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§ 1. Introduction

In this paper we study the arithmetic of the minimal splitting field of the Alexander polynomial of a knot and present two kinds of infinite families of knots, one is a family of knots which satisfying Heilbronn conjecture (Conjecture 1.3) and the other is a family of counterexamples to the conjecture.

For a knot K in $\mathbb{R}^3$ let us denote by $\Delta_K(t)$ the Alexander polynomial of K. In general, $\Delta_K(t)$ is defined up to $\pm t^k$. In this paper we assume that $\Delta_K(t)$ is normalized so that $\Delta_K(t)$ is a polynomial in $\mathbb{Z}[t]$ whose order at $t = 0$ is equal to 0. It is a difficult problem to determine the period of a knot K in terms of the Alexander polynomial $\Delta_K(t)$. However, some necessary conditions for a knot to have some period are known. For a positive integer $m \in \mathbb{Z}$, $m \geq 1$ let ζ_m be a primitive m-the root of unity in $\overline{\mathbb{Q}}$. Let F_f be the minimal splitting field of a polynomial $f \in \mathbb{Q}[t]$ over $\mathbb{Q}$.

Proposition 1.1 (Trotter [**7**]). *Let K be a fibered knot with* $\mathrm{disc}_t \Delta_K(t) \neq 0$. *If K has period m, then $\zeta_m \in F_{\Delta_K}$.*

Let p be a prime number. For a positive integer $\lambda \in \mathbb{Z}$, $\mathbb{Z} \geq 1$ and a polynomial $\mu \in \mathbb{Z}[t]$ let us denote by $M_p(\lambda, \mu)$ the set consisting of polynomials $f \in \mathbb{Z}[t]$ such that $f \equiv \pm t^k (1 + t + \cdots + t^{\lambda-1})^{p-1} \mu^p \pmod{p}$ for some integer $k \in \mathbb{Z}$.

Proposition 1.2 (Murasugi [**4**]). *If K is a knot with prime period p, then $\Delta_K(t) \in M_p(\lambda, \mu)$ for a positive integer $\lambda \in \mathbb{Z}, \lambda \geq 1$ and a polynomial $\mu \in \mathbb{Z}[t]$.*

Let $H(F)$ be the Hilbert class field of a finite number field F.

Conjecture 1.3 (Heilbronn (cf. [**5**])). *For a (fibered) knot K, if $\Delta_K(t) \in M_p(\lambda, 1)$ for a positive integer $\lambda \geq 1$, then $\zeta_p \in H(F_{\Delta_K})$.*

We have already obtained some counterexamples to the Heilbronn's conjecture (e.g., Example 2.3). Morishita gave a question which revises the Heilbronn's conjecture. For a polynomial $f \in \mathbb{Q}[t]$ we say that f satisfies the condition (U) if $\zeta_p \in H(F_f)$.

Question 1.4 (Morishita (cf. [**5**]))**.** *What is a condition so that the Alexander polynomial $\Delta_K(t)$ with degree $p-1$ of a knot K satisfies the condition* (U) *?*

In this paper we show the following theorem by constructing the explicit Alexander polynomials of knots.

Theorem 1.5 (Corollary 3.4)**.** *For every odd prime number $p > 3$ (resp. $p = 3$) there exist infinitely many knots K whose Alexander polynomials $\Delta_K(t)$ are monic (resp. non-monic) of degree $p-1$ and satisfy the two conditions $\Delta_K(t) \in M_p(2,1)$ and* (U).

Theorem 1.6 (Corollary 3.8)**.** *For every odd prime number $p > 3$ (resp. $p = 3$) there exist infinitely many knots K whose Alexander polynomials $\Delta_K(t)$ are monic (resp. non-monic) of degree $p-1$ and satisfy the conditions $\Delta_K(t) \in M_p(2,1)$ but fail the condition* (U).

Remark 1.7. The Alexander polynomial of a fibered knot is monic. For the Alexander polynomial $\Delta_K(t)$ of a knot K, if $\Delta_K(t)$ is monic, then there exists a fibered knot K' so that $\Delta_{K'}(t) = \Delta_K(t)$ (Burde [**1**]).

