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Abstract

Reaction-diffusion system approximations to the classical two-phase Stefan problem are considered in the present study. A reaction-diffusion system approximation to the Stefan problem has been proposed by Hilhorst et al. from an ecological point of view, and they have given convergence results for the system. In the present study, a new reaction-diffusion system approximation to the Stefan problem is proposed based on regularization of the enthalpy-temperature constitutive relation. The rates of convergence for each reaction-diffusion system are investigated in order to compare these two systems.

Key words: Stefan problem, reaction-diffusion systems, regularization, convergence rates

1 Introduction

Heat transfer problems involving phase change arise in a number of important physical and industrial contexts. A typical model of such problems, the classical two-phase Stefan problem, which describes the melting or freezing of a solid, in a rather simplified way, by accounting for the heat diffusion in each phase and the exchange of latent heat at the phase interface, is considered.

Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) be a bounded domain with smooth boundary $\partial\Omega$. The domain Ω is divided into liquid and solid phases by unknown interface $\Gamma(t)$ at time $t \in (0,T)$, where T is a positive constant. These phases are denoted by $\Omega_u(t)$ and $\Omega_v(t)$, respectively. Heat flow occurs in both the liquid and solid

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phases:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u & \text{in} \quad \Omega_u := \cup_{0<t<T} \Omega_u(t) \times \{t\}, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v & \text{in} \quad \Omega_v := \cup_{0<t<T} \Omega_v(t) \times \{t\}, \end{cases} \tag{1}$$

where d_1 and d_2 are given diffusion coefficients. The functions $u > 0$ and $-v < 0$ represent temperatures of the liquid and solid, respectively. Here, the melting temperature is taken to be zero. In addition, temperature is zero on the interface:

$$u = v = 0 \quad \text{on} \quad \Gamma := \cup_{0<t<T} \Gamma(t) \times \{t\}. \tag{2}$$

The energy balance between the two phases leads to the following Stefan condition:

$$\lambda V_n = -d_1 \frac{\partial u}{\partial n} - d_2 \frac{\partial v}{\partial n} \quad \text{on} \quad \Gamma, \tag{3}$$

where λ is the latent heat coefficient that represents the latent heat of crystallization per unit mass, n is the unit normal vector on $\Gamma(t)$ oriented from $\Omega_u(t)$ to $\Omega_v(t)$, and V_n is the normal speed of the interface. Equations (1)–(3) constitute the classical formulation of the classical two-phase Stefan problem.

The Stefan problem can be interpreted as an ecological problem. The problem can be stated as a free boundary problem for two competing species that are regionally segregated. Functions u and v are the densities of the competing species, which move by the diffusions described by (1). The regions Ω_u and Ω_v denote the habitats for u and v, respectively. The condition (2) shows that the regional segregation occurs for two competing species. The Stefan condition (3) is not described herein. Although the two species are completely segregated in this problem, the competition between the two species is regarded as being very strong. This leads to a reaction-diffusion system with a sufficiently small parameter ε, as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u - \dfrac{1}{\varepsilon} uv & \text{in} \quad Q := \Omega \times (0, T), \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v - \dfrac{1}{\varepsilon} uv & \text{in} \quad Q. \end{cases} \tag{4}$$

The two species coexist everywhere in Ω, while ε is a positive constant. Then, the following question arises: Is there any relation between (4) and the Stefan problem? Dancer et al. have answered the question [3,4]. In a sense, the system (4) is similar to the Stefan problem without the latent heat. In this case, the Stefan condition (3) is replaced by

$$0 = -d_1 \frac{\partial u}{\partial n} - d_2 \frac{\partial v}{\partial n} \quad \text{on} \quad \Gamma.$$

