# On High-Discrepancy Sequences

Tezuka, Shu
Faculty of Mathematics, Kyushu University

https://hdl.handle.net/2324/3385

出版情報：Kyushu Journal of Mathematics. 61 (2), pp.431-441, 2007-09. Faculty of Mathematics, Kyushu University
バージョン：
権利関係：

# MHF Preprint Series

**Kyushu University**
**21st Century COE Program**
**Development of Dynamic Mathematics with**
**High Functionality**


# On high-discrepancy sequences


## S. Tezuka



**MHF 2006-11**

( Received March 10, 2006 )



Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

# On High-Discrepancy Sequences

Shu Tezuka

*Faculty of Mathematics, Kyushu University*

*6-10-1 Hakozaki, Higashi-ku, Fukuoka-shi, Fukuoka-ken, Japan 812-8581*

**Abstract**

First, it is pointed out that the uniform distribution of points in $[0,1]^d$ is not always a necessary condition for every function in a proper subset of the class of all Riemann integrable functions to have the arithmetic mean of function values at the points converging to its integral over $[0,1]^d$ as the number of points goes to infinity. We introduce a formal definition of the $d$-dimensional *high-discrepancy sequences*, which are not uniformly distributed in $[0,1]^d$, and present motivation for the application of these sequences to high-dimensional numerical integration. Then, we prove that there exist non-uniform $(\infty, d)$-sequences which provide the convergence rate $O(N^{-1})$ for the integration of a certain class of $d$-dimensional Walsh function series, where $N$ is the number of points.

*Key words:* discrepancy, high dimensional integrals, Monte Carlo and quasi-Monte Carlo methods, $(t,d)$-sequences, uniform distribution, Walsh functions

## 1 Introduction

It is well known [3] that the uniform distribution of points $X_n, n = 0, 1, ...,$ in $[0,1]^d$ is a necessary and sufficient condition for every function $f(x_1, ..., x_d)$ in the class of all Riemann integrable functions over $[0,1]^d$, hereafter denoted by $\mathfrak{R}_d$, to satisfy

$$I(f) := \int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(X_n). \quad (1)$$

It is also known [3] that for the class of all continuous functions over $[0,1]^d$, which is a proper subset of $\mathfrak{R}_d$, the uniform distribution is a necessary and suf-

*Email address:* tezuka@math.kyushu-u.ac.jp (Shu Tezuka).

ficient condition for every function in the class to satisfy the equation (1). From the Information-Based Complexity (IBC) viewpoint, Woźniakowski [9,10] analyzed the average case complexity with respect to the Wiener sheet measure for the integration of this class of functions. The well-known Koksma-Hlawka theorem implies that the class of all functions over $[0,1]^d$ of bounded variation in the sense of Hardy and Krause is another proper subset of $\mathfrak{R}_d$ for which the uniform distribution is also a necessary and sufficient condition for every function in the class to satisfy the equation (1).

However, little is known about what happens to any other proper subsets of $\mathfrak{R}_d$. Although the uniform distribution of sample points is a sufficient condition for every function in any proper subset of $\mathfrak{R}_d$ to satisfy the equation (1), it is not always a necessary condition. The purpose of this paper is to explore a possibility to develop faster algorithms for a proper subset of $\mathfrak{R}_d$ based on non-uniform sample points than Monte Carlo and/or quasi-Monte Carlo methods. In the next section, we first introduce a formal definition of the $d$-dimensional *high-discrepancy sequences*, which are not uniformly distributed in $[0,1]^d$, and present motivation for the application of these sequences to high-dimensional numerical integration. Then, we prove that there exist non-uniform $(\infty, d)$-sequences which provide the convergence rate $O(N^{-1})$ for the integration of a certain class of $d$-dimensional Walsh function series, where $N$ is the number of points. In the last section, we discuss the significance of this result and future research directions.

## 2 Main Result

### *2.1 Definition and motivation for high-discrepancy sequences*

First, we introduce the definition of high-discrepancy sequences:

**Definition 1** *If a sequence of points $X_n, n = 0, 1, ...,$ in $[0,1]^d$ satisfies that for all $N > 1$, the star discrepancy $D_N^{(d)}$ of the first $N$ points is given by*

$$D_N^{(d)} = O(1), \tag{2}$$

*then we call it a high-discrepancy sequence, where the implied constant in the $O$ notation depends only on the dimension $d$.*

Notice that if the righthand side in the equation (2) is replaced by $O((\log N)^d/N)$, then it is the definition of a low-discrepancy sequence. High-discrepancy sequences are those sequences which are not uniformly distributed, and vice versa.

