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Abstract

First, it is pointed out that the uniform distribution of points in $[0,1]^d$ is not always a necessary condition for every function in a proper subset of the class of all Riemann integrable functions to have the arithmetic mean of function values at the points converging to its integral over $[0,1]^d$ as the number of points goes to infinity. We introduce a formal definition of the d-dimensional *high-discrepancy sequences*, which are not uniformly distributed in $[0,1]^d$, and present motivation for the application of these sequences to high-dimensional numerical integration. Then, we prove that there exist non-uniform (∞, d)-sequences which provide the convergence rate $O(N^{-1})$ for the integration of a certain class of d-dimensional Walsh function series, where N is the number of points.

Key words: discrepancy, high dimensional integrals, Monte Carlo and quasi-Monte Carlo methods, (t,d)-sequences, uniform distribution, Walsh functions

1 Introduction

It is well known [3] that the uniform distribution of points $X_n, n = 0, 1, ...,$ in $[0,1]^d$ is a necessary and sufficient condition for every function $f(x_1, ..., x_d)$ in the class of all Riemann integrable functions over $[0,1]^d$, hereafter denoted by $\mathfrak{R}_d$, to satisfy

$$I(f) := \int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(X_n). \tag{1}$$

It is also known [3] that for the class of all continuous functions over $[0,1]^d$, which is a proper subset of $\mathfrak{R}_d$, the uniform distribution is a necessary and suf-

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ficient condition for every function in the class to satisfy the equation (1). From the Information-Based Complexity (IBC) viewpoint, Woźniakowski [9,10] analyzed the average case complexity with respect to the Wiener sheet measure for the integration of this class of functions. The well-known Koksma-Hlawka theorem implies that the class of all functions over $[0,1]^d$ of bounded variation in the sense of Hardy and Krause is another proper subset of $\mathfrak{R}_d$ for which the uniform distribution is also a necessary and sufficient condition for every function in the class to satisfy the equation (1).

However, little is known about what happens to any other proper subsets of $\mathfrak{R}_d$. Although the uniform distribution of sample points is a sufficient condition for every function in any proper subset of $\mathfrak{R}_d$ to satisfy the equation (1), it is not always a necessary condition. The purpose of this paper is to explore a possibility to develop faster algorithms for a proper subset of $\mathfrak{R}_d$ based on non-uniform sample points than Monte Carlo and/or quasi-Monte Carlo methods. In the next section, we first introduce a formal definition of the d-dimensional *high-discrepancy sequences*, which are not uniformly distributed in $[0,1]^d$, and present motivation for the application of these sequences to high-dimensional numerical integration. Then, we prove that there exist non-uniform (∞, d)-sequences which provide the convergence rate $O(N^{-1})$ for the integration of a certain class of d-dimensional Walsh function series, where N is the number of points. In the last section, we discuss the significance of this result and future research directions.

2 Main Result

2.1 Definition and motivation for high-discrepancy sequences

First, we introduce the definition of high-discrepancy sequences:

Definition 1 *If a sequence of points $X_n, n = 0, 1, ...,$ in $[0,1]^d$ satisfies that for all $N > 1$, the star discrepancy $D_N^{(d)}$ of the first N points is given by*

$$D_N^{(d)} = O(1), \tag{2}$$

then we call it a high-discrepancy sequence, where the implied constant in the O notation depends only on the dimension d.

Notice that if the righthand side in the equation (2) is replaced by $O((\log N)^d/N)$, then it is the definition of a low-discrepancy sequence. High-discrepancy sequences are those sequences which are not uniformly distributed, and vice versa.

We have the following proposition:

Proposition 1 *A* (t,d)*-sequence in base two for which the first rows of its generator matrices are all identical to* $(1,0,0...)$ *is a high-discrepancy sequence.*

PROOF. Since the sequence is an (∞,d)-sequence which is distributed in only two sub-domains, $[0,\frac{1}{2}]^d$ and $[\frac{1}{2},1]^d$, the proof follows.

Note that there are two types of (∞,d)-sequences, i.e., uniform or non-uniform. The sequence introduced in Proposition 1 is a non-uniform (∞,d)-sequence.

We give the following theorems, which motivate us to investigate the application of high-discrepancy sequences defined in the above to high-dimensional numerical integration.

