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The considered problem is uniform convergence of sequences of hypergeometric series. To determine uniform convergence, we use the Weierstrass M-test: uniformly dominated convergence implies uniform convergence. We give necessary and sufficient conditions for uniformly dominated convergence of proper sequences of hypergeometric series. These conditions can be checked algorithmically.

1   Introduction

In this paper we study uniform convergence of sequences of hypergeometric series. We consider the sequences $\mathcal{U}(n) = \sum_{k=0}^{\infty} u(n,k)$ of hypergeometric series such that $u(n,k)$ is a proper hypergeometric term in n, k. We refer to these sequences as proper sequences of hypergeometric series. We assume that the individual series $\mathcal{U}(n)$ are non-terminating for large enough n. The underlying field is the complex numbers.

Recall [AP02] that a bivariate sequence $u(n,k)$ is a *hypergeometric term* if both quotients $u(n+1,k)/u(n,k)$ and $u(n,k+1)/u(n,k)$ can be realized as rational functions of n, k.

A bivariate sequence $u(n,k)$ is a *proper term* if there exist: non-negative integers p, q; complex constants ξ, θ; $b_1, \ldots, b_p$; $d_1, \ldots, d_q$; integers $\alpha_1, \ldots, \alpha_p$; $\beta_1, \ldots, \beta_p$; $\gamma_1, \ldots, \gamma_q$; $\delta_1, \ldots, \delta_q$; and a polynomial $P(n,k)$ such that

$$u(n,k) = P(n,k) \, \frac{(b_1)_{\alpha_1 n + \beta_1 k} \, \cdots \, (b_p)_{\alpha_p n + \beta_p k}}{(d_1)_{\gamma_1 n + \delta_1 k} \, \cdots \, (d_q)_{\gamma_q n + \delta_q k} \, k!} \, \xi^n \theta^k, \tag{1.1}$$

where $(a)_m$ is the Pochhammer symbol:

$$(a)_m = \begin{cases} a\,(a+1)\cdots(a+m-1), & \text{if } m > 0, \\ 1, & \text{if } m = 0, \\ 1/(a-1)\cdots(a-|m|), & \text{if } m < 0. \end{cases} \quad (1.2)$$

For those n, k for which (1.1) is undefined due to division by 0, we set $u(n,k) = 0$.

A bivariate sequence $u(n,k)$ is *holonomic* if the generating function $\sum_{n,k\geq 0} u(n,k)\,x^n y^k$ and all its partial derivatives generate a finite-dimensional vector space over the field of rational functions in x, y.

Proper terms are hypergeometric and holonomic [AP02, Theorem 3]. Since our coefficient field is algebraically closed, any holonomic hypergeometric term is *conjugate* to a proper term [AP02, Theorem 14]. This means, the quotients $u(n+1,k)/u(n,k)$ and $u(n,k+1)/u(n,k)$ can be realized by rational functions $f(n,k)$, $g(n,k)$, respectively, which would also satisfy $\widetilde{u}(n+1,k) = f(n,k)\,\widetilde{u}(n,k)$ and $\widetilde{u}(n,k+1) = g(n,k)\,\widetilde{u}(n,k)$ for some proper term $\widetilde{u}(n,k)$.

If a term $u(n,k)$ is holonomic, then it satisfies difference equations (in one or both variables) whose coefficients are dependent only on n [PWZ, Chapter 4]. If the term $u(n,k)$ is proper, and for any n the sum $\mathcal{U}(n) = \sum_{k=0}^{\infty} u(n,k)$ is terminating, Zeilberger's algorithm gives a recurrence relation with respect to n for $\mathcal{U}(n)$. The crucial step in Zeilberger's algorithm is to derive a recurrence relation

$$L(n)\,u(n,k) = R(n,k+1) - R(n,k), \quad (1.3)$$

where $L(n)$ is a linear difference operator with coefficients in n only, and $R(n,k)$ is a hypergeometric term. The linear recurrence is derived by summing (1.3) over all k; the right hand-side simplifies due to telescoping summation.

When generalizing Zeilberger's algorithm to non-terminating hypergeometric series, one needs to make sure that the series $\mathcal{U}(n) = \sum_{k=0}^{\infty} u(n,k)$ converges uniformly, so to justify manipulation of (1.3). This paper gives criteria to decide uniform convergence of $\mathcal{U}(n)$. In [VK02] these criteria are used for the Zeilberger type algorithms for non-terminating hypergeometric series.

2 Basic preliminary results

Throughout the paper, let $\mathbf{Z}_+$ denote the set of non-negative integers. We make the convention that $0^0 = 1$, which is the proper continuous limit of the function $|x|^x$.

As the criterium for uniform convergence of function series, we use the Weierstrass M-test formulated here below. (We apply it with $E = \mathbf{Z}_+$.)

Lemma 2.1 *Let $f_0(x), f_1(x), f_2(x), \ldots$ be a sequence of complex valued functions from a set E. If there exists a sequence $M_0, M_1, M_2, \ldots$ of real constants such that $|f_j(x)| < M_j$ for any $x \in E$ and all $j \in \mathbf{Z}_+$, and the series $\sum_{j=0}^{\infty} M_j$ converges, then the function series $\sum_{j=0}^{\infty} f_j(x)$ converges uniformly on E.*

We refer to a function series that satisfies the sufficient condition of this criterium as a *uniformly dominated convergent* series. In plain terms, the condition is that the series is uniformly bounded (or majorated) by an absolutely convergent series.

The main result of this paper is the sufficient and necessary conditions for uniformly dominated convergence of proper sequences of hypergeometric series. We present this result in Section 6. In the current Section, we present the main technical Lemma that we use to determine uniformly dominanted convergence, and provide a few asymptotic expressions for the gamma function. In Section 3 we provide a few other intermediate results. In Sections 4 and 5 we specify the form of hypergeometric series under consideration, and define the notation we use.

The following Lemma gives us a strategy to determine uniformly dominated convergence of sequences of non-terminating hypergeometric series.

Lemma 2.2 *Let $u(n, k)$ denote a hypergeometric term in n, k. We assume that the hypergeometric series $\mathcal{U}(n) = \sum_{k=0}^{\infty} u(n, k)$ is non-terminating for large enough n. The series sequence $\mathcal{U}(n)$ is uniformly dominated convergent if and only if the following conditions hold:*

(a) *For any $n \geq 0$, the series $\mathcal{U}(n)$ converges absolutely.*

(b) *The termwise limit $\sum_{k=0}^{\infty} \lim_{n \to \infty} u(n, k)$ exists and converges absolutely.*

(c) *For any function $N : \mathbf{Z}_+ \to \mathbf{Z}_+$ such that $N(k) \sim C_0 \, k^p$ for some real $p > 1$ and C_0, the series $\sum_{k=0}^{\infty} u(N(k), k)$ converges absolutely.*

(d) *For any function $N : \mathbf{Z}_+ \to \mathbf{Z}_+$ such that $N(k) \sim C_0 \, k^p$ for some real $p \in (0, 1)$ and C_0, the series $\sum_{k=0}^{\infty} u(N(k), k)$ converges absolutely.*

(e) *For any function $N : \mathbf{Z}_+ \to \mathbf{Z}_+$ such that $N(k) = \lambda \, k + \omega(k)$, with $\omega(k) = O(1)$ or $\omega(k) \sim C_0 \, k^p$ for some real $p \in (0, 1)$, C_0, and $\lambda > 0$, the series $\sum_{k=0}^{\infty} u(N(k), k)$ converges absolutely.*

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Proof. We may assume that the series $\mathcal{U}(n)$ actually depend on n. The conditions are necessary because an uniformly bounding series would be a majorant for the indicated series as well.

To prove the sufficiency, let $z(k) = \sup_{n \geq 0} |u(n,k)|$. Then the series $\sum_{k=0}^{\infty} z(k)$ is a precise uniform majorant for $\mathcal{U}(n)$. The series sequence $\mathcal{U}(n)$ is uniformly dominant convergent if and only if the series $\sum_{k=0}^{\infty} z(k)$ converges.

Let $h(\nu, \kappa)$ be the rational function of two complex variables equal to $u(\nu+1, \kappa)/u(\nu, \kappa)$ for positive integer values of ν and κ. Note that $h(\nu, \kappa)$ is non-zero and well-defined for positive integers κ and large enough integers ν, because $u(n,k) = 0$ would imply that the hypergeometric series are terminating or undefined for large enough n. The function $h(\nu, \kappa)$ may be complex valued, but the variables ν, κ are assumed to be real.

For each non-negative integer k, we have that either $z(k) = \lim_{n \to \infty} |u(n,k)|$, or

$$z(k) = |u(n_0, k)| \text{ and } |h(n_0, k)| \leq 1, |h(n_0 - 1, k)| \geq 1 \text{ for some integer } n_0.$$

In the later case, the rational function $|h(\nu, k)|$ of ν (with k fixed) has either a pole on the interval $\nu \in [n_0 - 1, n_0]$, or it is continuous and therefore achieves the value 1 on the same interval.

Let $\widehat{h}(\nu, \kappa)$ be the denominator of $|h(\nu, \kappa)|$. Since the series $\mathcal{U}(n)$ are not the same for all n, we have that $|h(\nu, k)|$ is not a constant function of ν for all large enough k. Let $\nu_1(\kappa), \ldots, \nu_m(\kappa)$ be the positive real algebraic functions, which are solutions of the algebraic equations $|h(\nu, \kappa)| = 1$, $\widehat{h}(\nu, \kappa) = 0$, and are defined for large enough κ. For $j = 1, \ldots, m$, let $N_j(k)$ be the integer-valued function

$$N_j(k) = \begin{cases} \lfloor \nu_j(k) \rfloor, & \text{if } |h(\lfloor \nu_j(k) \rfloor, k)| \leq 1, \\ \lceil \nu_j(k) \rceil, & \text{if } |h(\lfloor \nu_j(k) \rfloor, k)| > 1. \end{cases}$$

All these functions satisfy the assumption of one of the last three conditions. For large enough k, the candidates for $z(k)$ are $|u(0,k)|$, $\lim_{n \to \infty} |u(n,k)|$, and $|u(N_j(k), k)|$ for $j = 1, \ldots, m$. Note that each $N_j(k)$ is either bounded and we can apply condition *(a)*, or we can apply one of the conditions *(c)–(e)*. The sum of all candidates gives a series which is a uniform majorant for $\mathcal{U}(n)$. QED.

We will use the following asymptotic expressions for the gamma function. It will be convenient for us to uniformize all gamma expressions with a linear argument in $m \to \infty$ to expressions with the only one gamma value $\Gamma(m)$. Some corollaries are formulated with lesser generality than possible, so to indicate their application more stressfully.

Lemma 2.3 *Let λ be a real number, and let $\ell \in \mathbb{C}$.*

- *If $\lambda > 0$ then*

$$\Gamma(\lambda\, m + \ell) \sim (2\pi)^{\frac{1-\lambda}{2}}\, \lambda^{\ell - 1/2}\, m^{\ell + \frac{\lambda-1}{2}}\, \lambda^{\lambda m}\, \Gamma(m)^{\lambda} \qquad \text{as real } m \to \infty. \qquad (2.1)$$

- *If $\lambda < 0$, $\ell \notin \mathbb{Z}$, and m runs through a set of real numbers such that $\lambda\, m \in \mathbb{Z}$, then*

$$\Gamma(\lambda\, m + \ell) \sim \frac{(2\pi)^{\frac{1-\lambda}{2}}\, |\lambda|^{\ell - 1/2}}{2\, \sin(\pi\ell)}\, m^{\ell + \frac{\lambda-1}{2}}\, \lambda^{\lambda m}\, \Gamma(m)^{\lambda} \qquad \text{as } m \to \infty. \qquad (2.2)$$

Proof. The first statement follows from Stirling's asymptotic formula [AAR99, Theorem 1.4.1]:

$$\frac{\Gamma(\lambda\, m + \ell)}{\Gamma(m)^{\lambda}} \sim \frac{\sqrt{2\pi}\, (\lambda m + \ell)^{\lambda m + \ell - 1/2}\, \exp(-\lambda\, m - \ell)}{(2\pi)^{\lambda/2}\, m^{\lambda m - \lambda/2}\, \exp(-\lambda\, m)}$$

$$\sim (2\pi)^{\frac{1-\lambda}{2}}\, \lambda^{\lambda m + \ell - 1/2}\, m^{\ell + \frac{\lambda-1}{2}} \left(1 + \frac{\ell}{\lambda\, m}\right)^{\ell - 1/2} \left(1 + \frac{\ell}{\lambda\, m}\right)^{\lambda m} \exp(-\ell).$$

Note that

$$\lim_{m \to \infty} \left(1 + \frac{\ell}{\lambda\, m}\right)^{\ell - 1/2} = 1 \qquad \text{and} \qquad \lim_{m \to \infty} \left(1 + \frac{\ell}{\lambda\, m}\right)^{\lambda m} = \exp(\ell).$$

Formula (2.1) follows.

To prove the second statement we use Euler's reflection formula [AAR99, Theorem 1.2.1]:

$$\Gamma(\lambda\, m + \ell) = \frac{(-1)^{\lambda m}\, \pi}{\sin \pi\ell}\, \frac{1}{\Gamma\left(|\lambda| m + 1 - \ell\right)}. \qquad (2.3)$$

Now we apply the first statement to $\Gamma\left(|\lambda| m + 1 - \ell\right)$ and obtain (2.2). $\qquad$ QED.

Corollary 2.4 *Let λ be a nonzero real number, and let $\ell \in \mathbb{C}$. We assume that m runs through a set of real numbers such that $\lambda\, m$ is an integer. If $\lambda < 0$ then we additionally assume that $\ell \notin \mathbb{Z}$. Under these assumptions there is a constant $C_0 \in \mathbb{C}$ such that*

$$\Gamma(\lambda\, m + \ell) \sim C_0\, m^{\ell + \frac{\lambda-1}{2}}\, \lambda^{\lambda m}\, \Gamma(m)^{\lambda} \qquad (2.4)$$

as $m \to \infty$.

Corollary 2.5 *Let λ, N be integers, and let $\ell \in \mathbb{C}$. We assume that $\lambda \neq 0$. If $\lambda < 0$ we additionally assume that $\ell \notin \mathbb{Z}$. Then, as integer $m \to \infty$,*

$$\Gamma(\lambda m + N + \ell) \;\sim\; C_0\,(2\pi)^{\frac{1-\lambda}{2}}\,|\lambda|^{\ell - 1/2}\,\lambda^N\,m^{\ell + \frac{\lambda-1}{2}}\,\lambda^{\lambda m}\,\Gamma(m)^\lambda, \tag{2.5}$$

where

$$C_0 = \begin{cases} 1, & \text{if } \lambda > 0, \\ \frac{1}{2\sin \pi\ell}, & \text{if } \lambda < 0. \end{cases} \tag{2.6}$$

Proof. For $\lambda < 0$, the simplification is $|\lambda|^N / \sin \pi(\ell + N) = \lambda^N / \sin \pi\ell$. $\qquad$ QED.