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§ 2. Known results

For a positive integer n let A_n be the set of the Alexander polynomials $\Delta_K(t)$ of knots K with degree n. We define three conditions (A.1) to (A.3) for a polynomial $f(t) \in \mathbb{Z}[t]$ by

(A.1) the degree n of $f(t)$ is even,

(A.2) $f(t) = t^n f(t^{-1})$,

(A.3) $f(1) = \pm 1$.

Proposition 2.1 (Seifert [**6**]). *For a polynomial $f(t) \in \mathbb{Z}[t]$ with degree n, it holds that $f(t) \in A_n$ if and only if $f(t)$ satisfies all of the three conditions* (A.1) *to* (A.3).

Let p be a prime number. For a positive integer $\lambda \in \mathbb{Z}$, $\mathbb{Z} \geq 1$ and a polynomial $\mu \in \mathbb{Z}[t]$ let $M_p(\lambda, \mu)$ be the set as in Introduction, that is, the set consisting of polynomials $f \in \mathbb{Z}[t]$ such that $f \equiv \pm t^k(1 + t + \cdots + t^{\lambda-1})^{p-1}\mu^p \pmod{p}$ for some integer $k \in \mathbb{Z}$. If $f(t) \in A_n \cap M_p(\lambda, 1)$, then $t^k(1 + t + \cdots + t^{\lambda-1})^{p-1} \equiv f(t) = t^n f(t^{-1}) \equiv t^{n-k}(1 + t^{-1} + \cdots + t^{-(\lambda-1)})^{p-1} \pmod{p}$. By considering the degrees, one has that $k + (\lambda - 1)(p - 1) = n - k$, which shows $n = (\lambda - 1)(p - 1) + 2k$. Note that $k \geq 0$ for $f(t) \in \mathbb{Z}[t]$. Let us calculate the Hilbert class field $H(F_{\Delta_K})$ of the Alexander polynomials $\Delta_K(t)$ of some knots K and determine whether or not the Heilbronn's conjecture holds.

Example 2.2 (cf. [**5**]). For $\Delta_K(t) = 4 - 7t + 4t^2 \in A_2 \cap M_3(2, 1)$, we have $F_{\Delta_K} = \mathbb{Q}(\sqrt{-15})$. One can see that $H(F_{\Delta_K}) = F_{\Delta_K}(\zeta_3) \ni \zeta_3$. Thus Heilbronn's conjecture is true for the case $\Delta_K(t) = 4 - 7t + 4t^2$.

Example 2.3 (cf. [**5**]). When $\Delta_K(t) = 1 - 6t + 11t^2 - 6t^3 + t^4 \in A_4 \cap M_5(2, 1)$, we have $F_{\Delta_K} = \mathbb{Q}(\sqrt{5})$ since $\Delta_K(t) = (1 - 3t + t^2)^2$. Note that $H(\mathbb{Q}(\sqrt{5})) = \mathbb{Q}(\sqrt{5})$. This means that $\zeta_5 \notin H(F_{\Delta_K})$, that is, Heilbronn's conjecture is not true for the case $\Delta_K(t) = 1 - 6t + 11t^2 - 6t^3 + t^4$.

We define four conditions (U), (U_1), (U_2) and (U_3) for a finite Galois extension F of $\mathbb{Q}$ such that

(U) $\zeta_p \in H(F)$,

(U_1) $F(\zeta_p)/F$ is unramified,

(U_2) $F(\zeta_p)/F$ is unramified at all the prime ideals $\mathfrak{p}$ of F above p,

(U_3) for every prime ideal $\mathfrak{p}$ of F above p, the ramification index of $\mathfrak{p}$ in the extension $F/\mathbb{Q}$ is a multiple of $p - 1$.

The ramification theory of algebraic number theory implies

Lemma 2.4. *Assume that p is odd. If F is totally real, then $F(\zeta_p)/F$ is ramified at all infinite places of F, in particular, $\zeta_p \notin H(F)$. When F is totally imaginary, the four conditions* (U), (U$_1$), (U$_2$) *and* (U$_3$) *are equivalent to each other.*

Morishita and Taguchi showed a sufficient condition to hold (U). For a polynomial $g(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{Z}[t]$ we call $g(t)$ a p-Eisenstein polynomial if the coefficients a_i satisfy that $p \nmid a_n$, $p \mid a_i$ for $0 \le i \le n-1$ and $p^2 \nmid a_0$. In this paper we do not assume that a p-Eisenstein polynomial is monic.