Because the latent heat coefficient is zero in this case, the following question arises: Are there any reaction diffusion system approximations to the Stefan problem with positive latent heat? To answer this question, Hilhorst et al. considered the following reaction-diffusion system [7,8]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u - \dfrac{1}{\varepsilon}(uv + \lambda pu) & \text{in} \quad Q, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v - \dfrac{1}{\varepsilon}(uv + \lambda vw) & \text{in} \quad Q, \\ \dfrac{\partial w}{\partial t} = \dfrac{1}{\varepsilon}(pu - vw) & \text{in} \quad Q, \\ \dfrac{\partial p}{\partial t} = \dfrac{1}{\varepsilon}(-pu + vw) & \text{in} \quad Q \end{cases}$$

with the initial data satisfying

$$w_0(x) = \begin{cases} 1 & \text{if} \quad u_0(x) > 0, \\ 0 & \text{if} \quad u_0(x) = 0, \end{cases} \qquad p_0(x) = \begin{cases} 1 & \text{if} \quad v_0(x) > 0, \\ 0 & \text{if} \quad v_0(x) = 0 \end{cases}$$

for $x \in \Omega$ and

$$w_0 + p_0 = 1 \quad \text{a.e. in } \Omega. \tag{5}$$

The initial data w_0 and p_0 are the characteristic functions of the initial habitats for u and v, respectively. The relation (5) indicates that the initial distributions of u and v are completely segregated, and the initial interface $\Gamma(0)$ is a hypersurface. This system is interpreted ecologically as follows. Functions u and v represent the densities of two competing species that move by diffusion, and w and p are the characteristic-like functions of the habitats Ω_u and Ω_v, respectively. The constant λ denotes the cost rate when species u attacks the habitat of species v, and vice versa. There are two different types of interactions between u and v in this problem. One is direct competitive interaction (uv), and the other type is conflict type interactions (λpu and λwv), where species u (v) tries to invade the habitat of species v (u), or species v (u) tries to prevent species u (v) from invading their own habitat.

Note that $(w+p)_t = 0$, that is,

$$w(x,t) + p(x,t) = w_0(x) + p_0(x) = 1.$$

Function λw is transformed into $\tilde{w}$, which is denoted as w again, and λp is transformed into $\tilde{p}$. The initial data u_0, v_0 and w_0 are approximated by u_0^ε, v_0^ε and w_0^ε, respectively. In this paper, the homogeneous Neumann boundary conditions are assumed. The problem then yields the following reaction-diffusion system:

$$
(\mathrm{RD})_0 \begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u - \dfrac{1}{\varepsilon}(uv + (\lambda - w)u) & \text{in} \quad Q, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v - \dfrac{1}{\varepsilon}(uv + vw) & \text{in} \quad Q, \\ \dfrac{\partial w}{\partial t} = \qquad \dfrac{1}{\varepsilon}((\lambda - w)u - vw) & \text{in} \quad Q, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0^\varepsilon, \ v(\cdot, 0) = v_0^\varepsilon, \ w(\cdot, 0) = w_0^\varepsilon & \text{in} \quad \Omega, \end{cases}
$$

where ν is the outward normal unit vector to the boundary $\partial\Omega$. Hereafter, the following assumptions are imposed on the initial data:

$$
\begin{cases} u_0^\varepsilon, \ v_0^\varepsilon \in C(\overline{\Omega}), \quad w_0^\varepsilon \in L^\infty(\Omega), \\ 0 \leq u_0^\varepsilon \leq M, \quad 0 \leq v_0^\varepsilon \leq M, \quad 0 \leq w_0^\varepsilon \leq \lambda \quad \text{in} \quad \Omega \end{cases} \tag{6}
$$

for some positive constant M independent of ε. Hilhorst et al. [7,8] studied a number of relations between $(\mathrm{RD})_0$ and the Stefan problem. In stating their results for $(\mathrm{RD})_0$, another formulation of the classical two-phase Stefan problem in terms of enthalpy, which is the sum of sensible and latent heats, should be introduced.

The enthalpy formulation of the classical two-phase Stefan problem can be formulated as follows:

$$
(\mathrm{SP}) \begin{cases} z_t = \Delta d(\phi(z)) & \text{in} \quad Q, \\ \dfrac{\partial d(\phi(z))}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ z(x, 0) = z_0(x) & \text{for} \quad x \in \Omega, \end{cases}
$$

where z_0 is a given function, and the functions d and ϕ are defined as

$$
d(r) = \begin{cases} d_1 r \ \text{if} \ r \geq 0, \\ d_2 r \ \text{if} \ r < 0, \end{cases} \qquad \phi(r) = \begin{cases} r - \lambda & \text{if} \ r > \lambda, \\ 0 & \text{if} \ 0 \leq r \leq \lambda, \\ r & \text{if} \ r < 0 \end{cases}
$$