3 Other preliminary results

Here we continue with more asymptotic formulas for the gamma function and Pochhammer symbols. Lemma 3.7 is used only in the auxiliary Section 8.

We introduce the following function:

$$\Theta(x) = \frac{1+x}{x}\,\log(1+x) - 1. \tag{3.1}$$

Lemma 3.1 *Let $\omega(m)$ denote a real-valued function defined for large enough $m \in \mathbb{R}$, such that $\omega(m) = o(m)$ as $m \to \infty$. Then*

$$\Gamma(m + \omega(m)) \sim m^{\omega(m)}\,\exp\left(\omega(m)\,\Theta\left(\frac{\omega(m)}{m}\right)\right)\,\Gamma(m) \qquad as \quad m \to \infty. \tag{3.2}$$

Proof. By Stirling's asymptotic formula:

$$\frac{\Gamma(m + \omega(m))}{\Gamma(m)} \;\sim\; \frac{(m + \omega(m))^{m + \omega(m) - 1/2}}{m^{m - 1/2}}\,\exp(-\omega(m))$$

$$\sim\; m^{\omega(m)}\left(1 + \frac{\omega(m)}{m}\right)^{m + \omega(m) - 1/2}\,\exp(-\omega(m))$$

$$\sim\; m^{\omega(m)}\,\exp\left((m + \omega(m))\log\left(1 + \frac{\omega(m)}{m}\right) - \omega(m)\right). \tag{3.3}$$

The result follows. $\qquad$ QED.

Corollary 3.2 *Let $\omega(m)$ denote a real-valued function defined for large enough $m \in \mathbb{R}$, such that $\omega(m) = o(m)$ as $m \to \infty$. Then*

$$\Gamma(m + \omega(m)) \sim m^{\omega(m)} \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j(j+1)} \frac{\omega(m)^{j+1}}{m^j}\right) \Gamma(m). \qquad (3.4)$$

Proof. On the interval $x \in (-1, 1)$ we have

$$\Theta(x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j(j+1)} x^j. \qquad (3.5)$$

QED.

As a direct consequence, we obtain the following well known asymptotics:

$$(\ell)_m = \frac{\Gamma(m + \ell)}{\Gamma(m)} \sim m^{\ell}, \qquad \text{as} \quad m \to \infty \qquad (\ell \in \mathbb{C}). \qquad (3.6)$$

Lemma 3.3 *Let λ be a nonzero real number, and let $\ell \in \mathbb{C}$. If $\lambda < 0$ we additionally assume that $\ell \notin \mathbb{Z}$. Let $N = N(m)$ denote a function $N : \mathbf{Z}_+ \to \mathbf{Z}_+$ such that $N(m) - \lambda\, m = o(m)$ as $m \to \infty$. Let $\omega(m)$ denote the difference $N(m) - \lambda\, m$. Then there is a constant $C_1 \in \mathbb{C}$ such that*

$$(\ell)_{N(m)} \sim C_1\, m^{\ell + \frac{\lambda-1}{2}}\, \lambda^{N(m)}\, m^{\omega(m)} \exp\left(\omega(m)\, \Theta\left(\frac{\omega(m)}{\lambda\, m}\right)\right) \Gamma(m)^{\lambda} \qquad \text{as} \quad m \to \infty. \quad (3.7)$$

Proof. We have $N(m) = \lambda\, m + \omega(m)$ and $(\ell)_N = \Gamma(N + \ell)/\Gamma(\ell)$. Applying Corollary 2.4,

$$\Gamma(\lambda\, m + \omega(m) + \ell) = \Gamma\left(\lambda\left(m + \frac{\omega(m)}{\lambda}\right) + \ell\right)$$

$$\sim C_0\, m^{\ell + (\lambda-1)/2}\, \lambda^{\lambda\, m + \omega(m)}\, \Gamma\left(m + \frac{\omega(m)}{\lambda}\right)^{\lambda}.$$

Then we apply Lemma 3.2 to the last factor. QED.

Corollary 3.4 *Let λ be a nonzero integer, and let $\ell \in \mathbb{C}$. If $\lambda < 0$, we assume that $\ell \notin \mathbb{Z}$. Let $\omega(m)$ denote a function $\omega : \mathbf{Z}_+ \to \mathbf{Z}_+$ such that $\omega(m) = o(m)$ as $m \to \infty$. Then there is a constant $C_0 \in \mathbb{C}$ such that*

$$(\lambda\, m + \ell)_{\omega(m)} \sim C_0\, \lambda^{\omega(m)}\, m^{\omega(m)} \exp\left(\omega(m)\, \Theta\left(\frac{\omega(m)}{\lambda\, m}\right)\right) \qquad \text{as} \quad m \to \infty. \qquad (3.8)$$

Proof. We have $(\lambda m + \ell)_{\omega(m)} = \Gamma(\lambda m + \omega(m) + \ell)/\Gamma(\lambda m + \ell)$. Lemma 3.3 can be applied to the numerator and the denominator. QED.

Lemma 3.5 *Let λ be a non-zero integer, and let $\ell \in \mathbb{C}$. If $\lambda < 0$, we assume that $\ell \notin \mathbb{Z}$. Let $\omega(m)$ denote a function $\omega : \mathbf{Z}_+ \to \mathbb{R}$ such that $\lambda \omega(m) \in \mathbb{Z}$ whenever $m \in \mathbb{Z}_+$. We assume that $\omega(m)$ approaches $+\infty$, $-\infty$ or a finite limit as $m \to \infty$. Let us denote*

$$\varepsilon = \begin{cases} 1, & \text{if } \omega(m) \to +\infty \text{ or bounded as } m \to \infty, \\ -1, & \text{if } \omega(m) \to -\infty \text{ as } m \to \infty, \end{cases} \tag{3.9}$$

Then there is a constant $C_0 \in \mathbb{C}$ such that

$$(\ell)_{\lambda \omega(m)} \sim C_0 \, |\omega(m)|^{\ell - \frac{1}{2} + \frac{1}{2}\varepsilon\lambda} \, (\varepsilon\lambda)^{\lambda \omega(m)} \, \Gamma(|\omega(m)|)^{\varepsilon\lambda}. \tag{3.10}$$

Proof. When $\omega(m) \to +\infty$, by Corollary 2.4 we have

$$\Gamma(\lambda \omega(m) + \ell) \sim \widetilde{C}_0 \, \omega(m)^{\ell + \frac{\lambda-1}{2}} \, \lambda^{\lambda \omega(m)} \, \Gamma(\omega(m))^{\lambda}.$$

When $\omega(m) \to -\infty$, we apply Corollary 2.4 to $\Gamma(-\lambda\,|\omega(m)| + \ell)$. The result is

$$\Gamma(\lambda \omega(m) + \ell) \sim \widehat{C}_0 \, |\omega(m)|^{\ell - \frac{\lambda+1}{2}} \, (-\lambda)^{\lambda \omega(m)} \, \Gamma(|\omega(m)|)^{-\lambda}.$$

The Pochhammer symbol grows accordingly. If $\omega(m)$ is bounded or approaching a finite limit, so is the Pochhammer symbol. QED.

Lemma 3.6 *Let $Z = \sum_{j=0}^{\infty} v_j$ be a series, ϱ be a positive real number, and let*

$$w(j) = \frac{\log |v_j|}{j^{\varrho}} \qquad \text{for} \quad j = 0, 1, 2, \ldots.$$

- *If $\lim_{j \to \infty} w(j) = -\infty$ or $\lim_{j \to \infty} \sup w(j) < 0$, then the series Z converges absolutely.*

- *If $\lim_{j \to \infty} \sup w(j) = \infty$ or $\lim_{j \to \infty} \sup w(j) > 0$, then the series Z diverges.*

Proof. The case $p = 1$ is equivalent to the standard convergence criteria involving $\lim \sup |v_j|^{1/j}$; see [Rud74]. To prove the first statement in general, we choose a positive real number K such that $w(j) < -K$ for large enough j. Then $\log |v_j| < -Kj^{\varrho} < -2 \log j$ for large enough j. Therefore a tail of the series Z can be majorated by $\sum j^{-2}$, so converges absolutely.

To prove the second statement we choose a positive real K such that $w(j) > K$ for large enough j. Then $\log |v_j| > Kj^{\varrho}$, so v_j is unbounded. Hence the series Z diverges. QED.

Lemma 3.7 *Let a, b, p be real numbers. Assume that $a > 0$ and $0 < p < 1$. Consider the sequence $v_j = (aj + b)\, p^j$, with $j = 0, 1, 2, \ldots$. Then*

$$v_{j+1} < v_j \qquad if \qquad j > \frac{p}{1-p} - \frac{b}{a}.$$

Proof. A straightforward computation. QED.

4 Notation

We start with a hypergeometric term represented as in (1.1):

$$u(n, k) = P(n, k)\, \frac{(b_1)_{\alpha_1 n + \beta_1 k} \, \cdots \, (b_p)_{\alpha_p n + \beta_p k}}{(d_1)_{\gamma_1 n + \delta_1 k} \, \cdots \, (d_r)_{\gamma_q n + \delta_q k} \, k!} \, \xi^n \theta^k. \tag{4.1}$$

We can assume that:

- For $i = 1, \ldots, p$, either $\beta_j > 0$ or $\beta_j = 0$, $\alpha_j > 0$.

- For $i = 1, \ldots, q$, either $\delta_j > 0$ or $\delta_j = 0$, $\gamma_j > 0$.

- The b_j's and d_j's are not zero or negative integers. For those indices i for which $\alpha_j < 0$ or $\gamma_j < 0$ we require that the respective b_j or d_j is not an integer.

Under these conditions, the series $\mathcal{U}(n) = \sum_{k=0}^{\infty} u(n, k)$ is well defined and non-terminating for large enough n. Note that we allow $P(n, k)$ to have linear factors in n, k, so possibly $P(n, k) = 0$ for infinitely many integer pairs (n, k).

We will introduce a lot of notation for the expressions we need to check in order to determine uniformly dominant convergence for the series $\mathcal{U}(n)$. All of this notation is much less complicated if the hypergeometric series is written in the canonical form

$$H(n)\, {}_sF_r\left(\begin{array}{c} a_1 + \alpha_1\, n, \; \ldots, \; a_s + \alpha_s\, n \\ c_1 + \gamma_1\, n, \; \ldots, \; c_r + \gamma_r\, n \end{array} \middle| \; z \right), \tag{4.2}$$

where $H(n)$ is a hypergeometric term dependant on n only. Written in the general form (4.1), hypergeometric term (1.1) has the expression

$$H(n)\, \frac{(c_1)_{\gamma_1 n} \, \cdots \, (c_r)_{\gamma_r n}}{(a_1)_{\alpha_1 n} \, \cdots \, (a_s)_{\alpha_s n}} \, \frac{(a_1)_{\alpha_1 n + k} \, \cdots \, (a_s)_{\alpha_s n + k}}{(c_1)_{\gamma_1 n + k} \, \cdots \, (c_r)_{\gamma_r n + k} \, k!} \, z^k. \tag{4.3}$$

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In the other direction, the canonical form is easily achieved if the polynomial $P(n,k)$ factors nicely into linear factors, and all nonzero β_j's and δ_j's are equal to 1. If some β_j (or similarly, γ_j) is greater than 1, then that single Pochhammer symbol gives β_j upper parameters in the canonical form by the formula

$$(\ell)_{\lambda m} = \lambda^{\lambda m} \left(\frac{\ell}{\lambda}\right)_m \left(\frac{\ell+1}{\lambda}\right)_m \cdots \left(\frac{\ell+\lambda-1}{\lambda}\right)_m. \tag{4.4}$$

Then several α_j's in the canonical form may be rational numbers with the denominator β_j.

As an exercise, one may reformulate the notation and main result for the canonical form (4.2). Eventually, typical asymptotic expressions we have to consider have the form like

$$\sim \Gamma(k)^{s-r-1}\,\Gamma(n)^{S-R}\,z_1^k\,\zeta_0^n\,k^{A_0-1}\,n^{A_\infty^*}\,\exp\left(k\,\Phi_\infty\left(\frac{k}{n}\right)\right). \tag{4.5}$$

The variables in (4.8)–(4.9) appear in the powers of $\Gamma(k)$ and $\Gamma(n)$; the expressions in (4.10)–(4.14), (4.18)–(4.19) and (4.24) appear in the powers of k and n; the expressions in (5.1)–(5.3) appear with the exponents of k and n; the functions in (5.13)–(5.15) appear in the additional exponent.

For those j with, respectively, $\beta_j \neq 0$ or $\delta_j \neq 0$ (hence positive) we introduce

$$\widehat{a}_j = b_j + \frac{\beta_j-1}{2}, \qquad \widehat{c}_j = d_j + \frac{\delta_j-1}{2}. \tag{4.6}$$

For those j with, respectively, $\alpha_j \neq 0$ or $\gamma_j \neq 0$ we set

$$\widetilde{a}_j = b_j + \frac{\alpha_j-1}{2}, \qquad \widetilde{c}_j = d_j + \frac{\gamma_j-1}{2}. \tag{4.7}$$

Let us also introduce the following notation:

$$s = \sum_{j=1}^p \beta_j, \qquad r = \sum_{i=1}^q \delta_j, \qquad \widetilde{s} = \sum_{\alpha_j \neq 0} \beta_j, \qquad \widetilde{r} = \sum_{\gamma_j \neq 0} \delta_j, \tag{4.8}$$

$$S = \sum_{j=1}^p \alpha_j, \qquad R = \sum_{i=1}^q \gamma_j, \qquad \widetilde{S} = \sum_{\beta_j \neq 0} \alpha_j, \qquad \widetilde{R} = \sum_{\delta_j \neq 0} \gamma_j. \tag{4.9}$$

In the last two sums, we imply the summation range of all β_j's or γ_j's for which $\alpha_j \neq 0$, $\gamma_j \neq 0$, respectively. In the rest of the paper, summation or product ranges are implied by the range of definition of involved variables and by indicated conditions. For example,

$\sum_{\alpha_j=0} \widehat{a}_j$ is a summation over those j for which $\beta_j > 0$ and $\alpha_j = 0$. With this convention we define:

$$A_0 = \sum \widehat{a}_j - \sum \widehat{c}_j + \deg_k P(n,k), \tag{4.10}$$

$$A_\infty^* = \sum \widetilde{a}_j - \sum \widetilde{c}_j + \deg_n P(n,k), \tag{4.11}$$

$$A_0^* = \sum_{\alpha_j=0} \widehat{a}_j - \sum_{\gamma_j=0} \widehat{c}_j + \deg_k Q(k), \tag{4.12}$$

$$A_1 = \sum \widehat{a}_j - \sum \widehat{c}_j + \sum_{\beta_j=0} \widetilde{a}_j - \sum_{\delta_j=0} \widetilde{c}_j + \deg_{\{n,k\}} P(n,k), \tag{4.13}$$

$$A_1^* = \sum_{\alpha_j=0} \widehat{a}_j - \sum_{\gamma_j=0} \widehat{c}_j + \sum \widetilde{a}_j - \sum \widetilde{c}_j + \deg_{\{n,k\}} P(n,k). \tag{4.14}$$

In (4.12), we denote

$$Q(k) := \text{the leading coefficient of } P(n,k) \text{ with respect to } n. \tag{4.15}$$

Thus $Q(k)$ is a polynomial in k.