Proposition 2.5 (Morishita–Taguchi (cf. [**5**])). *If $g(t) \in \mathbb{Z}[t]$ is a p-Eisenstein polynomial of degree $p-1$, then $\zeta_p \in H(F_g)$.*

Example 2.6 (cf. [**5**]). If $g(t) = t^4 + 5t^3 - 40t^2 + 70t - 35$, then $g(t+1) = t^4 + 9t^3 - 19t^2 + 9t + 1$, which is the Alexander polynomial of some knot. One can see that F_g is a $\mathcal{D}_4$-extension of $\mathbb{Q}$ where $\mathcal{D}_4$ is the dihedral group of degree 4 with order 8. Note that F_g has no subfields which are cyclic quartic fields since $\mathcal{D}_4$ has no normal subgroups N such that $\mathcal{D}_4/N$ are isomorphic to the cyclic group $\mathcal{C}_4$ of order 4. Thus we have $\zeta_5 \notin F_{\Delta_K}$. Proposition 1.1 implies that K does not have period 5. On the other hand, Proposition 2.5 yields $\zeta_5 \in H(F_{\Delta_K})$.

For the use of Proposition 2.5 one needs a criterion whether or not a p-Eisenstein polynomial $g(t)$ can become the Alexander polynomial $\Delta_K(t)$ of a knot K by some translation $t \mapsto at+b$ for integers a and $b \in \mathbb{Z}$ with $a \neq 0$, that is, $g(at+b) = \Delta_K(t)$. However, such a criterion is not easy in general. By using a characterization of the Alexander polynomials we present a family of the Alexander polynomials which become p-Eisenstein polynomials in the next section.

§ 3. Construction of the Alexander polynomials

For a positive and even integer $n = 2r$ let B_n be the set of polynomials $f(t)$ in $\mathbb{Z}[t]$ such that $f(t) = t^n f(t^{-1})$. Note that $A_n = \{f(t) \in B_n | \deg_t f(t) = n, f(1) = \pm 1\}$.

Lemma 3.1 (cf. Crowell-Fox [**3**]). *The set B_n is a $\mathbb{Z}$-module with free basis $\{(t^2-2t+1)^i t^{r-i} | 0 \le i \le r\}$, that is, $B_n = \{\sum_{0 \le i \le r} b_i (t^2 - 2t + 1)^i t^{r-i} | b_i \in \mathbb{Z}\}$.*

Let p be a prime number and $\lambda \geq 1$ a positive integer such that $n \geq (\lambda-1)(p-1)$. We define a polynomial $\alpha_\lambda(t) = \alpha_{n,p,\lambda}(t)$ by

$$\alpha_\lambda(t) = t^k(1+t+\cdots+t^{\lambda-1})^{p-1} - (\lambda^{p-1}-1)t^r$$

where $k = r - (\lambda-1)(p-1)/2$. For integers $c_i \in \mathbb{Z}$ with $1 \leq i \leq r$ let us denote $\alpha_\lambda(t) + p\sum_{1\leq i\leq r} c_i(t^2-2t+1)^i t^{r-i}$ by $f_\lambda(\mathfrak{c}, t)$ where $\mathfrak{c} = (c_1, c_2, \ldots, c_r)$. We put $S_r(\mathbb{Z}) = \mathbb{Z}^r$ and $S_r^\times(\mathbb{Z}) = \{\mathfrak{c} = (c_1, c_2, \ldots, c_r) \in S_r(\mathbb{Z}) | c_r \neq 0\}$.

Lemma 3.2. *The intersection set $A_n \cap M_p(\lambda, 1)$ consists of the polynomials $\pm f_\lambda(\mathfrak{c}, t)$, $\mathfrak{c} = (c_1, c_2, \ldots, c_r)$ where $\mathfrak{c} \in S_r(Z)$ if $n = (\lambda-1)(p-1)$ and $\mathfrak{c} \in S_r^\times(Z)$ otherwise.*