Theorem 1 *If an* L_2 *function in* d *dimensions is written as*

$$f(x_1,...,x_d) = \sum_{k_1,...,k_d \geq 0} c_{k_1,...,k_d} \cos(2\pi k_1 x_1) \cdots \cos(2\pi k_d x_d),$$

then we have

$$\int_{[0,1]^d} f(x_1,...,x_d)dx_1...dx_d$$

$$= 2^{d-1}\left(\int_{[0,\frac{1}{2}]^d} f(x_1,...,x_d)dx_1...dx_d + \int_{[\frac{1}{2},1]^d} f(x_1,...,x_d)dx_1...dx_d\right).$$

PROOF. Since

$$\int_0^1 \cos(2\pi kx)dx = 0$$

for any $k \geq 1$, we have

$$\int_{[0,1]^d} f(x_1,...,x_d)dx_1...dx_d = c_{0,...,0}.$$

And since

$$\int_0^{1/2} \cos(2\pi kx)dx = \int_{1/2}^1 \cos(2\pi kx)dx = 0$$

for any $k \geq 1$, we have

$$\int_{[0,\frac{1}{2}]^d} f(x_1,...,x_d)dx_1...dx_d = \frac{c_{0,...,0}}{2^d},$$

and

$$\int_{[\frac{1}{2},1]^d} f(x_1,...,x_d)dx_1...dx_d = \frac{c_{0,...,0}}{2^d}.$$

Thus, the proof is complete.□

Theorem 2 *Suppose that the dimension d is odd. If an L_2 function in d dimensions is written as*

$$f(x_1, ..., x_d) = c_0 + \sum_{k_1,...,k_d \geq 1} c_{k_1,...,k_d} \sin(2\pi k_1 x_1) \cdots \sin(2\pi k_d x_d),$$

then we have

$$\int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d$$

$$= 2^{d-1} \left(\int_{[0,\frac{1}{2}]^d} f(x_1, ..., x_d) dx_1 ... dx_d + \int_{[\frac{1}{2},1]^d} f(x_1, ..., x_d) dx_1 ... dx_d \right).$$

PROOF. Since

$$\int_0^1 \sin(2\pi k x) dx = 0$$

for any $k \geq 1$, we have

$$\int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d = c_0.$$

And since

$$\int_0^{1/2} \sin(2\pi k x) dx = \int_{1/2}^1 \sin(2\pi k x) dx = 0$$

for any even $k \geq 2$, and

$$\int_0^{1/2} \sin(2\pi k x) dx = -\int_{1/2}^1 \sin(2\pi k x) dx = \frac{1}{k\pi}$$

for any odd $k \geq 1$, we have

$$\int_{[0,\frac{1}{2}]^d} f(x_1, ..., x_d) dx_1 ... dx_d = \frac{c_0}{2^d} + \sum_{\substack{k_1,...,k_d \geq 1 \\ \text{all } k_1, ..., k_d \text{ are odd}}} \frac{c_{k_1,...,k_d}}{k_1 \cdots k_d \pi^d},$$

and

$$\int_{[\frac{1}{2},1]^d} f(x_1, ..., x_d) dx_1 ... dx_d = \frac{c_0}{2^d} + (-1)^d \sum_{\substack{k_1,...,k_d \geq 1 \\ \text{all } k_1, ..., k_d \text{ are odd}}} \frac{c_{k_1,...,k_d}}{k_1 \cdots k_d \pi^d}.$$

Since d is odd, the proof is complete.□

In what follows, we show that there exist non-uniform (∞, d)-sequences which provide the convergence rate $O(N^{-1})$ for the integration of a class of d-dimensional Walsh function series.

The Walsh functions are defined by

$$\mathrm{wal}(0, x) = 1, \text{ for } x \in [0, 1),$$

and for an integer $k \geq 1$,

$$\mathrm{wal}(k, x) = (-1)^{\sum_{j=1}^{\infty} b_{j-1} a_j} = (-1)^{(\mathbf{k}, X)}$$

where $k = b_0 + b_1 2 + \cdots$, and $x = a_1 2^{-1} + a_2 2^{-2} + \cdots$ in the binary expansion, and $\mathbf{k} = (b_0, b_1, ...)$ and $X = (a_1, a_2, ...)$ are the binary vector representation of k and x, respectively. Then, we have the following theorem.