Now we define the function

$$\varphi(p) = \max_{\substack{f: \text{a monomial} \\ \text{of } P(n,k)}} \left(\deg_k f + p \deg_n f \right). \tag{4.16}$$

Therefore $\varphi(p)$ is the degree of the polynomial $P(n,k)$ if we give the weight 1 to the variable k and the weight $p > 0$ to the variable n. We have the following properties.

Lemma 4.1 *(i) For a function $N(k) : \mathbb{Z}_+ \to \mathbb{R}$ such that $N(k) \sim C_0 k^p$ as $k \to \infty$ for some non-zero constant C_0 and real $p > 0$, we have $P(N,k) = O(k^{\varphi(p)})$. For a general such function $N(k)$, there is a non-zero constant C_1 such that $P(N,k) \sim C_1 k^{\varphi(p)}$.*

(ii) The function $\varphi(p)$ is a continuous piecewise linear function on the real interval $[0,\infty)$, monotone non-decreasing. The linear slope of $\varphi(p)$ can only increase as p increases, as well.

(iii) For large enough p, we have $\varphi(p) = p \deg_n P(n,k) + \deg_k Q(k)$.

Proof. The first part is clear; its second statement holds with general C_0 for any fixed p.

Let $\mathbf{P}$ denote the Newton polygon of $P(n,k)$, that is, the convex hull in $\mathbb{R}^2$ of all half-lines from (C,D) to $(C,-\infty)$ and $(-\infty,D)$ for each monomial $k^C m^D$ of $P(n,k)$. Let $\{(C_i,D_i)\}_{i=1}^m$ be the sequence of the vertices of $\mathbf{P}$, ordered by increasing D_i. Then

$$\varphi(p) = \begin{cases} D_1 p + C_1, & \text{if } 0 \leq p \leq \frac{C_1-C_2}{D_2-D_1}, \\ D_i\, p + C_i, & \text{for } 1 < i < m \text{ and } \frac{C_{i-1}-C_i}{D_i-D_{i-1}} \leq p \leq \frac{C_i-C_{i+1}}{D_{i+1}-D_i}, \\ D_m p + C_m, & \text{if } p \geq \frac{C_{m-1}-C_m}{D_m-D_{m-1}}. \end{cases} \qquad (4.17)$$

The last two claims follow. QED.

Consequently, we introduce the two functions:

$$\psi_0(p) \;=\; \sum \widehat{a}_j - \sum \widehat{c}_j + \left(\sum_{\beta_j=0} \widetilde{a}_j - \sum_{\delta_j=0} \widetilde{c}_j \right) p + \varphi(p), \qquad (4.18)$$

$$\psi_\infty(p) \;=\; \sum_{\alpha_j=0} \widehat{a}_j - \sum_{\gamma_j=0} \widehat{c}_j + \left(\sum \widetilde{a}_j - \sum \widetilde{c}_j \right) p + \varphi(p). \qquad (4.19)$$

We will consider $\psi_0(p)$ on the interval $[0,1]$, and the function $\psi_\infty(p)$ on the interval $[1,\infty)$. We have the following properties.

Lemma 4.2   *(i) The real parts of $\psi_0(p)$ and $\psi_\infty(p)$ are continuous piecewise linear functions on the real interval $[0,\infty)$. Their linear slopes can only increase as p increases.*

(ii) On any interval $[A,B] \subset [0,\infty)$, the real parts of $\psi(p)$ and $\psi^*(p)$ achieve their maximum on $[A,B]$ at an end point, A or B.

(iii) Let (A,B) be a subinterval $[0,\infty)$, so possibly $B=\infty$. If the linear slope of $\mathrm{Re}\,\psi_0(p)$ or $\mathrm{Re}\,\psi_\infty(p)$ is zero or negative as $p \to B$ from the left, then the supremum of $\mathrm{Re}\,\psi_0(p)$ or $\mathrm{Re}\,\psi_\infty(p)$ on (A,B) is approached as $p \to A$.

(iv) $\psi_0(0) = A_0$, $\psi_0(1) = A_1$, and $\psi_\infty(1) = A_1^*$.

(v) For large enough p, we have $\psi_\infty(p) = p\,A_\infty^* + A_0^*$.

Proof. The first part follows from Lemma 4.1 *(ii)*. Since the slopes can only increase, on each interval $[A,B]$ the real parts of $\psi_0(p)$ and $\psi_\infty(p)$ are either monotone functions, or there is one locally extremal value inside the interval and that value is a local minimum. This

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shows the second part. In part *(iii)*, the function $\mathrm{Re}\ \psi_0(p)$ or $\mathrm{Re}\ \psi_\infty(p)$ does not increase on (A, B). The last two parts are straightforward. $\hspace{2cm}$ QED.

Let us define the set

$$\Omega = \left\{-\frac{\beta_j}{\alpha_j}\right\}_{\alpha_j \neq 0} \bigcup \left\{-\frac{\delta_j}{\gamma_j}\right\}_{\gamma_j \neq 0}, \tag{4.20}$$

and the family of polynomials

$$P_\lambda^\star(n, k) := P(\lambda k + n, k). \tag{4.21}$$

We assume that the polynomial $P_\lambda^\star(n, k)$ is expanded whenever we implicitly use it for some λ. Similarly as in (4.16), we define the family of functions

$$\varphi_\lambda^\star(p) = \max_{\substack{f:\ \text{a monomial} \\ \text{of } P_\lambda^\star(n,k)}} \left(\deg_k f + p \deg_n f\right). \tag{4.22}$$

We may need to consider these functions on the interval $p \in [0, 1]$.

Lemma 4.3 *(i) We have $\varphi_\lambda^\star(1) = \varphi(1)$, and $\varphi_\lambda^\star(p) \leq \varphi(1)$ for $p \in [0, 1]$. If*

$$\deg_k P(\lambda\, k, k) = \deg_{\{n,k\}} P(n, k), \tag{4.23}$$

then $\varphi_\lambda^\star(p) = \varphi(1)$ for any $p \in [0, 1]$.

(ii) The function $\varphi_\lambda^\star(p)$ is a continuous piecewise linear function on the real interval $[0, 1]$, monotone non-decreasing. The linear slope of $\varphi_\lambda^\star(p)$ can only increase as p increases, as well.

(iii) Let $N(k) : \mathbb{Z}_+ \rightarrow \mathbb{R}$ denote a function such that $N(k) \sim \lambda\, k + C_0\, k^p$ as $k \rightarrow \infty$ for some non-zero constants $C_0 \neq 0$, $\lambda \neq 0$ and $p \in [0, 1)$. Then $P(N, k) \sim C_1 k^{\varphi_\lambda^\star(p)}$ for some non-zero constant C_1.

Proof. For the first part, note that $\deg_{\{n,k\}} P_\lambda^\star(\lambda\, k, k) = \deg_{\{n,k\}} P(n, k)$. If (4.23) is satisfied, then the coefficient of $P_\lambda^\star(n, k)$ to $k^{\varphi(1)}$ is non-zero. (Non-generically, we may have $\deg_k P(\lambda\, k, k) < \deg_{\{n,k\}} P(n, k)$.)

The other two parts follow similarly as parts *(ii)* and *(i)* of Lemma 4.1, respectively. QED.

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We introduce a variation of $\psi_0(p)$ as well:

$$\psi_\lambda^\star(p) = \sum_{\alpha_j\lambda+\beta_j\neq0}\left(\widehat{a}_j+\frac{\alpha_j}{2}\right) - \sum_{\gamma_j\lambda+\delta_j\neq0}\left(\widehat{c}_j+\frac{\gamma_j}{2}\right) + \sum_{\beta_j=0}\widetilde{a}_j - \sum_{\delta_j=0}\widetilde{c}_j$$
$$+ \left(\sum_{\alpha_j\lambda+\beta_j=0}\left(b_j-\tfrac{1}{2}\right) - \sum_{\gamma_j\lambda+\delta_j=0}\left(d_j-\tfrac{1}{2}\right)\right)p + \varphi_\lambda^\star(p). \qquad (4.24)$$

Note that the linear coefficient to p is zero if $\lambda \notin \Omega$.

Lemma 4.4 *(i) For generic λ, the function $\psi_\lambda^\star(p)$ is a constant:*

$$\psi_\lambda^\star(p) = A_1 + \frac{\widetilde{S}-\widetilde{R}}{2} = A_1^* + \frac{\widetilde{s}-\widetilde{r}}{2}. \qquad (4.25)$$

(ii) For any λ, the real part of $\psi_\lambda^\star(p)$ is a continuous piecewise linear function on the real interval $[0,1]$. Its linear slope can only increase as p increases.

(iii) On any interval $[A,B] \subset [0,1]$, the real part of $\psi_\lambda^\star(p)$ achieve its maximum on $[A,B]$ at an end point, A or B. If the linear slope of $\mathrm{Re}\ \psi_\lambda^\star(p)$ is zero or negative as $p \to B$ from the left, then the supremum of $\mathrm{Re}\ \psi_\lambda^\star(p)$ on (A,B) is approached as $p \to A$.

Proof. In the first part, the generic λ are those $\lambda \notin \Omega$ which satisfy (4.23). Other two parts follow similarly as parts *(i)–(iii)* of Lemma 4.2. QED.

5 Further notation

The notation of the previous Section adds up the the parameters $\alpha_j, \beta_j, \gamma_j, \delta_j, b_j, d_j$. Here we introduce some "multiplicative" notation. Recall the convention $0^0 = 1$.

We introduce the following constants:

$$z_0 = \theta\,\frac{\prod\beta_j^{\beta_j}}{\prod\delta_j^{\delta_j}}, \qquad z_1 = \theta\,\frac{\prod_{\alpha_j\neq0}\alpha_j^{\beta_j}\,\prod_{\alpha_j=0}\beta_j^{\beta_j}}{\prod_{\gamma_j\neq0}\gamma_j^{\delta_j}\,\prod_{\gamma_j=0}\delta_j^{\delta_j}}, \qquad z_\infty = \theta\,\frac{\prod_{\alpha_j\neq0}\alpha_j^{\beta_j}}{\prod_{\gamma_j\neq0}\gamma_j^{\delta_j}}, \qquad (5.1)$$

$$\zeta_0 = \xi\,\frac{\prod\alpha_j^{\alpha_j}}{\prod\gamma_j^{\gamma_j}}, \qquad \zeta_1 = \xi\,\frac{\prod_{\beta_j\neq0}\beta_j^{\alpha_j}\,\prod_{\beta_j=0}\alpha_j^{\alpha_j}}{\prod_{\delta_j\neq0}\delta_j^{\gamma_j}\,\prod_{\delta_j=0}\gamma_j^{\gamma_j}}. \qquad (5.2)$$

Besides, we define the function

$$g(t) = |\theta| \, |\xi|^t \frac{\prod |\beta_j + \alpha_j t|^{\beta_j + \alpha_j t}}{\prod |\delta_j + \gamma_j t|^{\delta_j + \gamma_j t}}. \tag{5.3}$$

We have the following properties of $g(t)$.

Lemma 5.1 (i) *The function $g(t)$ is continuous on the whole real axis. It can be expressed as follows:*

$$g(t) = |z_1| \, |\zeta_0|^t \frac{\prod_{\alpha_j \neq 0} \left| t + \frac{\beta_j}{\alpha_j} \right|^{\beta_j + \alpha_j t}}{\prod_{\gamma_j \neq 0} \left| t + \frac{\delta_j}{\gamma_j} \right|^{\delta_j + \gamma_j t}}. \tag{5.4}$$

(ii) *$g(t)$ is continuously differentiable on $\mathbb{R} \setminus (\{0\} \cup \Omega)$, and*

$$\exp \frac{g'(t)}{g(t)} \; = \; |\xi| \, \exp(S - R) \, \frac{\prod |\beta_j + \alpha_j t|^{\alpha_j}}{\prod |\delta_j + \gamma_j t|^{\gamma_j}} \tag{5.5}$$

$$= \; |\zeta_0| \, \exp(S - R) \, \frac{\prod_{\alpha_j \neq 0} \left| t + \frac{\beta_j}{\alpha_j} \right|^{\alpha_j}}{\prod_{\gamma_j \neq 0} \left| t + \frac{\delta_j}{\gamma_j} \right|^{\gamma_j}}. \tag{5.6}$$

(iii) *A point $\lambda \in \{0\} \cup \{-\beta_j/\alpha_j\}_{\alpha_j \neq 0} \cup \{-\delta_j/\gamma_j\}_{\gamma_j \neq 0}$ is a genuine point of discontinuity of the derivative $g'(t)$ if and only if $\sum_{\beta_j + \alpha_j \lambda = 0} \alpha_j \neq \sum_{\delta_j + \gamma_j \lambda = 0} \gamma_j$. If this is the case, then the tangent line to $g(t)$ approaches the vertical line as $t \to \lambda$.*

(iv) *$g(0) = |z_0|$.*

(v) *$g(t) \sim |z_1| \, \exp(\widetilde{s} - \widetilde{r}) \, |\zeta_0|^t \, t^{(S-R)\,t + \widetilde{s} - \widetilde{r}}$ as $t \to \infty$.*

Proof. Consider the function

$$f(x) = \begin{cases} |x|^x, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases} \tag{5.7}$$

We can write $f(x) = \exp(x \log |x|)$ for non-zero x. It is a standard analysis exercise that $f(x)$ is a continuous function. Since $f'(x) = (1 + \log |x|) \, f(x)$, the function $f(x)$ is continuously differentiable on $\mathbb{R} \setminus \{0\}$. Expressions (5.4)–(5.6) routinely follow.

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For part *(iii)*, we compute that as $t \to \lambda$,

$$\exp \frac{g'(t)}{g(t)} \sim |\xi| \frac{\prod_{\beta_j + \alpha_j \lambda \neq 0} |\beta_j + \alpha_j \lambda|^{\alpha_j}}{\prod_{\delta_j + \gamma_j \lambda \neq 0} |\delta_j + \gamma_j \lambda|^{\gamma_j}} \frac{\prod_{\beta_j + \alpha_j \lambda = 0} |\alpha_j|^{\alpha_j}}{\prod_{\delta_j + \gamma_j \lambda = 0} |\gamma_j|^{\gamma_j}} \exp(S - R)$$
$$\times |t - \lambda|^{\sum_{\beta_j + \alpha_j \lambda = 0} \alpha_j - \sum_{\delta_j + \gamma_j \lambda = 0} \gamma_j}. \tag{5.8}$$

Hence, as $t \to \lambda$,

$$g'(t) \sim \left(C_0 + \left(\sum_{\beta_j + \alpha_j \lambda = 0} \alpha_j - \sum_{\delta_j + \gamma_j \lambda = 0} \gamma_j \right) \log |t - \lambda| \right) g(\lambda) \tag{5.9}$$

for a constant C_0. Part *(iii)* is evident.

