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Abstract

This paper presents explicit expressions for algebraic Gauss hypergeometric functions. We consider solutions of hypergeometric equations with the tetrahedral, octahedral and icosahedral monodromy groups. Conceptually, we pull-back such a hypergeometric equation onto its *Darboux curve* so that the pull-backed equation has a cyclic monodromy group. Minimal degree of the pull-back coverings is 4, 6 or 12 (for the three monodromy groups, respectively). In explicit terms, we replace the independent variable by a rational function of degree 4, 6 or 12, and transform hypergeometric functions to radical functions.

1 Introduction

Algebraic Gauss hypergeometric functions were studied by many authors; see for example [Sch72, Fuc75, Bri77, Kle78, Pep81, Bou98, Kat72, BD79, SU93, vdPU98]. This aim of this paper is to present satisfying explicit forms for these functions.

Let $H(e_0, e_1, e_\infty)$ denote the hypergeometric equation

$$\frac{d^2Y(Z)}{dZ^2} + \left(\frac{1-e_0}{Z} + \frac{1-e_1}{Z-1}\right)\frac{dY(Z)}{dZ} + \frac{(1-e_0-e_1)^2 - e_\infty^2}{4\,Z\,(Z-1)}\,Y(Z) = 0. \qquad (1)$$

This is a Fuchsian equation on $\mathbb{P}^1$ with 3 regular singular points $Z = 0,\ 1,\ \infty$. The local exponent differences at these points are equal (up to the sign) to e_0, e_1, e_∞, respectively.

Hypergeometric equation (1) has a basis of algebraic solutions if and only if its monodromy group is finite. The following hypergeometric equations (and their fractional-linear transformations; see Appendix 5.4) have this property and are called *standard hypergeometric equations* with algebraic solutions:

- $H(1, 1/n, 1/n)$, where n is a positive integer. The hypergeometric equation degenerates to a Fuchsian equation with two singular points. Its monodromy group is the cyclic group with n elements.

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- $H(1/2, 1/2, 1/n)$, where n is an integer, $n \geq 2$. The monodromy group of this equation is the dihedral group with $2n$ elements.
- $H(1/2, 1/3, 1/3)$. The monodromy group is the tetrahedral group, isomorphic to A_4.
- $H(1/2, 1/3, 1/4)$. The monodromy group is the octahedral group, isomorphic to S_4.
- $H(1/2, 1/3, 1/5)$. The monodromy group is the icosahedral group, isomorphic to A_5.

The celebrated theorem of Klein [Kle77] states that if a second order linear homogeneous differential equation only has algebraic solutions, then that equation is a *pull-back transformation* of a standard hypergeometric equation from the list above. Explicitly, if the Fuchsian equation has coefficients in $\mathbb{C}(X)$, the pull-back transformation changes the variable Z in (1) to a rational function $\varphi(X)$. In geometric terms, we have a finite covering $\varphi : \mathbb{P}^1_X \to \mathbb{P}^1_Z$ between two projective lines, and we *pull-back* the standard hypergeometric equation from $\mathbb{P}^1_Z$ onto $\mathbb{P}^1_X$.

Particularly, the theorem of Klein implies that if a hypergeometric equation H_1 only has algebraic solutions, then it is a pull-back transformation of a standard hypergeometric equation H_0 with algebraic solutions. The proof of Klein implies that the monodromy group of H_1 is either cyclic, or dihedral, or A_4, S_4 or A_5, and that H_0 can be chosen to be a standard hypergeometric equation with the same monodromy group.

Hypergeometric equations with finite monodromy group were first classified by Schwartz in [Sch72]. Disregarding hypergeometric equations with a cyclic monodromy group, Schwartz gave a list of 15 types of these hypergeometric equations. One type consists of hypergeometric equations with a dihedral monodromy group. The other types are represented by the following hypergeometric equations:

- $H(1/2, 1/3, 1/3)$, $H(1/3, 1/3, 2/3)$. The monodromy group is the tetrahedral group.
- $H(1/2, 1/3, 1/4)$, $H(2/3, 1/4, 1/4)$. The monodromy group is the octahedral group.
- $H(1/2, 1/3, 1/5)$, $H(1/2, 1/3, 2/5)$, $H(1/2, 1/5, 2/5)$, $H(1/3, 1/3, 2/5)$, $H(1/3, 2/3, 1/5)$, $H(2/3, 1/5, 1/5)$, $H(1/3, 2/5, 3/5)$, $H(1/3, 1/5, 3/5)$, $H(1/5, 1/5, 4/5)$, $H(2/5, 2/5, 2/5)$. The monodromy group is the icosahedral group.

We refer to Schwartz type of hypergeometric equations with algebraic solutions by the triple of the parameters e_0, e_1, e_∞ of these representative equations. (Usually, Schwartz type is denoted by a roman numeral from I to XV.) We refer to the listed 14 hypergeometric equations as *main representatives* of the Schwartz types.

Hypergeometric equations of the same Schwartz type are characterized by the property that their hypergeometric solutions are *contiguous* to hypergeometric solutions of the main representative. We refer to Appendix Section 5 for short introductions to pull-back transformations, Fuchsian equations, contiguous relations and other relevant topics.

Algebraic solutions of differential equations can be represented by minimal polynomial equations their satisfy [SU93], or (if the Galois group is solvable) by nested radical expressions. In [BvHW03], [Ber04, Chapter 1], [vHW] an algorithm is developed to represent algebraic solutions of second order linear differential equations using Klein's theorem. The representation form is

$$\theta(x)\, H(\phi(x)), \tag{2}$$

where $\phi(x), \theta(x)$ define Klein's morphism in the notation of (98), and $H(z)$ is a solution of a corresponding standard hypergeometric equation.

We propose to pull-back a hypergeometric equation with a finite monodromy groups to a Fuchsian equation with a cyclic monodromy group. Then some hypergeometric solutions are transformed to rather simple radical functions. We call a pull-back covering $\phi : D \to \mathbb{P}^1$ of this kind a *Darboux covering.* The covering curve D is called a *Darboux curve.* Identification of algebraic Gauss hypergeometric functions with radical functions on an algebraic curve offers satisfying geometric intuition, especially when the Darboux covering has low degree. We use the term *Darboux evaluation* to refer to the aspired identification of hypergeometric functions with radical functions.

Theory of Darboux coverings is developed in Section 3. It turns out that Darboux coverings for hypergeometric equations of the same Schwartz type are identical. Therefore we have finitely many different Darboux coverings and Darboux curves. We compute all Darboux coverings of minimal degree, which turns out to be 4, 6 or 12 for the tetrahedral, octahedral and icosahedral types, respectively. The corresponding Darboux curves have genus 0 or (for some icosahedral types) genus 1. For each Schwartz type, we use Darboux coverings of minimal degree and compute Darboux evaluations for 2 hypergeometric solutions of the main representative equation, and for 2 more hypergeometric functions of that Schwartz type. The evaluations are presented in Section 2. Using these formulas and contiguous relations, one can compute Darboux evaluations for 2 different hypergeometric solutions of any hypergeometric equation with the tetrahedral, octahedral or icosahedral monodromy.

Klein evaluations of [BvHW03], [Ber04], [vHW] and Darboux evaluations have comparable visual complexity, especially when Darboux evaluation for the standard hypergeometric function $H(z)$ in (2) is used. Algorithmically, our proposal looks superior if we restrict ourselves to hypergeometric equations. In [vHW], [BvHW03], invariants or semi-invariants of minimal degree have to be computed by solving a symmetric power of the hypergeometric equation. When the hypergeometric equation has large local exponent differences (in absolute values), solving the symmetric power equation apparently involves a linear algebra problem of proportional size. We propose to use the "data base" of Darboux evaluations in Section 2 and contiguous relations. We do not need to solve any differential equations, except in computing the data base. Computational aspects of contiguous relations are mentioned in Appendix 5.6.

Darboux curves and coverings are introduced also in PhD thesis [Vid99, Section 4.2]. Most Darboux coverings of minimal degree, and all corresponding Darboux curves are presented there. Only Darboux coverings for the types $(2/3, 1/5, 1/5)$, $(1/3, 2/5, 3/5)$, $(1/3, 1/5, 3/5)$, $(1/5, 1/5, 4/5)$, $(2/5, 2/5, 2/5)$ are not computed explicitly there.

2 Hypergeometric evaluations

In this Section we present a few Darboux evaluations for each Schwartz type of hypergeometric functions with the tetrahedral, octahedral and icosahedral monodromy group. We use Darboux coverings of minimal possible degree, which is, respectively, 4, 6 and 12 for the three monodromy groups; see Lemma 3.2.

For each Schwartz type we evaluate 4 hypergeometric functions. In each evaluated group, the first and the third functions are solutions of the representative hypergeometric equation of that Schwartz type, as listed in Section 1. The second function is contiguous to the first one, and the fourth function is contiguous to the third one. With these evaluations, one can evaluate 2 independent hypergeometric solutions of any hypergeometric equation of the same Schwartz type, by using contiguous relations.

For the tetrahedral and octahedral Schwartz types, Darboux coverings are evident from the arguments of the hypergeometric functions. Darboux coverings for icosahedral Schwartz types are given in (19), (28), (34), (44), (54), (60). Some of these Darboux coverings are valid for two different Schwartz types. For 7 icosahedral types, the Darboux curve has genus 1 rather than 0. Weierstrass equations $\xi^2 = x\,(1+\alpha\,x+\beta\,x^2)$, with $\alpha,\beta\in\mathbb{C}$, for these curves are given in (33), (43), (53), (59).

A method to compute Darboux evaluations is presented in Section 4.1. Most attention is paid to describing complicated computations on genus 1 Darboux curves. Computer package Maple was used in the computations. The Darboux evaluations hold locally around $x=0$ or, if the Darboux curve has genus 1, around the point $(x,\xi)=(0,0)$. The simplest way to check each evaluation is to expand both sides in power series around $x=0$. If the Darboux curve has genus 1, one has to replace ξ by the respective $\sqrt{x}\,\sqrt{1+\alpha\,x+\beta\,x^2}$, and expand the power series in $\sqrt{x}$.

2.1 Tetrahedral hypergeometric equations

For the Schwartz type (1/2, 1/3, 1/3) we give the following evaluations:

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 1/4,\,-1/12 \\ 2/3 \end{array}\middle|\,\frac{x\,(x+4)^3}{4(2x-1)^3}\right) = \big(1-2x\big)^{-1/4}. \tag{3}$$

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 5/4,\,-1/12 \\ 5/3 \end{array}\middle|\,\frac{x\,(x+4)^3}{4(2x-1)^3}\right) = \frac{1+x}{\big(1+\frac{1}{4}x\big)^2}\,\big(1-2x\big)^{-1/4}. \tag{4}$$

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 1/4,\;7/12 \\ 4/3 \end{array}\middle|\,\frac{x\,(x+4)^3}{4(2x-1)^3}\right) = \frac{1}{1+\frac{1}{4}x}\,(1-2x)^{3/4}. \tag{5}$$

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 1/4,\,-5/12 \\ 1/3 \end{array}\middle|\,\frac{x\,(x+4)^3}{4(2x-1)^3}\right) = \big(1+\tfrac{5}{2}x\big)\,\big(1-2x\big)^{-5/4}. \tag{6}$$

For the Schwartz type (1/3, 1/3, 2/3) we give the following evaluations.

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 1/2,\,-1/6 \\ 2/3 \end{array}\middle|\,\frac{x\,(x+2)^3}{(2x+1)^3}\right) = \big(1+2x\big)^{-1/2}. \tag{7}$$

$$
{}_2F_1\left(\left.\begin{array}{c}1/2,\ 5/6\\ 2/3\end{array}\right|\frac{x\,(x+2)^3}{(2x+1)^3}\right) = \frac{1}{(1-x)^2}\,\big(1+2x\big)^{3/2}. \tag{8}
$$

$$
{}_2F_1\left(\left.\begin{array}{c}1/6,\ 5/6\\ 4/3\end{array}\right|\frac{x\,(x+2)^3}{(2x+1)^3}\right) = \frac{1}{1+\frac{1}{2}x}\,\big(1+2x\big)^{1/2}\big(1+x\big)^{1/3}. \tag{9}
$$

$$
{}_2F_1\left(\left.\begin{array}{c}1/6, -1/6\\ 1/3\end{array}\right|\frac{x\,(x+2)^3}{(2x+1)^3}\right) = \big(1+2x\big)^{-1/2}\big(1+x\big)^{1/3}. \tag{10}
$$

2.2 Octahedral hypergeometric equations

For the Swartz type (1/2, 1/3, 1/4) we give the following evaluations.

$$
{}_2F_1\left(\left.\begin{array}{c}7/24,\ -1/24\\ 3/4\end{array}\right|\frac{108\,x\,(x-1)^4}{(x^2+14x+1)^3}\right) = \big(1+14x+x^2\big)^{-1/8}. \tag{11}
$$

$$
{}_2F_1\left(\left.\begin{array}{c}7/24,\ 23/24\\ 7/4\end{array}\right|\frac{108\,x\,(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1+2x-\frac{1}{11}x^2}{(1-x)^3}\,\big(1+14x+x^2\big)^{7/8}. \tag{12}
$$

$$
{}_2F_1\left(\left.\begin{array}{c}5/24,\ 13/24\\ 5/4\end{array}\right|\frac{108\,x\,(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1}{1-x}\,\big(1+14x+x^2\big)^{5/8}. \tag{13}
$$

$$
{}_2F_1\left(\left.\begin{array}{c}5/24,\ -11/24\\ 1/4\end{array}\right|\frac{108\,x\,(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1-22x-11x^2}{\big(1+14x+x^2\big)^{11/8}}. \tag{14}
$$

For the Swartz type (1/4, 1/4, 2/3) we give the following evaluations.

$$
{}_2F_1\left(\left.\begin{array}{c}7/12,\ -1/12\\ 3/4\end{array}\right|\frac{27\,x\,(x+1)^4}{2(x^2+4x+1)^3}\right) = \frac{\big(1+\frac{1}{2}x\big)^{1/4}}{(1+4x+x^2)^{1/4}}. \tag{15}
$$

$$
{}_2F_1\left(\left.\begin{array}{c}7/12,\ 11/12\\ 7/4\end{array}\right|\frac{27\,x\,(x+1)^4}{2(x^2+4x+1)^3}\right) = \frac{(1+\frac{1}{2}x)^{1/4}\,(1+4x+x^2)^{7/4}}{(1+x)^3}. \tag{16}
$$

$$
{}_2F_1\left(\left.\begin{array}{c}1/6,\ 5/6\\ 5/4\end{array}\right|\frac{27\,x\,(x+1)^4}{2(x^2+4x+1)^3}\right) = \frac{\big(1+2x\big)^{1/4}\big(1+4x+x^2\big)^{1/2}}{1+x}. \tag{17}
$$

$$
{}_2F_1\left(\left.\begin{array}{c}1/6,\ -1/6\\ 1/4\end{array}\right|\frac{27\,x\,(x+1)^4}{2(x^2+4x+1)^3}\right) = \frac{\big(1+2x\big)^{1/4}}{\big(1+4x+x^2\big)^{1/2}}. \tag{18}
$$

2.3 Icosahedral hypergeometric equations

The Darboux covering for hypergeometric equations of the Schwartz type $(1/2, 1/3, 1/5)$ is

$$
\varphi_1(x) = \frac{1728\;x\;(x^2-11x-1)^5}{(x^4+228x^3+494x^2-228x+1)^3} \tag{19}
$$