Proof. Let $\mathfrak{c}$ be an element in $S_r(\mathbb{Z})$ if $n = (\lambda-1)(p-1)$ and in $S_r^\times(\mathbb{Z})$ otherwise. It follows from the definition that $\deg_t f_\lambda(\mathfrak{c}, t) \leq n$. If $n = (\lambda-1)(p-1)$, then the coefficient of t^{p-1} in $f_\lambda(\mathfrak{c}, t)$ is equal to $1 + pc_r$. Since $c_r \in \mathbb{Z}$, one has that $1 + pc_r \neq 0$ and $\deg_t f_\lambda(\mathfrak{c}, t) = n$. When $n > (\lambda-1)(p-1)$, the coefficient of t^{p-1} in $f_\lambda(\mathfrak{c}, t)$ is equal to pc_r. Thus it holds that $\deg_t f_\lambda(\mathfrak{c}, t) = n$ for $c_r \neq 0$. It is easy to see that $\pm f_\lambda(\mathfrak{c}, t)$ satisfies the conditions (A.2) and (A.3). Hence the polynomial $\pm f_\lambda(\mathfrak{c}, t)$ belongs to $A_n \cap M_p(\lambda, 1)$. Let $f(t)$ be a polynomial in $A_n \cap M_p(\lambda, 1)$. For a non-negative integer $k \in \mathbb{Z}$ with $k = r - (\lambda-1)(p-1)/2$, one has $\pm t^k(1+t+\cdots+t^{\lambda-1})^{p-1} \in B_n$. Since B_n is a $\mathbb{Z}$-molude, we have $f(t) \mp t^k(1+t+\cdots+t^{\lambda-1})^{p-1} \in B_n$ and $(f(t) \mp t^k(1+t+\cdots+t^{\lambda-1})^{p-1})/p \in B_n$ if $f(1) = \pm 1$, respectively. Lemma 3.1 implies that $f(t) = \pm t^k(1+t+\cdots+t^{\lambda-1})^{p-1} + p\sum_{0\leq i\leq r} c_i(t^2-2t+1)^i t^{r-i}$ where $c_i \in \mathbb{Z}$ for $0 \leq i \leq r$. It follows from $f(1) = \pm 1$ that $\pm\lambda^{p-1} + pc_0 = \pm 1$ and $c_0 = \mp(\lambda^{p-1}-1)/p \in \mathbb{Z}$. Hence $f(t)$ is of the form $\pm f_\lambda(\mathfrak{c}, t)$. □

For considering Question 1.4 we have $n = p - 1 \geq 2$, which means that $\lambda = 1$ or 2. Here one has that $\alpha_1(t) = t^{(p-1)/2}$ and $\alpha_2(t) = (1+t)^{p-1} - (2^{p-1}-1)t^{(p-1)/2}$. Let us define a number $\varepsilon_p \in \mathbb{Q}$ by $\varepsilon_p = -(2^{p-1}-1)/(4p)$. Note that $v_p(\varepsilon_p) \geq 0$.

Theorem 3.3. *Let $h(X) = \sum_{j=0}^{(p-3)/2} s_j X^j \in \mathbb{Z}[X]$ be a polynomial of degree less than $(p-1)/2$ such that $s_0 \not\equiv \varepsilon_p \pmod{p}$ and $s_0 > \varepsilon_p$. Then*

$$f(t) = \alpha_2(t) + pt^{(p-1)/2}(t + t^{-1} - 2)h(t + t^{-1} + 2)$$

is a polynomial in $A_{p-1} \cap M_p(2,1)$ *satisfying* $\zeta_p \in H(F_f)$. *The polynomial* $f(t)$ *is monic if and only if* $s_{(p-3)/2} = 0$.

Proof. It is easy to check that $f(t) \in A_{p-1} \cap M_p(2,1)$. In fact, it holds that $f(t) = f_2(\mathfrak{c}, t)$ for an element $\mathfrak{c} = (c_1, c_2, \ldots, c_{(p-1)/2}) \in S_{(p-1)/2}(\mathbb{Z})$ satisfying $\sum_{i=1}^{(p-1)/2} c_i(t + t^{-1} - 2)^{i-1} = \sum_{j=0}^{(p-3)/2} s_j(t + t^{-1} + 2)^j$. One has that $f(t-1) \equiv t^{p-1} \pmod{p}$ and $f(-1) = (-1)^{(p+1)/2}(2^{p-1} - 1 + 4ps_0)$. The condition $s_0 \not\equiv \varepsilon_p \pmod{p}$ is equivalent to $v_p(f(-1)) = 1$. Thus $f(t-1)$ is a p-Eisenstein polynomial of degree $p-1$. Now put $X = t + t^{-1} + 2$. Then we have $f(t)/t^{(p-1)/2} = X^{(p-1)/2} - (2^{p-1} - 1) + p(X-4)h(X)$, which is denoted by $\widetilde{f}(X)$. Here it is seen that $\widetilde{f}(0) = -(2^{p-1} - 1) - 4ps_0$ and $\widetilde{f}(4) = 1$. It follows from $s_0 > \varepsilon_p$ that $\widetilde{f}(0) < 0$. This shows that $\widetilde{f}(X) = 0$ has a real solution x with $0 < x < 4$. For a complex number $z \in \mathbb{C}$ with $z + z^{-1} \in \mathbb{R}$, the condition $0 < z + z^{-1} + 2 < 4$ holds if and only if z is not real. This means that $f(t)$ has a non-real zero in $\mathbb{C}$. Thus F_f is not totally real but totally imaginary. Hence Lemma 2.4 verifies that $\zeta_p \in H(F_f)$. $\square$