Part *(iv)* is obvious. To show the asymptotic expression of part *(v)*, we use (5.4) to derive

$$g(t) = |z_1| \, |\zeta_0|^t \, t^{(S-R)\, t + \widetilde{s} - \widetilde{r}} \frac{\prod_{\alpha_j \neq 0} \left| 1 + \frac{\beta_j}{\alpha_j t} \right|^{\alpha_j t}}{\prod_{\gamma_j \neq 0} \left| 1 + \frac{\delta_j}{\gamma_j t} \right|^{\gamma_j t}}. \tag{5.10}$$

Whether $\lambda > 0$ or $\lambda < 0$, we have $\left(1 + \frac{\ell}{\lambda t} \right)^{\lambda t} \to \exp(\ell)$ as $t \to \infty$. QED.

For completeness, one can compute that

$$\exp \frac{g'(t)}{g(t)} \sim |\zeta_1| \exp(S - R) \, t^{(S - \widetilde{S}) - (R - \widetilde{R})} \qquad \text{as } t \to +0. \tag{5.11}$$

$$\exp \frac{g'(t)}{g(t)} \sim |\zeta_0| \exp(S - R) \, t^{S - R} \qquad \text{as } t \to \infty. \tag{5.12}$$

The first expression is a special case of (5.8).

At the last, we introduce the family of functions

$$\Phi_\lambda(x) = \sum_{\alpha_j \lambda + \beta_j \neq 0} \frac{\alpha_j^2 \, x}{\alpha_j x + \alpha_j \lambda + \beta_j} - \sum_{\gamma_j \lambda + \delta_j \neq 0} \frac{\gamma_j^2 \, x}{\gamma_j x + \gamma_j \lambda + \delta_j} \tag{5.13}$$

In particular,

$$\Phi_0(x) = \sum_{\beta_j \neq 0} \frac{\alpha_j^2 \, x}{\alpha_j x + \beta_j} - \sum_{\delta_j \neq 0} \frac{\gamma_j^2 \, x}{\gamma_j x + \delta_j}. \tag{5.14}$$

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We also introduce

$$\Phi_\infty(x) = \sum_{\alpha_j \neq 0} \frac{\beta_j^2 \, x}{\beta_j x + \alpha_j} - \sum_{\gamma_j \neq 0} \frac{\delta_j^2 \, x}{\delta_j x + \gamma_j}. \tag{5.15}$$

This is almost all notation we will need to describe the constants we have to check to determine uniformly dominated convergence of $\mathcal{U}(n)$.

6 The main result

Our main result is the following.

Theorem 6.1 *The series* $\mathcal{U}(n) = \sum_k u(n,k)$ *is uniformly bounded by an absolutely convergent series only if the following restrictions are satisfied:*

(i) $s \leq r+1$ *and* $S \leq R$.

(ii) If $s = r+1$ *then one of the following two conditions must hold:*

- $|z_0| < 1$.
- $|z_0| = 1$, $\operatorname{Re} A_0 < 0$ *and* $\widetilde{S} \leq \widetilde{R}$.

(iii) If $S = R$ *then one of the following three conditions must hold:*

- $|\zeta_0| < 1$.
- $|\zeta_0| = 1$, $\operatorname{Re} A_\infty^* < 0$ *and* $\widetilde{s} \leq \widetilde{r}$.
- $\zeta_0 = 1$, $A_\infty^* = 0$ *and* $\widetilde{s} \leq \widetilde{r}$.

These conditions are sufficient for uniformly dominated convergence if $s < r+1$ *or* $S < R$. *Otherwise, that is when*

$$s = r+1 \qquad and \qquad S = R, \tag{6.1}$$

the series $\mathcal{U}(n)$ *are bounded by an absolutely convergent series if and only if:*

(iv) $g(t) \leq 1$ *for all* $t > 0$.

(v) For those $t > 0$ *which satisfy* $g(t) = 1$, *we have* $\operatorname{Re} \psi_t^\star(0) < 0$,

$$\sum_{\alpha_j t + \beta_j = 0} \alpha_j = \sum_{\gamma_j t + \delta_j = 0} \gamma_j, \tag{6.2}$$

and one of the following two conditions holds:

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- $\Phi_t(x) \equiv 0$; and $\operatorname{Re} \psi_t^\star(1) \leq 0$.
- $\Phi_t(x) = v_m x^m + O(x^{m+1})$ around $x = 0$, where m is a positive odd integer, $v_m < 0$; and $\operatorname{Re} \psi_t^\star\left(\frac{m}{m+1}\right) < 0$.

(vi) If $|z_0| = 1$ and $\widetilde{S} = \widetilde{R}$, then one of the following conditions holds:

- $|\zeta_1| < 1$.
- $|\zeta_1| = 1$; $\Phi_0(x) \equiv 0$; and $\operatorname{Re} A_1 \leq 0$.
- $|\zeta_1| = 1$; $\Phi_0(x) = v_m x^m + O(x^{m+1})$ around $x = 0$ for some positive integer m and negative real v_m; and $\operatorname{Re} \psi_0\left(\frac{m}{m+1}\right) < 0$.

(vii) If $|\zeta_0| = 1$ and $\tilde{s} = \tilde{r}$, then one of the following conditions holds:

- $|z_1| < 1$.
- $|z_1| = 1$; $\Phi_\infty(x) \equiv 0$; and $\operatorname{Re} A_1^* < 0$.
- $|z_1| = 1$; $\Phi_\infty(x) \equiv 0$; $\operatorname{Re} A_1^* = 0$; and either $\operatorname{Re} A_\infty^* < 0$ or

$$\deg_{\{n,k\}} P(n,k) > \deg_n P(n,k) + \deg_k Q(k). \tag{6.3}$$

- $|z_1| = 1$; $\Phi_\infty(x) = v_m x^m + O(x^{m+1})$ around $x = 0$ for some positive integer m and negative real v_m; $\operatorname{Re} \psi_\infty\left(\frac{m+1}{m}\right) < 0$; and if $A_\infty^* = 0$ then $\operatorname{Re} A_0^* < 0$.

If these conditions are satisfied, then the limit series $\lim_{n \to \infty} \mathcal{U}(n)$ is equal to:

- If $S < R$, $|\zeta_0| < 1$ or $\operatorname{Re} A_\infty^* < 0$, then 0.
- If $S = R$, $\zeta_0 = 1$, $A_\infty^* = 0$ and $\widetilde{s} < \widetilde{r}$, then $H_0 Q(0)$.
- If $S = R$, $\zeta_0 = 1$, $A_\infty^* = 0$ and $\widetilde{s} = \widetilde{r}$, then

$$\lim_{n \to \infty} \mathcal{U}(n) = H_0 \sum_{k=0}^{\infty} Q(k) \frac{\prod_{\alpha_j=0} (b_j)_{\beta_j k}}{\prod_{\gamma_j=0} (d_j)_{\delta_j k}} \frac{z_\infty^k}{k!}, \tag{6.4}$$

where H_0 is the following constant:

$$H_0 = (2\pi)^{\sum_{\alpha_j \neq 0} \frac{1-\alpha_j}{2} - \sum_{\gamma_j \neq 0} \frac{1-\gamma_j}{2}} \prod_{\alpha_j \neq 0} \frac{|\alpha_j|^{b_j - \frac{1}{2}}}{\Gamma(b_j)} \prod_{\gamma_j \neq 0} \frac{\Gamma(d_j)}{|\gamma_j|^{d_j - \frac{1}{2}}} \frac{\prod_{\gamma_j < 0} 2 \sin \pi d_j}{\prod_{\alpha_j < 0} 2 \sin \pi b_j}. \tag{6.5}$$

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We prove this Theorem in the following Section.

Here we make a few comments. We reformulate some conditions, or indicate some possible or typically effective simplifications. We keep some redundancy in notation or formulation to make the proof more smooth, or to make non-typical complications better understandable.

- Condition (6.3) means that among the monomials of $P(n,k)$ of the highest degree in n, k there are no monomials of the highest degree in n. Recall that $Q(k)$ is defined in (4.15).

- Equality (6.2) is trivially satisfied if $t \notin \Omega$. Recall that the set Ω is defined in (4.20).

- Let B_1 denote the constant implied in (4.25). If $t \notin \Omega$, then the function $\psi_t^\star(p)$ in condition *(v)* is rather simple:

$$\psi_t^\star(p) = B_1 - \varphi(1) + \varphi_t^\star(p). \tag{6.6}$$

  From part *(i)* of Lemma 4.3 it follows that $\psi_t^\star(p) \leq B_1$ for $p \in [0,1]$, and $\psi_t^\star(1) = B_1$. For generic p we have $\psi_t^\star(p) = B_1$ for $p \in [0,1]$.

  If $t \notin \Omega$, $g(t) = 1$ and $\Phi_t(x) \equiv 0$, the condition *(v)* can be replaced by the following restriction: either $\operatorname{Re} B_1 < 0$, or $\operatorname{Re} B_1 = 0$ and $\deg_k P(tk, k) < \deg_{\{n,k\}} P(n,k)$. If we can apply this simpler restriction to at least one $t \notin \Omega$, then the points $t \notin \Omega$ with $g(t) = 1$, $\Phi_t(x) \not\equiv 0$ can only strengthen $\operatorname{Re} B_1 = 0$ to $\operatorname{Re} B_1 < 0$ and add conditions on $\Phi_t(x)$.

- In definition (4.24) of $\psi_t^\star(p)$, we can replace the summations of $b_j - \frac{1}{2}$, $d_j - \frac{1}{2}$ by respective summations of $\widetilde{a}_j$, $\widetilde{c}_j$, because these summations are relevant only when (6.2) holds. We can also replace these summations by respective summations of $\widehat{a}_j$, $\widehat{c}_j$, because condition (6.2) is equivalent to $\sum_{\alpha_j t + \beta_j = 0} \beta_j = \sum_{\gamma_j t + \delta_j = 0} \delta_j$.

- All conditions of Theorem 6.1 can be checked algorithmically. The only less straightforward part is checking the condition $g(t) \leq 1$ for $t > 0$, and identifying the points with $g(t) = 1$. We consider this issue in Section 8.

- Suppose that the polynomial $P(n,k)$ has a linear factor $\widetilde{\alpha} n + \widetilde{\beta} k + \ell$ with $\widetilde{\alpha}, \widetilde{\beta} \in \mathbb{Z}$. The linear factor can be expressed as $\ell(\ell+1)_{\widetilde{\alpha} n + \widetilde{\beta} k} / (\ell)_{\widetilde{\alpha} n + \widetilde{\beta} k}$. Notice that all conditions, in particular *(v)*, are stable if we rewrite expression (4.1) of $u(n,k)$ by replacing in $P(n,k)$ the linear factor by the constant ℓ, and appending the two Pochhammer terms to the products of p and q Pochhammer symbols.

- The polynomial $Q(k)$ occurs only in (6.3) and in the expressions for $\lim_{n\to\infty} \mathcal{U}(n)$. The constant z_∞ occurs only in (6.4). The constants A_0^* occurs in the last case of condition *(vii)*. The set Ω does not explicitly appear in the formulater Theorem.

- Notice that $H_0 = \prod_{\gamma_j \neq 0} \Gamma(d_j) \big/ \prod_{\alpha_j \neq 0} \Gamma(b_j)$ is all nonzero α_j's, γ_j's are equal to 1.

7 Proof of the main theorem

Here we prove Theorem 6.1. The strategy is outlined by Lemma 2.2.

With our notation and summation/product conventions, we may split the hypergeometric summand $u(n,k)$ in the following ways. Firstly, we can switch to variables in (4.6)–(4.7) as follows:

$$u(n,k) = \frac{\prod_{\beta_j=0}\left(\widetilde{a}_j + \frac{1-\alpha_j}{2}\right)_{\alpha_j n}}{\prod_{\delta_j=0}\left(\widetilde{c}_j + \frac{1-\gamma_j}{2}\right)_{\gamma_j n}} \, \xi^n \, P(n,k) \, \frac{\prod\left(\widehat{a}_j + \frac{1-\beta_j}{2}\right)_{\beta_j k+\alpha_j n}}{\prod\left(\widehat{c}_j + \frac{1-\delta_j}{2}\right)_{\delta_j k+\gamma_j n}} \, \frac{\theta^k}{k!}. \tag{7.1}$$

Alternatively, we can split the Pochhammer symbols in other wa and obtain the following expression for $u(n,k)$:

$$\frac{\prod\left(\widetilde{a}_j + \frac{1-\alpha_j}{2}\right)_{\alpha_j n}}{\prod\left(\widetilde{c}_j + \frac{1-\gamma_j}{2}\right)_{\gamma_j n}} \, \xi^n \, P(n,k) \, \frac{\prod_{\alpha_j\neq 0}\left(\widehat{a}_j + \frac{1-\beta_j}{2} + \alpha_j n\right)_{\beta_j k}}{\prod_{\gamma_j\neq 0}\left(\widehat{c}_j + \frac{1-\delta_j}{2} + \gamma_j n\right)_{\delta_j k}} \, \frac{\prod_{\alpha_j=0}\left(\widehat{a}_j + \frac{1-\beta_j}{2}\right)_{\beta_j k}}{\prod_{\gamma_j=0}\left(\widehat{c}_j + \frac{1-\delta_j}{2}\right)_{\delta_j k}} \, \frac{\theta^k}{k!}. \tag{7.2}$$

Note that here the first two terms do not depend on k, and the last two terms do not depend on n. We will use these expressions in different cases of Lemma 2.2.

Condition *(a)* of Lemma 2.2 is satisfied under the following necessary and sufficient restrictions:

(a1) $s \leq r + 1$.

(a2) If $s = r + 1$, then $|z_0| \leq 1$.

(a3) If $s = r + 1$, $|z_0| = 1$, then $\widetilde{S} \leq \widetilde{R}$ and $\operatorname{Re} A_0 < 0$.

because for fixed general n we have

$$u(n,k) \;\sim\; C(n) \, k^{\deg_k P(n,k) + \sum(\widehat{a}_j + \alpha_j n) - \sum(\widehat{c}_j + \gamma_j n)} \, \frac{\prod \beta_j^{\beta_j k}}{\prod \delta_j^{\delta_j k}} \, \Gamma(k)^{\sum \beta_j - \sum \delta_j} \, \frac{\theta^k}{k!} \tag{7.3}$$

$$\sim\; C(n) \, k^{\left(\widetilde{S} - \widetilde{R}\right) n + A_0 - 1} \, z_0^k \, \Gamma(k)^{s-r-1}. \tag{7.4}$$

Recall that $k! = k\,\Gamma(k)$. These conditions are general convergence conditions for hypergeometric series; see [AAR99, Theorems 2.1.1–2].