Theorem 3 *A function in the d-dimensional L_2 space can be written as*

$$f(x_1, ..., x_d) = \sum_{k_1,...,k_d \geq 0} c_{k_1,...,k_d} \mathrm{wal}(k_1, x_1) \cdots \mathrm{wal}(k_d, x_d).$$

We have

$$\int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d$$

$$= 2^{d-1} \left(\int_{[0,\frac{1}{2}]^d} f(x_1, ..., x_d) dx_1 ... dx_d + \int_{[\frac{1}{2},1]^d} f(x_1, ..., x_d) dx_1 ... dx_d \right) - E_d,$$

where

$$E_d = \sum_{\substack{0 \leq k_1,...,k_d \leq 1 \\ (k_1,...,k_d) \neq (0,...,0) \\ k_1 + \cdots + k_d = 0 \pmod 2}} c_{k_1,...,k_d}. \tag{3}$$

PROOF. First, we have

$$\int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d = c_{0,...,0}.$$

Since

$$\int_0^{1/2} \mathrm{wal}(1, x) dx = -\int_{1/2}^1 \mathrm{wal}(1, x) dx = \frac{1}{2},$$

and

$$\int_0^{1/2} \mathrm{wal}(k, x) dx = \int_{1/2}^1 \mathrm{wal}(k, x) dx = 0$$

for any $k \geq 2$, we have

$$\int_{[0,\frac{1}{2}]^d} f(x_1, ..., x_d) dx_1 ... dx_d = \frac{1}{2^d} \sum_{0 \leq k_1,...,k_d \leq 1} c_{k_1,...,k_d},$$

and

$$\int_{[\frac{1}{2},1]^d} f(x_1, ..., x_d) dx_1 ... dx_d = \frac{1}{2^d} \sum_{0 \le k_1, ..., k_d \le 1} (-1)^{k_1 + \cdots + k_d} c_{k_1, ..., k_d}.$$

Thus, the proof is complete.□

We now consider a class of functions for which $E_d = 0$. First, we give some definitions.

Definition 2 *Let $m \geq 1$ be an integer and let u be a nonempty subset $\{j_1, ..., j_{|u|}\} \subseteq \{1, ..., d\}$. We define*

$$\phi_{u,m}(x_1, ..., x_d) = \prod_{i=1}^{|u|} \mathrm{wal}(k_{j_i}^{(m)}, x_{j_i}),$$

for $2^{m-1} \leq k_{j_1}^{(m)}, ..., k_{j_{|u|}}^{(m)} < 2^m$.

Definition 3 *We define a class $\mathfrak{F}_d$ which consists of functions*

$$f(x_1, ..., x_d) = c_0 + \sum_{\substack{1 \le |u| \le d \\ |u| \text{ is odd}}} \sum_{m=1}^{\infty} c_{u,m} \phi_{u,m}(x_1, ..., x_d),$$

where c_0 and $c_{u,m}, m = 1, 2, ...,$ are constants satisfying

$$|c_0| + \sum_{\substack{1 \le |u| \le d \\ |u| \text{ is odd}}} \sum_{m=1}^{\infty} |c_{u,m}| 2^{m-1} \leq M < \infty.$$

Here, M is a constant.

We have the following theorem.

Theorem 4 *For any function $f(x_1, ..., x_d)$ in $\mathfrak{F}_d$, we have*

$$\int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d$$

$$= 2^{d-1} \left(\int_{[0,\frac{1}{2}]^d} f(x_1, ..., x_d) dx_1 ... dx_d + \int_{[\frac{1}{2},1]^d} f(x_1, ..., x_d) dx_1 ... dx_d \right).$$

PROOF. From the equation (3) and Definition 3, it follows that the quantity E_d in Theorem 3 becomes zero for the case of $\mathfrak{F}_d$. Thus, the proof is complete.□

Hereafter, we denote by $\mathfrak{S}_d$ a class of (t,d)-sequences in base two whose generator matrices are nonsingular and lower-triangular. Note that this class is a special subset of the class of non-uniform (∞,d)-sequences introduced in Proposition 1. We prove the following theorem.

Theorem 5 *For any sequence* $\mathbf{s}_n, n=0,1,...,$ *in the class* $\mathfrak{S}_d$*, the integration error* e_N *of any function* f *in* $\mathfrak{F}_d$ *is given by*

$$e_N(\mathbf{s}_n,f) := \left|I(f)-\frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{s}_n)\right| < \frac{1}{N}\sum_{\substack{1\le|u|\le d\\ |u| \text{ is odd}}}\sum_{m=1}^{\infty}|c_{u,m}|\min(2^{m-1},N).$$

From the condition in Definition 3, we have $e_N(\mathbf{s}_n,f)=O(N^{-1})$, where the asymptotic constant is M.

In order to prove the above theorem, we need the following three lemmas, which are generalization of Lemmas 2, 3 and 4 in [8]. Let $s_n, n=0,1,...,$ be a sequence in $\mathfrak{S}_1$, and let $s_{n,m}$ denote the m-th bit of s_n.