We have the following proposition:

**Proposition 1** *A $(t, d)$-sequence in base two for which the first rows of its generator matrices are all identical to $(1, 0, 0...)$ is a high-discrepancy sequence.*

**PROOF.** Since the sequence is an $(\infty, d)$-sequence which is distributed in only two sub-domains, $[0, \frac{1}{2}]^d$ and $[\frac{1}{2}, 1]^d$, the proof follows.

Note that there are two types of $(\infty, d)$-sequences, i.e., uniform or non-uniform. The sequence introduced in Proposition 1 is a non-uniform $(\infty, d)$-sequence.

We give the following theorems, which motivate us to investigate the application of high-discrepancy sequences defined in the above to high-dimensional numerical integration.

**Theorem 1** *If an $L_2$ function in $d$ dimensions is written as*

$$f(x_1, ..., x_d) = \sum_{k_1,...,k_d \geq 0} c_{k_1,...,k_d} \cos(2\pi k_1 x_1) \cdots \cos(2\pi k_d x_d),$$

*then we have*

$$\int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d$$
$$= 2^{d-1} \left( \int_{[0,\frac{1}{2}]^d} f(x_1, ..., x_d) dx_1 ... dx_d + \int_{[\frac{1}{2},1]^d} f(x_1, ..., x_d) dx_1 ... dx_d \right).$$

**PROOF.** Since

$$\int_0^1 \cos(2\pi kx) dx = 0$$

for any $k \geq 1$, we have

$$\int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d = c_{0,...,0}.$$

And since

$$\int_0^{1/2} \cos(2\pi kx) dx = \int_{1/2}^1 \cos(2\pi kx) dx = 0$$

for any $k \geq 1$, we have

$$\int_{[0,\frac{1}{2}]^d} f(x_1, ..., x_d) dx_1 ... dx_d = \frac{c_{0,...,0}}{2^d},$$

and

$$\int_{[\frac{1}{2},1]^d} f(x_1, ..., x_d) dx_1 ... dx_d = \frac{c_{0,...,0}}{2^d}.$$

Thus, the proof is complete.□

**Theorem 2** *Suppose that the dimension $d$ is odd. If an $L_2$ function in $d$ dimensions is written as*

$$f(x_1, ..., x_d) = c_0 + \sum_{k_1,...,k_d \geq 1} c_{k_1,...,k_d} \sin(2\pi k_1 x_1) \cdots \sin(2\pi k_d x_d),$$

*then we have*

$$\int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d$$

$$= 2^{d-1} \left( \int_{[0,\frac{1}{2}]^d} f(x_1, ..., x_d) dx_1 ... dx_d + \int_{[\frac{1}{2},1]^d} f(x_1, ..., x_d) dx_1 ... dx_d \right).$$

**PROOF.** Since

$$\int_0^1 \sin(2\pi kx) dx = 0$$

for any $k \geq 1$, we have

$$\int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d = c_0.$$

And since

$$\int_0^{1/2} \sin(2\pi kx) dx = \int_{1/2}^1 \sin(2\pi kx) dx = 0$$

for any even $k \geq 2$, and

$$\int_0^{1/2} \sin(2\pi kx) dx = - \int_{1/2}^1 \sin(2\pi kx) dx = \frac{1}{k\pi}$$

for any odd $k \geq 1$, we have

$$\int_{[0,\frac{1}{2}]^d} f(x_1, ..., x_d) dx_1 ... dx_d = \frac{c_0}{2^d} + \sum_{\substack{k_1,...,k_d \geq 1 \\ \text{all } k_1, ..., k_d \text{ are odd}}} \frac{c_{k_1,...,k_d}}{k_1 \cdots k_d \pi^d},$$

and

$$\int_{[\frac{1}{2},1]^d} f(x_1, ..., x_d) dx_1 ... dx_d = \frac{c_0}{2^d} + (-1)^d \sum_{\substack{k_1,...,k_d \geq 1 \\ \text{all } k_1, ..., k_d \text{ are odd}}} \frac{c_{k_1,...,k_d}}{k_1 \cdots k_d \pi^d}.$$

Since $d$ is odd, the proof is complete.□

In what follows, we show that there exist non-uniform $(\infty, d)$-sequences which provide the convergence rate $O(N^{-1})$ for the integration of a class of $d$-dimensional Walsh function series.