Corollary 3.4 (Theorem 1.5)**.** *For every odd prime number* $p > 3$ (*resp.* $p = 3$) *there exist infinitely many knots* K *whose Alexander polynomials* $\Delta_K(t)$ *are monic* (*resp. non-monic*) *and satisfy the two conditions* $\Delta_K(t) \in A_{p-1} \cap M_p(2,1)$ *and* (U).

Proof. For example, one may take an arbitrary integer $s \in \mathbb{Z}$ satisfying $s \not\equiv \varepsilon_p \pmod{p}$ and $s > \varepsilon_p$ for the polynomial $h(X)$ in Theorem 3.3. $\square$

Example 3.5 (Examples 2.2 and 2.6). For the case that $p = 3$ and $h(X) = 1$ in Theorem 3.3, one has $f(X) = 4t^2 - 7t + 4$. Since $\varepsilon_3 = -1/4 \equiv 2 \pmod{3}$, we have $\zeta_3 \in H(F_f)$. If $p = 5$ and $h(X) = 1$, then $f(t) = t^4 + 9t^3 - 19t^2 + 9t + 1$ and $\zeta_5 \in H(F_f)$ for $\varepsilon_5 = -3/4 \equiv 3 \pmod{5}$.

Theorem 3.6. *Assume* $p \geq 5$. *Let* $h(X) = \sum_{i=0}^{(p-3)/2} s_i X^i \in \mathbb{Z}[X]$ *be a polynomial of degree less than* $(p-1)/2$. *If* $s_0 \equiv \varepsilon_p \pmod{p}$ *and* $s_1 \not\equiv \varepsilon_p/4 \pmod{p}$, *then* $f(t) = \alpha_2(t) + pt^{(p-1)/2}(t + t^{-1} - 2)h(t + t^{-1} + 2)$ *is a polynomial in* $A_{p-1} \cap M_p(2,1)$ *such that* $\zeta_p \notin H(F_f)$. *The polynomial* $f(t)$ *is monic if and only if* $s_{(p-3)/2} = 0$.

Proof. In the same way as in the proof of Theorem 3.3 one sees $f(t) \in A_{p-1} \cap M_p(2,1)$. For $0 \leq j \leq p-1$ let $\gamma_j \in \mathbb{Z}$ be integers such that $f(t-1) = \sum_{j=0}^{p-1} \gamma_j t^j$.

It is calculated that

$$
\begin{aligned}
\gamma_0 &= (-1)^{(p+1)/2} 4p(s_0-\varepsilon_p),\\
\gamma_1 &= (-1)^{(p-1)/2} 2p(p-1)(s_0-\varepsilon_p),\\
\gamma_2 &= (-1)^{(p-1)/2} 4p\Big(s_1-\frac{p^2-4p+5}{8}s_0-\frac{(2^{p-1}-1)(p-1)(p-3)}{32p}\Big).
\end{aligned}
$$

By the condition $s_0 \equiv \varepsilon_p \pmod p$ we have that $v_p(\gamma_0)=v_p(\gamma_1)\geq 2$ and $\gamma_2 \equiv (-1)^{(p-1)/2}4p(s_1-\varepsilon_p/4) \pmod{p^2}$. If $s_1 \not\equiv \varepsilon_p/4 \pmod p$, then $v_p(\gamma_2)=1$. Since $f(t-1)\equiv t^{p-1} \pmod p$, it holds that $v_p(\gamma_j)\geq 1$ for $3\leq j\leq p-2$. Using the Newton polygon method (cf. [**2**]), one can show that the ramification index of every prime ideal of F_f above p is equal to $p-3$. Hence we have $\zeta_p \notin H(F_f)$ for Lemma 2.4. □