Lemma 1 *Suppose* $m\ge 1$*. For all* $n=h\,2^m, h=0,1,...,$ *and for all* $0\le \ell < 2^{m-1}$*, we have*

$$s_{n+\ell,m} = 1 - s_{n+\ell+2^{m-1},m}.$$

PROOF. If we compare $n+\ell$ and $n+\ell+2^{m-1}$, only the coefficient of 2^{m-1} in their binary representation is different while all the others are identical. Since the generator matrix is nonsingular and lower-triangular, this completes the proof. □

Lemma 2 *Suppose* $m\ge 1$*. For all* $n=h\,2^m, h=0,1,...,$ *and for all* $0\le \ell < 2^{m-1}$*, we have*

$$\mathrm{wal}(k,s_{n+\ell}) = -\mathrm{wal}(k,s_{n+\ell+2^{m-1}}),$$

where k *is any integer with* $2^{m-1}\le k < 2^m$*.*

PROOF. Since the generator matrix is nonsingular and lower-triangular, we have $s_{n+\ell,e} = s_{n+\ell+2^{m-1},e}$ for all $1\le e < m$. From the assumption that $k = b_0 + b_1 2 + \cdots + b_{m-1}2^{m-1}$ with $b_{m-1}=1$ and Lemma 1, the proof is complete. □

For d dimensions, we denote a sequence in $\mathfrak{S}_d$ by $\mathbf{s}_n = (s_n^{(1)},...,s_n^{(d)}), n = 0,1,...,$ Then we have

Lemma 3 *Suppose $m \geq 1$. Let u be any nonempty subset $\{j_1, ..., j_{|u|}\} \subseteq \{1, ..., d\}$. If the cardinality $|u|$ is odd, then for all $n = h\ 2^m, h = 0, 1, ...,$ and for all $0 \leq \ell < 2^{m-1}$, we have*

$$\phi_{u,m}(\mathbf{s}_{n+\ell}) = -\phi_{u,m}(\mathbf{s}_{n+\ell+2^{m-1}}).$$

PROOF. Denote

$$\phi_{u,m}(\mathbf{s}_{n+\ell}) = \prod_{i=1}^{|u|} \text{wal}(k_{j_i}^{(m)}, s_{n+\ell}^{(j_i)}),$$

where $2^{m-1} \leq k_{j_1}^{(m)}, ..., k_{j_{|u|}}^{(m)} < 2^m$. Without loss of generality, we assume that the number of $j_i, 1 \leq i \leq |u|$, with $\text{wal}(k_{j_i}^{(m)}, s_{n+\ell}^{(j_i)}) = -1$ is odd. Then, from Lemma 2, the number of $j_i, 1 \leq i \leq |u|$, with $\text{wal}(k_{j_i}^{(m)}, s_{n+\ell+2^{m-1}}^{(j_i)}) = -1$ is even because $|u|$ is odd. Thus, the proof is complete. $\square$

Note that $I(\phi_{u,m}) = 0$ for any $u \neq \emptyset$ and $m \geq 1$. From Lemma 3, it follows that the integration error e_N of $\phi_{u,m}$ with $|u|$ odd for any sequence $\mathbf{s}_n, n = 0, 1, ...,$ in $\mathfrak{S}_d$ is given by

$$e_N(\mathbf{s}_n, \phi_{u,m}) = \left| \frac{1}{N} \sum_{n=0}^{N-1} \phi_{u,m}(\mathbf{s}_n) \right| = \left| \frac{1}{N} \sum_{n=N-N_m}^{N-1} \phi_{u,m}(\mathbf{s}_n) \right| \leq \frac{\min(2^{m-1}, N)}{N},$$

where N_m is the residue of N modulo 2^m.

We are now ready to prove Theorem 5.

$$e_N(\mathbf{s}_n, f) = \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{s}_n) - I(f) \right| = \left| \frac{1}{N} \sum_{n=0}^{N-1} (f(\mathbf{s}_n) - c_0) \right|$$

$$= \left| \frac{1}{N} \sum_{n=0}^{N-1} \sum_{\substack{1 \leq |u| \leq d \\ |u| \text{ is odd}}} \sum_{m=1}^{\infty} c_{u,m} \phi_{u,m}(\mathbf{s}_n) \right| \leq \sum_{\substack{1 \leq |u| \leq d \\ |u| \text{ is odd}}} \sum_{m=1}^{\infty} \left| \frac{c_{u,m}}{N} \sum_{n=0}^{N-1} \phi_{u,m}(\mathbf{s}_n) \right|$$

$$\leq \frac{1}{N} \sum_{\substack{1 \leq |u| \leq d \\ |u| \text{ is odd}}} \sum_{m=1}^{\infty} |c_{u,m}| \min(2^{m-1}, N).$$

Thus, the proof is complete.