For condition *(b)* of Lemma 2.2, we fix general k and use (7.2), Corollary 2.5:

$$
u(n,k) \;\sim\; n^{\deg_n P(n,k)+\sum(\widetilde{a}_j+\beta_j k)-\sum(\widetilde{c}_j+\delta_j k)} \frac{\prod \alpha_j^{\alpha_j\, n}}{\prod \gamma_j^{\gamma_j\, n}}\, \Gamma(n)^{\sum \alpha_j - \sum \gamma_j}\, \xi^n
$$

$$
\times\, H_0\, Q(k)\, \frac{\prod_{\alpha_j\neq 0} \alpha_j^{\beta_j\, k}}{\prod_{\gamma_j\neq 0} \gamma_j^{\delta_j\, k}}\, \frac{\prod_{\alpha_j=0} \left(\widehat{a}_j + \frac{1-\beta_j}{2}\right)_{\beta_j\, k}}{\prod_{\gamma_j=0} \left(\widehat{c}_j + \frac{1-\delta_j}{2}\right)_{\delta_j\, k}}\, \frac{\theta^k}{k!}. \tag{7.5}
$$

Here H_0 and $Q(k)$ are the same as in (6.5), (4.15). The first line of the right-hand side can be rewritten as $n^{(\widetilde{s}-\widetilde{r})\,k+A_\infty^*}\,\zeta_0^n\,\Gamma(n)^{S-R}$. The second line is independent of n. For the existence of the termwise limit we first check whether $u(n,k)$ is bounded as $k\to\infty$, and whether the limit $\lim_{n\to\infty} u(n,0)$ exists:

(b1) $S \leq R$.

(b2) If $S = R$, then $|\zeta_0| \leq 1$.

(b3) If $S = R$, $|\zeta_0| = 1$, then $\widetilde{s} \leq \widetilde{r}$ and $\operatorname{Re} A_\infty^* \leq 0$.

(b4) If $S = R$, $|\zeta_0| = 1$, $\operatorname{Re} A_\infty^* = 0$, then $\zeta_0 = 1$ and $A_\infty^* = 0$.

Under these conditions the termwise limit $\lim_{n\to\infty} \mathcal{U}(n)$ is the zero series if $S < R$, $|\zeta_0| < 1$ or $\operatorname{Re} A_\infty^* < 0$. Otherwise condition *(b4)* applies. Then (recalling Lemma 2.5) the termwise limit is $H_0\, Q(0)$ if $\widetilde{s} < \widetilde{r}$, and it is equal to (6.4) if $\widetilde{s} = \widetilde{r}$. In these cases, asymptotics (7.5) can be rewritten, up to a constant factor, as $k^{A_0^*-1}\, z_1^k\, \Gamma(k)^{s-r-(\widetilde{s}-\widetilde{r})-1}$. Additional conditions for the convergence of the limit series are the following:

(b5) If $S = R$, $\zeta_0 = 1$, $\widetilde{s} = \widetilde{r}$, $A_\infty^* = 0$, then $s \leq r+1$.

(b6) If $S = R$, $\zeta_0 = 1$, $\widetilde{s} = \widetilde{r}$, $A_\infty^* = 0$, $s = r+1$, then $|z_1| \leq 1$.

(b7) If $S = R$, $\zeta_0 = 1$, $\widetilde{s} = \widetilde{r}$, $A_\infty^* = 0$, $s = r+1$, $|z_1| = 1$, then $\operatorname{Re} A_0^* < 0$.

Now we check condition *(c)* of Lemma 2.2. We assume that $N = N(k)$ is an integer-valued function such that $N(k) \sim C_0\, k^p$ as $k\to\infty$, with $p > 1$ and $C_0 > 0$ real constants.

Using formula (7.2), Corollaries 2.4 and 3.4 we get the following asymptotic expression as $k \to \infty$:

$$
\begin{aligned}
u(N,k) \;\sim\; & C_1 \, N^{\sum \widetilde{a}_j - \sum \widetilde{c}_j} \, \frac{\prod \alpha_j^{\alpha_j N}}{\prod \gamma_j^{\gamma_j N}} \, \Gamma(N)^{\sum \alpha_j - \sum \gamma_j} \, \xi^N \, P(N,k) \\
& \times \frac{\prod_{\alpha_j \neq 0} \alpha_j^{\beta_j k}}{\prod_{\gamma_j \neq 0} \gamma_j^{\delta_j k}} \, N^{\left(\sum_{\alpha_j \neq 0} \beta_j - \sum_{\gamma_j \neq 0} \delta_j \right) k} \, \exp \left(k \, \widetilde{\Phi}_\infty \left(\frac{k}{N} \right) \right) \\
& \times k^{\sum_{\alpha_j = 0} \widehat{a}_j - \sum_{\gamma_j = 0} \widehat{c}_j} \, \frac{\prod_{\alpha_j = 0} \beta_j^{\beta_j k}}{\prod_{\gamma_j = 0} \delta_j^{\delta_j k}} \, \Gamma(k)^{\sum_{\alpha_j = 0} \beta_j - \sum_{\gamma_j = 0} \delta_j} \, \frac{\theta^k}{k!},
\end{aligned} \tag{7.6}
$$

for some $C_1 \in \mathbb{R}$ and

$$
\widetilde{\Phi}_\infty(x) = \sum_{\alpha_j \neq 0} \beta_j \Theta \left(\frac{\beta_j x}{\alpha_j} \right) - \sum_{\gamma_j \neq 0} \delta_j \Theta \left(\frac{\delta_j x}{\gamma_j} \right). \tag{7.7}
$$

We rearrange as

$$
\begin{aligned}
u(N,k) \;\sim\; & C_1 \, \Gamma(N)^{S-R} \, \zeta_0^N \, \Gamma(k)^{s-r-1} \left(\frac{N^k}{\Gamma(k)} \right)^{\widetilde{s} - \widetilde{r}} z_1^k \, \exp \left(k \, \Phi_1 \left(\frac{k}{N} \right) \right) \\
& \times P(N,k) \, k^{\left(\sum \widetilde{a}_j - \sum \widetilde{c}_j \right) p + \sum_{\alpha_j = 0} \widehat{a}_j - \sum_{\gamma_j = 0} \widehat{c}_j - 1}.
\end{aligned} \tag{7.8}
$$

We compute that, as $k \to \infty$,

$$
\begin{aligned}
\frac{\log |u(N,k)|}{k} = \; & (S - R) \frac{N \log N - N}{k} + \frac{N}{k} \log |\zeta_0| + (s - r - 1)(\log k - 1) \\
& + (\widetilde{s} - \widetilde{r}) \left(\log \frac{N}{k} + 1 \right) + \log |z_1| + o(1).
\end{aligned} \tag{7.9}
$$

Note that $\log(N/k) \sim (p - 1) \log k + O(1)$.

To investigate absolute convergence of $\sum_{k=0}^{\infty} s(N,k)$, we first look at formula (7.9) and use Lemma 3.6 with $\varrho = 1$. The series must converge absolutely for all relevant $N = N(k)$. The most subtle case is when the expression in (7.9) is $o(1)$. Eventually we get the following list of conditions:

(c1) $S \leq R$.

(c2) If $S = R$, then $|\zeta_0| \leq 1$.

(c3) If $S = R$, $|\zeta_0| = 1$, then $\tilde{s} \le \tilde{r}$ and $s \le r + 1$.

(c4) If $S = R$, $|\zeta_0| = 1$, $s = r + 1$, $\tilde{s} = \tilde{r}$, then $|z_1| \le 1$.

(c5) If $S = R$, $|\zeta_0| = 1$, $s = r + 1$, $\tilde{s} = \tilde{r}$, $|z_1| = 1$, then $\operatorname{Re} A_\infty^* \le 0$ and one of the following conditions holds:

    (c5A) $\widetilde{\Phi}_\infty(x) \equiv 0$, and $\operatorname{Re} A_1^* < 0$.

    (c5B) $\widetilde{\Phi}_\infty(x) \equiv 0$, $\operatorname{Re} A_1^* = 0$, and either $\deg_{\{n,k\}} P(n, k) > \deg_n P(n, k) + \deg_k Q(k)$ or $\operatorname{Re} A_\infty^* < 0$.

    (c5C) $\widetilde{\Phi}_\infty(x) = v_m x^m + O(x^{m+1})$ around $x = 0$ for some positive integer m and negative real v_m, and $\operatorname{Re} \psi_\infty\left(\frac{m+1}{m}\right) < 0$.

Here we comment the case when the expression in (7.9) is $o(1)$ as $k \to \infty$. Formula (7.8) becomes then, for general $N(k)$ by the first two parts of Lemma 4.1,

$$u(N, k) \sim \widetilde{C}_1 \, \exp\left(k \, \widetilde{\Phi}_\infty\left(\frac{k}{N} \right) \right) k^{\psi_\infty(p) - 1}, \tag{7.10}$$

for some $\widetilde{C}_1 \in \mathbb{R}$. To have convergence for large p, we must have $\operatorname{Re} A_\infty^* \le 0$. If $\widetilde{\Phi}_\infty \equiv 0$ we must have $\operatorname{Re} \psi_\infty(p) < 0$ for all $p \in (1, \infty)$. By part *(ii)* of Lemma 4.2, the real part of $\psi_\infty(p)$ approaches its supremum with $p \mapsto 1$. The condition $\operatorname{Re} \psi_\infty(p) < 0$ is ensured in Case *(c5A)*. The Case *(c5B)* occurs when the supremum is not achieved inside the interval $(1, \infty)$. If $\widetilde{\Phi}_\infty \not\equiv 0$, then the exponential factor in (7.10) is asymptotic to

$$\exp\left(\frac{v_m}{C_0^m} \, k^{1 - (p-1)\,m} \right). \tag{7.11}$$

For $p \ge \frac{m+1}{m}$ then the exponential factor is asymptotically a constant. Then we must have $\operatorname{Re} \psi_\infty(p) < 0$ for all $p \in \left[\frac{m+1}{m}, \infty \right)$; by part *(iii)* of Lemma 4.2 we have to check the value $\operatorname{Re} \psi_\infty\left(\frac{m+1}{m} \right)$. If $p < \frac{m+1}{m}$ then the exponential factor determines convergence; the condition on v_m follows from Lemma 3.6 with $\varrho = 1 - (p - 1)\,m$.

Now we check condition *(d)* of Lemma 2.2. We assume that $N(k) \sim C_0 \, k^p$, where $p \in (0, 1)$ and $C_0 > 0$ are real constants. Using formula (7.1), Corollary 2.4 and Lemma 3.3,

we get the following asymptotic expression as $k \to \infty$:

$$u(N,k) \sim C_1 N^{\sum_{\beta_j=0} \widetilde{a}_j - \sum_{\delta_j=0} \widetilde{c}_j} \frac{\prod_{\beta_j=0} \alpha_j^{\alpha_j N}}{\prod_{\delta_j=0} \gamma_j^{\gamma_j N}} \Gamma(N)^{\sum_{\beta_j=0} \alpha_j - \sum_{\delta_j=0} \gamma_j} \xi^N$$

$$\times \frac{\prod_{\beta_j\neq 0} \beta_j^{\alpha_j N}}{\prod_{\delta_j\neq 0} \delta_j^{\gamma_j N}} \frac{\prod \beta_j^{\beta_j k}}{\prod \delta_j^{\delta_j k}} k^{\left(\sum_{\beta_j\neq 0} \alpha_j - \sum_{\delta_j\neq 0} \gamma_j\right) N} \exp\left(N \widetilde{\Phi}_0\left(\frac{N}{k}\right)\right)$$

$$\times P(N,k) k^{\sum \widehat{a}_j - \sum \widehat{c}_j} \Gamma(k)^{\sum \beta_j - \sum \delta_j} \frac{\theta^k}{k!}, \tag{7.12}$$

where $C_1 \in \mathbb{R}$ and

$$\widetilde{\Phi}_0(x) = \sum_{\beta_j\neq 0} \alpha_j \Theta\left(\frac{\alpha_j x}{\beta_j}\right) - \sum_{\delta_j\neq 0} \gamma_j \Theta\left(\frac{\gamma_j x}{\delta_j}\right). \tag{7.13}$$

We rearrange as

$$u(N,k) \sim C_0 \Gamma(k)^{s-r-1} z_0^k \Gamma(N)^{S-R} \left(\frac{k^N}{\Gamma(N)}\right)^{\widetilde{S}-\widetilde{R}} \zeta_1^N \exp\left(N \widetilde{\Phi}_0\left(\frac{N}{k}\right)\right)$$

$$\times P(n,k) k^{\sum \widehat{a}_j - \sum \widehat{c}_j + \left(\sum_{\beta_j=0} \widetilde{a}_j - \sum_{\delta_j=0} \widetilde{c}_j\right) p - 1}, \tag{7.14}$$

We compute that

$$\frac{\log|s(N,k)|}{k} = (s-r-1)(\log k - 1) + \log|z_0| + (S-R)\frac{N}{k}\left(p \log k - 1\right)$$

$$+ \left(\widetilde{S} - \widetilde{R}\right)\frac{N}{k}\left((1-p)\log k + 1\right) + \frac{N}{k} \log|\zeta_1| + o\left(k^{-1+p}\right). \tag{7.15}$$

The last two expressions can be conveniently compared with (7.8)–(7.9). Currently, $k \gg N$. Like in the previous case, first we consider formula (7.15) and use Lemma 3.6 with $\varrho = p$. We get a similiar set of conditions:

(d1) $s \leq r+1$.

(d2) If $s = r+1$ then $|z_0| \leq 1$.

(d3) If $s = r+1$, $|z_0| = 1$, then $S \leq R$ and $\widetilde{S} \leq \widetilde{R}$.

(d4) If $s = r+1$, $|z_0| = 1$, $S = R$, $\widetilde{S} = \widetilde{R}$, then $|\zeta_1| \leq 1$.

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(d5) If $s = r+1$, $|z_0| = 1$, $S = R$, $\widetilde{S} = \widetilde{R}$, $|\zeta_1| = 1$, then $\operatorname{Re} A_0 < 0$ and one of the following conditions holds:

(d5A) $\widetilde{\Phi}_2(x) \equiv 0$ and $\operatorname{Re} A_1 \leq 0$.

(d5B) $\widetilde{\Phi}_2(x) = v_m x^m + O(x^{m+1})$ around $x = 0$ for some positive integer m and negative real v_m, and $\operatorname{Re} \psi_0\left(\frac{m}{m+1}\right) < 0$.

In condition *(d5)*, we may consider possibilities for $\operatorname{Re} A_0 = 0$, but this is unnecessary because of condition *(a3)*. In condition *(d5A)*, the case $\operatorname{Re} A_1 = 0$ ought to be supplemented by conditions that $\operatorname{Re} \psi_0(p) \neq 0$ for all $p < 1$; but this is obsolete, since if the linear slope of $\operatorname{Re} \psi_0(p)$ immediately to the left of $p = 1$ is zero, then the supremum is approached with $p \to 0$ by part *(iii)* of Lemma 4.2.