for $r \in \mathbb{R}$. Functions z and $\phi(z)$ represent, physically, the enthalpy and the temperature, respectively. In the solid phase, $\phi(z) < 0$, and in the liquid phase, $\phi(z) > 0$. These two phases are separated by the zero level set of $\phi(z)$. In this formulation, the interface disappears as an explicit unknown. For example, the interface is recovered as the zero level set of the temperature $\phi(z)$. Next, the functions $u := \phi_1(z) := \phi(z)^+$, $v := \phi_2(z) := \phi(z)^-$ and $w := \phi_3(z) := z - \phi(z)$, are defined, where ϕ^+ indicates the positive part of ϕ and ϕ^- indicates the negative part of ϕ. Then, u, $-v$ and w represent, physically, the temperature of the liquid, the temperature of the solid and the latent heat. Or, for an ecological system, u, v and w represent the densities of the two competing species and the characteristic-like function of the habitat for u, respectively.

This problem should be understood in a weak sense.

Definition 1 *A function $z \in L^\infty(Q_T)$ is a weak solution of* (SP) *with an initial datum $z_0 \in L^\infty(\Omega)$ if it satisfies $d(\phi(z)) \in L^2(0,T;H^1(\Omega))$ and*

$$\int_0^T \langle z, \zeta_t \rangle + \langle z_0, \zeta(\cdot,0) \rangle = \int_0^T \langle \nabla d(\phi(z)), \nabla \zeta \rangle \tag{7}$$

for all functions $\zeta \in \mathcal{K} := \{\zeta \in H^1(Q) \mid \zeta(\cdot,T) = 0\}$. Here, $\langle \cdot,\cdot \rangle$ denotes the inner product of $L^2(\Omega)$.

The unique existence of the weak solution of the problem and a relation between the classical form and the enthalpy form are known (see for example [2,5,7,9,12] and the references therein).

The results for $(\mathrm{RD})_0$ reported by Hilhorst et al. can now be discussed.

Theorem 2 (Hilhorst, Iida, Mimura & Ninomiya [7,8]) *Suppose the initial data satisfy (6). Then, there exists a unique solution $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ of* $(\mathrm{RD})_0$ *in Q.*

Let z be a unique weak solution of (SP) *with an initial datum $z_0 \in L^\infty(\Omega)$. Assume that*

$$u_0^\varepsilon \to \phi_1(z_0),\ v_0^\varepsilon \to \phi_2(z_0),\ w_0^\varepsilon \to \phi_3(z_0) \quad \text{weakly in } L^2(\Omega) \text{ as } \varepsilon \to 0$$

and (6). Then,

$$\begin{aligned} &\left.\begin{aligned} u^\varepsilon &\to \phi_1(z) \\ v^\varepsilon &\to \phi_2(z) \end{aligned}\right\} \text{strongly in } L^2(Q) \text{ and weakly in } L^2(0,T;H^1(\Omega)), \\ &z^\varepsilon \to z \qquad\quad \text{weakly in } L^2(Q) \end{aligned}$$

as $\varepsilon \to 0$. Here, $z^\varepsilon := u^\varepsilon - v^\varepsilon + w^\varepsilon$.

Thus, Hilhorst et al. have given a reaction diffusion system approximation to the Stefan problem from an ecological point of view and have proved the above convergence results.

The diffusion vanishes in (SP), where $z \in (0,\lambda)$. Vanishing diffusion characterizes the presence of a free boundary Γ, and the solution exhibits a lack of regularity across Γ. Thus, the solution of the Stefan problem generally has low regularity properties. Consequently, regularizations of the enthalpy-temperature constitutive relation are sometimes used for the Stefan problem (see for example [1,6,14,15]). Next, the combined use of a reaction diffusion system approximation with a regularization procedure is considered. Then, a new reaction-diffusion system $(\mathrm{RD})_\xi$ is proposed (see the following section) that approximates the Stefan problem. Under certain assumptions, convergence results similar to Theorem 2 can be obtained for $(\mathrm{RD})_\xi$. However, the reaction-diffusion system $(\mathrm{RD})_\xi$ is expected to give a better approximation than $(\mathrm{RD})_0$. Therefore, rates of convergence with respect to ε are investigated for each reaction-diffusion system.

In the next section, a reaction-diffusion system and the theoretical results for the rates of convergence of the system $(\mathrm{RD})_0$ and those of the proposed system are presented. In Sections 3 and 4, the results for the system $(\mathrm{RD})_0$ and the proposed system, respectively, are presented.