We should notice that a sequence $\mathbf{s}_n, n = 1, 2, ...,$ in $\mathfrak{S}_d$ is distributed in the sub-domain $[\frac{1}{2^m}, \frac{1}{2^{m-1}}]^d$ when n is a multiple of 2^{m-1} but not of 2^m for some $m \geq 1$. Therefore, high-discrepancy sequences employed in the above are *essentially non-uniform* because they are distributed in only a part of the diagonal domain $[0, \frac{1}{2}]^d \bigcup [\frac{1}{2}, 1]^d$.

3 Discussion

For a function $f(x_1, ..., x_d)$ which satisfies

$$\int_{[0,1]^d} f(x_1, ..., x_d) dx_1 ... dx_d$$

$$= 2^{d-1} \left(\int_{[0,\frac{1}{2}]^d} f(x_1, ..., x_d) dx_1 ... dx_d + \int_{[\frac{1}{2},1]^d} f(x_1, ..., x_d) dx_1 ... dx_d \right),$$

there are three approaches to its numerical integration when the dimension d is large.

- Monte Carlo methods: We have two types of Monte Carlo methods: uniform random sampling over the whole domain $[0, 1]^d$ or over two sub-domains, $[0, \frac{1}{2}]^d$ and $[\frac{1}{2}, 1]^d$. For both cases, the convergence rate is $O(N^{-1/2})$.
- Quasi-Monte Carlo or low-discrepancy methods: This approach provides the convergence rate $O((\log N)^d)/N)$ for functions of bounded variation [1,4–7].
- High-discrepancy methods: There are two types of high-discrepancy methods: The first type is to apply low-discrepancy sequences to each of the two sub-domains, $[0, \frac{1}{2}]^d$ and $[\frac{1}{2}, 1]^d$, as the whole domain. The class of (∞, d)-sequences introduced in Proposition 1 can be used for such purpose. In this case, we get the convegence rate $O((\log N)^d)/N)$ for functions of bounded variation. Note that this type is closely related to Hlawka-Mück approach [2]. The second type uses essentially non-uniform sequences such as the class $\mathfrak{S}_d$. For the particular case of $\mathfrak{F}_d$, we have shown the $O(N^{-1})$ convergence rate, which is better than Monte Carlo and quasi-Monte Carlo methods. However, at least at present, nothing is known for functions besides $\mathfrak{F}_d$.

An important consequence from our result is that the "low-discrepancy" approach is not the only way for accelerating the computation of high dimensional numerical integration. If one considers a class of functions for which the uniform distribution is not a necessary condition to satisfy the equation (1), there is a possibility to develop faster algorithms based on the "high-discrepancy" approach than Monte Carlo and/or quasi-Monte Carlo (low-discrepancy) methods. There are several applications in practice involving a class of functions in high dimensions which are localized in a very limited domain. Commonly, we have used the so-called importance sampling techniques including Markov Chain Monte Carlo (MCMC) methods for these applications. Unfortunately, all Monte Carlo techniques suffer from the slow convergence rate $O(N^{-1/2})$, while our high discrepancy approach, which is deterministic, has a potential of the convergence rate $O(N^{-1})$. Therefore, we can consider the high-discrepancy approach as a possible alternative. However, there are many topics to be explored in this new area.

References

[1] M. Drmota and R. F. Tichy, *Sequences, Discrepancies and Applications*, Lecture Notes in Mathematics, Vol. 1651, Springer, 1997.

[2] E. Hlawka and R. Mück, A Transformation of Equidistributed Sequences, In S. K. Zaremba (Ed.), *Applications of Number Theory to Numerical Analysis*, Academic Press, New York (1972), 371-388.

[3] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, John-Wiley & Sons, 1974.

[4] J. Matoušek, *Geometric Discrepancy: An Illustrated Guide,* Springer, 1999.

[5] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, CBMS-NSF Regional Conference Series in Applied Mathematics, No. 63, SIAM, 1992.

[6] I. H. Sloan and S. Joe, *Lattice Methods for Multiple Integration,* Claredon Press, Oxford, 1994.

[7] S. Tezuka, *Uniform Random Numbers: Theory and Practice*, Kluwer Academic Publishers, 1995.

[8] S. Tezuka, On the Necessity of Low-Effective Dimension, *Journal of Complexity*, **21** (2005), 710-721.

[9] J. F. Traub and A. G. Werschulz, *Complexity and Information*, Cambridge Univ. Press, 1998.

[10] H. Woźniakowski, Average Case Complexity of Multidimensional Integration, *Bull. Amer. Math. Soc.* , **24** (1991), 185-194.

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