The Walsh functions are defined by

$$\mathrm{wal}(0,x)=1, \text{ for } x\in[0,1),$$

and for an integer $k\geq 1$,

$$\mathrm{wal}(k,x)=(-1)^{\sum_{j=1}^{\infty} b_{j-1}a_j}=(-1)^{(\mathbf{k},X)}$$

where $k=b_0+b_1 2+\cdots$, and $x=a_1 2^{-1}+a_2 2^{-2}+\cdots$ in the binary expansion, and $\mathbf{k}=(b_0,b_1,...)$ and $X=(a_1,a_2,...)$ are the binary vector representation of $k$ and $x$, respectively. Then, we have the following theorem.

**Theorem 3** *A function in the d-dimensional $L_2$ space can be written as*

$$f(x_1,...,x_d)=\sum_{k_1,...,k_d\geq 0} c_{k_1,...,k_d}\mathrm{wal}(k_1,x_1)\cdots\mathrm{wal}(k_d,x_d).$$

*We have*

$$\int_{[0,1]^d} f(x_1,...,x_d)dx_1...dx_d$$

$$=2^{d-1}\left(\int_{[0,\frac{1}{2}]^d} f(x_1,...,x_d)dx_1...dx_d+\int_{[\frac{1}{2},1]^d} f(x_1,...,x_d)dx_1...dx_d\right)-E_d,$$

*where*

$$E_d=\sum_{\substack{0\leq k_1,...,k_d\leq 1\\ (k_1,...,k_d)\neq(0,...,0)\\ k_1+\cdots+k_d=0 \pmod 2}} c_{k_1,...,k_d}. \tag{3}$$

**PROOF.** First, we have

$$\int_{[0,1]^d} f(x_1,...,x_d)dx_1...dx_d=c_{0,...,0}.$$

Since

$$\int_0^{1/2}\mathrm{wal}(1,x)dx=-\int_{1/2}^1\mathrm{wal}(1,x)dx=\frac{1}{2},$$

and

$$\int_0^{1/2}\mathrm{wal}(k,x)dx=\int_{1/2}^1\mathrm{wal}(k,x)dx=0$$

for any $k\geq 2$, we have

$$\int_{[0,\frac{1}{2}]^d} f(x_1,...,x_d)dx_1...dx_d=\frac{1}{2^d}\sum_{0\leq k_1,...,k_d\leq 1} c_{k_1,...,k_d},$$

and

$$\int_{[\frac{1}{2},1]^d} f(x_1,...,x_d)dx_1...dx_d = \frac{1}{2^d}\sum_{0\le k_1,...,k_d\le 1}(-1)^{k_1+\cdots+k_d}c_{k_1,...,k_d}.$$

Thus, the proof is complete.□

We now consider a class of functions for which $E_d = 0$. First, we give some definitions.

**Definition 2** *Let $m \ge 1$ be an integer and let $u$ be a nonempty subset $\{j_1,...,j_{|u|}\} \subseteq \{1,...,d\}$. We define*

$$\phi_{u,m}(x_1,...,x_d) = \prod_{i=1}^{|u|} \mathrm{wal}(k_{j_i}^{(m)}, x_{j_i}),$$

*for $2^{m-1} \le k_{j_1}^{(m)},...,k_{j_{|u|}}^{(m)} < 2^m$.*

**Definition 3** *We define a class $\mathfrak{F}_d$ which consists of functions*

$$f(x_1,...,x_d) = c_0 + \sum_{\substack{1\le|u|\le d\\ |u| \text{ is odd}}}\sum_{m=1}^{\infty} c_{u,m}\phi_{u,m}(x_1,...,x_d),$$

*where $c_0$ and $c_{u,m}, m=1,2,...,$ are constants satisfying*

$$|c_0| + \sum_{\substack{1\le|u|\le d\\ |u| \text{ is odd}}}\sum_{m=1}^{\infty}|c_{u,m}|2^{m-1} \le M < \infty.$$

*Here, $M$ is a constant.*

We have the following theorem.

**Theorem 4** *For any function $f(x_1,...,x_d)$ in $\mathfrak{F}_d$, we have*

$$\int_{[0,1]^d} f(x_1,...,x_d)dx_1...dx_d$$

$$= 2^{d-1}\left(\int_{[0,\frac{1}{2}]^d} f(x_1,...,x_d)dx_1...dx_d + \int_{[\frac{1}{2},1]^d} f(x_1,...,x_d)dx_1...dx_d\right).$$

**PROOF.** From the equation (3) and Definition 3, it follows that the quantity $E_d$ in Theorem 3 becomes zero for the case of $\mathfrak{F}_d$. Thus, the proof is complete.□

Hereafter, we denote by $\mathfrak{S}_d$ a class of $(t,d)$-sequences in base two whose generator matrices are nonsingular and lower-triangular. Note that this class is a special subset of the class of non-uniform $(\infty,d)$-sequences introduced in Proposition 1. We prove the following theorem.

