TAGAMI & Masahisa TABATA
Numerical computations of a melting glass convection in the furnace
- MHF2005-16 Raimundas VIDŪNAS
Normalized Leonard pairs and Askey-Wilson relations
- MHF2005-17 Raimundas VIDŪNAS
Askey-Wilson relations and Leonard pairs
- MHF2005-18 Kenji KAJIWARA & Atsushi MUKAIHIRA
Soliton solutions for the non-autonomous discrete-time Toda lattice equation
- MHF2005-19 Yuu HARIYA
Construction of Gibbs measures for 1-dimensional continuum fields
- MHF2005-20 Yuu HARIYA
Integration by parts formulae for the Wiener measure restricted to subsets in \mathbb{R}^d
- MHF2005-21 Yuu HARIYA
A time-change approach to Kotani's extension of Yor's formula
- MHF2005-22 Tadahisa FUNAKI, Yuu HARIYA & Mark YOR
Wiener integrals for centered powers of Bessel processes, I
- MHF2005-23 Masahisa TABATA & Satoshi KAIZU
Finite element schemes for two-fluids flow problems
- MHF2005-24 Ken-ichi MARUNO & Yasuhiro OHTA
Determinant form of dark soliton solutions of the discrete nonlinear Schrödinger equation
- MHF2005-25 Alexander V. KITAEV & Raimundas VIDŪNAS
Quadratic transformations of the sixth Painlevé equation
- MHF2005-26 Toru FUJII & Sadanori KONISHI
Nonlinear regression modeling via regularized wavelets and smoothing parameter selection

- MHF2005-27 Shuichi INOKUCHI, Kazumasa HONDA, Hyen Yeal LEE, Tatsuro SATO,
Yoshihiro MIZOGUCHI & Yasuo KAWAHARA
On reversible cellular automata with finite cell array
- MHF2005-28 Toru KOMATSU
Cyclic cubic field with explicit Artin symbols
- MHF2005-29 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Kaori NAGATOU
A computational approach to constructive a priori and a posteriori error
estimates for finite element approximations of bi-harmonic problems
- MHF2005-30 Kaori NAGATOU, Kouji HASHIMOTO & Mitsuhiro T. NAKAO
Numerical verification of stationary solutions for Navier-Stokes problems
- MHF2005-31 Hidefumi KAWASAKI
A duality theorem for a three-phase partition problem
- MHF2005-32 Hidefumi KAWASAKI
A duality theorem based on triangles separating three convex sets
- MHF2005-33 Takeaki FUCHIKAMI & Hidefumi KAWASAKI
An explicit formula of the Shapley value for a cooperative game induced from
the conjugate point
- MHF2005-34 Hideki MURAKAWA
A regularization of a reaction-diffusion system approximation to the two-phase
Stefan problem
- MHF2006-1 Masahisa TABATA
Numerical simulation of Rayleigh-Taylor problems by an energy-stable finite
element scheme
- MHF2006-2 Ken-ichi MARUNO & G R W QUISPEL
Construction of integrals of higher-order mappings
- MHF2006-3 Setsuo TANIGUCHI
On the Jacobi field approach to stochastic oscillatory integrals with quadratic
phase function
- MHF2006-4 Kouji HASHIMOTO, Kaori NAGATOU & Mitsuhiro T. NAKAO
A computational approach to constructive a priori error estimate for finite
element approximations of bi-harmonic problems in nonconvex polygonal
domains
- MHF2006-5 Hidefumi KAWASAKI
A duality theory based on triangular cylinders separating three convex sets in
 R^n
- MHF2006-6 Raimundas VIDŪNAS
Uniform convergence of hypergeometric series

- MHF2006-7 Yuji KODAMA & Ken-ichi MARUNO
N-Soliton solutions to the DKP equation and Weyl group actions
- MHF2006-8 Toru KOMATSU
Potentially generic polynomial
- MHF2006-9 Toru KOMATSU
Generic sextic polynomial related to the subfield problem of a cubic polynomial
- MHF2006-10 Shu TEZUKA & Anargyros PAPAGEORGIOU
Exact cubature for a class of functions of maximum effective dimension
- MHF2006-11 Shu TEZUKA
On high-discrepancy sequences
- MHF2006-12 Raimundas VIDŪNAS
Detecting persistent regimes in the North Atlantic Oscillation time series
- MHF2006-13 Toru KOMATSU
Tamely Eisenstein field with prime power discriminant
- MHF2006-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCO
Generating function associated with the Hankel determinant formula for the solutions of the Painlevé IV equation
- MHF2006-15 Raimundas VIDŪNAS
Darboux evaluations of algebraic Gauss hypergeometric functions
- MHF2006-16 Masato KIMURA & Isao WAKANO
New mathematical approach to the energy release rate in crack extension
- MHF2006-17 Toru KOMATSU
Arithmetic of the splitting field of Alexander polynomial