2 A reaction-diffusion system and main results

The following reaction-diffusion system with sufficiently small parameter ε and ξ is proposed:

$$(\mathrm{RD})_\xi \begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u - \dfrac{1}{\varepsilon}(uv + (\eta - w)u) & \text{in} \quad Q, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v - \dfrac{1}{\varepsilon}(uv + vw) & \text{in} \quad Q, \\ \dfrac{\partial w}{\partial t} = \xi \Delta w + \dfrac{1}{\varepsilon}((\eta - w)u - vw) & \text{in} \quad Q, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0^{\varepsilon,\xi}, \ v(\cdot,0) = v_0^{\varepsilon,\xi}, \ w(\cdot,0) = w_0^{\varepsilon,\xi} & \text{in} \quad \Omega, \end{cases}$$

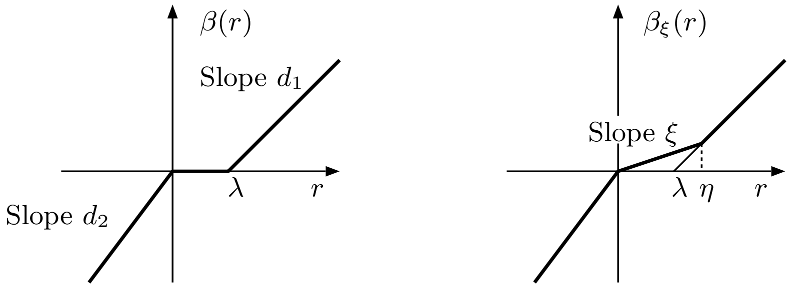
where $\eta = d_1\lambda/(d_1 - \xi)$.

When ξ is zero, this system is obviously reduced to the system $(\mathrm{RD})_0$. That is, the system $(\mathrm{RD})_\xi$ is an extension of $(\mathrm{RD})_0$.

For fixed ξ, the system $(\mathrm{RD})_0$ can be regarded as an approximation to a weak solution of equation

$$\begin{cases} \dfrac{\partial z_\xi}{\partial t} = \Delta \beta_\xi(z_\xi), & \text{in} \quad Q, \\ \dfrac{\partial \beta_\xi(z_\xi)}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ z_\xi(x,0) = z_{\xi 0}(x) & \text{for} \quad x \in \Omega, \end{cases} \tag{8}$$

Fig. 1. Constitutive relation β and regularized relation β_ξ.



where $z_{\xi 0} \in C(\overline{\Omega})$, β_ξ is defined as (see Figure 1)

$$\beta_\xi(r) = \begin{cases} d_1(r-\lambda) & \text{if } r > \eta, \\ \xi r & \text{if } 0 \le r \le \eta, \\ d_2 r & \text{if } r < 0. \end{cases}$$

The weak solution of (8) is Hölder continuous in $\overline{Q}$ (see [10]), although the weak solution of (SP) is generally discontinuous. In this sense, $(\mathrm{RD})_\xi$ can be interpreted as the combination of a reaction diffusion system approximation and a regularization procedure.

The following assumption is made regarding the initial data:

$$\begin{cases} u_0^{\varepsilon,\xi},\ v_0^{\varepsilon,\xi},\ w_0^{\varepsilon,\xi} \in C(\overline{\Omega}), \\ 0 \le u_0^{\varepsilon,\xi} \le M, \quad 0 \le v_0^{\varepsilon,\xi} \le M, \quad 0 \le w_0^{\varepsilon,\xi} \le \eta \quad \text{in} \quad \Omega \end{cases} \tag{9}$$

for some positive constant M independent of ε and ξ.

Then, the existence and uniqueness of the solution of $(\mathrm{RD})_\xi$ follows from Lunardi [11, Proposition 7.3.2].