The simplest evaluations for this Schwartz type are these:

$$
{}_2F_1\left(\left.\begin{array}{c}19/60,\ -1/60\\ 4/5\end{array}\right|\varphi_1(x)\right) = \big(1-228x+494x^2+228x^3+x^4\big)^{-1/20}. \tag{20}
$$

$$ {}_2F_1\left(\left.\begin{array}{c} 19/60,\ 59/60 \\ 4/5 \end{array}\right| \varphi_1(x)\right) = \frac{(1+66x-11x^2)\,(1-228x+494x^2+228x^3+x^4)^{19/20}}{(1+x^2)\,(1+522x-10006x^2-522x^3+x^4)}. \quad (21) $$

$$ {}_2F_1\left(\left.\begin{array}{c} 11/60,\ 31/60 \\ 6/5 \end{array}\right| \varphi_1(x)\right) = \frac{(1-228x+494x^2+228x^3+x^4)^{11/20}}{1+11x-x^2}. \quad (22) $$

$$ {}_2F_1\left(\left.\begin{array}{c} 11/60,\ -29/60 \\ 1/5 \end{array}\right| \varphi_1(x)\right) = \frac{1+435x-6670x^2-3335x^4-87x^5}{(1-228x+494x^2+228x^3+x^4)^{29/20}}. \quad (23) $$

The Darboux covering for the Schwartz type $(1/2, 1/3, 2/5)$ is the same, $\varphi_1(x)$. Here are evaluations:

$$ {}_2F_1\left(\left.\begin{array}{c} 13/60,\ -7/60 \\ 3/5 \end{array}\right| \varphi_1(x)\right) = \frac{1-7x}{\left(1-228x+494x^2+228x^3+x^4\right)^{7/20}}. \quad (24) $$

$$ {}_2F_1\left(\left.\begin{array}{c} 13/60,\ 53/60 \\ 3/5 \end{array}\right| \varphi_1(x)\right) = \frac{(1+119x+187x^2+17x^3)(1-228x+494x^2+228x^3+x^4)^{13/20}}{(1+x^2)\,(1+522x-10006x^2-522x^3+x^4)}. \quad (25) $$

$$ {}_2F_1\left(\left.\begin{array}{c} 17/60,\ 37/60 \\ 7/5 \end{array}\right| \varphi_1(x)\right) = \frac{\left(1+\frac{1}{7}x\right)(1-228x+494x^2+228x^3+x^4)^{17/20}}{\left(1+11x-x^2\right)^2}. \quad (26) $$

$$ {}_2F_1\left(\left.\begin{array}{c} 17/60,\ -23/60 \\ 2/5 \end{array}\right| \varphi_1(x)\right) = \frac{(1+207x-391x^2+1173x^3+46x^4)}{(1-228x+494x^2+228x^3+x^4)^{23/20}}. \quad (27) $$

The Darboux covering for Schwartz type $(1/2, 1/5, 2/5)$ is the following:

$$ \varphi_2(x) = \frac{64\,x\,(x^2-x-1)^5}{(x^2-1)\,(x^2+4x-1)^5}. \quad (28) $$

The simplest evaluations are the following:

$$ {}_2F_1\left(\left.\begin{array}{c} 7/20,\ -1/20 \\ 4/5 \end{array}\right| \varphi_2(x)\right) = \frac{(1+x)^{7/20}}{(1-x)^{1/20}\,(1-4x-x^2)^{1/4}}. \quad (29) $$

$$ {}_2F_1\left(\left.\begin{array}{c} 7/20,\ 19/20 \\ 4/5 \end{array}\right| \varphi_2(x)\right) = \frac{(1+3x)\,(1+x)^{7/20}\,(1-x)^{19/20}\,(1-4x-x^2)^{7/4}}{(1+x^2)\,(1+22x-6x^2-22x^3+x^4)}. \quad (30) $$

$$ {}_2F_1\left(\left.\begin{array}{c} 3/20,\ 11/20 \\ 6/5 \end{array}\right| \varphi_2(x)\right) = \frac{(1+x)^{3/20}\,(1-x)^{11/20}\,(1-4x-x^2)^{3/4}}{1+x-x^2}. \quad (31) $$

$$ {}_2F_1\left(\left.\begin{array}{c} 3/20,\ -9/20 \\ 1/5 \end{array}\right| \varphi_2(x)\right) = \frac{(1+12x-6x^2-2x^3-9x^4)\,(1+x)^{3/20}}{(1-x)^{9/20}\,(1-4x-x^2)^{9/4}}. \quad (32) $$

Darboux curves for other icosahedral Schwartz types have genus 1.

The Darboux curve for hypergeometric equations of the Schwartz type $(1/3, 1/3, 2/5)$ is given by the equation

$$ E_3: \qquad \xi^2 = x\,(1+33x-9x^2). \quad (33) $$

The Darboux covering is

$$ \varphi_3(x,\xi) = \frac{144\,\xi\,\left(1+33x-9x^2\right)^2\,(1-9\xi+54x)}{\left(1+21\xi-117x+9x\xi-234x^2\right)^3} \quad (34) $$

Here are simplest evaluations for this Schwartz type:

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 3/10,\ -1/30 \\ 3/5 \end{array} \middle| \varphi_3(x,\xi) \right) = \frac{(1-9\xi+54x)^{1/30}}{(1+21\xi-117x+9x\xi-234x^2)^{1/10}}. \tag{35} $$

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 3/10,\ 29/30 \\ 3/5 \end{array} \middle| \varphi_3(x,\xi) \right) = $$

$$ \frac{(1+21\xi-117x+9x\xi-234x^2)^{9/10}(1+9x)^2(1+198x-99x^2)}{(1-9\xi+54x)^{29/30}\,(1-21\xi-117x-9x\xi-234x^2)^2}. \tag{36} $$

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 7/10,\ 11/30 \\ 7/5 \end{array} \middle| \varphi_3(x,\xi) \right) = \frac{\left(1+21\xi-117x+9x\xi-234x^2\right)^{11/10}}{(1-9\xi+54x)^{11/30}\,(1+33x-9x^2)}. \tag{37} $$

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} -3/10,\ 11/30 \\ 2/5 \end{array} \middle| \varphi_3(x,\xi) \right) = \frac{(1-9\xi+54x)^{19/30}\,(1-15\xi-72x-54x^2)}{(1+21\xi-117x+9x\xi-234x^2)^{9/10}\,(1+9x)}. \tag{38} $$

The Darboux curve and covering for the Schwartz type $(1/3, 2/3, 1/5)$ are the same as in (33) and (34). The simplest evaluations are:

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} -1/10,\ 17/30 \\ 4/5 \end{array} \middle| \varphi_3(x,\xi) \right) = \frac{(1-9\xi+54x)^{13/30}}{(1+21\xi-117x+9x\xi-234x^2)^{3/10}}. \tag{39} $$

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 9/10,\ 17/30 \\ 9/5 \end{array} \middle| \varphi_3(x,\xi) \right) = \frac{\left(1+21\xi-117x+9x\xi-234x^2\right)^{17/10}\left(1+\frac{3}{7}x\right)}{(1-9\xi+54x)^{17/30}\,\left(1+33x-9x^2\right)^2}. \tag{40} $$

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 1/10,\ 23/30 \\ 6/5 \end{array} \middle| \varphi_3(x,\xi) \right) = \frac{\left(1+21\xi-117x+9x\xi-234x^2\right)^{3/10}(1-9\xi+54x)^{7/30}(\xi+5x)}{\xi\,(1+9x)}. \tag{41} $$

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 1/10,-7/30 \\ 1/5 \end{array} \middle| \varphi_3(x,\xi) \right) = \frac{(1-9\xi+54x)^{7/30}\,(1-21x)}{(1+21\xi-117x+9x\xi-234x^2)^{7/10}}. \tag{42} $$

The Darboux curve for the Schwartz type $(2/3,\ 1/5,\ 1/5)$ is given by

$$ E_4: \qquad \xi^2 = x\,(1+5x-5x^2). \tag{43} $$

The Darboux covering is given by

$$ \varphi_4(x,\xi) = \frac{432\,x\,\left(1-\frac{7}{5}\xi-9x-x^2\right)^5\,(1+50x-125\xi^2+450x\xi-500x^2)}{(5\xi+57x)\,\left(1+\frac{18}{5}\xi-16x+x^2\right)^5\,(1+50x-125\xi^2-450x\xi-500x^2)}. \tag{44} $$

The simplest evaluations are:

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 1/6,-1/30 \\ 4/5 \end{array} \middle| \varphi_4(x,\xi) \right) = \frac{\left(1-\frac{3}{5}\xi-\frac{34}{5}x\right)^{1/6}}{(1+3\xi-20x)^{1/6}\,(1+50x-125\xi^2-450x\xi-500x^2)^{1/30}} \tag{45} $$

$$ {}_2\mathrm{F}_1\left(\begin{array}{c} 1/6, 29/30 \\ 4/5 \end{array} \middle| \varphi_4(x,\xi) \right) = $$

$$\frac{(1+3\xi-20x)^{5/6}\left(1-\frac{3}{5}\xi-\frac{34}{5}x\right)^{1/6}\left(1-\frac{35}{4}\xi-\frac{101}{4}x\right)}{(1+50x-125\xi^2-450x\xi-500x^2)^{1/30}\left(1-\frac{95}{4}\xi+\frac{83}{4}x+\frac{21}{4}\xi^2+\frac{475}{4}x\xi+10x^2\right)}. \quad (46)$$

$${}_2\mathrm{F}_1\left(\begin{matrix}1/6,\ 11/30\\ 6/5\end{matrix}\middle|\,\varphi_4(x,\xi)\right) = \left(1+50x-125\xi^2-450x\xi-500x^2\right)^{11/30} \times$$
$$\frac{(1+3\xi-20x)^{5/6}\left(1-\frac{3}{5}\xi-\frac{34}{5}x\right)^{1/6}\left(1+\frac{21}{4}\xi+\frac{41}{4}x\right)}{(1-9x)\left(1-\frac{7}{4}\xi-\frac{15}{2}x\right)(1+5\xi+10x)}. \quad (47)$$

$${}_2\mathrm{F}_1\left(\begin{matrix}1/6,\ 11/30\\ 1/5\end{matrix}\middle|\,\varphi_4(x,\xi)\right) = \left(1+50x-125\xi^2-450x\xi-500x^2\right)^{11/30} \times$$
$$\frac{(1+3\xi-20x)^{5/6}\left(1-\frac{3}{5}\xi-\frac{34}{5}x\right)^{1/6}\left(1+\frac{21}{4}\xi+\frac{41}{4}x\right)}{\left(1-\frac{95}{4}\xi+\frac{83}{4}x+\frac{21}{4}\xi^2+\frac{475}{4}x\xi+10x^2\right)(1+5\xi+10x)}. \quad (48)$$

The Darboux curve and covering for the Schwartz type $(1/3, 2/5, 3/5)$ are the same as in (43) and (44). The simplest evaluations are:

$${}_2\mathrm{F}_1\left(\begin{matrix}-1/6,\ 13/30\\ 3/5\end{matrix}\middle|\,\varphi_4(x,\xi)\right) = \frac{(1+50x-125\xi^2-450x\xi-500x^2)^{13/30}(1-3\xi+2x)}{(1+3\xi-20x)^{5/6}\left(1-\frac{3}{5}\xi-\frac{34}{5}x\right)^{1/6}(1+5\xi+10x)}. \quad (49)$$

$${}_2\mathrm{F}_1\left(\begin{matrix}5/6,\ 13/30\\ 3/5\end{matrix}\middle|\,\varphi_4(x,\xi)\right) = (1+50x-125\xi^2-450x\xi-500x^2)^{13/30} \times$$
$$\frac{(1+3\xi-20x)^{13/6}\left(1-\frac{7}{4}\xi+\frac{25}{2}x-\frac{245}{4}x^2\right)\left(1-\frac{7}{20}\xi-\frac{79}{20}x\right)}{\left(1-\frac{3}{5}\xi-\frac{34}{5}x\right)^{1/6}\left(1-\frac{95}{4}\xi+\frac{83}{4}x+\frac{21}{4}\xi^2+\frac{475}{4}x\xi+10x^2\right)^2(1-5x)^2}. \quad (50)$$

$${}_2\mathrm{F}_1\left(\begin{matrix}5/6,\ 7/30\\ 7/5\end{matrix}\middle|\,\varphi_4(x,\xi)\right) = \left(1+50x-125\xi^2-450x\xi-500x^2\right)^{7/30} \times$$
$$\frac{\left(1+\frac{18}{5}\xi-16x+x^2\right)^{7/6}\left(1+\frac{1}{25}x\right)^{5/6}(1+5\xi+10x)}{\left(1-\frac{7}{5}\xi-9x-x^2\right)^2(1-5x)^{7/6}}. \quad (51)$$

$${}_2\mathrm{F}_1\left(\begin{matrix}-1/6,\ 7/30\\ 2/5\end{matrix}\middle|\,\varphi_4(x,\xi)\right) = \frac{\left(1+50x-125\xi^2-450x\xi-500x^2\right)^{7/30}\left(1-\frac{27}{5}\xi+\frac{58}{5}x-2x^2\right)}{\left(1+\frac{18}{5}\xi-16x+x^2\right)^{5/6}(1+\frac{1}{25}x)^{1/6}(1-5x)^{7/6}}. \quad (52)$$

The Darboux curve for the Schwartz type $(1/3,\ 1/5,\ 3/5)$ is given by

$$E_5: \qquad \xi^2 = x\,(1+x)\,(1+16x). \quad (53)$$

The Darboux covering is given by

$$\varphi_5(x,\xi) = -\frac{54\,(\xi+5x)^3\,(1-2\xi+6x)^5}{(1-16x^2)\,(\xi-5x)^2\,(1-2\xi-14x)^5}. \quad (54)$$

$${}_2\mathrm{F}_1\left(\begin{matrix}-1/15,\ 8/15\\ 4/5\end{matrix}\middle|\,\varphi_5(x,\xi)\right) = \frac{(1+4x)^{8/15}\,(\xi+5x)^{1/6}\,x^{1/15}}{(1-2\xi-14x)^{1/3}\,(\xi-3x)^{3/10}}. \quad (55)$$

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 14/15,\ 8/15 \\ 9/5 \end{matrix} \,\middle|\, \varphi_5(x,\xi) \right) = $$

$$ \frac{(1-2\xi-14x)^{8/3}\left(1+\frac{2}{3}\xi+\frac{2}{3}x-\frac{16}{3}x^2\right)(1+4x)^{8/15}\,(\xi-5x)^2\,x^{1/15}}{(1-2\xi+6x)^4\,(\xi+5x)^{11/6}\,(\xi-3x)^{3/10}}. \quad (56) $$

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 2/15,\ 11/15 \\ 6/5 \end{matrix} \,\middle|\, \varphi_5(x,\xi) \right) = \frac{(1-\xi+x)(1-2\xi-14x)^{2/3}(\xi+5x)^{1/6}(\xi-3x)^{13/10}}{(1+\xi+x)\,(1-2\xi+6x)\,(1+4x)^{13/15}\,x^{11/15}}. \quad (57) $$

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 2/15, -4/15 \\ 1/5 \end{matrix} \,\middle|\, \varphi_5(x,\xi) \right) = \frac{(1+3\xi+x)\,(1+4x)^{2/15}\,(\xi+5x)^{7/6}\,(\xi-3x)^{3/10}}{(1+\xi+x)\,(1-2\xi-14x)^{4/3}\,x^{11/15}}. \quad (58) $$

The Darboux curve for the Schwartz type (1/5, 1/5, 4/5) is given by

$$ E_6: \qquad \xi^2 = x\,(1+x-x^2). \quad (59) $$

The Darboux covering is given by

$$ \varphi_6(x,\xi) = \frac{16\,\xi\,\left(1+x-x^2\right)^2\,(1-\xi)^2}{(1+\xi+2x)\,(1+\xi-2x)^5}. \quad (60) $$

The simplest evaluations are:

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 7/10, -1/10 \\ 4/5 \end{matrix} \,\middle|\, \varphi_6(x,\xi) \right) = \frac{(1-\xi+2x)^{1/15}\,(1-\xi)^{3/5}}{(1+\xi+2x)^{7/30}\,\sqrt{1+\xi-2x}} \quad (61) $$

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 7/10,\ 9/10 \\ 9/5 \end{matrix} \,\middle|\, \varphi_6(x,\xi) \right) = \frac{(1-\xi+2x)^{1/15}\,(1+\xi+2x)^{23/30}\,(1+\xi-2x)^{7/2}}{(1-\xi)^{7/5}\,(1+x-x^2)^2} \quad (62) $$