The case when the expression in (7.15) is $o(k^{-1+p})$ is similar to the consideration of $o(1)$ in (7.9). Formula (7.14) becomes then, for general $N(k)$,

$$u(N, k) \sim \widetilde{C}_1 \exp\left(N \, \widetilde{\Phi}_0\left(\frac{N}{k}\right) \right) k^{\psi_0(p)-1}, \tag{7.16}$$

for some $\widetilde{C}_1 \in \mathbb{R}$. If $\widetilde{\Phi}_0 \equiv 0$ we must have $\operatorname{Re} \psi_0(p) < 0$ for all $p \in (0, 1)$. By part *(ii)* of Lemma 4.2, we have to check the behavior of $\psi_0(p)$ near the end-points $p = 0$ and $p = 1$. If $\widetilde{\Phi}_0(x) \not\equiv 0$, then the exponential factor in (7.16) is asymptotic to

$$\exp\left(v_m C_0^{m+1} k^{p-(1-p)\,m} \right). \tag{7.17}$$

For $p \leq \frac{m}{m+1}$ then the exponential factor is asymptotically a constant. Then we must have $\operatorname{Re} \psi_0(p) < 0$ for all $p \in \left(0, \frac{m}{m+1}\right]$; by part *(iii)* of Lemma 4.2 we have to check the values $\psi_0\left(\frac{m}{m+1}\right)$ and $\psi_0(0)$. If $p > \frac{m}{m+1}$ then the exponential factor determines convergence; the condition on v_m follows from Lemma 3.6 with $\varrho = p - (1-p)\,m$.

It remains to check condition *(e)* of Lemma 2.2. Let us define the family of functions:

$$\widetilde{\Phi}_\lambda(x) = \sum_{\alpha_j \lambda + \beta_j \neq 0} \alpha_j \, \Theta\left(\frac{\alpha_j \, x}{\alpha_j \, \lambda + \beta_j} \right) - \sum_{\gamma_j \lambda + \delta_j \neq 0} \gamma_j \, \Theta\left(\frac{\gamma_j \, x}{\gamma_j \, \lambda + \delta_j} \right). \tag{7.18}$$

We split condition *(e)* into two cases:

(⋆) $N(k) = tk + \omega(k)$ with real positive $t \notin \Omega$, and either $\omega(k) = O(1)$ or $\omega(k) \sim C_0 k^p$ for some real $p \in (0, 1)$ and C_0.

$(\star\star)$ $N(k) = tk + \omega(k)$ with real positive $t \in \Omega$, and either $\omega(k) = O(1)$ or $\omega(k) \sim C_0 k^p$ for some real $p \in (0,1)$ and C_0.

Recall that Ω is defined in (4.20).

For case $(\star)$ we use formula (4.1) and Lemma 3.3 to derive the following asymptotic expression as $k \to \infty$:

$$
\begin{aligned}
u(N,k) \quad \sim \quad & C_1 \, \Gamma(k)^{\sum(\beta_j + \alpha_j t) - \sum(\delta_j + \gamma_j t)} \, \frac{\theta^k}{k!} \, \xi^N \, P(N,k) \\
& \times \frac{\prod (\beta_j + \alpha_j t)^{\beta_j k + \alpha_j N}}{\prod (\delta_j + \gamma_j t)^{\delta_j k + \gamma_j N}} \, k^{(\sum \alpha_j - \sum \gamma_j) \omega(k)} \, \exp\left(\omega(k) \, \widetilde{\Phi}_t \left(\frac{\omega(k)}{k} \right) \right) \\
& \times k^{\sum \left(\widehat{a}_j + \frac{1}{2} \alpha_j t \right) - \sum \left(\widehat{c}_j + \frac{1}{2} \gamma_j t \right) + \sum_{\beta_j = 0} \left(\widetilde{a}_j + \frac{t-1}{2} \alpha_j \right) - \sum_{\delta_j = 0} \left(\widetilde{c}_j + \frac{t-1}{2} \gamma_j \right)}.
\end{aligned}
\tag{7.19}
$$

for some $C_1 \in \mathbb{C}$. We arrange as follows:

$$
\begin{aligned}
|u(N,k)| \quad \sim \quad & \widetilde{C}_1 \, \Gamma(k)^{(\sum \alpha_j - \sum \gamma_j) t + \sum \beta_j - \sum \delta_j} \, \frac{|P(N,k)|}{k!} \, \exp\left(\omega(k) \, \widetilde{\Phi}_t \left(\frac{\omega(k)}{k} \right) \right) \\
& \times \left(|\theta| |\xi|^t \frac{\prod |\beta_j + \alpha_j t|^{\beta_j + \alpha_j t}}{\prod |\delta_j + \gamma_j t|^{\delta_j + \gamma_j t}} \right)^k \left(|\xi| \, k^{\sum \alpha_j - \sum \gamma_j} \frac{\prod |\beta_j + \alpha_j t|^{\alpha_j}}{\prod |\delta_j + \gamma_j t|^{\gamma_j}} \right)^{\omega(k)} \\
& \times k^{\mathrm{Re}\left(\sum \widehat{a}_j - \sum \widehat{c}_j + \sum_{\beta_j=0} \widetilde{a}_j - \sum_{\delta_j=0} \widetilde{c}_j \right) + \frac{t-1}{2} (\sum \alpha_j - \sum \gamma_j) + \frac{1}{2} \left(\sum_{\beta_j \neq 0} \alpha_j - \sum_{\delta_j \neq 0} \gamma_j \right)}.
\end{aligned}
\tag{7.20}
$$

for some $\widetilde{C}_1 \in \mathbb{R}$. Using (4.24), (5.3), (5.5), we rewrite:

$$
\begin{aligned}
|u(N,k)| \quad \sim \quad & \Gamma(k)^{(S-R) t + s - r - 1} \left(\frac{k^{\omega(k) + \frac{t-1}{2}}}{\exp(1)} \right)^{S-R} g(t)^k \, \exp\left(\omega(k) \, \frac{g'(t)}{g(t)} \right) \\
& \times \exp\left(\omega(k) \, \widetilde{\Phi}_t \left(\frac{\omega(k)}{k} \right) \right) k^{\mathrm{Re}\, \psi_t^\star(p) - 1}.
\end{aligned}
\tag{7.21}
$$

Here we set $p = 0$ if $\omega(k) = O(1)$. Recall that $\psi_t^\star(p)$ is a monotone non-decreasing function by part *(ii)* of Lemma 4.3.

Notice that we have $s \le r + 1$ and $S \le R$ by conditions *(a1)* and *(b1)*. Case $(\star)$ gives additional conditions if $S = R$ and $s = r + 1$. Firstly, we must have $g(t) \le 1$ for all positive $t \in \mathbb{R} \setminus \Omega$. If this is the case, and $g(t_0) = 1$ for some positive $t_0 \in \mathbb{R} \setminus \Omega$, then $g'(t_0) = 0$. Indeed, $g(t_0) \neq 0$ would imply $g(t_1) > 1$ for some $t_1 \in \mathbb{R} \setminus \Omega$ in a neighborhood of t_0. Therefore we may ignore the exponential factor with $g'(t)$. At these points t_0 we have to consider the last two terms in (7.21). Eventually we get the following conditions for the case $(\star)$:

(e1) If $S = R$, $s = r + 1$, then $g(t) \leq 1$ for all positive $t \in \mathbb{R} \setminus \Omega$.

(e2) If $S = R$, $s = r + 1$, and $g(t) = 1$ for some positive $t \in \mathbb{R} \setminus \Omega$, then for any $t_0 \in \mathbb{R} \setminus \Omega$ where $g(t_0) = 1$, we must have $\operatorname{Re} \psi_{t_0}^{\star}(0) < 0$ and one of the following two conditions satisfied:

(e2A) $\widetilde{\Phi}_{t_0}(x) \equiv 0$, and $\operatorname{Re} \psi_{t_0}^{\star}(1) \leq 0$.

(e2B) $\widetilde{\Phi}_{t_0}(x) = v_m x^m + O(x^{m+1})$ around $x = 0$, where m is a positive **odd** integer, $v_m < 0$, and $\operatorname{Re} \psi_{t_0}^{\star}\left(\frac{m}{m+1}\right) < 0$.

Here we comment the situations when condition *(e2)* applies. We have $\operatorname{Re} \psi_{t_0}^{\star}(0) < 0$ because the power of k in (7.21) determines the convergence when $\omega(k) = O(1)$. If $\widetilde{\Phi}_t(x) \equiv 0$, we must have $\operatorname{Re} \psi_t^{\star}(p) < 0$ for all $p \in [0, 1)$. By part *(iii)* of Lemma 4.4, it is enough to have $\operatorname{Re} \psi_{t_0}^{\star}(1) \leq 0$. If $\widetilde{\Phi}_t(x) \not\equiv 0$, then the exponential factor is asymptotic to $\exp\left(v_m C_0^{m+1} k^{p-(1-p)m}\right)$; it is relevant when $p \in \left(\frac{m}{m+1}, 1\right)$. If m is even, the exponential factor is unbounded either when $C_0 > 0$ or when $C_0 < 0$. Hence m must be odd. Then Lemma 3.6 with $\varrho = p - (1-p)m$ gives the restriction $v_m < 0$. The power of k factor must be restricted for $p \in \left[\frac{m}{m+1}, 1\right]$. By part *(iii)* of Lemma 4.4, it is enough to have $\operatorname{Re} \psi_{t_0}^{\star}\left(\frac{m}{m+1}\right) \leq 0$.

Now we consider the case $(\star\star)$, with $t \in \Omega$. Formula (7.19) should be modified as follows:

- The sums and products should be supplemented by the conditions $\beta_j + \alpha_j t \neq 0$ or $\delta_j + \gamma_j t \neq 0$. This is unnecessary for the sums in the power of $\Gamma(k)$, and eventually in some products (since $0^0 = 1$). Note that these conditions are already indicated in definition (7.18) of $\widetilde{\Phi}_t(x)$.

- By Lemma 3.5, we have to append

$$|\omega(k)|^{\sum^*(b_j - \frac{1}{2}) - \sum_*(d_j - \frac{1}{2})} \frac{\prod^*(\varepsilon\alpha_j)^{\alpha_j \omega(k)}}{\prod_*(\varepsilon\gamma_j)^{\gamma_j \omega(k)}} \left(\sqrt{|\omega(k)|} \,\Gamma(|\omega(k)|)\right)^{\varepsilon\left(\sum^* \alpha_j - \sum_* \gamma_j\right)}, \quad (7.22)$$

where the range of the summations $\sum^*$ and the product $\prod^*$ are those j with $\beta_j + \alpha_j t = 0$, and the range of the summations $\sum_*$ and the product $\prod_*$ are those j with $\delta_j + \gamma_j t = 0$.

With these modifications, asymptotic expression (7.21) can be written eventually as

$$|u(N, k)| \;\sim\; \Gamma(k)^{(S-R)t+s-r-1}\left(\sqrt{|\omega(k)|}\,\Gamma(|\omega(k)|)\right)^{\varepsilon\left(\sum^* \alpha_j - \sum_* \gamma_j\right)} k^{(S-R-\sum^* \alpha_j + \sum^* \gamma_j)\omega(k)}$$

$$\times g(t)^k \left(|\xi| \frac{\prod_{\beta_j + \alpha_j t \neq 0} |\beta_j + \alpha_j t|^{\alpha_j} \prod^* |\alpha_j|^{\alpha_j}}{\prod_{\delta_j + \gamma_j t \neq 0} |\delta_j + \gamma_j t|^{\gamma_j} \prod_* |\gamma_j|^{\gamma_j}}\right)^{\omega(k)} \exp\left(\omega(k) \,\widetilde{\Phi}_t\left(\frac{\omega(k)}{k}\right)\right)$$

$$\times k^{\operatorname{Re} \psi_t^{\star}(p) + (S-R)\frac{t-1}{2} - 1}. \tag{7.23}$$

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Here we set $p = 0$ if $\omega(k) = O(1)$.

As in the case $(\star)$, we have additional conditions if $s = r + 1$ and $S = R$. Then we have:

$$\log|u(N, k)| = \left(\sum^* \alpha_j - \sum_* \gamma_j\right)\left(\log|\omega(k)| - \log k - 1\right)\omega(k)$$
$$+ k \log g(t) + O(w(k) + \log k). \tag{7.24}$$

In general, the dominant term is $k \log g(t)$; hence we must have $g(t) \leq 1$.

Suppose that $g(t_0) = 1$ for some $t_0 \in \Omega$. If $\Sigma^*\alpha_j \neq \Sigma_*\gamma_j$ (where the range conditions $*$ apply to the point t_0), then the first term approaches $+\infty$ for those $\omega(k) \sim C_0 k^p$ with p close to 1 and $C_0 > 0$ or $C_0 < 0$ correspondingly. Hence $\Sigma^*\alpha_j = \Sigma_*\gamma_j$ if $g(t_0) = 1$ for some $t_0 \in \Omega$.

If $\Sigma^*\alpha_j = \Sigma_*\gamma_j$, then $g(t)$ is actually differentiable at t_0 by part *(iii)* of Lemma 5.1. The value of the derivative can be derived from (5.6) or (5.8). If $s = r + 1$, $S = R$, $g(t_0) = 1$ and $\Sigma^*\alpha_j = \Sigma_*\gamma_j$, we can rewrite (7.23) as follows:

$$|u(N, k)| \sim \exp\left(\omega(k)\frac{g'(t_0)}{g(t_0)}\right)\exp\left(\omega(k)\,\widetilde{\Phi}_{t_0}\left(\frac{\omega(k)}{k}\right)\right)k^{\operatorname{Re}\psi_t^\star(p) - 1}. \tag{7.25}$$

If $g'(t_0) \neq 0$, then condition *(e1)* is contradicted for some point $t \in \mathbb{R} \setminus \Omega$ in a neighborhood of t_0. Hence we may assume $g'(t_0) = 0$. Eventually we get the following conditions:

(e3) If $S = R$, $s = r + 1$, then $g(t) \leq 1$ for all positive $t \in \Omega$.

(e4) If $S = R$, $s = r + 1$, $g(t_0) = 1$ for some positive $t_0 \in \Omega$, then for any $t_0 \in \mathbb{R} \setminus \Omega$ where $g(t_0) = 1$ we must have $\sum^* \alpha_j = \sum_* \gamma_j$, $\operatorname{Re}\psi_{t_0}^\star(0) < 0$, and one of the following two conditions satisfied:

    (e4A) $\widetilde{\Phi}_{t_0}(x) \equiv 0$, and $\operatorname{Re}\psi_{t_0}^\star(1) \leq 0$.

    (e4B) $\widetilde{\Phi}_{t_0}(x) = v_m x^m + O(x^{m+1})$ around $x = 0$, where m is a positive odd integer, $v_m < 0$, and $\operatorname{Re}\psi_{t_0}^\star\left(\frac{m}{m+1}\right) < 0$.