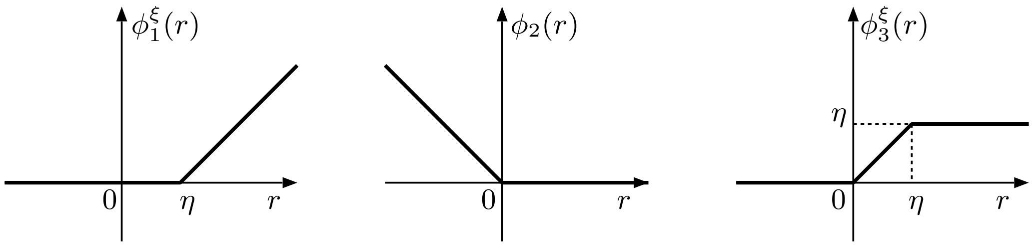
Next, the results for the convergence rates of reaction-diffusion systems $(\mathrm{RD})_\xi$, $\xi \ge 0$ are presented.

Theorem 3 *Let z be a unique weak solution of* (SP) *with an initial datum $z_0 \in L^\infty(\Omega)$, and let $(u^\varepsilon,\ v^\varepsilon, w^\varepsilon)$ be a unique solution of* $(\mathrm{RD})_0$ *with the initial data satisfying assumption (6). Then, there exists a positive constant C independent of ε such that*

$$\|u - u^\varepsilon\|_{L^2(Q)} + \|v - v^\varepsilon\|_{L^2(Q)} + \left\| \int_0^t (\theta - \theta^\varepsilon) \right\|_{L^\infty(0,T;H^1(\Omega))} + \|z - z^\varepsilon\|_{L^\infty(0,T;(H^1(\Omega))^*)} \le C(\varepsilon^{1/2} + \sigma(\varepsilon))^{1/2},$$

where $u := \phi_1(z)$, $v := \phi_2(z)$, $\theta := \beta(z) := d(\phi(z))$, $\theta^\varepsilon := d_1 u^\varepsilon - d_2 v^\varepsilon$, $z^\varepsilon := u^\varepsilon - v^\varepsilon + w^\varepsilon$, $\sigma(\varepsilon) := \|z_0 - z_0^\varepsilon\|^2_{L^2(\Omega)}$ and $z_0^\varepsilon = u_0^\varepsilon - v_0^\varepsilon + w_0^\varepsilon$.

Fig. 2. Functions ϕ_1^ξ, ϕ_2 and ϕ_3^ξ.



The rate of convergence is $\mathcal{O}(\varepsilon^{1/4})$ if σ is of order $\varepsilon^{1/2}$.

Theorem 4 *Let z be a unique weak solution of* (SP) *with an initial datum $z_0 \in L^\infty(\Omega)$, and let $(u_\xi^\varepsilon,\ v_\xi^\varepsilon, w_\xi^\varepsilon)$ be a unique solution of* $(\mathrm{RD})_\xi$ *with initial data satisfying the assumption given in (9). Then, there exists a positive constant C independent of ε and ξ such that*

$$
\begin{aligned}
&\|u-u_\xi^\varepsilon\|_{L^2(Q)} + \|v - v_\xi^\varepsilon\|_{L^2(Q)} + \xi^{1/2}\|w - w_\xi^\varepsilon\|_{L^2(Q)} \\
&\quad + \left\| \int_0^t (\theta - \theta_\xi^\varepsilon) \right\|_{L^\infty(0,T;H^1(\Omega))} + \|z - z_\xi^\varepsilon\|_{L^\infty(0,T;(H^1(\Omega))^*)} \\
&\quad \le C(\xi\,|A_\xi(z)| + \xi^2 + \frac{\varepsilon}{\xi} + \sigma(\varepsilon,\xi))^{1/2},
\end{aligned}
$$

where $u = \phi_1(z)$, $v = \phi_2(z)$, $w = \phi_3(z)$, $\theta = \beta(z)$, $\theta_\xi^\varepsilon := d_1 u_\xi^\varepsilon - d_2 v_\xi^\varepsilon + \xi w_\xi^\varepsilon$, $z_\xi^\varepsilon := u_\xi^\varepsilon - v_\xi^\varepsilon + w_\xi^\varepsilon$, $\sigma(\varepsilon,\xi) := \|z_0 - z_0^{\varepsilon,\xi}\|_{L^2(\Omega)}^2$, $z_0^{\varepsilon,\xi} := u_0^{\varepsilon,\xi} - v_0^{\varepsilon,\xi} + w_0^{\varepsilon,\xi}$ and $A_\xi(z) := \{(x,t) \in Q \mid 0 \le \beta(z(x,t)) \le \eta\xi\}$.