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 1/10,\ 9/10 \\ 6/5 \end{matrix} \,\middle|\, \varphi_6(x,\xi) \right) = \frac{(\xi+2x+x^2)\,(1+\xi)^{1/10}\,(1-\xi)^{3/10}}{\xi\,(1-\xi+2x)^{1/30}(1+\xi+2x)^{2/15}\sqrt{1+\xi-2x}}. \quad (63) $$

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 1/10, -1/10 \\ 1/5 \end{matrix} \,\middle|\, \varphi_6(x,\xi) \right) = \frac{(1+\xi)^{1/10}\,(1-\xi)^{3/10}}{(1-\xi+2x)^{1/30}\,(1+\xi+2x)^{2/15}\,\sqrt{1+\xi-2x}}. \quad (64) $$

The Darboux curve and covering for the Schwartz type $(2/5, 2/5, 2/5)$ are the same as in (59) and (60). The simplest evaluations are:

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 3/10, -1/10 \\ 3/5 \end{matrix} \,\middle|\, \varphi_6(x,\xi) \right) = \frac{(1-\xi+2x)^{2/15}\,(1+\xi+2x)^{1/30}\,(1-\xi)^{1/5}}{\sqrt{1+\xi-2x}}. \quad (65) $$

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 3/10,\ 9/10 \\ 8/5 \end{matrix} \,\middle|\, \varphi_6(x,\xi) \right) = $$

$$ \frac{(1-\xi+2x)^{2/15}(1+\xi+2x)^{1/30}(1-\xi)^{1/5}(1+\xi-2x)^{3/2}\left(1+\frac{1}{2}\xi+\frac{1}{2}x\right)}{(1+x-x^2)\,(1-\xi+x-x^2)}. \quad (66) $$

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 3/10,\ 7/10 \\ 7/5 \end{matrix} \,\middle|\, \varphi_6(x,\xi) \right) = \frac{(1+\xi+2x)^{7/30}\,(1+\xi)^{1/5}\,(1+\xi-2x)^{3/2}}{(1-\xi+2x)^{1/15}\,(1-\xi)^{2/5}\,(1+x-x^2)}. \quad (67) $$

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 3/10, -3/10 \\ 2/5 \end{matrix} \,\middle|\, \varphi_6(x,\xi) \right) = \frac{(1+\xi+2x)^{7/30}\,(1+\xi)^{1/5}\,(1-3\xi+4x-2x^2)}{(1-\xi+2x)^{1/15}\,(1-\xi)^{2/5}\,(1+\xi-2x)^{3/2}}. \quad (68) $$

3 Darboux curves

Classical definitions [Dar78], [Oll01] in integration theory of vector fields are the following. Consider a polynomial vector field on $\mathbb{C}^2$ given by a derivation $\mathcal{L} = f(x,y)\,\partial/\partial x + g(x,y)\,\partial/\partial y$ with $f, g \in \mathbb{C}[x,y]$. A polynomial $p(x,y)$ is called a *Darboux polynomial* for the vector field (or the derivation $\mathcal{L}$) if $p(x,y)$ divides $\mathcal{L}p(x,y)$ in the ring $\mathbb{C}[x,y]$. An algebraic curve defined as the zero set of a Darboux polynomial is called a *Darboux curve.* A Darboux curve is infinitesimally invariant under the vector field. Hence an alternative term is *invariant algebraic curve*, as in [CLPZ] for example.

In differential Galois theory we have the following definition [Wei95], [Sin92]. Let K be a differential field, and let $R = K[y_1, \ldots, y_n]$ be a differential ring. Let $\mathcal{D}$ denote the derivation on R, and suppose that it extends the derivation on K. Then $p \in R$ is a *Darboux polynomial* for $\mathcal{D}$ if p divides $\mathcal{D}p$ in R. For example, consider differential equation

$$y^{(n)} + a_{n-1}\, y^{(n-1)} + \ldots + a_1\, y' + a_0\, y = 0, \qquad \text{with} \quad a_0, a_1, \ldots, a_{n-1} \in K. \tag{69}$$

Let $\mathcal{D}$ be the derivation on $K[y, y', \ldots, y^{(n-1)}]$ defined by $\mathcal{D}y = y'$, $\mathcal{D}y' = y''$, ..., $\mathcal{D}y^{(n-2)} = y^{(n-1)}$, $\mathcal{D}y^{(n-1)} = -a_{n-1}y^{(n-1)} - \ldots - a_1 y' - a_0 y$. Darboux polynomials for this derivation correspond to semi-invariants of the differential Galois group for (69), see [Wei95, Theorem 38]. If the order n of (69) is equal to 2, then one considers Darboux polynomials in $K[u]$ for the derivation $\mathcal{D}$ defined by $\mathcal{D}u = -u^2 - a_1 u - a_0$. (See [Wei94, UW96], where Darboux polynomials are called *special polynomials.*)

As one may notice, the terminology “Darboux polynomials”, “Darboux curves” is not consistently accepted. We wish to use the term “Darboux curves” because we consider the curves geometrically, with little reference to their differential and algebraic properties. We offer the following definition of Darboux curves and Darboux coverings.

Definition 3.1 Let C denote an algebraic curve (see Appendix 5.1). We suppose that the function field $\mathbb{C}(C)$ is a differential field. Consider differential equation (69) on $\mathbb{P}^1$, assuming $K = \mathbb{C}(\mathbb{P}^1)$. We say that a finite covering $\phi : C \to \mathbb{P}^1$ is a *Darboux covering* for (69), if a pull-back of (69) with respect to ϕ has a solution Y such that:

(i) Its logarithmic derivative $u = Y'/Y$ is in $\mathbb{C}(C)$;

(ii) The algebraic degree of u over K is precisely the degree of ϕ.

In the described situation, C is called a *Darboux curve* for (69).

To see connection with previous definitions, let us assume that the order n of (69) is equal to 2. We assume that $K = \mathbb{C}(z)$ with $z' = 1$. Let d denote the degree of ϕ. Then we have the following facts:

- The logarithmic derivative u is an algebraic solution of the associated Riccati equation $u' + u^2 + a_1 u + a_0 = 0$. Let $P(u) = 0$ denote the minimal monic polynomial equation defining u over K. The polynomial P has degree d.

- The minimal polynomial P is a Darboux polynomial for the mentioned derivation $\mathcal{D}u = -u^2 - a_1 u - a_0$; see [UW96, Lemma 2.4].

- The expression $y^{n+1}P(y'/y)$ is a homogeneous polynomial in $K[y,y']$. It is a Darboux polynomial for the specified derivation on $K[y,y']$.
- Assume that $a_1, a_0 \in \mathbb{C}[z]$, and consider the vector field $\partial/\partial z - (u^2 + a_1 u + a_0)\,\partial/\partial u$. Then P is a Darboux polynomial according the first definition above.

Besides, we have the following facts.

- The polynomial P is a defining equation for the Darboux curve C, and $\mathbb{C}(C) = \mathbb{C}(z,u)$.
- Suppose that $P = u^m + \sum_{j=0}^{d-1} b_j\, u^j$. Then $-b_{d-1}$ is the logarithmic derivative of an exponential solution of the d-th symmetric power of (69); see [UW96, Theorem 2.1]. All other coefficients b_j are determined by b_{d-1} and (69); see [SU93, Section 3.2].
- The mentioned exponential solution (which can be expressed as $\exp\int -b_{d-1}$) is a degree d semi-invariant of the differential Galois group [UW96, Section 1.2].
- Inside a Picard-Vessiot extension for (69), the field $\mathbb{C}(C)$ is fixed by a 1-reducible subgroup of the differential Galois group for (69) of finite index [SU93, Lemma 3.1].
- If the differential Galois group of (69) is finite, the 1-reducible subgroups are cyclic subgroups [UW96, Lemma 1.5].

As we see, Darboux coverings correspond to algebraic Riccati solutions and to semi-invariants of the differential Galois group. We defined Darboux curves and coverings for general linear differential equations. It appears to be useful to parameterize Darboux curves of genus 0, or express Darboux curves of higher genus by convenient birational models. Solving pull-backed equations gives Darboux evaluations for solutions of an original equation; these can be handy and satisfying expressions. For example, Darboux evaluations for solutions of hypergeometric equations with dihedral monodromy groups can be recognized in [Erd53, 2.5.5] or [Vid04, Section 6]. In general, Darboux coverings and Darboux evaluations can be computed from minimal polynomials for Riccati equation and by direct solution of pull-backed equations. This routine does not look more effective than algorithms in [BvHW03], [Ber04], [vHW]. For hypergeometric equations, we can utilize contiguous relations.

We consider hypergeometric equations with the tetrahedral, octahedral or icosahedral monodromy groups. The differential Galois group is the same. Pull-backed equations on Darboux curves have cyclic monodromy groups. The following lemma categorizes possible Darboux coverings.

Lemma 3.2 *Suppose that the differential Galois group G of a hypergeometric equation* (1) *is either tetrahedral A_4, or octahedral S_4, or icosahedral A_5.*

- *If $G \cong A_4$ (tetrahedral group), Darboux coverings for* (1) *have degree* 4, 6 *or* 12.
- *If $G \cong S_4$ (octahedral group), Darboux coverings have degree* 6, 8, 12 *or* 24.
- *If $G \cong A_5$ (icosahedral group), Darboux coverings have degree* 12, 20, 30 *or* 60.

- *Darboux coverings of each degree are unique up to fractional-linear transformations of the variable z and automorphisms of the Darboux curve.*

Proof. According to [UW96, Corollary 1.7], these are possible degrees of minimal polynomials for solutions of the Riccati equation associated to the normalized equation (100). Normalization (99) does not change the degree of Riccati solutions, so these are also the degrees of Darboux coverings for the hypergeometric equation.

In the tetrahedral case, there are 2 minimal polynomials of degree 4 for Riccati solutions since there are 2 semi-invariants (up to scalar multiplication) of degree 4. The two semi-invariants are interchanged by a change of solution basis. For example, in the form presented in [SU93, Section 4.3.1], the basis change is $(y_1, y_2) \mapsto (\sqrt{-1}\, y_1, y_2)$ or $(y_2, \sqrt{-1}\, y_1)$. One can check connection formulas for Gauss hypergeometric series [IKSY91, Chapter 4] that such a basis can be realized by 2 pairs of hypergeometric series; by a basis of hypergeometric series around one singularity, and by a basis of hypergeometric series around other singularity. The fractional-linear transformation which permutes those 2 singularities acts on the solution basis as noted above. The conclusion is that this fractional-linear transformation permutes the 2 semi-invariants, and hence the 2 minimal polynomials for Riccati solutions. Hence the 2 Darboux coverings are isomorphic.

For other listed degrees, minimal polynomials for Riccati solutions are unique, except for the maximal degrees 12, 24 and 60. There are infinitely many minimal polynomials of maximal degree, but their all generate the same Piccard-Vessiot extension for the hypergeometric series. Hence Darboux coverings are unique in all cases, up to isomorphism. $\square$

3.1 Darboux coverings for standard hypergeometric equations

Here are the main facts about Darboux coverings for standard hypergeometric equations.

Lemma 3.3 *Let H denote a standard hypergeometric equation with tetrahedral, octahedral or icosahedral differential Galois group G. Let $\varphi : D \to \mathbb{P}^1$ be a Darboux covering for H of degree m. Then:*

(i) The Darboux curve D has genus zero.

(ii) Let $\widetilde{\varphi} : \widetilde{D} \to \mathbb{P}^1$ be a Darboux covering for H of the maximal degree $|G|$. Then we have a factorization $\widetilde{\varphi} = \gamma \circ \varphi$, where the covering $\gamma : \widetilde{D} \to D$ is given (up to fractional-linear transformations of D and $\widetilde{D}$) by $x \mapsto x^{|G|/m}$.

(iii) Let $X \in \mathbb{P}^1$ be a regular singular point of the hypergeometric equation. Assume that the local exponent difference at X has denominator k. Then there are $\lfloor m/k \rfloor$ points above X with branching index k, and other points above X are unramified.

Proof. First we prove this Theorem for the Darboux covering $\widetilde{\varphi}$ of maximal degree $m = |G|$. The first statement follows classically from [Kle84]. In particular, there is an action of G on $\widetilde{D} \cong \mathbb{P}^1$, and the projection $\mathbb{P}^1 \to \mathbb{P}^1/G \cong \mathbb{P}^1$ is precisely $\widetilde{\varphi}$. This map is also the inverse of a Schwartz map for the hypergeometric equation. At each fiber of $\widetilde{\varphi}$ the points have the same branching index, since $\mathbb{C}(\widetilde{D}) \supset \mathbb{C}[z]$ is a Galois extension. This extension is also the Piccard-Vessiot extension for H, so a suitable pull-back of H with respect to $\widetilde{\varphi}$ has trivial Galois group. Hence all local exponent differences of any pull-back of H with respect to $\widetilde{\varphi}$ are integers. As a consequence, in the situation of part *(iii)* the branching indices should be integer multiples of k. Since $g(\widetilde{D}) = 0$, Hurwitz theorem leaves only one possibility which is described in part *(iii)*. Part *(ii)* is trivial in the considered case.

Now we consider a general Darboux covering $\varphi : D \to \mathbb{P}^1$. Let u be a corresponding Riccati solution of degree m, so that $\mathbb{C}(D) \cong \mathbb{C}(z, u)$. Let $K \supset \mathbb{C}(z)$ denote the the Piccard-Vessiot extension of (101). We have $K \cong \mathbb{C}(\widetilde{D})$ as just above. Consider the tower of field extensions $K \supset \mathbb{C}(D) \supset \mathbb{C}(z)$. Here $K \supset \mathbb{C}(D)$ is the Piccard-Vessiot extension for the differential equation $y' = uy$, so its Galois group must be a cyclic subgroup of G of index m. The existence of the corresponding covering $\gamma : \widetilde{D} \to D$ implies *(i)*. Hurwitz theorem implies that there are exactly two branching points of γ. This implies *(ii)*. We see that the equation $y' = uy$ has exactly two singular points, so in the situation of part *(iii)* branching indices are equal to either k or 1, and that there in total only two unramified points above the three regular singular points of (101). This information gives part *(iii)*. $\square$

Note that the rational functions for the Darboux coverings $\widetilde{\varphi}$ of maximal degree are invariants of A_4, S_4, A_5 with respect to the classical action these groups on $\mathbb{C}[x]$ (or on homogeneous polynomials in $\mathbb{C}[x, y]$); see [Kle84].

Here are explicit expressions for Darboux coverings for a standard tetrahedral equation (up to permutations of the regular singular points on the $\mathbb{P}^1$ below and fractional-linear transformations of the Darboux curve):

$$x \mapsto z = \frac{x\,(x+4)^3}{4\,(2x-1)^3}, \quad x \mapsto z = \frac{\left(x^2-6\,x-3\right)^3}{\left(x^2+6\,x-3\right)^3}, \quad x \mapsto z = \frac{x^3\left(x^3+4\right)^3}{4\,\left(2x^3-1\right)^3}. \tag{70}$$

Darboux coverings of degree 6, 8, 12 for a standard octahedral equation are given by:

$$\frac{27\,x\,(x+1)^4}{2\,(x^2+4x+1)^3}, \quad \frac{(x^2+20x-8)^4}{256\,x\,(x+1)^3\,(x-8)^3}, \quad \frac{27(x-1)^4(x^2+6x+1)^4}{(x^2-10x+1)^3\,(3x^2+2x+3)^3}. \tag{71}$$

Darboux coverings of degree 12, 20, 30 for a standard icosahedral equation are given by:

$$\frac{1728\,x\,(x^2-11x-1)^5}{(x^4+228x^3+494x^2-228x+1)^3}, \quad \frac{64\,(x^4+55x^3-165x^2-275x+25)^5}{125x(x^2+5x+40)^3(x^2-40x-5)^3(8x^2-5x+5)^3},$$
$$\frac{27\,(x^2+2x+5)^5\,(x^4+20x^3-210x^2+100x+25)^5}{(3x^2-10x+15)^3(x^4+70x^2+25)^3(x^4-60x^3-370x^2-300x+25)^3}. \tag{72}$$

Expressions for the Darboux covering of maximal degree 60 can be obtained by substituting, respectively, $x \mapsto x^5$, $x \mapsto x^3$ or $x \mapsto x^2$ into the functions in (72). The obtained 3 expressions are related by fractional-linear transformations of x. This relation can be used to compute other Darboux coverings once a Darboux covering of the minimal (or maximal) degree is known. A similar remark applies to Darboux coverings for standard octahedral or tetrahedral equations.