**Theorem 5** *For any sequence $\mathbf{s}_n, n=0,1,...,$ in the class $\mathfrak{S}_d$, the integration error $e_N$ of any function $f$ in $\mathfrak{F}_d$ is given by*

$$e_N(\mathbf{s}_n,f) := \left| I(f) - \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{s}_n)\right| < \frac{1}{N}\sum_{\substack{1\le|u|\le d\\ |u| \text{ is odd}}}\sum_{m=1}^{\infty}|c_{u,m}|\min(2^{m-1},N).$$

From the condition in Definition 3, we have $e_N(\mathbf{s}_n,f)=O(N^{-1})$, where the asymptotic constant is $M$.

In order to prove the above theorem, we need the following three lemmas, which are generalization of Lemmas 2, 3 and 4 in [8]. Let $s_n, n=0,1,...,$ be a sequence in $\mathfrak{S}_1$, and let $s_{n,m}$ denote the $m$-th bit of $s_n$.

**Lemma 1** *Suppose $m\ge 1$. For all $n=h\,2^m, h=0,1,...,$ and for all $0\le \ell < 2^{m-1}$, we have*

$$s_{n+\ell,m} = 1 - s_{n+\ell+2^{m-1},m}.$$

**PROOF.** If we compare $n+\ell$ and $n+\ell+2^{m-1}$, only the coefficient of $2^{m-1}$ in their binary representation is different while all the others are identical. Since the generator matrix is nonsingular and lower-triangular, this completes the proof. □

**Lemma 2** *Suppose $m\ge 1$. For all $n=h\,2^m, h=0,1,...,$ and for all $0\le \ell < 2^{m-1}$, we have*

$$\mathrm{wal}(k,s_{n+\ell}) = -\mathrm{wal}(k,s_{n+\ell+2^{m-1}}),$$

*where $k$ is any integer with $2^{m-1}\le k<2^m$.*

**PROOF.** Since the generator matrix is nonsingular and lower-triangular, we have $s_{n+\ell,e}=s_{n+\ell+2^{m-1},e}$ for all $1\le e<m$. From the assumption that $k=b_0+b_1 2+\cdots+b_{m-1}2^{m-1}$ with $b_{m-1}=1$ and Lemma 1, the proof is complete. □

For $d$ dimensions, we denote a sequence in $\mathfrak{S}_d$ by $\mathbf{s}_n=(s_n^{(1)},...,s_n^{(d)}), n=0,1,...,$ Then we have

**Lemma 3** *Suppose $m \geq 1$. Let $u$ be any nonempty subset $\{j_1, ..., j_{|u|}\} \subseteq \{1, ..., d\}$. If the cardinality $|u|$ is odd, then for all $n = h\ 2^m, h = 0, 1, ...,$ and for all $0 \leq \ell < 2^{m-1}$, we have*

$$\phi_{u,m}(\mathbf{s}_{n+\ell}) = -\phi_{u,m}(\mathbf{s}_{n+\ell+2^{m-1}}).$$

**PROOF.** Denote

$$\phi_{u,m}(\mathbf{s}_{n+\ell}) = \prod_{i=1}^{|u|} \mathrm{wal}(k_{j_i}^{(m)}, s_{n+\ell}^{(j_i)}),$$

where $2^{m-1} \leq k_{j_1}^{(m)}, ..., k_{j_{|u|}}^{(m)} < 2^m$. Without loss of generality, we assume that the number of $j_i, 1 \leq i \leq |u|$, with $\mathrm{wal}(k_{j_i}^{(m)}, s_{n+\ell}^{(j_i)}) = -1$ is odd. Then, from Lemma 2, the number of $j_i, 1 \leq i \leq |u|$, with $\mathrm{wal}(k_{j_i}^{(m)}, s_{n+\ell+2^{m-1}}^{(j_i)}) = -1$ is even because $|u|$ is odd. Thus, the proof is complete. □

Note that $I(\phi_{u,m}) = 0$ for any $u \neq \emptyset$ and $m \geq 1$. From Lemma 3, it follows that the integration error $e_N$ of $\phi_{u,m}$ with $|u|$ odd for any sequence $\mathbf{s}_n, n = 0, 1, ...,$ in $\mathfrak{S}_d$ is given by

$$e_N(\mathbf{s}_n, \phi_{u,m}) = \left| \frac{1}{N} \sum_{n=0}^{N-1} \phi_{u,m}(\mathbf{s}_n) \right| = \left| \frac{1}{N} \sum_{n=N-N_m}^{N-1} \phi_{u,m}(\mathbf{s}_n) \right| \leq \frac{\min(2^{m-1}, N)}{N},$$

where $N_m$ is the residue of $N$ modulo $2^m$.