The subcases of *(e4)* are derived similarly as the subcases of *(e2)*. Compared with conditions *(e1)–(e2)*, we additionally have the condition $\sum^* \alpha_j = \sum_* \gamma_j$ in *(e4)*. An implicit difference between cases $(\star)$ and $(\star\star)$ is that the functions $\psi_t^\star(p)$ and $\widetilde{\Phi}_t(x)$ can be defined simpler in case $(\star)$.

Before summarizing up the derived conditions, we remark that the nonzero Taylor coefficients (3.5) of $\Theta(x)$ have the same signs as the Taylor coefficients of the rational function $x/(1 + x) = \sum_{j=1}^\infty (-1)^{j+1} x^j$. The corresponding coefficients differ the positive factor

$j(j+1)$. If we replace each occurrence of $\Theta(x)$ by $x/(1+x)$ in definitions (7.7), (7.13), (7.18) of $\widetilde{\Phi}_\infty(x)$, $\widetilde{\Phi}_0(x)$, $\widetilde{\Phi}_\lambda(x)$, respectively, we get the rational functions $\Phi_\infty(x)$, $\Phi_0(x)$, $\Phi_\lambda(x)$ defined in (5.15), (5.14), (5.13), respectively. The Taylor coefficients around $x = 0$ of the rational functions differ by the positive factor $j(j+1)$ from the respective coefficients of the corresponding $\widetilde{\Phi}$-functions. Therefore we may replace in conditions *(c5), (d5), (e2), (e4)* the functions $\widetilde{\Phi}_\infty(x)$, $\widetilde{\Phi}_0(x)$, $\widetilde{\Phi}_\lambda(x)$ by the rational functions $\Phi_\infty(x)$, $\Phi_0(x)$, $\Phi_\lambda(x)$, respectively.

Now we summarize the conditions *(a1)–(a3), (b1)–(b7), (c1)–(c5), (d1)–(d5), (e1)–(e4)*. Note that

$$(a1) \Rightarrow (b5) \ \& \ (d1), \qquad (a2) \Rightarrow (d2), \qquad (b1) \Rightarrow (c1), \qquad (b2) \Rightarrow (c2),$$
$$(c4) \Rightarrow (b6), \qquad (a1) \ \& \ (b3) \Rightarrow (c3), \qquad (a3) \ \& \ (b1) \Rightarrow (d3).$$

Therefore we may discard the conditions *(b5)–(b6), (c1)–(c3), (d1)–(d3)*. Because of *(a3)*, we can drop the restriction $\operatorname{Re} A_0 < 0$ in *(d5)*. Because of *(b3)*, we can drop the restriction $\operatorname{Re} A_\infty^* \leq 0$ in *(c5)*. Besides, in cases *(c5A)* and *(c5B)* we can drop condition *(b7)*, because $\operatorname{Re}(A_\infty^* + A_0^*) \leq \operatorname{Re} A_1^*$.

We have the following correspondence between the conditions:

$$(a1) \ \& \ (b1) \Rightarrow (i), \qquad (a2)-(a3) \Leftrightarrow (ii), \qquad (b2)-(b4) \Leftrightarrow (iii),$$
$$(e1) \ \& \ (e3) \Leftrightarrow (iv), \qquad (e2) \ \& \ (e4) \Leftrightarrow (v), \qquad (d4)-(d5) \Leftrightarrow (vi),$$
$$(c4)-(c5) \ \& \ (b7) \Leftrightarrow (vii).$$

The limit $\lim_{n\to\infty} \mathcal{U}(n)$ is discussed right after the conditions *(b1)–(b4)* here above.   QED.

8   The properties of $g(t)$

As we already mentioned, all conditions of Theorem 6.1 can be determined algorithmically. The only less straightforward part is dealing with the function $g(t)$ in parts *(iv)–(v)*. This is fitting when $s = r + 1$ and $S = R$. Key properties of $g(t)$ are presented in Lemma 5.1.

Lemma 8.1 *In the context of Sections 4 and 5, suppose that $s = r + 1$, $S = R$, and that conditions (ii)–(iii), (vii) of Theorem 6.1 hold. Then $g(t) \leq 1$ for all $t > 0$ if and only if the following conditions hold:*

- *For all $t \notin \Omega$ such that*

$$|\zeta_0| \prod_{\alpha_j \neq 0} \left| t + \frac{\beta_j}{\alpha_j} \right|^{\alpha_j} = \prod_{\gamma_j \neq 0} \left| t + \frac{\delta_j}{\gamma_j} \right|^{\gamma_j} \tag{8.1}$$

  we have

$$|z_1| \prod_{\alpha_j \neq 0} \left| t + \frac{\beta_j}{\alpha_j} \right|^{\beta_j} \leq \prod_{\gamma_j \neq 0} \left| t + \frac{\delta_j}{\gamma_j} \right|^{\delta_j}. \tag{8.2}$$

- *For all $t \in \Omega$ such that equality (6.2) holds, we have $g(t) \leq 1$.*

If these conditions are satisfied, then $g(t) = 1$ are those points $t \notin \Omega$ where equalities in (8.1) and (8.2) hold, and possibly some points $t \in \Omega$ where equality (6.2) holds.

Proof. By parts *(i)–(ii)* of Lemma 5.1, the function $g(t)$ is continuous on $\mathbb{R}$, and it is continuously differentiable on $\mathbb{R} \setminus \Omega$. We need to investigate the behavior of $g(t)$ as t approaches $+\infty$, 0 or singularities of $g'(t)$, and find local extremuma of $g(t)$.

As $t \to 0$, then $g(t) \to |z_0|$ by part *(iv)* of Lemma 5.1. But $|z_0| \leq 1$ by part *(ii)* of Theorem 6.1. As $t \to \infty$, then $g(t) \sim |z_1| \exp(\widetilde{s} - \widetilde{r}) |\zeta_0|^t \, t^{\widetilde{s} - \widetilde{r}}$ by part *(iv)* of Lemma 5.1. The chain of possible restrictions $|\zeta_0| \leq 1$, $\widetilde{s} \leq \widetilde{r}$, $|z_1| \leq 1$ is implied by parts *(iii)*, *(vii)* of Theorem 6.1. By part *(iv)* of Lemma 5.1, genuine points of discontinuity of $g'(t)$ are not local extremuma.

It remains to check the local extremuma at those $t > 0$ where $g'(t)$ is actually continuous. For these points, either $t \notin \Omega$, or $t \in \Omega$ and equality (6.2) holds. Condition (8.1) is just reformulation of $g'(t) = 0$, following expression (5.6). Recall that we assume $S = R$. Inequality (8.2) is equivalent to $g(t) \leq 1$ if condition (8.1) is satisfied.

If $g(t) \leq 1$ for all $t > 0$, then the points with $g(t) = 1$ are local extremuma. If $t \in \Omega$ and $g'(t) = 0$, then the quotient of the left and right hand sides of (8.2) is equal to $g(t)$. QED.

Here are a few general remarks.

- If all α_j's and γ_j's are even, then condition (8.1) is actually a polynomial equation for t. If there are some odd α_j's or γ_j's, we can square both sides of (8.1) and get a polynomial equation for t as well. We have to find real positive roots of these equations. The numeric or algebraic roots of the polynomial equations can be found algorithmically. On the other hand, the equations might have inappropriately high degree. It might be useful to have some estimates of the number and location of relevant solutions.

- The two conditions for $g(t) \leq 1$ can be formulated in a single statement, if we add conditions $\alpha_j t + \beta_j \neq 0$ or $\gamma_j t + \delta_j \neq 0$ to the products in (8.1) and (8.2), or make the convention that the both-side factors $|t - \lambda|$ with $\lambda = t$ in these formulas cancel out if $t \in \Omega$ and equality (6.2) holds. The unified statement is: For all $t > 0$ such that equalities (6.2) and (8.1) hold, we must have (8.2). Identification of the points $g(t) = 1$ can be similarly unified. From algorithmic point of view, the single equation (8.1) with simplified or cancelled-out powers of $|t - \lambda|$ determines all local extremuma.

- Let us denote $h(t) = g'(t)/g(t)$. Using formula (5.5) we derive

$$h'(t) = \sum \frac{\alpha_j^2}{\alpha_j t + \beta_j} - \sum \frac{\gamma_j^2}{\gamma_j t + \delta_j}. \tag{8.3}$$

If we compute the zeroes and poles of this rational function, and (signs of) values of $h(t)$ there, we can determine intervals where zeroes of $h(t)$ lie. Since $g'(t)$ has the same sign as $h(t)$ for any $t \notin \Omega$, those are also intervals for the zeroes of $g'(t)$, or extremuma of $g(t)$.

- Lemma 8.1 formally holds in the case when $g(t)$ is a constant function as well. Of course, in that case condition *(iv)* of Theorem 6.1 is straightforward.

In the rest of this Section, we explicitly consider a simple case of the $g(t)$-function:

$$\widehat{g}(t) = \frac{|\alpha\, t + 1|^{\alpha t + 1}\ |\gamma\, t|^{\gamma t}}{|\alpha\, t|^{\alpha t}\ |\gamma\, t + 1|^{\gamma t + 1}}, \tag{8.4}$$

This case naturally occurs with sequences of hypergeometric functions of the form

$${}_sF_r\left(\begin{array}{c} a_1 + \alpha\, n,\, a_2,\, \ldots,\, a_s \\ c_1 + \gamma\, n,\, c_2,\, \ldots,\, c_r \end{array} \,\middle|\, z \right). \tag{8.5}$$

There may be more upper and lower parameters dependant on n, if they cancel each other out in the expression of $g(t)$. Knowledge of the function $\widehat{g}(t)$ may help to arrive at effective estimates for more complicated functions $g(t)$, by splitting them into a product of $\widehat{g}(t)$'s.

In the following Lemma, we present basic properties of $\widehat{g}(t)$. We assume here that $\gamma > 0$, but allow t to be both positive and negative. If $\gamma < 0$ in (8.4), then Lemma 8.2 can be applied by considering $\gamma \mapsto -\gamma$, $t \mapsto -t$, $\alpha \mapsto -\alpha$, so that $\gamma > 0$ and $t < 0$.

Lemma 8.2 *Assume that α, γ are integers, $\alpha \neq \gamma$ and $\gamma > 0$.*

(i) *The function $\widehat{g}(t)$ is continuous on the whole real axis, and is differentiable everywhere except the points $x \in \{0, -1/\alpha, -1/\gamma\}$. These three points are not local extremuma.*

(ii) *$\widehat{g}(0) = 1$, and $\lim_{t \to \pm\infty} \widehat{g}(t) = |\alpha/\gamma|$.*

(iii) *$\sup_{t>0} \widehat{g}(t) = \max(1, |\alpha/\gamma|)$.*

(iv) *The global supremum of $\widehat{g}(t)$ is achieved for a negative t, and it is the only local extrema which satisfies $\widehat{g}(t) > 1$ and $\widehat{g}(t) > |\alpha/\gamma|$.*

Proof. The first part follows from parts *(i)–(iii)* of Lemma 5.1. The value $\widehat{g}(0)$ is trivial. We have

$$\lim_{t \to \infty} \widehat{g}(t) = \lim_{t \to \infty} \frac{|\alpha\,t + 1|}{|\gamma\,t + 1|} \left|1 + \frac{1}{\alpha\,t}\right|^{\alpha\,t} \Big/ \left|1 + \frac{1}{\gamma\,t}\right|^{\gamma\,t} = \left|\frac{\alpha}{\gamma}\right|,$$

and similarly for $\lim_{t \to -\infty} \widehat{g}(t)$.

Let us consider

$$h(t) := \frac{\widehat{g}'(t)}{\widehat{g}(t)} = \alpha \log|1 + \alpha\,t| - \alpha \log|\alpha\,t| + \gamma \log|\gamma\,t| - \gamma \log|1 + \gamma\,t|. \tag{8.6}$$

The local extremuma of $\widehat{g}(t)$ are determined by $h(t) = 0$. We have:

$$h'(t) = \frac{\gamma - \alpha}{t\,(1 + \alpha\,t)\,(1 + \gamma\,t)}, \tag{8.7}$$

We conclude that $h(t)$ and $\widehat{g}'(t)$ are monotone on the intervals separated by points 0, $-1/\alpha$ and $-1/\gamma$. Here are some relevant limits:

$$\lim_{t \to -1/\gamma} h(t) = \infty, \qquad \lim_{t \to \pm\infty} h(t) = 0,$$

$$\lim_{t \to -1/\alpha} h(t) = \begin{cases} -\infty, & \text{if } \alpha > 0, \\ \infty, & \text{if } \alpha < 0. \end{cases} \qquad \lim_{t \to 0} h(t) = \begin{cases} \infty, & \text{if } \alpha > \gamma, \\ -\infty, & \text{if } \gamma > \alpha. \end{cases}$$

We distinguish the following cases:

- If $0 < \alpha < \gamma$, then $\widehat{g}(t)$ has a local maximum on the interval $(-1/\gamma, 0)$, which is greater than $\widehat{g}(0) = 1 > \alpha/\gamma$. There is a local minimum on $(-1/\alpha, -1/\gamma)$, which is less than $\widehat{g}(-\infty) = \alpha/\gamma < 1$. For positive t the function $\widehat{g}(t)$ decreases from 1 to α/γ. See the first graph in Figure 1.

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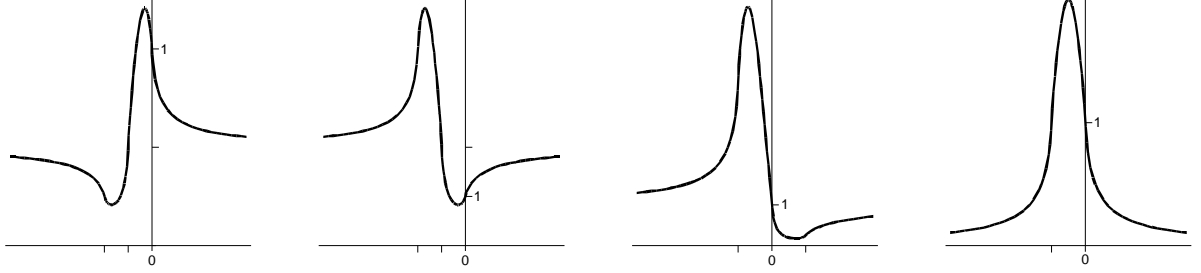
Figure 1: The function $\widehat{g}(t)$

- If $0 < \gamma < \alpha$, then $\widehat{g}(t)$ has a local maximum on the interval $(-1/\gamma, -1/\alpha)$, which is greater than $\widehat{g}(-\infty) = \alpha/\gamma > 1$. There is a local minimum on $(-1/\alpha, 0)$, which is less than $\widehat{g}(0) = 1 < \alpha/\gamma$. For positive t function $\widehat{g}(t)$ increases from 1 to α/γ. See the second graph in Figure 1.