All coverings in (70)–(72) can be computed from scratch by part *(iii)* of Lemma 3.3, which determines the branching pattern for the Darboux coverings, and by using Algorithm 1 in [Vid05, Section 3].

3.2 Properties of Darboux curves

The following Lemma is the key for effectiveness of our proposal to use Darboux evaluations. It implies that there are only finitely many different Darboux coverings for all hypergeometric equations.

Lemma 3.4 *Let H_1, H_2 denote two hypergeometric equations of the same Schwartz type. Suppose that $\phi_1 : D_1 \to \mathbb{P}^1$ and $\phi_2 : D_2 \to \mathbb{P}^1$ are Darboux coverings for H_1, H_2 respectively of the same degree. Then the Darboux curves D_1, D_2 are isomorphic, and the coverings ϕ_1, ϕ_2 are the same (up to automorphisms of $\mathbb{P}^1$ and of the Darboux curve).*

Proof. Let y_1 be a hypergeometric solution of H_1. Since H_1 and H_2 have the same Schwartz type, there is a hypergeometric solution y_2 which is contiguous to y_1. Note that y_1' is contiguous to y_1. Therefore there is a contiguous relation $y_2 = ay_1 + by_1'$, with $a, b \in \mathbb{C}(z)$. There is also a contiguity relation $y_2' = cy_1 + dy_1'$, with $c, d \in \mathbb{C}(z)$. Let u_1 denote the Riccati solution y_1'/y_1 for H_1, and let u_2 denote the Riccati solution y_2'/y_2 for H_2. Then $u_2 = (c + du_1)/(a + bu_1)$. This implies that the function fields $\mathbb{C}(z, u_1)$ and $\mathbb{C}(z, u_2)$ are isomorphic, and so are the corresponding Darboux curves. □

The following lemma allows us to compute all necessary Darboux coverings once Darboux coverings for standard hypergeometric equations (from Subsection 3.1) are known. For this purpose we usually take H_1 to be a main representative equation of a chosen Schwartz type, and we usually take H_0 to be the corresponding standard equation.

Lemma 3.5 *Let H_0, H_1 denote two linear differential equations. Suppose that H_1 is a pull-back of H_0 with respect to a covering $\psi : \mathbb{P}^1 \to \mathbb{P}^1$. Suppose that $\phi_0 : D_0 \to \mathbb{P}^1$ is a Darboux covering for H_0. Then the fiber product D_1 of $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ and $\phi_0 : D_0 \to \mathbb{P}^1$ is a Darboux curve for H_1, and the projection $\phi_1 : D_1 \to \mathbb{P}^1$ (to the source of ψ) is a Darboux covering for H_1 of the same degree as ϕ_0.*

Proof. Let z denote a projective parameter for the $\mathbb{P}^1$ below, and let x denote a projective parameter of the other $\mathbb{P}^1$, so that the covering ψ corresponds to the extension $\mathbb{C}(x) \supset \mathbb{C}(z)$. Suppose that u_0 be a Riccati solution for H_0 which determines D_0, so that $\mathbb{C}(D_0) \cong \mathbb{C}(z, u_0)$. Then there is a solution y for H_0 which satisfies $y' = u_0 y$. The logarithmic derivative of the pull-back of y with respect to (98) is $u_0\psi' + \theta'/\theta$. It is a Riccati solution for H_1, and it lies in $\mathbb{C}(x, u_0) \cong \mathbb{C}(x) \otimes_{\mathbb{C}(z)} \mathbb{C}(x, u_0)$ which is the function field for the fiber product of ψ and ϕ_0. Since the degree of the projection to $\mathbb{P}^1$ is equal to the degree of ψ, the claim follows.□

The simplest coverings for Klein pull-backs ψ from standard hypergeometric equations to other main representatives were first computed in [Sch72]. These Klein coverings are familiar from classical transformations of hypergeometric series (see for instance [AAR99] or

Denominators	Schwartz	Klein	$g(D)$, when $\deg\gamma : \widetilde{D} \to D$ is equal to			
k, ℓ, m	type	degree	m	ℓ	k	1
2, 3, 3	(1/2, 1/3, 1/3)	1	0	0	0	0
	(1/3, 1/3, 2/3)	2	0	0	1	1
2, 3, 4	(1/2, 1/3, 1/4)	1	0	0	0	0
	(2/3, 1/4, 1/4)	2	0	1	2	3
2, 3, 5	(1/2, 1/3, 1/5)	1	0	0	0	0
	(1/2, 1/3, 2/5)	7	0	0	0	0
	(1/2, 1/5, 2/5)	3	0	2	2	4
	(1/3, 1/3, 2/5)	2	1	1	3	5
	(1/3, 2/3, 1/5)	6	1	1	3	5
	(2/3, 1/5, 1/5)	2	1	3	5	9
	(1/3, 2/5, 3/5)	10	1	3	5	9
	(1/3, 1/5, 3/5)	4	1	3	5	9
	(1/5, 1/5, 4/5)	6	1	5	7	13
	(2/5, 2/5, 2/5)	6	1	5	7	13

Table 1: Genus of Darboux curves

[Vid04, Section 5]), except the ones for the Schwartz types $(1/2, 1/3, 2/5)$, $(1/3, 2/3, 1/5)$, $(1/3, 2/5, 3/5)$. Here are rational functions that define Klein pull-backs for their main representatives:

$$\frac{x^2\,(189-64x)^5}{(3584x^2+2457x-2916)^3},\quad \frac{4\,x\,(25x-9)^5}{27(x-1)(125x+3)^3},\quad \frac{3125\,x^2(x-1)^3(5x+27)^5}{4(625x^3-2875x^2+675x-729)^3}. \tag{73}$$

Table 1 shows the genus of all Darboux curves; this information can be computed using Lemma 3.5, part (iii) of Lemma 3.3, remarks in Appendixes 5.7 and 5.2. The third column shows the degree of Klein morphism for the main representatives of each Schwartz type. We explicitly compute and utilize Darboux coverings of minimal degree.

3.3 Hypergeometric functions on Darboux curves

First we describe radical solutions of a Fucshian equation with cyclic monodromy.

Lemma 3.6 *Let C be an algebraic curve. Consider a second order Fuchsian differential equation* (69) *on C. Suppose that its differential Galois group G is finite and cyclic. Suppose that there are singular points with non-integer local exponent differences. Then:*

(i) There exist exactly two independent radical solutions f_1, f_2.

(ii) For any regular singular point $P \in C$ of (69) *where the local exponent difference is not an integer, the $\mathbb{Q}$-valuations of f_1 and f_2 are different.*

Proof. Let $K_{PV} \supset \mathbb{C}(C)$ be the Piccard-Vessiot extension of the differential equation (69), and let V denote its space of solutions in K_{PV}. The monodromy group G acts does not act on V by scalar multiplication, because otherwise the quotient of two independent solutions would be in $\mathbb{C}(C)$. Hence the representation of G on V splits into two irreducible representations. Let $f_1, f_2 \in V$ be generators of the two G-invariant subspaces. If $P \in C$ is a point where the local exponent difference is not an integer, f_1, f_2 are in different spaces invariant under the the local monodromy group. Hence they have well-defined $\mathbb{Q}$-valuations, and the valuations should be different. $\square$

Hypergeometric functions which pull-back to radical solutions under a Darboux covering are characterized as follows.

Lemma 3.7 *Let H denote a hypergeometric equation. Suppose that its monodromy group G is tetrahedral, octahedral or icosahedral. Let $\phi : D \to \mathbb{P}^1$ denote a Darboux morphism for H, of degree d. Let $\theta(z)\, {}_2\mathrm{F}_1\left(\left. {A, B \atop C} \right| f(z) \right)$ denote a hypergeometric solution of H, with $f(z)$ a fractional-linear function, and $\theta(z)$ a radical factor. Assume that the denominator of the lower parameter C is equal to $|G|/d$. Then the hypergeometric function is pull-backed to a radical function. Conversely, each radical solution of the pull-backed eqaution represents (up to a scalar multiple) a hypergeometric equation with the assumed property.*

Proof. Set $m = |G|/d$. Suppose that the pull-backed equation is normalized so that its monodromy group is the cyclic group of order m. Then all non-integer local exponents of the pull-backed equation have denominator m. We may assume that $f(z) = z$. There is a point P above $z = 0$ such that the local exponent difference δ for the pull-backed equation at P is non-integer. The denominator of δ is m. Let λ_1, λ_2 denote the local exponents at P, and let t denote a local parameter at P. For each local exponent λ_j there is a unique power series solution of the form $t^{\lambda_j}(1 + \alpha_1 t + \alpha_2 t + \ldots)$. By part *(ii)* of Lemma 3.6, both power series represent radical functions.

The hypergeometric function is pull-backed, up to a constant multiple, to one of the two mentioned power series. Hence the pull-back is a radical function. Conversely, push-forwards of the two power series must me hypergeometric series. $\square$

Lemma 3.8 *Let F_1, F_2 denote two contiguous algebraic (but not rational) Gauss hypergeometric functions. Let D denote their common Darboux curve. Then the pull-back of F_1/F_2 is a rational function on D.*

Proof. The logarithmic derivative F_1'/F_1 is a rational function D. Up to a factor in $C(D)$, F_1' is Gauss hypergeometric function contiguous to F_1. The contiguous relation between F_1, F_1' and F_2 has coefficients in $\mathbb{C}(x)$, hence the claim follows. $\square$

Note that if we take any pair of contiguous evaluations in Section 2, the quotient of the right-hand sides is a rational (rather then radical) function on the Darboux curve, as suggested by Lemma 3.8. This allows us to express other contiguous evaluations conveniently as product of a fixed radical function and some rational function on the Darboux curve.

4 Computation of Darboux evaluations

This is an outline of actual computations that led us to the list of Darboux evaluations in Section 2. If the Darboux curve has genus 0, computations are quite straightforward. Subsections 4.2 and 4.3 are devoted to difficulties of expressing rational and radical functions on genus 1 Darboux curves.

4.1 Computation of Darboux curves and coverings

Darboux coverings for the 3 standard Schwartz types are considered in Subsection 3.1. Lemma 3.5 was used to compute other Darboux coverings and curves and curves. Let us fix an Schwartz type which is not standard, and let H denote its representative hypergeometric equation as listed in Section 1. Suppose that $\varphi : \mathbb{P}^1_z \mapsto \mathbb{P}^1_Z$ is a Darboux covering (of minimal degree) for the corresponding standard hypergeometric equation. Suppose that $\psi : \mathbb{P}^1_X \mapsto \mathbb{P}^1_Z$ is a Klein covering for H. Then the Darboux curve is a fiber product of φ and ψ. An equation for it is given by the equation $\psi(X) = \phi(z)$; see Subsection 5.7.

If the Darboux curve has genus 0, then a parameterization of it immediately gives the Darboux covering. As an example, consider the icosahedral type $(1/2, 1/3, 2/5)$. We apply Lemma 3.5 with $H_0 = H(1/5, 1/2, 1/3)$ and $H_1 = H(2/5, 1/2, 1/3)$. By formulas (19) and (73) we get the following equation for the Darboux curve:

$$\frac{X^2\,(189-64X)^5}{(3584X^2+2457X-2916)^3} = \frac{1728\,z\,(z^2-11z-1)^5}{(z^4+228z^3+494z^2-228z+1)^3}. \tag{74}$$

This is a rational curve. It can be parameterized by standard algorithms and computer algebra packages such as Maple. Here is a parameterization:

$$X = \frac{1728\,x\,(x^2-11x-1)^5}{(x^4+228x^3+494x^2-228x+1)^3}, \qquad z = \frac{(7x-1)^5}{x^2\,(x+7)^5}. \tag{75}$$

(Recall that parameterizations are unique up to fractional-linear transformations on $\mathbb{P}^1_x$.) The parametric expression for X gives the Darboux covering for $H(2/5, 1/2, 1/3)$. We recognize that this is the same Darboux covering (19) as for the Schwartz type $(1/2, 1/3, 1/5)$.

With a computer algebra package and standard algorithms [vH94], [Kov86] at hand, it is straightforward to pull-back a hypergeometric equation to a Darboux curve of genus 0, and solve the pull-backed differential equation. Lemma 3.7 characterizes hypergeometric functions which have to be identified with radical solutions. Since there are only 2 radical solutions by Lemma 3.6, computer algebra systems should return them. (Otherwise we may consider a simplified version of the procedure in Subsection 4.3.)

If the Darboux curve D has genus 1, we first wish to compute a convenient Weierstrass model from the equation $\psi(X) = \varphi(z)$. With such a model at hand, we identify D with the elliptic curve $(D, \mathcal{O})$, where $\mathcal{O}$ denotes the point at infinity. We intend to have the point $(x, \xi) = (0, 0)$ on D above $X = 0$, so to allow easy power series verification of the evaluations in (35)–(68). This intention eventuates conveniently.

The Darbox covering is given by the X-component of an isomorphism between the elliptic curve and the model $\psi(X) = \varphi(z)$. We wish to express the covering function in such a way that multiplicities of its zeroes and poles would be well visible. This is not a straightforward problem; it is discussed in Subsection 4.2. Pull-backs of hypergeometric equations onto elliptic curves and finding radical solutions of those pull-backs are discussed in Subsection 4.3. In the rest of this Subsection, we derive the four elliptic curves E_3, E_4, E_5, E_6, introduced in (33), (43), (53), (59) as Darboux curves for some icosahedral Schwartz types.

For the Schwartz type $(1/3, 1/3, 2/5)$, we use Lemma 3.5 with $H_0 = H(1/5, 1/2, 1/3)$ and $H_1 = H(2/5, 1/3, 1/3)$. This gives the equation $X^2/4(X-1) = \varphi_1(z)$ for the Darboux curve. After applying the fractional-linear transformation $F \mapsto F/(F-1)$ to both sides we get:

$$\frac{X^2}{(X-2)^2} = -\frac{1728\, z\, (z^2-11z-1)^5}{(z^2+1)^2\,(z^4-522z^3-10006z^2+522z+1)^2}. \tag{76}$$

We collect full squares onto the left-hand side and observe that the Darboux curve is isomorphic to the genus 1 curve

$$\widetilde{\xi}^2 = -1728\, z\, (z^2-11z-1). \tag{77}$$

This curve is isomorphic to E_3 via the isomorphism $(z, \widetilde{\xi}) \mapsto (3x, 72\xi)$. The Darboux covering is given by the X-component of the isomorphism between E_3 and (76). We have

$$\frac{X}{X-2} = \frac{72\,\xi\,(9x^2-33x-1)^2}{(9x^2+1)(81x^4-14094x^3-90054x^2+1566x+1)},$$

so

$$\varphi_3(x,\xi) = 2\Big/\left(1+\frac{(9x^2+1)(81x^4-14094x^3-90054x^2+1566x+1)}{72\,\xi\,(9x^2-33x-1)^2}\right). \tag{78}$$

Expression (34) is derived by methods of Subsection 4.2.

For the Schwartz type $(1/3, 2/3, 1/5)$, we use Lemma 3.5 with $H_0 = H(2/5, 1/2, 1/3)$ and $H_1 = H(4/5, 1/3, 1/3)$. Then we get the same equation $X^2/4(X-1) = \varphi_1(z)$. Hence the Darboux curve is the same as for the type $(1/3, 2/3, 1/5)$.