We are now ready to prove Theorem 5.

$$e_N(\mathbf{s}_n, f) = \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{s}_n) - I(f) \right| = \left| \frac{1}{N} \sum_{n=0}^{N-1} (f(\mathbf{s}_n) - c_0) \right|$$

$$= \left| \frac{1}{N} \sum_{n=0}^{N-1} \sum_{\substack{1 \leq |u| \leq d \\ |u| \text{ is odd}}} \sum_{m=1}^{\infty} c_{u,m} \phi_{u,m}(\mathbf{s}_n) \right| \leq \sum_{\substack{1 \leq |u| \leq d \\ |u| \text{ is odd}}} \sum_{m=1}^{\infty} \left| \frac{c_{u,m}}{N} \sum_{n=0}^{N-1} \phi_{u,m}(\mathbf{s}_n) \right|$$

$$\leq \frac{1}{N} \sum_{\substack{1 \leq |u| \leq d \\ |u| \text{ is odd}}} \sum_{m=1}^{\infty} |c_{u,m}| \min(2^{m-1}, N).$$

Thus, the proof is complete.

We should notice that a sequence $\mathbf{s}_n, n = 1, 2, ...,$ in $\mathfrak{S}_d$ is distributed in the sub-domain $[\frac{1}{2^m}, \frac{1}{2^{m-1}}]^d$ when $n$ is a multiple of $2^{m-1}$ but not of $2^m$ for some $m \geq 1$. Therefore, high-discrepancy sequences employed in the above are *essentially non-uniform* because they are distributed in only a part of the diagonal domain $[0, \frac{1}{2}]^d \bigcup [\frac{1}{2}, 1]^d$.

## 3 Discussion

For a function $f(x_1, ..., x_d)$ which satisfies

$$\int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d$$

$$= 2^{d-1} \left( \int_{[0,\frac{1}{2}]^d} f(x_1, ..., x_d) dx_1 ... dx_d + \int_{[\frac{1}{2},1]^d} f(x_1, ..., x_d) dx_1 ... dx_d \right),$$

there are three approaches to its numerical integration when the dimension $d$ is large.

- Monte Carlo methods: We have two types of Monte Carlo methods: uniform random sampling over the whole domain $[0,1]^d$ or over two sub-domains, $[0, \frac{1}{2}]^d$ and $[\frac{1}{2}, 1]^d$. For both cases, the convergence rate is $O(N^{-1/2})$.
- Quasi-Monte Carlo or low-discrepancy methods: This approach provides the convergence rate $O((\log N)^d)/N)$ for functions of bounded variation [1,4–7].
- High-discrepancy methods: There are two types of high-discrepancy methods: The first type is to apply low-discrepancy sequences to each of the two sub-domains, $[0, \frac{1}{2}]^d$ and $[\frac{1}{2}, 1]^d$, as the whole domain. The class of $(\infty, d)$-sequences introduced in Proposition 1 can be used for such purpose. In this case, we get the convegence rate $O((\log N)^d)/N)$ for functions of bounded variation. Note that this type is closely related to Hlawka-Mück approach [2]. The second type uses essentially non-uniform sequences such as the class $\mathfrak{S}_d$. For the particular case of $\mathfrak{F}_d$, we have shown the $O(N^{-1})$ convergence rate, which is better than Monte Carlo and quasi-Monte Carlo methods. However, at least at present, nothing is known for functions besides $\mathfrak{F}_d$.

An important consequence from our result is that the "low-discrepancy" approach is not the only way for accelerating the computation of high dimensional numerical integration. If one considers a class of functions for which the uniform distribution is not a necessary condition to satisfy the equation (1), there is a possibility to develop faster algorithms based on the "high-discrepancy" approach than Monte Carlo and/or quasi-Monte Carlo (low-discrepancy) methods. There are several applications in practice involving a class of functions in high dimensions which are localized in a very limited domain. Commonly, we have used the so-called importance sampling techniques including Markov Chain Monte Carlo (MCMC) methods for these applications. Unfortunately, all Monte Carlo techniques suffer from the slow convergence rate $O(N^{-1/2})$, while our high discrepancy approach, which is deterministic, has a potential of the convergence rate $O(N^{-1})$. Therefore, we can consider the high-discrepancy approach as a possible alternative. However, there are many topics to be explored in this new area.