- If $\alpha < 0$, then $\widehat{g}(t)$ has a local maximum on the interval $(-1/\gamma, 0)$, which is greater than $\widehat{g}(-\infty) = |\alpha/\gamma|$ and $\widehat{g}(0) = 1$. There is a local minimum on $(0, -1/\alpha)$, which is less than 1 and $|\alpha/\gamma|$. The supremum of $\widehat{g}(t)$ over positive t is achieved as $t \to 0$ or $t \to \infty$. See the third graph in Figure 1.

- If $\alpha = 0$, then $g(t)$ has a local maximum on the interval $(-1/\gamma, 0)$, which is greater than $\widehat{g}(0) = 1$. There are no other extremuma in this case. For positive t the function $\widehat{g}(t)$ decreases from 1 to 0. See the last graph in Figure 1.

This analysis proves parts *(iii)–(iv)* of the Lemma.   QED.

Corollary 8.3 *Suppose that $\alpha \neq \gamma$. If $\gamma > 0$, then the supremum of $\widehat{g}(t)$ over $t > 0$ is achieved either as $t \to 0$ or $t \to \infty$. If $\gamma < 0$, then the supremum of $\widehat{g}(t)$ over $t > 0$ is achieved for some $t \in \left(0, \frac{1}{|\gamma|}\right)$.*

Proof. If $\gamma > 0$, we use parts *(ii)–(iii)* of Lemma 8.2. If $\gamma < 0$ then we apply Lemma 8.2 after changing the signs $\gamma \mapsto -\gamma$, $t \mapsto -t$, $\alpha \mapsto -\alpha$.   QED.

To estimate how high is the maximum of $\widehat{g}(t)$ over those t with $\gamma\,t < 0$, we need this Lemma.

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Lemma 8.4 *Suppose that $x \geq 1$. The equation*

$$\frac{y^x}{x^x} = \frac{(y+1)^{x-1}}{(x-1)^{x-1}} \tag{8.8}$$

has a unique root y such that $y \geq 1$.

Let $y(x)$ denote the unique root as a function of x. Asymptotically,

$$y(x) \sim \tau\, x - \frac{\tau+1}{2} - \frac{(\tau+1)\,(\tau-2)}{24\,\tau}\frac{1}{x} + \dots, \qquad as\ x \to \infty, \tag{8.9}$$

where τ is the real solution of $\log(\tau) = 1 + 1/\tau$:

$$\tau \approx 3.591121476668622136649222925741634842210\dots. \tag{8.10}$$

For $x \geq 1$ we have

$$\tau\,(x-1) + 1 < y(x) < \tau\,(x-1) + \frac{\tau-1}{2}. \tag{8.11}$$

Proof. Let us consider the logarithm of the ratio of both sides of (8.8):

$$\Psi(x,y) = x\,\log y - x\,\log x - (x-1)\,\log(y+1) + (x-1)\,\log(x-1). \tag{8.12}$$

For fixed $x \geq 1$, we have to find solutions of $\Psi(x,y) = 0$ with $y \geq 1$. We have:

$$\frac{\partial \Psi(x,y)}{\partial y} = \frac{y+x}{y(y+1)}. \tag{8.13}$$

Hence, as a function of y, $\Psi(x,y)$ is continuous increasing function on the interval $(1, \infty)$. There can be at most one root $y \geq 1$. We may check

$$\Psi(x,1) = (x-1)\log\frac{x-1}{2} - x\log x, \tag{8.14}$$

$$\Psi(x,y) \sim \log y + O(1) \quad \text{as } y \to \infty. \tag{8.15}$$

Since $\Psi(x,1) < 0$, and $\Psi(x,y) \to \infty$ as $t \to \infty$, there exists a root $y \geq 1$ indeed.

A straightforward attempt to solve $\Psi(x,y) = 0$ asymptotically gives (8.9).

To prove the inequalities in (8.11), we show

$$\Psi\big(x, \tau\,(x-1) + 1\big) < 0, \qquad \Psi\left(x, \tau\,(x-1) + \frac{\tau-1}{2}\right) > 0. \tag{8.16}$$

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Then the monotonicity of $y(x)$ will imply (8.11).

First we show the second inequality. We substitute $y = \tau x - (\tau + 1)/2$ into $\Psi(x, y)$:

$$
\begin{aligned}
\Psi(x, y) &= \log(\tau) + x \log\left(1 - \frac{\tau + 1}{2\,\tau\,x}\right) - (x - 1)\log\left(\frac{x}{x - 1}\left(1 - \frac{\tau - 1}{2\,\tau\,x}\right)\right) \\
&= \sum_{j=1}^{\infty} \frac{1}{j\,(j+1)}\left(1 - j\left(\frac{\tau + 1}{2\,\tau}\right)^{j+1} - \frac{\tau\,j + j + 2\,\tau}{2\,\tau}\left(\frac{\tau - 1}{2\,\tau}\right)^{j}\right)\frac{1}{x^j}. \quad (8.17)
\end{aligned}
$$

The power series converges for $x > 1$, since a tail of it can be majorated by $\sum_j \frac{1}{j\,(j+1)} x^{-j}$. The series terms are positive for large enough j. The first terms of (8.17) are

$$
\frac{0.03017\ldots}{x^2} + \frac{0.03017\ldots}{x^3} + \frac{0.02564\ldots}{x^4} + \ldots
$$

After applying Lemma 3.7 twice with $p = (\tau \pm 1)/2\tau$, we conclude that all terms in the series are positive. Hence the second inequality in (8.16) follows.

If $y = \tau x - \tau + 1$, then

$$
\begin{aligned}
\Psi(x, y) &= \log(\tau) + x \log\left(1 - \frac{\tau - 1}{\tau\,x}\right) - (x - 1)\log\left(\frac{x}{x - 1}\left(1 - \frac{\tau - 2}{\tau\,x}\right)\right) \\
&= \sum_{j=1}^{\infty} \frac{1}{j\,(j+1)}\left(1 - j\left(\frac{\tau - 1}{\tau}\right)^{j+1} - \frac{2\,j + \tau}{\tau}\left(\frac{\tau - 2}{\tau}\right)^{j}\right)\frac{1}{x^j}. \quad (8.18)
\end{aligned}
$$

The power series converges for $x > 1$, just as (8.17). The series terms are positive for large enough j. The first terms of (8.18) are

$$
-\frac{0.10522\ldots}{x} - \frac{0.02770\ldots}{x^2} - \frac{0.00378\ldots}{x^3} + \frac{0.00466\ldots}{x^4} + \ldots
$$

Applying lemma 3.7 twice with $p = (\tau - 1)/\tau$ and $p = (\tau - 2)/\tau$ we conclude that starting with the power x^{-4} the coefficients are positive. Hence the first three terms in (8.18) are negative, and all remaining terms in (8.18) are positive. Let us consider the function

$$
\Psi_1(x) = x^3\,\Psi(x,\, -\tau\,x + \tau + 1). \quad (8.19)
$$

The Laurent series of the derivative of this function at $x = \infty$ is:

$$
\frac{d\,\Psi_1(x)}{dx} = -0.21044\ldots x - 0.02770\ldots - \frac{0.00466\ldots}{x^2} + \ldots
$$

The information about the signs of the coefficients in (8.18) implies that all non-zero terms in the Laurent series are negative. Therefore $\Psi_1(x)$ is a decreasing function on the interval $(1, \infty)$. Further, $\lim_{x \to 1+} \Psi_1(x) = 0$, since $\Psi(x, y)$ is continuous and $\Psi(1, 1) = 0$. Therefore $\Psi_1(x) < 0$ for $x \in (1, \infty)$. Consequently, the first inequality in (8.16) follows as well. QED.

The main result about the function $\widehat{g}(t)$ defined in (8.4) is the following.

Theorem 8.5 *Suppose that α, γ are integers. Then*

$$\sup_{t>0} \widehat{g}(t) = \begin{cases} 1, & \text{if } \alpha = \gamma, \\ \infty, & \text{if } \gamma = 0, \ \alpha \neq 0, \\ \max(|\frac{\alpha}{\gamma}|, 1), & \text{if } \gamma > 0, \\ y(\frac{\alpha}{\gamma}), & \text{if } \alpha < \gamma < 0, \\ 1 + 1/y(\frac{\gamma}{\gamma - \alpha}), & \text{if } \gamma < \alpha < 0, \\ 2, & \text{if } \gamma < 0, \ \alpha = 0, \\ 1 + y(\frac{\gamma - \alpha}{\gamma}), & \text{if } \gamma < 0 < \alpha. \end{cases} \tag{8.20}$$

where the function $y(x)$ is defined in Lemma 8.4.

Proof. If $\alpha = \gamma$, then $\widehat{g}(t) \equiv 1$. If $\gamma \neq 0$, then

$$\widehat{g}(t) = \left| 1 + \frac{1}{\alpha t} \right|^{\alpha t} |\alpha t + 1|,$$

and $\widehat{g}(t) \sim \exp(1) |\alpha| t$ as $t \to \infty$. If $\gamma > 0$, we apply part *(iii)* of Lemma 8.2.

From now on we assume $\gamma < 0$, $\alpha \neq \gamma$. We use Lemma 8.2 with the flipped signs of γ, t and α. By part *(iv)*, the supremum is a local extremum, so it is achieved for some $t = t_{\sup}$ (dependent on α and γ) satisfying $\widehat{g}'(t_{\sup}) = 0$. Expression (8.6) gives the following equation for $t_{\sup}$:

$$\frac{|\alpha t_{\sup} + 1|^{\alpha} |\gamma t_{\sup}|^{\gamma}}{|\alpha t_{\sup}|^{\alpha} |\gamma t_{\sup} + 1|^{\gamma}} = 1. \tag{8.21}$$

Hence,

$$\widehat{g}(t_{\sup}) = \frac{|\alpha t_{\sup} + 1|}{|\gamma t_{\sup} + 1|}. \tag{8.22}$$

Let us define the function

$$\widetilde{y}(\alpha, \gamma) = \frac{\alpha t_{\sup} + 1}{\gamma t_{\sup} + 1}, \tag{8.23}$$

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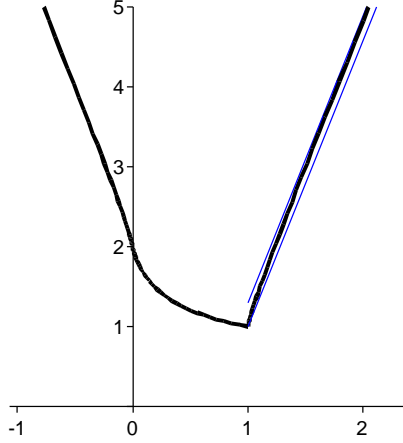
Figure 2: The supremum of $\widehat{g}(t)$ for $\gamma\, t < 0$, as a function of $\dfrac{\alpha}{\gamma}$.

so that $\widehat{g}(t_{\mathrm{sup}}) = |\widetilde{y}(\alpha,\gamma)|$. We have:

$$
\begin{aligned}
\frac{|\widetilde{y}(\alpha,\gamma)|^{\alpha}}{|\widetilde{y}(\alpha,\gamma) - 1|^{\alpha-\gamma}}
&= \frac{|\alpha\, t_{\mathrm{sup}} + 1|^{\alpha}\, |t_{\mathrm{sup}}|^{\gamma-\alpha}}{|\gamma\, t_{\mathrm{sup}} + 1|^{\gamma}\, |\alpha - \gamma|^{\alpha-\gamma}} \\
&= \frac{|\alpha|^{\alpha}}{|\gamma|^{\gamma}\, |\alpha - \gamma|^{\alpha-\gamma}},
\end{aligned}
\tag{8.24}
$$

where the second equality holds because of (8.21). Formula (8.24) implies that $\widetilde{y}(\alpha,\gamma)$ is a real solution of

$$
\frac{|\widetilde{y}|^{x}}{|x|^{x}} = \frac{|\widetilde{y} - 1|^{x-1}}{|x - 1|^{x-1}}, \qquad \text{where} \quad x = \frac{\alpha}{\gamma}, \quad \widetilde{y} = \widetilde{y}(\alpha,\gamma)
\tag{8.25}
$$

Conversely, if (8.25) holds, then expression (8.23) is also true provided that t_{sup} is well-defined, which is not the case only when $\widetilde{y}(\alpha,\gamma) = \alpha/\gamma$. It follows that all solutions of (8.25) except $\widetilde{y} = x$ correspond to local extremuma of $\widehat{g}(t)$. We need a solution of (8.25) whose absolute value is greater than $\max(1, |x|)$.

The cases $x = 1$ and $x = 0$ can be proved by solving the equation (8.25) directly.

If $x > 1$, we have two possibilities: either $\widetilde{y} > 1$ or $\widetilde{y} < -1$ for the relevant solution of (8.25). But if $\widetilde{y} > 0$ and $\widetilde{y} > x$, then the left-hand side of (8.25) is always bigger than the right-hand side. Hence the relevant solution has $\widetilde{y} < -1$. Then $y = |\widetilde{y}|$ satisfies (8.8), so the supremum for $\alpha < \gamma < 0$ is equal to $y\,(\alpha/\gamma)$.

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If $x < 0$ then the transformation $x \mapsto 1 - x$, $\widetilde{y} \mapsto 1 - \widetilde{y}$ transforms equation (8.25) to the same equation with $x > 1$. Since $\widetilde{y} = -y\,(\alpha/\gamma)$ for $x > 1$, we get the result for $\gamma < 0 < \alpha$.

Similarly, if $x \in (0, 1)$ then the transformation $x \mapsto 1/(1 - x)$, $\widetilde{y} \mapsto 1/(1 - \widetilde{y})$ transforms equation (8.25) to the same equation with $x > 1$. The inverse transformation on $\widetilde{y}$ for $x > 1$ is $1 - 1/\widetilde{y}$, with $\widetilde{y} = -y\,(\alpha/\gamma)$ again. Hence the remaining case $\gamma < \alpha < 0$ follows.   QED.

Figure 2 gives the graph of $\sup_{t>0} \widehat{g}(t)$ for $\lambda < 0$ as a function of α/γ, as specified by Theorem 8.5. The continuous graph is piecewise defined on the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$. On the interval $(1, \infty)$, the function is identical to the function $y(x)$ of Lemma 8.4. The blue lines are the bounding lines in (8.11). As we see, the function approaches the asymptotic straight line very fast. The function can be transformed between the three intervals by the fractional-linear transformations implied in Theorem 8.5. The tangent slopes at $\alpha/\gamma = 1$ (from the right) and at $\alpha/\gamma = 0$ (from both sides) are actually vertical. To see this at $\alpha/\gamma = 1$, compute dy/dx from $\Psi(x, y) = 0$ as in (8.12). The tangent slope at $\alpha/\gamma = 1$ from the left is equal to $-1/\tau$.

As we see, the graph in Figure 2 grows rather fast with $|\alpha/\gamma|$. If one tries to estimate the supremum of $g(t)$ by expressing it as a product of $\widehat{g}(t)$'s, the negative γ_j's should be preferably paired with negative α_j's of similar magnitude, so that the respective quotients α/γ would be close to 1.

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