For the Schwartz type $(2/3, 1/5, 1/5)$, we use Lemma 3.5 with $H_0 = H(1/5, 1/2, 1/3)$ and $H_1 = H(1/5, 1/5, 2/3)$. This gives the equation $4X(1-X) = \varphi_1(z)$ for the Darboux curve. After applying the fractional-linear transformation $F \mapsto 1-F$ to both sides we get:

$$(1-2X)^2 = \frac{(z^2+1)^2\,(z^4-522z^3-10006z^2+522z+1)^2}{(z^4+228z^3+494z^2-228z+1)^3}. \tag{79}$$

We collect full squares to the left-hand side and observe that the Darboux curve is isomorphic to the genus 1 curve

$$\widetilde{\xi}^2 = z^4+228z^3+494z^2-228z+1. \tag{80}$$

This curve is isomorphic to E_4 by the isomorphism

$$(z, \widetilde{\xi}) \mapsto \left(\frac{57x-5\xi}{x+25}, \frac{25\,(\xi^2-570\xi-380x^2+248x+25)}{(x+25)^2}\right). \tag{81}$$

Like with $\varphi_3(x,\xi)$, we identify $\varphi_4(x,\xi)$ with the X-component of the isomorphism between E_4 and (79), and apply methods of Lemma 4.2 to get expression (44).

Elliptic curve	Mordell-Weil group	Rational points
E_3	$\mathbb{Z}/6\mathbb{Z}$	$\mathcal{O}$, $(0,0)$, $(-\frac{1}{9},\frac{5}{9})$, $(-\frac{1}{9},-\frac{5}{9})$, $(1,-5)$, $(1,5)$
E_4	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\mathcal{O}$, $\mathcal{O}^*=(0,0)$, $A_n=n\left(\frac{1}{5},\frac{3}{5}\right)$, $\widetilde{A}_n=-n\left(\frac{1}{5},\frac{3}{5}\right)$,
		$A_n^*=n\left(\frac{1}{5},\frac{3}{5}\right)+\mathcal{O}^*$, $\widetilde{A}_n^*=-n\left(\frac{1}{5},\frac{3}{5}\right)+\mathcal{O}^*$
E_5	$\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\mathcal{O}$, $(0,0)$, $(-\frac{1}{4},-\frac{3}{4})$, $(-\frac{1}{4},-\frac{3}{4})$,
		$(-1,0)$, $(-\frac{1}{16},0)$, $(\frac{1}{4},-\frac{5}{4})$, $(\frac{1}{4},\frac{5}{4})$
E_6	$\mathbb{Z}/6\mathbb{Z}$	$\mathcal{O}$, $(0,0)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$, $(1,1)$

Table 2: Rational points on elliptic curves

For the Schwartz type $(1/3,2/5,3/5)$, we use Lemma 3.5 with $H_0=H(2/5,1/2,1/3)$ and $H_1=H(2/5,2/5,2/3)$. Then we get the same equation $4X(1-X)=\varphi_1(z)$. Hence the Darboux curve is the same as for the type $(2/3,1/5,1/5)$.

For the Schwartz type $(1/3,1/5,3/5)$, we use Lemma 3.5 with $H_0=H(1/5,1/2,1/3)$ and $H_1=H(1/5,1/3,3/5)$. This gives the equation $-64X\big/(X-1)(9X-1)^3=\varphi_1(z)$ for the Darboux curve. This curve is isomorphic to E_5, though it is not straightforward to compute a handy isomorphism with current computer algebra packages. The package algcurves of Maple 9.0 can be used to obtain a Weierstrass form and an isomorphism. The isomorphism ought to be simplified using methods of Subsection 4.2. Eventually, we obtain an isomorphism given by $z=-2(\xi-3x)^2\big/(\xi+3x)(4x+1)$ and $X=\varphi_5(x,\xi)$ as in (54).

For the Schwartz type $(1/5,1/5,4/5)$, we use Lemma 3.5 with $H_0=H(1/5,1/2,2/5)$ and $H_1=H(1/5,1/5,4/5)$. This gives the equation $4X(1-X)=\varphi_2(z)$ for the Darboux curve. After applying the fractional-linear transformation $F\mapsto 1-F$ to both sides we get

$$(1-2X)^2=-\frac{64\,x\,(x^2-x-1)^5}{(x^2+1)^2(x^4-22x^3-6x^2+22x+1)^2}. \tag{82}$$

We collect full squares to the left-hand side and easily observe that the Darboux curve is isomorphic E_6.

For the Schwartz type $(2/5,2/5,2/5)$, we use Lemma 3.5 with $H_0=H(2/5,1/2,1/5)$ and $H_1=H(2/5,2/5,2/5)$. Then we get the same equation $4X(1-X)=\varphi_2(z)$. Hence the Darboux curve is the same as for the type $(1/5,1/5,4/5)$.

Notice that the elliptic curves E_3, E_4, E_5, E_6 are defined over $\mathbb{Q}$. It is useful to know rational points on them. Table 2 gives this arithmetic information [Sil86]. It was computed using Maple package Apecs [Con]. Recall that by $\mathcal{O}$ we denote the point at infinity. As we see, only the curve E_4 has infinitely many rational points. In Table 2 we introduce the notation $\mathcal{O}^*$, A_n, $\widetilde{A}_n$, A_n^*, $\widetilde{A}_n^*$ (with positive $n\in\mathbb{Z}$) for the rational points on E_4.

4.2 Representing functions on genus 1 curves

Here we consider the problem of representation of rational functions on elliptic curves. Foremost, we use techniques of this Subsection to compute expressions (34), (44), (54), (60) for Darboux coverings from genus 1 Darboux curves. Subsection 4.3 extends these techniques for computation of expressions on the right-hand sides of (35)–(42), (45)–(52), (55)–(58), (61)–(68).

A canonical way to represent a rational functions F on a (hyper)elliptic curve $\xi^2 = G(x)$ (with $G(x) \in \mathbb{C}[x]$) is the sum $f_1(x) + \xi f_2(x)$, with $f_1(x), f_2(x) \in \mathbb{C}(x)$. This representation suits well algebraic computations, but it gives little geometric information about the function. For example, the principal divisor for a function can be much simpler than the degree of $f_1(x)$ and $f_2(x)$ may suggest. We would like to have a compact expression that reflects well multiplicities in the principal divisor. We do not give strict definitions or algorithms for an alternative representation. Rather, we give Tables of principal divisors on the elliptic curves E_3, E_4, E_5, E_6, and propose to combine those principal divisors to make the divisor for F. The corresponding multiplicative expression in $\mathbb{C}[x, \xi]$-polynomials from our Tables will give, up to a constant multiple, a compact expression for F that we hopefully seek. We need to compute only finitely many rational and radical functions on elliptic curves, and our Tables will be give enough information for these purposes.

Concretely, we start with Darboux covering $\varphi_3(x, \xi)$. Its principal divisor can be computed from (78) to be the following:

$$(0,0) + \mathcal{O} + 5\left(\tfrac{11+5\sqrt{5}}{6}, 0\right) + 5\left(\tfrac{11-5\sqrt{5}}{6}, 0\right) - 3R_1 - 3R_2 - 3R_3 - 3R_4. \tag{83}$$

Here the four points R_1, R_2, R_3, R_4 are defined by the equations

$$81x^4 + 6156x^3 + 4446x^2 - 684x + 1 = 0, \qquad 150\xi = 27x^3 + 1989x^2 + 741x - 7. \tag{84}$$

Table 3 gives a list of principal divisors on E_3. For $i \in \{1, 2, 3, 4\}$, by $\widetilde{R}_i$ we denote the inverse of R_i in the group structure of E_3. Divisor (83) can be rewritten as follows:

$$\begin{aligned}\Big\{(0,0) + \left(\tfrac{11+5\sqrt{5}}{6}, 0\right) + \left(\tfrac{11-5\sqrt{5}}{6}, 0\right) - 3\mathcal{O}\Big\} + 2\Big\{2\left(\tfrac{11+5\sqrt{5}}{6}, 0\right) + 2\left(\tfrac{11-5\sqrt{5}}{6}, 0\right) - 4\mathcal{O}\Big\} \\ + \Big\{3\left(-\tfrac{1}{9}, -\tfrac{5}{9}\right) - 3\mathcal{O}\Big\} - 3\Big\{R_1 + R_2 + R_3 + R_4 + \left(-\tfrac{1}{9}, -\tfrac{5}{9}\right) - 5\mathcal{O}\Big\}.\end{aligned}$$

Observe that each divisor in curled brackets is present in Table 3. We can immediately build the corresponding multiplicative combination of the functions ξ, $1 + 33x - 9x^2$, $1 - 9\xi + 54x$, $1 + 21\xi - 117x + 9x\xi - 234x^2$. Up to undetermined constant multiple, the multiplicative expression is (34). The constant multiple can be determined by evaluating the multiplicative expression and (78) a convenient point, say $(-\frac{1}{9}, \frac{5}{9})$.

As an extra exercise, one may consider the function $1 - \varphi_3$. Its divisor can be computed from (34) or (78) to be

$$3\widetilde{R}_1 + 3\widetilde{R}_2 + 3\widetilde{R}_3 + 3\widetilde{R}_4 - 3R_1 - 3R_2 - 3R_3 - 3R_4. \tag{85}$$

Function	Divisor
ξ	$(0,0)+\left(\frac{11+5\sqrt{5}}{6},0\right)+\left(\frac{11-5\sqrt{5}}{6},0\right)-3\mathcal{O}$
$1+33x-9x^2$	$2\left(\frac{11+5\sqrt{5}}{6},0\right)+2\left(\frac{11-5\sqrt{5}}{6},0\right)-4\mathcal{O}$
$1-9\xi+54x$	$3\left(-\frac{1}{9},-\frac{5}{9}\right)-3\mathcal{O}$
$1+9\xi+54x$	$3\left(-\frac{1}{9},\frac{5}{9}\right)-3\mathcal{O}$
$1+9x$	$\left(-\frac{1}{9},-\frac{5}{9}\right)+\left(-\frac{1}{9},\frac{5}{9}\right)-2\mathcal{O}$
$\xi+5x$	$(0,0)+(-\frac{1}{9},\frac{5}{9})+(1,-5)-3\mathcal{O}$
$1+21\xi-117x+9x\xi-234x^2$	$R_1+R_2+R_3+R_4+\left(-\frac{1}{9},-\frac{5}{9}\right)-5\mathcal{O}$
$1-21\xi-117x-9x\xi-234x^2$	$\widetilde{R}_1+\widetilde{R}_2+\widetilde{R}_3+\widetilde{R}_4+\left(-\frac{1}{9},\frac{5}{9}\right)-5\mathcal{O}$

Table 3: Principal divisors on E_3

A straightforward combinatorial work suggests the expression

$$1-\varphi_3(x,\xi)=\frac{(1-21\xi-117x-9x\xi-234x^2)^3\,(1-9\xi+54x)}{(1+21\xi-117x+9x\xi-234x^2)^3\,(1+9\xi+54x)}. \tag{86}$$

Our proposal boils down in building a sufficient table of principal divisors, and combining the known principal divisors to arrive at the principal divisor of a target function. In practise, both things are done in parallel. We start with the functions we wish to express, and compute their divisors. We look at $\mathbb{Q}$-rational points that occur, and use knowledge of the Mordell-Weil group (see Table 2) to foresee and compute suitable $\mathbb{C}[x,\xi]$-polynomials that vanish only on rational points. Then we distinguish $\mathbb{Q}$-irreducible divisor components of higher degree. For each such divisor component Γ, we use Gröbner bases to find $\mathbb{C}[x,\xi]$-polynomials of minimal degree that vanish on Γ with sufficient multiplicities. We choose those polynomials whose divisors enlarge Γ minimally or least awkwardly. We look at additional components that occur (usually they are rational points); if they are new, we introduce new polynomials that could compensate the additional components.

For example, consider the $\mathbb{Q}$-irreducible component $R_1+R_2+R_3+R_4$ defined by (84). A Gröbner basis gives the following quadratic polynomials that vanish on it:

$$9x\xi-234x^2+21\xi-117x+1, \quad 3\xi^2-2088x^2+150\xi-744x+7. \tag{87}$$

Other quadratic polynomials are obtained by linear combination. We have chosen the first polynomial in (87), and consequently we had to compensate its additional component $(-\frac{1}{9},-\frac{5}{9})$. A reasonable alternative is the quadratic polynomial $\xi^2-21x\xi-150x^2+\xi+25x$, whose divisor is $R_1+R_2+R_3+R_4+(0,0)+(1,-5)-6\mathcal{O}$.

Now consider computation of expression (44) for the Darboux covering $\varphi_4(x,\xi)$. This is the most complicated case, so our description of the computational method reaches deeper refinement level. Let us introduce the functions N and L on the rational points of E_4:

$$\begin{array}{lllll} N(A_n)=n, & N(A_n^*)=n, & N(\widetilde{A}_n)=-n, & N(\widetilde{A}_n^*)=-n, & N(\mathcal{O}^*)=0, \\ L(A_n)=0, & L(A_n^*)=1, & L(\widetilde{A}_n)=0, & L(\widetilde{A}_n^*)=1, & L(\mathcal{O}^*)=1. \end{array} \tag{88}$$

Principal divisors of $\mathbb{C}[x,\xi]$-polynomials have the form $\sum_{j=1}^n S_j - n\mathcal{O}$. If all points S_j are rational, by Lemma 5.1 we must have

$$\sum_{j=1}^{n} N(S_j)=0 \quad \text{and} \quad \sum_{j=1}^{n} L(S_j) \quad \text{even.} \tag{89}$$

A preliminary expression for $\varphi_4(x,\xi)$ can be computed by composing the obvious isomorphism between the curves in (79) and (80) with isomorphism (81). The principal divisor of $\varphi_4(x,\xi)$ is:

$$(0,0)+\mathcal{O}+5P_1+5P_2-\left(\tfrac{1}{125},\tfrac{57}{625}\right)-(-25,285)-5Q_1-5Q_2, \tag{90}$$

where

$$P_{1,2}=\left(-\tfrac{3}{2}\pm\tfrac{7}{2\sqrt{5}},7\mp 3\sqrt{5}\right), \qquad Q_{1,2}=\left(-4\pm\tfrac{9}{\sqrt{5}},-27\pm 12\sqrt{5}\right).$$

In the notation of Table 2, we have $\left(\frac{1}{125},\frac{57}{625}\right)=A_5$ and $(-25,285)=A_5^*$.

It seems convenient to consider the lines through P_1, P_2 and through Q_1, Q_2. Their equations are $7\xi=4-30x$ and $3\xi=20x-1$, respectively. The third points on these two lines are, respectively, $\widetilde{A}_6^*$ and $\widetilde{A}_4^*$. But if we add and subtract extra divisor terms $5\widetilde{A}_6^*$ and $5\widetilde{A}_4^*$ in (90), it is very cumbersome to compensate them due to (89). The $\mathbb{C}[x,\xi]$ polynomials for compensating principal divisors are expected to have very large coefficients.