## References


[1] M. Drmota and R. F. Tichy, *Sequences, Discrepancies and Applications*, Lecture Notes in Mathematics, Vol. 1651, Springer, 1997.

[2] E. Hlawka and R. Mück, A Transformation of Equidistributed Sequences, In S. K. Zaremba (Ed.), *Applications of Number Theory to Numerical Analysis*, Academic Press, New York (1972), 371-388.

[3] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, John-Wiley & Sons, 1974.

[4] J. Matoušek, *Geometric Discrepancy: An Illustrated Guide,* Springer, 1999.

[5] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, CBMS-NSF Regional Conference Series in Applied Mathematics, No. 63, SIAM, 1992.

[6] I. H. Sloan and S. Joe, *Lattice Methods for Multiple Integration,* Claredon Press, Oxford, 1994.

[7] S. Tezuka, *Uniform Random Numbers: Theory and Practice*, Kluwer Academic Publishers, 1995.

[8] S. Tezuka, On the Necessity of Low-Effective Dimension, *Journal of Complexity*, **21** (2005), 710-721.

[9] J. F. Traub and A. G. Werschulz, *Complexity and Information*, Cambridge Univ. Press, 1998.

[10] H. Woźniakowski, Average Case Complexity of Multidimensional Integration, *Bull. Amer. Math. Soc.* , **24** (1991), 185-194.

# List of MHF Preprint Series, Kyushu University

**21st Century COE Program**
**Development of Dynamic Mathematics with High Functionality**

MHF2003-1 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Yoshitaka WATANABE
A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems

MHF2003-2 Masahisa TABATA & Daisuke TAGAMI
Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients

MHF2003-3 Tomohiro ANDO, Sadanori KONISHI & Seiya IMOTO
Adaptive learning machines for nonlinear classification and Bayesian information criteria

MHF2003-4 Kazuhiro YOKOYAMA
On systems of algebraic equations with parametric exponents

MHF2003-5 Masao ISHIKAWA & Masato WAKAYAMA
Applications of Minor Summation Formulas III, Plücker relations, Lattice paths and Pfaffian identities

MHF2003-6 Atsushi SUZUKI & Masahisa TABATA
Finite element matrices in congruent subdomains and their effective use for large-scale computations

MHF2003-7 Setsuo TANIGUCHI
Stochastic oscillatory integrals - asymptotic and exact expressions for quadratic phase functions -

MHF2003-8 Shoki MIYAMOTO & Atsushi YOSHIKAWA
Computable sequences in the Sobolev spaces

MHF2003-9 Toru FUJII & Takashi YANAGAWA
Wavelet based estimate for non-linear and non-stationary auto-regressive model

MHF2003-10 Atsushi YOSHIKAWA
Maple and wave-front tracking — an experiment

MHF2003-11 Masanobu KANEKO
On the local factor of the zeta function of quadratic orders

MHF2003-12 Hidefumi KAWASAKI
Conjugate-set game for a nonlinear programming problem

MHF2004-1 Koji YONEMOTO & Takashi YANAGAWA
Estimating the Lyapunov exponent from chaotic time series with dynamic noise

MHF2004-2 Rui YAMAGUCHI, Eiko TSUCHIYA & Tomoyuki HIGUCHI
State space modeling approach to decompose daily sales of a restaurant into time-dependent multi-factors

MHF2004-3 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Cubic pencils and Painlevé Hamiltonians

MHF2004-4 Atsushi KAWAGUCHI, Koji YONEMOTO & Takashi YANAGAWA
Estimating the correlation dimension from a chaotic system with dynamic noise

MHF2004-5 Atsushi KAWAGUCHI, Kentarou KITAMURA, Koji YONEMOTO, Takashi YANAGAWA & Kiyofumi YUMOTO
Detection of auroral breakups using the correlation dimension

MHF2004-6 Ryo IKOTA, Masayasu MIMURA & Tatsuyuki NAKAKI
A methodology for numerical simulations to a singular limit

MHF2004-7 Ryo IKOTA & Eiji YANAGIDA
Stability of stationary interfaces of binary-tree type