The general rate of convergence is $\mathcal{O}(\varepsilon^{1/4})$ if ξ and σ are chosen to be of order $\varepsilon^{1/2}$. However, if the non-degeneracy property $|A_\xi(z)| \le C\xi$ for some positive constant C is valid, as investigated by Nochetto [15], and if ξ is chosen to be of order $\varepsilon^{1/3}$ and σ is of order $\varepsilon^{2/3}$, then the rate becomes $\mathcal{O}(\varepsilon^{1/3})$. In addition, the corresponding latent heat w_ξ^ε and the corresponding enthalpy z_ξ^ε converge strongly in $L^2(Q)$. The rate of convergence is $\mathcal{O}(\varepsilon^{1/6})$. Thus, better results are obtained for the reaction-diffusion system $(\mathrm{RD})_\xi$ than for $(\mathrm{RD})_0$.

The relationship between $(\mathrm{RD})_\xi$ and (8) for fixed $\xi > 0$ can be obtained as follows. Let functions ϕ_i^ξ $(i = 1,3)$ be defined as follows (see Figure 2):

$$
\phi_1^\xi(r) = \begin{cases} r - \eta & \text{if } \ r \ge \eta, \\ 0 & \text{if } \ r < \eta, \end{cases} \qquad \phi_3^\xi(r) = \begin{cases} \eta & \text{if } \ r \ge \eta, \\ r & \text{if } \ 0 < r < \eta, \\ 0 & \text{if } \ r \le 0 \end{cases}
$$

for $r \in \mathbb{R}$. Note that the relation $\beta_\xi = d_1\phi_1^\xi - d_2\phi_2 + \xi\phi_3^\xi$ holds.

Corollary 5 *Let z_ξ be a unique weak solution of (8) with an initial datum*

$z_0 \in L^\infty(\Omega)$, *and let* $(u^\varepsilon_\xi,\ v^\varepsilon_\xi, w^\varepsilon_\xi)$ *be a unique solution of* $(\mathrm{RD})_\xi$ *with initial data satisfying the assumption give in (9). Then, there exists a positive constant* C *independent of* ε *and* ξ *such that*

$$\begin{aligned}\|u_\xi - u^\varepsilon_\xi\|_{L^2(Q)} &+ \|v_\xi - v^\varepsilon_\xi\|_{L^2(Q)} + \|w_\xi - w^\varepsilon_\xi\|_{L^2(Q)} \\ &+ \left\| \int_0^t (\theta_\xi - \theta^\varepsilon_\xi) \right\|_{L^\infty(0,T;H^1(\Omega))} + \|z_\xi - z^\varepsilon_\xi\|_{L^\infty(0,T;(H^1(\Omega))^*)} \\ &\le C(\varepsilon + \sigma(\varepsilon))^{1/2},\end{aligned}$$

where $u_\xi = \phi^\xi_1(z_\xi)$, $v_\xi = \phi_2(z_\xi)$, $w_\xi = \phi^\xi_3(z_\xi)$, $\theta_\xi = \beta_\xi(z_\xi)$, $\theta^\varepsilon_\xi := d_1 u^\varepsilon_\xi - d_2 v^\varepsilon_\xi + \xi w^\varepsilon_\xi$, $z^\varepsilon_\xi := u^\varepsilon_\xi - v^\varepsilon_\xi + w^\varepsilon_\xi$, $\sigma(\varepsilon) := \|z_0 - z^{\varepsilon,\xi}_0\|^2_{L^2(\Omega)}$, *and* $z^{\varepsilon,\xi}_0 := u^{\varepsilon,\xi}_0 - v^{\varepsilon,\xi}_0 + w^{\varepsilon,\xi}_0$.

The rate of convergence is $\mathcal{O}(\varepsilon^{1/2})$ if σ is of order ε. Corollary 5 also implies that the solution of (4) converges to the weak solution of the Stefan problem without the latent heat and that its rate of convergence in $L^2(Q)$ is $\mathcal{O}(\varepsilon^{1/2})$.

3 A proof of Theorem 3

In this section, a proof of Theorem 3 is given. To this end, a number of lemmas are presented for the functions u^ε, v^ε and w^ε.