We may define functions in (88) on the divisors P_1+P_2 and Q_1+Q_2 and keep consistency with (89) by considering the two divisors as equivalents of A_6^* and A_4^*, respectively. Rather than introducing the sub-expression $P_1+P_2+\widetilde{A}_6^*-3\mathcal{O}$ in (90), we may try to work with the principal divisor $P_1+P_2+\widetilde{A}_4^*+\widetilde{A}_2-4\mathcal{O}$. Then $\widetilde{A}_4^*$ is compensated automatically. We can work out the following expressions of (90) as sums of principal divisors:

$$\begin{aligned} &5\left(P_1+P_2+\widetilde{A}_4^*+\widetilde{A}_2-4\mathcal{O}\right)+\left(A_5+\widetilde{A}_5^*+\mathcal{O}^*-3\mathcal{O}\right) \\ &-5\left(Q_1+Q_2+\widetilde{A}_4^*-3\mathcal{O}\right)-\left(5\widetilde{A}_2+2A_5-7\mathcal{O}\right)-\left(A_5^*+\widetilde{A}_5^*-2\mathcal{O}\right), \end{aligned}$$

and

$$\begin{aligned} &5\left(P_1+P_2+\widetilde{A}_4^*+\widetilde{A}_2-4\mathcal{O}\right)+\left(A_{10}+\widetilde{A}_{10}^*+\mathcal{O}^*-3\mathcal{O}\right) \\ &-5\left(Q_1+Q_2+\widetilde{A}_4^*-3\mathcal{O}\right)-\left(5\widetilde{A}_2+A_{10}-6\mathcal{O}\right)-\left(A_5+A_5^*+\widetilde{A}_{10}^*-3\mathcal{O}\right). \end{aligned}$$

Now we can build a table of $\mathbb{C}[x,\xi]$-polynomials of the involved principal divisors. Eventually, the two decompositions of (90) give the following expressions for $\varphi_4(x,\xi)$:

$$\frac{8208\,(1-7y+15x+15x^2)^5\;(-5\xi+57x)}{(1+3\xi-20x)^5(59375x^2\xi+12350x\xi+475\xi-166250x^3-49875x^2-3800x-19)(x+25)},$$

Function	Divisor	Function	Divisor
x	$2\,\mathcal{O}^* - 2\mathcal{O}$	$1-15x-5x^2$	$P_1+P_2+\widetilde{P}_1+\widetilde{P}_2-4\mathcal{O}$
$1-5x$	$A_1+\widetilde{A}_1-2\mathcal{O}$	$1-40x-5x^2$	$Q_1+Q_2+\widetilde{Q}_1+\widetilde{Q}_2-4\mathcal{O}$
$25+x$	$A_5^*+\widetilde{A}_5^*-2\mathcal{O}$	$5-7\xi-45x-5x^2$	$P_1+P_2+\widetilde{A}_5^*+\widetilde{A}_1-4\mathcal{O}$
$1-125x$	$A_5+\widetilde{A}_5-2\mathcal{O}$	$5+18\xi-80x+5x^2$	$Q_1+Q_2+\widetilde{A}_5^*+A_1-4\mathcal{O}$
$5\xi+57x$	$A_5^*+\widetilde{A}_5+\mathcal{O}^*-3\mathcal{O}$	$1-7y+15x+15x^2$	$P_1+P_2+\widetilde{A}_4^*+\widetilde{A}_2-4\mathcal{O}$
$-5\xi+57x$	$\widetilde{A}_5^*+A_5+\mathcal{O}^*-3\mathcal{O}$	$4-7\xi-30x$	$P_1+P_2+\widetilde{A}_6^*-3\mathcal{O}$
$1+5\xi+10x$	$2\widetilde{A}_1+A_2-3\mathcal{O}$	$1+3\xi-20x$	$Q_1+Q_2+\widetilde{A}_4^*-3\mathcal{O}$
$1-3\xi+2x$	$A_2+A_2^*+\widetilde{A}_4^*-3\mathcal{O}$	$1-8\xi+22x-15x^2$	$2A_1+A_2^*+\widetilde{A}_4^*-4\mathcal{O}$
$4+21\xi+41x$	$A_2+A_4^*+\widetilde{A}_6^*-3\mathcal{O}$	$4-35\xi-101x$	$A_1^*+A_5+\widetilde{A}_6^*-3\mathcal{O}$
$5-3\xi-34x$	$A_5^*+\widetilde{A}_4^*+\widetilde{A}_1-3\mathcal{O}$	$20-7\xi-79x$	$A_1+A_5^*+\widetilde{A}_6^*-3\mathcal{O}$

Function	Divisor
$1+50x-125\xi^2+450x\xi-500x^2$	$5A_1+\widetilde{A}_5-6\mathcal{O}$
$1+50x-125\xi^2-450x\xi-500x^2$	$5\widetilde{A}_1+A_5-6\mathcal{O}$
$25-570\xi+248x+\xi^2-380x^2$	$R_1+R_2+R_3+R_4+2\widetilde{A}_5^*-6\mathcal{O}$
$4+95\xi+83x+21\xi^2-475x\xi+40x^2$	$R_1+R_2+R_3+R_4+\widetilde{A}_6^*+\widetilde{A}_4^*-6\mathcal{O}$

Table 4: Principal divisors on E_4

Function	Divisor
$\xi+5x$	$2\left(\frac{1}{4},-\frac{5}{4}\right)+(0,0)-3\mathcal{O}$
$\xi-5x$	$2\left(\frac{1}{4},\frac{5}{4}\right)+(0,0)-3\mathcal{O}$
$\xi+3x$	$2\left(-\frac{1}{4},\frac{3}{4}\right)+(0,0)-3\mathcal{O}$
$\xi-3x$	$2\left(-\frac{1}{4},-\frac{3}{4}\right)+(0,0)-3\mathcal{O}$
$1+\xi+x$	$\left(-\frac{1}{4},-\frac{3}{4}\right)+\left(\frac{1}{4},-\frac{5}{4}\right)+(-1,0)$
$1-2\xi+6x$	$P_1+P_2+\left(\frac{1}{4},\frac{5}{4}\right)-3\mathcal{O}$
$1+12x+16x^2$	$P_1+P_2+\widetilde{P}_1+\widetilde{P}_2-4\mathcal{O}$
$1-2\xi-14x$	$Q_1+Q_2+\left(\frac{1}{4},-\frac{5}{4}\right)-3\mathcal{O}$
$1-28x+16x^2$	$Q_1+Q_2+\widetilde{Q}_1+\widetilde{Q}_2-4\mathcal{O}$
$1+8\xi-28x+8x\xi-104x^2$	$R_1+R_2+R_3+R_4+\left(\frac{1}{4},\frac{5}{4}\right)-5\mathcal{O}$

Table 5: Divisors on E_5

and, respectively,

$$\frac{432\,(1-7\xi+15x+15x^2)^5\,(49495\xi+292441x)}{(1+3\xi-20x)^5(2375\xi^2+7400x\xi+1150\xi-42000x^2-10200x-521)(2605\xi+29678x-475)}.$$

Compensation of $5\widetilde{A}_2$ look quite awkward in both formulas.

As the last attempt, we try to introduce $P_1+P_2+\widetilde{A}_5^*+\widetilde{A}_1-4\mathcal{O}$ and $Q_1+Q_2+\widetilde{A}_5^*+A_1-4\mathcal{O}$ in (90), forgetting divisors of the linear polynomials $7\xi+30x-4$ and $3\xi-20x+1$. A natural effort to compensate $5\widetilde{A}_1$ and $5A_1$ leads to the following expression decomposition of (90):

$$\begin{aligned} & 5\left(P_1+P_2+\widetilde{A}_5^*+\widetilde{A}_1-4\mathcal{O}\right)+\left(5A_1+\widetilde{A}_5-6\mathcal{O}\right)+(2\mathcal{O}^*-2\mathcal{O}) \\ -5&\left(Q_1+Q_2+\widetilde{A}_5^*+A_1-4\mathcal{O}\right)-\left(5\widetilde{A}_1+A_5-6\mathcal{O}\right)-\left(A_5^*+\widetilde{A}_5+\mathcal{O}^*-3\mathcal{O}\right). \end{aligned}$$

This expression gives formula (44). The most convenient principal divisors are listed in Table 4. The points R_1, R_2, R_3, R_4 on E_4 are the points in the fiber $z=1$ of $\varphi_4(x,\xi)$.

The principal divisor for $\varphi_5(x,\xi)$ turns out to be

$$(0,0)+\mathcal{O}+5P_1+5P_2-\left(-\tfrac{1}{4},\tfrac{3}{4}\right)-\left(-\tfrac{1}{4},-\tfrac{3}{4}\right)-5Q_1-5Q_2, \tag{91}$$

where

$$P_{1,2}=\left(\tfrac{-3\pm\sqrt{5}}{8},\tfrac{-5\pm3\sqrt{5}}{8}\right), \qquad Q_{1,2}=\left(\tfrac{7\pm3\sqrt{5}}{8},\tfrac{-45\mp21\sqrt{5}}{8}\right).$$

As mentioned in Subsection 4.1, computation of a preliminary expression for $\varphi_5(x,\xi)$ was not straightforward. A natural effort leads to the following expression decomposition of (91):

$$\begin{aligned} & 3\left(2\left(\tfrac{1}{4},-\tfrac{5}{4}\right)+(0,0)-3\mathcal{O}\right)+5\left(P_1+P_2+\left(\tfrac{1}{4},\tfrac{5}{4}\right)-3\mathcal{O}\right) \\ & \quad -\left(\left(\tfrac{1}{4},\tfrac{5}{4}\right)+\left(\tfrac{1}{4},-\tfrac{5}{4}\right)+\left(-\tfrac{1}{4},\tfrac{3}{4}\right)+\left(-\tfrac{1}{4},-\tfrac{3}{4}\right)-4\mathcal{O}\right) \\ & -2\left(2\left(\tfrac{1}{4},\tfrac{5}{4}\right)+(0,0)-3\mathcal{O}\right)-5\left(Q_1+Q_2+\left(\tfrac{1}{4},-\tfrac{5}{4}\right)-3\mathcal{O}\right). \end{aligned} \tag{92}$$

This expression gives formula (54). The most convenient principal divisors are listed in Table 5. The points R_1, R_2, R_3, R_4 on E_5 are the points in the fiber $z=1$ of $\varphi_5(x,\xi)$.

The principal divisor for $\varphi_6(x,\xi)$ turns out to be

$$(0,0)+\mathcal{O}+5P_1+5P_2-(1,1)-(-1,1)-5Q_1-5Q_2. \tag{93}$$

where

$$P_{1,2}=\left(\tfrac{1\pm\sqrt{5}}{2},0\right), \qquad Q_{1,2}=\left(-2\pm\sqrt{5},-5\pm2\sqrt{5}\right).$$

We naturally arrive at the following expression for (93):

$$\begin{aligned} & \left(P_1+P_2+(0,0)-\mathcal{O}\right)+2\left(2P_1+2P_2-4\mathcal{O}\right)+2\left(2(1,1)+(1,-1)-2\mathcal{O}\right) \\ & \qquad -\left(3(-1,1)-3\mathcal{O}\right)-5\left(Q_1+Q_2+(1,1)-3\mathcal{O}\right). \end{aligned}$$

This expression gives formula (60). The most convenient principal divisors are listed in Table 6. The points R_1, R_2 on E_6 are in the fiber $z=1$ of $\varphi_6(x,\xi)$.

Function	Divisor	Function	Divisor
ξ	$(0,0)+P_1+P_2-3\mathcal{O}$	$1+x-x^2$	$2P_1+2P_2-4\mathcal{O}$
$1-\xi$	$(-1,1)+2(1,1)-3\mathcal{O}$	$1-4x-x^2$	$Q_1+Q_2+R_1+R_2-4\mathcal{O}$
$1+\xi$	$(-1,-1)+2(1,-1)-3\mathcal{O}$	$\xi+2x+x^2$	$(0,0)+Q_1+Q_2-3\mathcal{O}$
$1+\xi+2x$	$3(-1,1)-3\mathcal{O}$	$1+\xi-2x$	$Q_1+Q_2+(1,1)-3\mathcal{O}$
$1-\xi+2x$	$3(-1,-1)-3\mathcal{O}$	$1-\xi-2x$	$R_1+R_2+(1,-1)-3\mathcal{O}$

Table 6: Divisors on E_6

4.3 Computation of hypergeometric evaluations

In principle, evaluations (35)–(42), (45)–(52), (55)–(58), (61)–(68) can be computed by pulling-back their hypergeometric equations onto E_3, E_4, E_5, E_6, respectively, and finding radical solutions of the pull-backed differential equations. We use divisors (on Darboux curves) with coefficients in $\mathbb{Q}$, introduced in Appendix 5.1. But standard computer algebra systems do not handle differential equations on higher genus curves.

The pull-backed equations are cumbersome as we will see. On the other hand, their singular points and local exponent differences are quite easy to see, see Appendix 5.5. The local exponents tell us possible coefficients in the principal divisors of the radical solutions. Possible principal divisors are restricted by Lemma 5.2 and Lemma 3.6. For simplest hypergeometric equations, we may end up with just 2 possible principal divisors for radical solutions; then we can find those solutions without computing the pull-back equation explicitly. Additional contiguous evaluations can be obtained by differentiating known solutions and contiguous relations, while respecting Lemma 3.8 and avoiding explicit computation with pull-back equation again.

Otherwise we may have several candidates for radical solutions, which we must check by substituting into the pull-back equation. If we have one undetermined simple zero for a radical solution, its location can be restricted by the arithmetical argument that the principal divisor should be invariant under the Galois action of $\overline{\mathbb{Q}}$.

We start with computation of evaluations (35)–(38). We use Riemann notation (102) and consider the icosahedral hypergeometric equation with the solution space

$$P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & -1/30 & z \\ 2/5 & 1/3 & 3/10 & \end{array}\right\}.$$

Its pull-back $z\mapsto\varphi_3(x,\xi)$, $y(z)\mapsto Y(\varphi_3(x,\xi))$ onto E_3 has the following regular singular points and local exponents:

$$\left\{ \begin{array}{cccccccc} (0,0) & \mathcal{O} & \left(\frac{11+5\sqrt{5}}{6},0\right) & \left(\frac{11-5\sqrt{5}}{6},0\right) & R_1 & R_2 & R_3 & R_4 \\ 0 & 0 & 0 & 0 & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ \frac{2}{5} & \frac{2}{5} & 2 & 2 & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} \end{array} \right\}$$

Condition *(i)* of Lemma 5.2 leaves only two candidates for the divisors of radical solutions:

$$\tfrac{2}{5}\,\mathcal{O} - \tfrac{1}{10}R_1 - \tfrac{1}{10}R_2 - \tfrac{1}{10}R_3 - \tfrac{1}{10}R_4, \qquad \tfrac{2}{5}\,(0,0) - \tfrac{1}{10}R_1 - \tfrac{1}{10}R_2 - \tfrac{1}{10}R_3 - \tfrac{1}{10}R_4. \tag{94}$$

They must represent divisors of the two solutions of Lemma (3.6) and, by identification of local solutions, the hypergeometric series (35) and (37). Since the point $x=0$ corresponds to $(0,0)\in E_3$, the first divisor obviously the series in (35). The right-hand side of that formula is easy to construct from the divisor by using Table 3 and the known value of left-hand side at $x=0$. Similarly, the second divisor in (94) implies the identity

$$\varphi_3(x,\xi)^{2/5}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c} 7/10, 11/30 \\ 7/5 \end{array}\right| \varphi_3(x,\xi)\right) = \frac{x^{1/5}\,(1-9\xi+54x)^{1/30}}{(1+21\xi-117x+9x\xi-234x^2)^{1/10}}.$$

This gives formula (37). We derived these identities without even computing the pull-back differential equation on E_3. Now we can derive formulas (36), (38) without any reference to pull-back equations as well. Contiguous companion (36) for (35) can be computed (using a computer algebra package) as follows:

1. Differentiate both sides of (36) and obtain an expression for ${}_2\mathrm{F}_1\left(\left.\begin{array}{c} 13/10, 29/30 \\ 8/5 \end{array}\right| \varphi_3\right)$. At this stage we are satisfied with any radical expression for this function.

2. Using contiguous relations, find some expressions for the hypergeometric series which are contiguous to the series in (35), and the integer differences between the corresponding upper and lower parameters are at most ± 1.

3. Find $\mathbb{Q}$-principal divisors of the just computed expressions. Choose one divisor which looks most convenient. For example, we chose the following divisor for ${}_2\mathrm{F}_1\left(\left.\begin{array}{c} 3/10, 29/30 \\ 3/5 \end{array}\right| \varphi_3\right)$:

$$\tfrac{9}{10}R_1 + \tfrac{9}{10}R_2 + \tfrac{9}{10}R_3 + \tfrac{9}{10}R_4 + S_1 + S_2 + \widetilde{S}_1 + \widetilde{S}_2 + \tfrac{2}{5}\mathcal{O} - 2\widetilde{R}_1 - 2\widetilde{R}_2 - 2\widetilde{R}_3 - 2\widetilde{R}_4,$$

where $S_1, S_2, \widetilde{S}_1, \widetilde{S}_2$ are the points with the x-coordinate equal to $1 \pm 10/3\sqrt{11}$. An example of other candidate is the following divisor for ${}_2\mathrm{F}_1\left(\left.\begin{array}{c} 3/10, 29/30 \\ 8/5 \end{array}\right| \varphi_3\right)$:

$$\tfrac{9}{10}R_1 + \tfrac{9}{10}R_2 + \tfrac{9}{10}R_3 + \tfrac{9}{10}R_4 + \widehat{P}_1 + \widehat{P}_2 + \widehat{P}_3 - \tfrac{3}{5}\mathcal{O} - 3\left(\tfrac{11+5\sqrt{5}}{6},0\right) - 3\left(\tfrac{11-5\sqrt{5}}{6},0\right),$$

where the points $\widehat{P}_1, \widehat{P}_2, \widehat{P}_3$ are defined by the equations $9x^3+378x^2+117x-4=0$, $5\xi = 3x^2+24x-2$.