MHF2004-8 Yuko ARAKI, Sadanori KONISHI & Seiya IMOTO
Functional discriminant analysis for gene expression data via radial basis expansion

MHF2004-9 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Hypergeometric solutions to the $q$ - Painlevé equations

MHF2004-10 Raimundas VIDŪNAS
Expressions for values of the gamma function

MHF2004-11 Raimundas VIDŪNAS
Transformations of Gauss hypergeometric functions

MHF2004-12 Koji NAKAGAWA & Masakazu SUZUKI
Mathematical knowledge browser

MHF2004-13 Ken-ichi MARUNO, Wen-Xiu MA & Masayuki OIKAWA
Generalized Casorati determinant and Positon-Negaton-Type solutions of the Toda lattice equation

MHF2004-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCO
Generating function associated with the determinant formula for the solutions of the Painlevé II equation

MHF2004-15 Kouji HASHIMOTO, Ryohei ABE, Mitsuhiro T. NAKAO & Yoshitaka WATANABE
Numerical verification methods of solutions for nonlinear singularly perturbed problem

MHF2004-16 Ken-ichi MARUNO & Gino BIONDINI
Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete versions

MHF2004-17 Ryuei NISHII & Shinto EGUCHI
Supervised image classification in Markov random field models with Jeffreys divergence

MHF2004-18 Kouji HASHIMOTO, Kenta KOBAYASHI & Mitsuhiro T. NAKAO
Numerical verification methods of solutions for the free boundary problem

MHF2004-19 Hiroki MASUDA
Ergodicity and exponential $\beta$-mixing bounds for a strong solution of Lévy-driven stochastic differential equations

MHF2004-20 Setsuo TANIGUCHI
The Brownian sheet and the reflectionless potentials

MHF2004-21 Ryuei NISHII & Shinto EGUCHI
Supervised image classification based on AdaBoost with contextual weak classifiers

MHF2004-22 Hideki KOSAKI
On intersections of domains of unbounded positive operators

MHF2004-23 Masahisa TABATA & Shoichi FUJIMA
Robustness of a characteristic finite element scheme of second order in time increment

MHF2004-24 Ken-ichi MARUNO, Adrian ANKIEWICZ & Nail AKHMEDIEV
Dissipative solitons of the discrete complex cubic-quintic Ginzburg-Landau equation

MHF2004-25 Raimundas VIDŪNAS
Degenerate Gauss hypergeometric functions

MHF2004-26 Ryo IKOTA
The boundedness of propagation speeds of disturbances for reaction-diffusion systems

MHF2004-27 Ryusuke KON
Convex dominates concave: an exclusion principle in discrete-time Kolmogorov systems

MHF2004-28 Ryusuke KON
Multiple attractors in host-parasitoid interactions: coexistence and extinction

MHF2004-29 Kentaro IHARA, Masanobu KANEKO & Don ZAGIER
Derivation and double shuffle relations for multiple zeta values

MHF2004-30 Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Generalized partitioned quantum cellular automata and quantization of classical CA

MHF2005-1 Hideki KOSAKI
Matrix trace inequalities related to uncertainty principle

MHF2005-2 Masahisa TABATA
Discrepancy between theory and real computation on the stability of some finite element schemes

MHF2005-3 Yuko ARAKI & Sadanori KONISHI
Functional regression modeling via regularized basis expansions and model selection

MHF2005-4 Yuko ARAKI & Sadanori KONISHI
Functional discriminant analysis via regularized basis expansions

MHF2005-5 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Point configurations, Cremona transformations and the elliptic difference Painlevé equations

MHF2005-6 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Construction of hypergeometric solutions to the $q$ - Painlevé equations

MHF2005-7 Hiroki MASUDA
Simple estimators for non-linear Markovian trend from sampled data:
I. ergodic cases

MHF2005-8 Hiroki MASUDA & Nakahiro YOSHIDA
Edgeworth expansion for a class of Ornstein-Uhlenbeck-based models

MHF2005-9 Masayuki UCHIDA
Approximate martingale estimating functions under small perturbations of dynamical systems

MHF2005-10 Ryo MATSUZAKI & Masayuki UCHIDA
One-step estimators for diffusion processes with small dispersion parameters from discrete observations

MHF2005-11 Junichi MATSUKUBO, Ryo MATSUZAKI & Masayuki UCHIDA
Estimation for a discretely observed small diffusion process with a linear drift

MHF2005-12 Masayuki UCHIDA & Nakahiro YOSHIDA
AIC for ergodic diffusion processes from discrete observations