Lemma 3.7. *Let us put $f(t)=(3s+1)t^2-(6s+1)t+3s+1$ for an integer $s\in\mathbb{Z}$. Then $f(t)$ is a polynomial in $A_2\cap M_3(2,1)$. The condition $\zeta_3\in H(F_f)$ holds if and only if $v_3(s+1/4)$ is even and s is greater than $-1/4$.*

Proof. The discriminant of the polynomial $f(t)$ is equal to $-12(s+1/4)$, that is, $F_f=\mathbb{Q}(\sqrt{-3(s+1/4)})$. The ramification index of 3 in $F_f/\mathbb{Q}$ is divisible by 2 if and only if $v_3(s+1/4)$ is even. The field F_f is totally imaginary if and only if $s>-1/4$. Hence the assertion holds. □

Theorem 3.6 and Lemma 3.7 imply

Corollary 3.8 (Theorem 1.6)**.** *For every odd prime number $p>3$ (resp. $p=3$) there exist infinitely many knots K whose Alexander polynomials $\Delta_K(t)$ are monic (resp. non-monic) and satisfy the conditions $\Delta_K(t)\in A_{p-1}\cap M_p(2,1)$ but fail the condition* (U).

Proof. When $p\geq 5$, for the polynomial $h(X)$ in Theorem 3.6 one may take a polynomial s_1t+s_0 where s_1 and $s_0\in\mathbb{Z}$ are integers such that $s_0\equiv\varepsilon_p \pmod p$ and $s_1\not\equiv\varepsilon_p/4 \pmod p$. In fact, $\varepsilon_5=-3/4\not\equiv 0 \pmod 5$, which means that we can have $s_{(p-3)/2}=s_1=0$ for the case $p=5$. For $p=3$ let us set $s=-(3u^2+3u+1)$ in Lemma 3.7 where $u\in\mathbb{Z}$ is an integer. Then one has $f(t)=-((3u+1)t-(3u+2))((3u+2)t-(3u+1))$, which implies that $F_f=\mathbb{Q}$. □

Example 3.9 (Example 2.3). When $p = 5$ and $h(X) = -2$, one has $f(t) = t^4 - 6t^3 + 11t^2 - 6t + 1$. Since $\varepsilon_5 = -3/4 \equiv 3 \pmod 5$, the polynomial $h(X) = s_1X + s_0 = -2$ satisfies that $s_0 \equiv \varepsilon_5 \pmod 5$ and $s_1 \not\equiv \varepsilon_5/4 \equiv 2 \pmod 5$. Theorem 3.6 shows that $\zeta_5 \notin H(F_{\Delta_K})$.

Example 3.10. For an integer $s \in \mathbb{Z}$ let us put $f_s(t) = t^4 + (5s+4)t^3 - (10s+9)t^2 + (5s+4)t + 1$, which is obtained as $p = 5$ and $h(X) = s$ in Theorems 3.3 and 3.6. It is calculated that the discriminant $\delta(s)$ of the polynomial $f_s(t)$ is equal to $-5^5(s+2)^2(5s+6)^2(4s+3)$. Here $f_{-3}(t)$ (resp. $f_{-1}(t)$) have four real (resp. four non-real) zeros. It holds that $f_{-2}(t) = (t - (3+\sqrt{5})/2)^2(t - (3-\sqrt{5})/2)^2$. The polynomial $f_0(t)$ has two real and two non-real zeros. Thus F_{f_s} is totally real (resp. totally imaginary) provided $s = -3, -2$ (resp. $s = -1, 0$). Note that there exist no zeros of $\delta(s)$ in the areas $s \leq -3$ or $s \geq 0$. Hence F_{f_s} is totally real (resp. totally imaginary) when $s \leq -2$ (resp. $s \geq -1$). Theorems 3.3 and 3.6 with the argument above imply that Heilbronn's conjecture is true for $f_s(t)$ if and only if $s \geq -1$ and $s \not\equiv 3 \pmod 5$.

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