4. Using Table 3 and techniques of Subsection 4.2, compute a convenient expression for a fractional-linear function with the chosen principal divisor. For example, we rewrite our chosen divisor as

$$\tfrac{9}{10}\left\{R_1+R_2+R_3+R_4+\left(-\tfrac19,-\tfrac59\right)-5\mathcal{O}\right\}-2\left\{\widetilde{R}_1+\widetilde{R}_2+\widetilde{R}_3+\widetilde{R}_4+\left(-\tfrac19,\tfrac59\right)-5\mathcal{O}\right\}$$
$$+\left\{S_1+S_2+\widetilde{S}_1+\widetilde{S}_2-4\mathcal{O}\right\}+2\left\{\left(-\tfrac19,-\tfrac59\right)+\left(-\tfrac19,\tfrac59\right)-2\mathcal{O}\right\}-\tfrac{29}{30}\left\{3\left(-\tfrac19,-\tfrac59\right)-3\mathcal{O}\right\}$$

and replace the five terms by the corresponding polynomials from Table 3.

For more comfortable calculations, we can express contiguous function in the form $G(x,\xi)\times {}_2\mathrm{F}_1\left(\left.\begin{array}{c}3/10,\,-1/30\\3/5\end{array}\right|\,\varphi_3\right)$ with $G(x,\xi)\in\mathbb{C}(E_3)$, following Lemma 3.8. A contiguous companion for (37), such as (38), can be found by a similar procedure.

Now we consider computation of evaluations (39)–(42). The pull-back $z\mapsto\varphi_3(x,\xi)$, $y(z)\mapsto Y(\varphi_3(x,\xi))$ of the icosahedral hypergeometric equation with the solution space

$$P\left\{\begin{array}{cccc}0&1&\infty&\\0&0&-1/10&z\\1/5&1/3&17/30&\end{array}\right\}.$$

is the following differential equation (with coefficients in a convenient form):

$$\begin{aligned}Y''\;&+\;\frac{3(3+47\xi+1974x-2051\xi^2+2676x\xi+54348x^2+33\xi^3-1002x\xi^2-1548x^2\xi)}{10\,\xi^2\,(1+21\xi-117x+9x\xi-234x^2)}\,Y'\\&+\;\frac{51\,\xi\,(1-9\xi+54x)\,(1-21\xi-117x-9x\xi-234x^2)}{25\,x^2\,(1+9x)\,(1+21\xi-117x+9x\xi-234x^2)^2}\,Y\;=\;0.\end{aligned}\tag{95}$$

Like in the previous case above, we know singularities and local exponents of this equation without cumbersome computations. Here they are:

$$\left\{\begin{array}{cccccc}(0,0)&\mathcal{O}&R_1&R_2&R_3&R_4\\0&0&-\frac{3}{10}&-\frac{3}{10}&-\frac{3}{10}&-\frac{3}{10}\\\frac15&\frac15&\frac{17}{10}&\frac{17}{10}&\frac{17}{10}&\frac{17}{10}\end{array}\right\}$$

Condition *(i)* of Lemma 5.2 gives the following candidates for the divisors of radical solutions:

$$\tfrac15\,\mathcal{O}+X-\tfrac{3}{10}R_1-\tfrac{3}{10}R_2-\tfrac{3}{10}R_3-\tfrac{3}{10}R_4,\quad \tfrac15\,(0,0)+Y-\tfrac{3}{10}R_1-\tfrac{3}{10}R_2-\tfrac{3}{10}R_3-\tfrac{3}{10}R_4.$$

Here X and Y are regular points of (95). By condition *(ii)* and the additional statement of the same lemma, these should be torsion points on E_3 defined over $\mathbb{Q}$. The possibilities for X and Y are: $(-\frac19,-\frac59)$, $(-\frac19,\frac59)$, $(1,-5)$, $(1,5)$. This gives 8 possible divisors of a radical solution. For each possibility, one has to construct a radical function (in any form) with that divisor, and to check whether it is a solution of (95). Alternatively, candidate solutions can be expanded in power series around $(x,\xi)=(0,0)$ and compared with the hypergeometric series in (39) and (41). Then one does not have to know explicit equation (95), but has to find enough power series terms of *all* candidate solutions (so that the right candidates could be selected). It turns out that actual solutions have $X = (-\frac{1}{9}, -\frac{5}{9})$ and $Y = (1, -5)$. These two solutions and evaluations (39), (41) can be expressed in a convenient form by using methods of subsection 4.2. Eventually, both companion evaluations (40), (42) can be obtained by the same four-step procedure as in the previous case.

In principle, one can find all our evaluations by combining the local exponents of the corresponding pull-back equations, and trying all candidate principal divisors of possible solutions. But the number of possibilities is usually very large. Needless to say, all our computations need assistance of computer algebra packages.

Now we consider evaluation of formulas (45)–(48). A pull-back of a hypergeometric equation with the local exponent differences $(1/5, 1/5, 2/3)$ has the following singularities and local exponents:

$$\left\{ \begin{array}{cccccccccc} \mathcal{O}^* & \mathcal{O} & R_1 & R_2 & R_3 & R_4 & A_5 & A_5^* & Q_1 & Q_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{30} & -\frac{1}{30} & -\frac{1}{6} & -\frac{1}{6} \\ \frac{1}{5} & \frac{1}{5} & 2 & 2 & 2 & 2 & \frac{1}{6} & \frac{1}{6} & \frac{5}{6} & \frac{5}{6} \end{array} \right\}$$

Condition *(i)* of Lemma 5.2 gives the following candidates for the divisors of radical solutions:

$$\begin{array}{rr} -\frac{1}{6}Q_1 - \frac{1}{6}Q_2 + \frac{1}{6}A_5 + \frac{1}{6}A_5^*, & \frac{1}{5}\,\mathcal{O} + \frac{1}{5}\,(0,0) - \frac{1}{6}Q_1 - \frac{1}{6}Q_2 - \frac{1}{30}A_5 - \frac{1}{30}A_5^*, \\ \frac{1}{5}\mathcal{O} - \frac{1}{6}Q_1 - \frac{1}{6}Q_2 + \frac{1}{6}A_5 - \frac{1}{30}A_5^*, & \frac{1}{5}(0,0) - \frac{1}{6}Q_1 - \frac{1}{6}Q_2 - \frac{1}{30}A_5 + \frac{1}{6}A_5^*, \\ \frac{1}{5}\mathcal{O} - \frac{1}{6}Q_1 - \frac{1}{6}Q_2 - \frac{1}{30}A_5 + \frac{1}{6}A_5^*, & \frac{1}{5}(0,0) - \frac{1}{6}Q_1 - \frac{1}{6}Q_2 + \frac{1}{6}A_5 - \frac{1}{30}A_5^*. \end{array}$$

The candidate divisors are grouped into possible pairs of divisors of actual solutions according to Lemma 3.6. The divisors in the first pair do not satisfy condition *(ii)* of Lemma 5.2. To decide the right pair, one may take one divisor from each of the one two pairs, construct an expression for a corresponding radical function, and compare its power series around $(0, 0)$. The last pair is the right one. Once we have the right divisors for (45) and (47), we can proceed similarly as in the pervious cases. Application of the methods of Subsection 4.2 may require some combinatorial creativeness. For example, here is a convenient splitting of the right divisor for (45):

$$\frac{1}{6}\left(\widetilde{A}_4^* + A_5^* + \widetilde{A}_1 - 3\mathcal{O}\right) - \frac{1}{6}\left(Q_1 + Q_2 + \widetilde{A}_4^* - 3\mathcal{O}\right) - \frac{1}{30}\left(5\widetilde{A}_1 + A_5 - 6\mathcal{O}\right).$$

Now we consider evaluation of formulas (49)–(52). We comment only the most complicated step of choosing the divisors of the actual solutions of the pull-back of a hypergeometric equation with the local exponent differences $(2/5, 3/5, 1/3)$. The singularities and local exponents are the following:

$$\left\{ \begin{array}{cccccccc} \mathcal{O}^* & \mathcal{O} & P_1 & P_2 & A_5 & A_5^* & Q_1 & Q_2 \\ 0 & 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{6} & -\frac{5}{6} & -\frac{5}{6} \\ \frac{2}{5} & \frac{2}{5} & 2 & 2 & \frac{13}{30} & \frac{13}{30} & \frac{13}{6} & \frac{13}{6} \end{array} \right\}$$

Condition *(i)* of Lemma 5.2 gives the following candidates for the divisors of radical solutions:

$$-\tfrac{1}{6}\,A_5-\tfrac{1}{6}\,A_5^*-\tfrac{5}{6}\,Q_1-\tfrac{5}{6}\,Q_2+2P_1, \qquad -\tfrac{1}{6}\,A_5-\tfrac{1}{6}\,A_5^*-\tfrac{5}{6}\,Q_1-\tfrac{5}{6}\,Q_2+2P_2,$$
$$\begin{array}{rl} -\tfrac{1}{6}\,A_5-\tfrac{1}{6}\,A_5^*-\tfrac{5}{6}\,Q_1-\tfrac{5}{6}\,Q_2+X+Y, & X+Y\in\{A_5,A_5^*\} \\ \tfrac{2}{5}\mathcal{O}+\tfrac{2}{5}\mathcal{O}^*+\tfrac{13}{30}\,A_5+\tfrac{13}{30}\,A_5^*-\tfrac{5}{6}\,Q_1-\tfrac{5}{6}\,Q_2, & \\ \tfrac{2}{5}\mathcal{O}+\tfrac{13}{30}\,A_5-\tfrac{1}{6}\,A_5^*-\tfrac{5}{6}\,Q_1-\tfrac{5}{6}\,Q_2+X, & X\in\{A_2,A_2^*\} \\ \tfrac{2}{5}\mathcal{O}-\tfrac{1}{6}\,A_5+\tfrac{13}{30}\,A_5^*-\tfrac{5}{6}\,Q_1-\tfrac{5}{6}\,Q_2+X, & X\in\{A_2,A_2^*\}, \\ \tfrac{2}{5}\mathcal{O}^*+\tfrac{13}{30}\,A_5-\tfrac{1}{6}\,A_5^*-\tfrac{5}{6}\,Q_1-\tfrac{5}{6}\,Q_2+X, & X\in\{A_2,A_2^*\} \\ \tfrac{2}{5}\mathcal{O}^*-\tfrac{1}{6}\,A_5+\tfrac{13}{30}\,A_5^*-\tfrac{5}{6}\,Q_1-\tfrac{5}{6}\,Q_2+X, & X\in\{A_2,A_2^*\}. \end{array}$$

Here the restrictions on the additional points X, Y follow from condition *(ii)* and the additional statement of Lemma 5.2. The first three possibilities can paired only with the fourth divisor as (divisors of) functions f_1, f_2 of Lemma 3.6. To refute them, one has to check only that a function with the fourth divisor (as the principal divisor) is not a solution of the pull-back equation. Other possibilities have to be paired and checked as we did for equation (95). The right divisors are these:

$$\tfrac{2}{5}\mathcal{O}+\tfrac{13}{30}\,A_5-\tfrac{1}{6}\,A_5^*-\tfrac{5}{6}\,Q_1-\tfrac{5}{6}\,Q_2+A_2^*, \quad \tfrac{2}{5}\mathcal{O}^*-\tfrac{1}{6}\,A_5+\tfrac{13}{30}\,A_5^*-\tfrac{5}{6}\,Q_1-\tfrac{5}{6}\,Q_2+A_2.$$

Now we can compute (49), (51) like in the previous cases, etc.

Computation of evaluations (55)–(58), (61)–(68) is similar and no more complicated: first find singularities and local exponents of the pull-backed equation (of a corresponding main hypergeometric equation); make a list of possible divisors for radical solutions of the pull-back; use Lemma 3.6 to make a short divisor list for necessary check, find radical functions for the candidate divisors from the short list; compare their power series around $(0,0)$ with the expansions of the hypergeometric series; take the right divisors and find a convenient expression for their functions (using methods of Subsection 4.2); and by the described four-step procedure find evaluations of the two contiguous companion hypergeometric series.

5 Appendix

Here we recall definitions and facts which are important to us. This material is widely known, but quite rarely presented in a way which is most convenient for our purposes. We concentrate the details that we use. For similar introductions, we refer to [vdW02], [Beu02], [Ber04].

5.1 Algebraic curves

For general theory of algebraic curves we refer to [Ful69] or to [Sha74]. We assume algebraic curves to be reduced, irreducible, smooth, complete (or projective), defined over $\mathbb{C}$. In particular, the projective line $\mathbb{P}^1$ is $\mathbb{C}\cup\{\infty\}$ set-theoretically. (All curves in this paper are defined over $\mathbb{Q}$, but we do not consider airthmetic properties here.)

Let C denote an algebraic curve. We denote the field of rational functions on C by $\mathbb{C}(C)$. It can be generated by 2 functions, since C is birationally isomorphic to a (possibly singular) curve in $\mathbb{P}^2$. The function field $\mathbb{C}(\mathbb{P}^1)$ can be generated by 1 function; such a generator is called a *rational parameter*.

If $P \in C$ and $f \in \mathbb{C}(C)$, then $\mathrm{ord}_P(f)$ denotes the *valuation* of f at P. If negative, this is the order of a pole of f at P; otherwise this is the vanishing order of f at P. A *local parameter* at P is a function $t_P \in \mathbb{C}(C)$ such that $\mathrm{ord}_P(t_P) = 1$. For example, if C is the projective line, then $x - \alpha$ is a local parameter at the point $x = \alpha$, and $1/x$ is a local parameter at $x = \infty$.

A *divisor* on C is a finite formal sum $\sum_{P \in C} a_P P$, with $a_P \in \mathbb{Z}$. The *degree* of such a divisor is the integer $\sum_{P \in C} a_P$. The divisors form a commutative group under addition. For a function $f \in \mathbb{C}(C)$ we have its *principal divisor* $\sum_{P \in C} \mathrm{ord}_P(f)\, P$, which has degree 0. Principal divisors form a subgroup of degree 0 divisors. The quotient of these two groups is a *Piccard group* of C; it is denoted by $\mathrm{Pic}(C)$. For example, $\mathrm{Pic}(\mathbb{P}^1)$ is the trivial group because all degree zero divisors on $\mathbb{P}^1$ are principal.

Explicit curves in this paper have either genus 0 (i.e., isomorphic to $\mathbb{P}^1$) or genus 1. Let E denote a curve of genus 1. It can be represented in a *Weierstrass form* $\xi^2 = G_3(x)$, where $G_3(x)$ is a cubic polynomial in $\mathbb{C}[x]$. The point at infinity in this model by $\mathcal{O}$. The Piccard group of E is isomorphic (set-theoretically) to E itself. As usual, we identify a point $P \in E$ with the element of $\mathrm{Pic}(E)$ represented by the divisor $P - \mathcal{O}$. Then the additive group law on E can be given by the known chord-and-tangent method. In particular, if three points of E lie on one line of $\mathbb{P}^2$, they add up to the neutral element $\mathcal{O}$. The curve E with this group law is an *elliptic curve* $(E, \mathcal{O})$. Recall that a *torsion point* on E is a point of finite order.

Lemma 5.1 *Let $(E, \mathcal{O})$ denote an elliptic curve, and let $T = \sum_{P \in E} a_P P$ be a divisor on E. Then T is a principal divisor if and only if $\sum_{P \in E} a_P P = \mathcal{O}$ in the additive group of $(E, \mathcal{O})$.*

Proof. Follows from the specified identification of $(E, \mathcal{O})$ with $\mathrm{Pic}(E)$. □

We also consider *radical functions* on C, that is, products of $\mathbf{Q}$-powers of functions from $\mathbb{C}(C)$. These are multi-valued functions, but their branching points are poles or zeroes with finitely many complex branches coming together. Valuations of those functions are well defined at any point, and have values in $\mathbb{Q}$ at the branching points. Accordingly, we consider their principal divisors $\sum_{P \in C} a_P P$ with coefficients $a_P \in \mathbb{Q}$.