MHF2005-13 Hiromichi GOTO & Kenji KAJIWARA
Generating function related to the Okamoto polynomials for the Painlevé IV equation

MHF2005-14 Masato KIMURA & Shin-ichi NAGATA
Precise asymptotic behaviour of the first eigenvalue of Sturm-Liouville problems with large drift

MHF2005-15 Daisuke TAGAMI & Masahisa TABATA
Numerical computations of a melting glass convection in the furnace

MHF2005-16 Raimundas VIDŪNAS
Normalized Leonard pairs and Askey-Wilson relations

MHF2005-17 Raimundas VIDŪNAS
Askey-Wilson relations and Leonard pairs

MHF2005-18 Kenji KAJIWARA & Atsushi MUKAIHIRA
Soliton solutions for the non-autonomous discrete-time Toda lattice equation

MHF2005-19 Yuu HARIYA
Construction of Gibbs measures for 1-dimensional continuum fields

MHF2005-20 Yuu HARIYA
Integration by parts formulae for the Wiener measure restricted to subsets in $\mathbb{R}^d$

MHF2005-21 Yuu HARIYA
A time-change approach to Kotani's extension of Yor's formula

MHF2005-22 Tadahisa FUNAKI, Yuu HARIYA & Mark YOR
Wiener integrals for centered powers of Bessel processes, I

MHF2005-23 Masahisa TABATA & Satoshi KAIZU
Finite element schemes for two-fluids flow problems

MHF2005-24 Ken-ichi MARUNO & Yasuhiro OHTA
Determinant form of dark soliton solutions of the discrete nonlinear Schrödinger equation

MHF2005-25 Alexander V. KITAEV & Raimundas VIDŪNAS
Quadratic transformations of the sixth Painlevé equation

MHF2005-26 Toru FUJII & Sadanori KONISHI
Nonlinear regression modeling via regularized wavelets and smoothing parameter selection

MHF2005-27 Shuichi INOKUCHI, Kazumasa HONDA, Hyen Yeal LEE, Tatsuro SATO, Yoshihiro MIZOGUCHI & Yasuo KAWAHARA
On reversible cellular automata with finite cell array

MHF2005-28 Toru KOMATSU
Cyclic cubic field with explicit Artin symbols

MHF2005-29 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Kaori NAGATOU
A computational approach to constructive a priori and a posteriori error estimates for finite element approximations of bi-harmonic problems

MHF2005-30 Kaori NAGATOU, Kouji HASHIMOTO & Mitsuhiro T. NAKAO
Numerical verification of stationary solutions for Navier-Stokes problems

MHF2005-31 Hidefumi KAWASAKI
A duality theorem for a three-phase partition problem

MHF2005-32 Hidefumi KAWASAKI
A duality theorem based on triangles separating three convex sets

MHF2005-33 Takeaki FUCHIKAMI & Hidefumi KAWASAKI
An explicit formula of the Shapley value for a cooperative game induced from the conjugate point

MHF2005-34 Hideki MURAKAWA
A regularization of a reaction-diffusion system approximation to the two-phase Stefan problem

MHF2006-1 Masahisa TABATA
Numerical simulation of Rayleigh-Taylor problems by an energy-stable finite element scheme

MHF2006-2 Ken-ichi MARUNO & G R W QUISPEL
Construction of integrals of higher-order mappings

MHF2006-3 Setsuo TANIGUCHI
On the Jacobi field approach to stochastic oscillatory integrals with quadratic phase function

MHF2006-4 Kouji HASHIMOTO, Kaori NAGATOU & Mitsuhiro T. NAKAO
A computational approach to constructive a priori error estimate for finite element approximations of bi-harmonic problems in nonconvex polygonal domains

MHF2006-5 Hidefumi KAWASAKI
A duality theory based on triangular cylinders separating three convex sets in $R^n$

MHF2006-6 Raimundas VIDŪNAS
Uniform convergence of hypergeometric series

MHF2006-7 Yuji KODAMA & Ken-ichi MARUNO
N-Soliton solutions to the DKP equation and Weyl group actions

MHF2006-8 Toru KOMATSU
Potentially generic polynomial

MHF2006-9 Toru KOMATSU
Generic sextic polynomial related to the subfield problem of a cubic polynomial

MHF2006-10 Shu TEZUKA & Anargyros PAPAGEORGIOU
Exact cubature for a class of functions of maximum effective dimension

MHF2006-11 Shu TEZUKA
On high-discrepancy sequences