Lemma 5.2 *Let $(E, \mathcal{O})$ denote an elliptic curve, and let $T = \sum_{P \in E} a_P P$ be a divisor with coefficients in $\mathbb{Q}$. Then T is the principal divisor for a radical function if and only if the following conditions hold:*

(i) The degree $\sum_{P \in E} a_P$ is zero.

(ii) Let $\sum_{P \in E} \widetilde{a}_P P$ be an integer multiple of T such that all coefficients $\widetilde{a}_P$ are integers. Then, in the additive group of E, the point $\sum_{P \in C} \widetilde{a}_P P$ must be a torsion point.

Proof. Under these conditions, an integer factor nT of T would be a principle divisor with integer coefficients. Then T is a divisor of $G^{1/n}$ for some $G \in \mathbb{C}(E)$.

On the other hand, if T is a divisor of a radical function f, then an integer power f^n is a rational function. The divisor nT sums up to $\mathcal{O}$ by Lemma 5.1. Other integer factors of T with integral coefficients may sum up to a torsion point. □

5.2 Finite coverings and pull-back transformations

Consider a finite covering $\phi : C \to D$ from C to other algebraic curve D. It induces an algebraic field extension $\mathbb{C}(C) \supset \mathbb{C}(D)$. We denote the degree of ϕ by $\deg \phi$. The genus $g(C)$ and $g(D)$ of both curves and branching data are related by the Hurwitz formula:

$$2\,g(C) - 2 = \big(2\,g(D) - 2\big)\,\deg \phi + \sum_{P \in C} (r_P - 1)\,. \tag{96}$$

Here r_P is the branching order at P. It is equal to $\operatorname{ord}_P(t_{\phi(P)} \circ \phi)$.

Now we convene what we mean by a pull-back of hypergeometric equation (1) with respect to a finite covering. Let C denote an algebraic curve. Suppose that the function field $\mathbb{C}(C)$ of C is generated by functions x, ξ. If C is a rational curve, we may assume that ξ is not used and x is a rational parameter of C.

Consider a finite covering $\phi : C \to \mathbb{P}^1$. Let z denote a rational parameter for $\mathbb{P}^1$. Then a pull-back of (1) with respect to ϕ is a differential equation is defined by transformation:

$$z \longmapsto \phi(x, \xi), \qquad y(z) \longmapsto Y(x, \xi) = \theta(x, \xi)\; y(\phi(x, \xi)). \tag{97}$$

Here $\theta(x, \xi)$ is a radical function. Note that such a function has the property that its logarithmic derivative $\theta'(x, \xi)/\theta(x, \xi)$ is in $\mathbb{C}(C)$. We use the derivation on $\mathbb{C}(C)$ that extends the usual derivative on $\mathbb{C}(x)$. If $C \cong \mathbb{P}^1$, then transformation (97) is the following:

$$z \longmapsto \phi(x), \qquad y(z) \longmapsto Y(x) = \theta(x)\; y(\phi(x)). \tag{98}$$

5.3 Differential Galois theory

A *differential field* K is a field with a derivation, i.e., a map $D : K \to K$ which satisfies $D(a+b) = D(a) + D(b)$ and the Leibnitz rule $D(ab) = aD(b) + bD(a)$. One usually denotes $D(a)$ by a'. An *extension* of the differential field K is a differential field L which contains K and whose derivation extends the derivation of K. The basic example of a differential field is the field $\mathbb{C}(z)$ of rational functions on $\mathbb{P}^1$ with the usual derivation. Other example is the field $\mathbb{C}(C)$ of rational functions on an algebraic curve. To give a derivation on $\mathbb{C}(C)$ one may consider a finite covering $\phi : C \to \mathbb{P}^1$ and the corresponding unique extension of the usual derivation of $\mathbb{C}(z)$.

Fix a differential field K and consider a linear homogeneous differential equation (69). Solutions in any extension of K form a linear space over the *constant field* $\{a \in K \mid a' = 0\}$. The dimension of the solution space is at most n. A *Piccard-Vessiot extension* $K_{PV} \supset K$ for (69) is, roughly speaking, a minimal extension of differential fields, such that the solutions of (69) in K_{PV} form a linear space of dimension n. The *differential Galois group* G of (69) is the group of autocoverings of K that fix the elements of K_{PV}. The action of G on the n-dimensional space of solutions in K_{PV} gives a faithful n-dimensional representation of G. Therefore the differential Galois group G is usually considered as an algebraic subgroup of $\mathrm{GL}(n,\mathbb{C})$.

In Section 3, we utilize the *Riccati equation* associated to (69). Solutions for the Riccati equation are precisely the logarithmic derivatives y'/y of solutions for (69). Explicitly, the Riccati equation for (69) with $n = 2$ is $u' + u^2 + a_1 u + a_0 = 0$. Rational or algebraic solutions of the Riccati equation are important in finding "closed form" solutions of the original equation (69), see [Kov86]. We refer to algebraic solutions of the Riccati equation in our working definition of Darboux curves.

Suppose that hypergeometric equation (1) has a finite monodromy group G. Then the differential Galois group is isomorphic to G. (More generally, the differential Galois group of a Fuchsian equation is isomorphic to the Zariski closure of a representation of the monodromy group.) The Piccard-Vessiot extension $K_{PV} \supset \mathbb{C}(z)$ is a finite Galois extension, the usual Galois group is isomorphic to G as well. If $y(z) \in K_{PV}$ is a solution of (1), then $K_{PV} = \mathbb{C}(z,y)$.

In most papers on differential Galois theory, second order differential equations are normalized to the form $y(z)'' = r(z)y(z)$, with $r(z) \in \mathbb{C}(z)$. Hypergeometric equation (1) can be normalized by the transformation

$$y(z) \longmapsto z^{(e_0-1)/2}\,(1-z)^{(e_1-1)/2}\,y(z). \tag{99}$$

The normalized equation is:

$$\frac{d^2y(z)}{dz^2} = \left(\frac{e_1^2-1}{4\,(z-1)^2} + \frac{e_0^2-1}{4\,z^2} + \frac{1+e_\infty^2-e_0^2-e_1^2}{4\,z\,(z-1)} \right) y(x). \tag{100}$$

If the monodromy group G of a hypergeometric equation is isomorphic to A_4, S_4 or A_5, then the differential Galois group of the normalized equation (100) is $G \times \{1,-1\}$. This does not change facts that are important to us. Algebraic degree of Riccati solutions for (1) is the same as of Riccati solutions for (100).

5.4 Hypergeometric equations

If we permute the parameters e_0, e_1, e_∞ in (1) or multiply some of them by -1, we get hypergeometric equations related by well-known fractional-linear transformations [AAR99]. In general, there are 24 hypergeometric equations related in this way, and they share the same (up to radical factors and fractional-linear change of the independent variable) 24 hypergeometric Kummer's solutions.

Hypergeometric equation (1) is usually written in the following form:

$$z\,(1-z)\,\frac{d^2y(z)}{dz^2} + \big(C-(A+B+1)\,z\big)\,\frac{dy(z)}{dz} - A\,B\,y(z) = 0. \tag{101}$$

The parameters are relates as follows: $e_0 = 1-C$, $e_1 = C-A-B$, $e_\infty = A-B$. The local exponents of (101) at $z=0$ are 0 and $1-C$. The local exponents at $z=1$ are 0 and

$C-A-B$. The local exponents at $z=\infty$ are A and B. We use the Riemann P-notation to denote the linear space of solutions for (101):

$$P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & A & z \\ 1-C & C-A-B & B & \end{array}\right\}. \tag{102}$$

As we see, the first row contains the regular singular points, and the other rows contain the local exponents and the variable z. This information determines a Fuchsian equation with three regular singular points. In general, a basis of solutions for (102) is

$${}_2\mathrm{F}_1\left(\left.\begin{array}{c} A,\ B \\ C \end{array}\right|\, z\right), \qquad z^{1-C}\ {}_2\mathrm{F}_1\left(\left.\begin{array}{c} A+1-C,\ B+1-C \\ 2-C \end{array}\right|\, z\right), \tag{103}$$

where ${}_2\mathrm{F}_1\left(\left.{A,B \atop C}\right| z\right) := 1+\frac{A\,B}{C\cdot 1!}\,z+\frac{A(A+1)B(B+1)}{C(C+1)\cdot 2!}\,z^2+\ldots$ is the Gauss hypergeometric series.

5.5 Fuchsian equations

All differential equations that we explicitly consider are *Fuchsian equations*. These equations have only regular singular points. For equation (69) this means the following: if $K=\mathbb{C}(C)$ for an algebraic curve C, then for any point $P\in C$ and for $i=1,\ldots,n$ we must have $\mathrm{ord}_P(a_i)\geq(n-i)\,(\mathrm{ord}_P(t_P{}')-1)$, where t_P is a local parameter at P. Local exponents at P can be defined as follows: substitute $y=t_P^{\mu}$ into the Fuchsian equation and consider the terms to the power $\mu+n((\mathrm{ord}_P(t_P{}')-1)$ of t_P as an equation in μ; the roots of that equation are precisely the local exponents. The local exponents at regular points are equal to $0,1,\ldots,n-1$.

In general, singularities and local exponents do not determine a Fuchsian equation uniquely. Hence we cannot always use the P-notation for general Fuchsian equations. However, in Section 4.3 we write down arrays of singularities and local exponents similar to (102).

5.6 Contiguous relations of Gauss hypergeometric functions

Two Gauss hypergeometric functions are called *contiguous* (or *associated* in [Erd53]) if they have the same argument z and their parameters a, b and c differ respectively by integers. As is known [AAR99, Section 2.5], for any three contiguous ${}_2\mathrm{F}_1$ functions there is a *contiguous relation*, which is a linear relation between the three functions where the coefficients are rational functions in the parameters a,b,c and the argument z. A straightforward (though not efficient) method to compute a contiguous expression for ${}_2\mathrm{F}_1\left(\left.{a+k,b+\ell \atop c+m}\right| z\right)$ in terms of ${}_2\mathrm{F}_1\left(\left.{a+1,b \atop c}\right| z\right)$ and ${}_2\mathrm{F}_1\left(\left.{a,b \atop c}\right| z\right)$ is the following. By using the contiguous relations

$$b\ {}_2\mathrm{F}_1\left(\left.\begin{array}{c} a,b+1 \\ c \end{array}\right|\, z\right) = (b-a)\ {}_2\mathrm{F}_1\left(\left.\begin{array}{c} a,b \\ c \end{array}\right|\, z\right)+a\ {}_2\mathrm{F}_1\left(\left.\begin{array}{c} a+1,b \\ c \end{array}\right|\, z\right), \tag{104}$$

$$(c-1)\ {}_2\mathrm{F}_1\left(\left.\begin{array}{c} a,\ b \\ c-1 \end{array}\right|\, z\right) = (c-a-1)\ {}_2\mathrm{F}_1\left(\left.\begin{array}{c} a,b \\ c \end{array}\right|\, z\right)+a\ {}_2\mathrm{F}_1\left(\left.\begin{array}{c} a+1,b \\ c \end{array}\right|\, z\right), \tag{105}$$

one eliminates the shifts in b and c, and then by using the contiguous relation

$$a(1-z)\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}a\!+1,b\\ c\end{matrix}\,\right|\,z\right)=(2a-c-az+bz)\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}a,b\\ c\end{matrix}\,\right|\,z\right)+(c-a)\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}a\!-1,b\\ c\end{matrix}\,\right|\,z\right) \qquad (106)$$

one gets an expression with two contiguous terms. Effective computation of contiguous relations is considered in [Vid03a]. They can be computed in $O(\log(\max(k,l,m))$ steps, but complexity of expressions in each such step grows exponentially, and the output is $O(\max(k,l,m))$.

One can rewrite contiguity conditions in terms of local exponent differences at $z=0$, 1, ∞ for the hypergeometric equation, since the parameters a, b, c determine the local exponent differences and vice versa (if the sign of local exponent differences is taken into account). The main hypergeometric solutions (103) of two hypergeometric equations (101) are contiguous if for each $X\in\{0,1,\infty\}$ the difference of signed local exponent differences at X of the two equations is an integer, and the sum of the three integer differences is even. Two hypergeometric equations have solutions contiguous to each other (or equivalently, they have the same Schwartz type) if one can choose a permutation of local exponent differences and their sign in such a way that the just described situation occurs.

For example, the parameters e_0,e_1,e_∞ of hypergeometric equations of the Schwartz type $(1/3,1/3,2/3)$ can be characterized as follows: they are rational numbers, their denominators are equal to 3, and the sum of their numerators is even.

5.7 Fiber products of curves

Let C_1 and C_2 denote two curves over $\mathbb{C}$. Let $\phi_1 : C_1 \to \mathbb{P}^1$ and $\phi_2 : C_2 \to \mathbb{P}^1$ be two finite coverings of degree m and n respectively. The fiber product of $\phi_1 : C_1 \to \mathbb{P}^1$ and $\phi_2 : C_2 \to \mathbb{P}^1$ is a curve B with two coverings $\psi_1 : B \to C_1$ and $\psi_2 : B \to C_2$ such that $\phi_1\circ\psi_1=\phi_2\circ\psi_2$, and for any other curve $\widetilde{B}$ with coverings $\widetilde{\psi}_1 : \widetilde{B}\to C_1$ and $\widetilde{\psi}_2 : \widetilde{B}\to C_2$ satisfying $\phi_1\circ\widetilde{\psi}_1=\phi_2\circ\widetilde{\psi}_2$ there is a unique covering $\xi : \widetilde{B}\to B$ such that $\widetilde{\psi}_1=\psi_1\circ\xi$ and $\widetilde{\psi}_2=\psi_2\circ\xi$. Then the following diagram commutes:

$$\begin{tikzcd}
\widetilde{B} \arrow[rr, "\zeta"] \arrow[dr, "\widetilde{\psi}_1"'] \arrow[drrr, "\widetilde{\psi}_2"] & & B \arrow[dl, "\psi_1"] \arrow[dr, "\psi_2"] & \\
& C_1 \arrow[dr, "\phi_1"] & & C_2 \arrow[dl, "\phi_2"] \\
& & \mathbb{P}^1 &
\end{tikzcd}$$

We have $\deg\psi_1=\deg\phi_2$ and $\deg\psi_2=\deg\phi_1$. On the level for function fields, we have $\mathbb{C}(B)=\mathbb{C}(C_1)\otimes_{\mathbf{C}(\mathbb{P}^1)}\mathbb{C}(C_2)$.

A birational model for B is the curve on $C_1\times C_2$ of those points $(X_1,X_2)\in C_1\times C_2$ which satisfy $\varphi_1(X_1)=\varphi_2(X_2)$. This is a singular model in general. A singular point corresponds to a pair $(X_1,X_2)\in B$ such that X_1 and X_2 have branching indices $r_1>1$, $r_2>1$

respectively (with respect to ϕ_1 and ϕ_2). Such a singularity is of type $x_1^{r_1} - x_2^{r_2}$; by resolving it we get $\gcd(r_1, r_2)$ points that correspond to (X_1, X_2) on a non-singular model for B; see [**?**]. This information allows us to compute the branching data for the projections $\psi : B \to C_1$ and $\psi : B \to C_2$ and the genus of B. For example, if $X_1 \in C_1$ has branching index r_1 (with respect to ϕ_1), and the branching data of ϕ_2 above $\phi_1(X_1)$ is $a_1 + \ldots + a_k$, then the branching data for ψ_1 above X_1 is the following: $\gcd(a_1, r_1) * \frac{\mathrm{lcm}(a_1, r_1)}{r_1} + \ldots + \gcd(a_k, r_1) * \frac{\mathrm{lcm}(a_k, r_1)}{r_1}$.

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