Lemma 6 *The functions* u^ε, v^ε *and* w^ε *are uniformly bounded with respect to* ε *in* $L^\infty(Q)$. *More precisely, the functions* u^ε, v^ε *and* w^ε *satisfy*

$$0 \le u^\varepsilon \le M, \quad 0 \le v^\varepsilon \le M, \quad 0 \le w^\varepsilon \le \lambda \quad in \quad Q.$$

Proof. The assertion follows from the maximum principle. □

Lemma 7 *There exists a positive constant* C *independent of* ε *such that*

$$\int_0^T \{\langle u^\varepsilon, v^\varepsilon\rangle + \langle \lambda - w^\varepsilon, u^\varepsilon\rangle + \langle v^\varepsilon, w^\varepsilon\rangle\} \le C\varepsilon.$$

Proof. Integration of the equation for u^ε in Q yields

$$\iint_Q \frac{1}{\varepsilon}(u^\varepsilon v^\varepsilon + (\lambda - w^\varepsilon)u^\varepsilon) = \int_\Omega (u^\varepsilon_0 - u^\varepsilon(\cdot, T)) \le M|\Omega|,$$

which implies the first and second terms of the desired estimate. Similarly, the third term can be shown by integrating the equation for v^ε. □

Lemma 8 *There exists a positive constant C independent of ε such that*

$$d_1\|\nabla u^\varepsilon\|_{L^2(Q)} + d_2\|\nabla v^\varepsilon\|_{L^2(Q)} \le C.$$

Proof. Multiplying the equation for u^ε by u^ε and integrating by parts on Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^\varepsilon)^2 + d_1\int_\Omega |\nabla u^\varepsilon|^2 + \int_\Omega \frac{1}{\varepsilon}((u^\varepsilon)^2 v^\varepsilon + (\lambda - w^\varepsilon)(u^\varepsilon)^2) = 0.$$

Integrating on $(0,T)$ and using Lemmas 6 and 7, we can obtain the first term. The second term can be similarly obtained. □

Lemmas 6 and 8 imply that the functions u^ε and v^ε are uniformly bounded with respect to ε in $L^2(0,T;H^1(\Omega))$.

From $(\mathrm{RD})_0$, we deduce that

$$\frac{\partial z^\varepsilon}{\partial t} = \Delta\theta^\varepsilon.$$

Multiplying this expression by a test function $\zeta \in \mathcal{K}$ and integrating by parts, we obtain the identity

$$\int_0^T \langle z^\varepsilon, \zeta_t\rangle + \langle z_0^\varepsilon, \zeta(\cdot,0)\rangle = \int_0^T \langle \nabla\theta^\varepsilon, \nabla\zeta\rangle. \tag{10}$$

After subtraction from (10) to (7), we obtain the relation

$$-\int_0^T \langle e_z, \zeta_t\rangle - \langle e_{z_0}, \zeta(\cdot,0)\rangle + \int_0^T \langle \nabla e_\theta, \nabla\zeta\rangle = 0 \tag{11}$$

for all $\zeta \in \mathcal{K}$. Here, errors are defined as

$$e_z := z - z^\varepsilon, \quad e_\theta := \theta - \theta^\varepsilon, \quad e_{z_0} := z_0 - z_0^\varepsilon.$$

The concept behind the proof is to use a stable test function in (11).

Proof of Theorem 3. For fixed $t_0 \in (0,T]$, the following test function is proposed:

$$\zeta(x,t) = \begin{cases} \displaystyle\int_t^{t_0} e_\theta(x,s)ds & \text{if} \quad 0 \le t < t_0, \\ 0 & \text{if} \quad t_0 \le t \le T. \end{cases}$$

The first term of the left-hand side of (11) yields

$$-\int_0^T \langle e_z, \zeta_t\rangle = \int_0^{t_0} \langle e_z, e_\theta\rangle.$$

The term $\langle e_z, e_\theta\rangle$ is then estimated. For a.e. $(x,t) \in Q$,

$$\begin{aligned}\langle e_z, e_\theta\rangle &= \langle e_z, \theta - \beta(z^\varepsilon)\rangle + \langle e_z, \beta(z^\varepsilon) - \theta^\varepsilon\rangle \\ &= d_1\langle e_z, u - \phi_1(z^\varepsilon)\rangle - d_2\langle e_z, v - \phi_2(z^\varepsilon)\rangle + \langle e_z, \beta(z^\varepsilon) - \theta^\varepsilon\rangle.\end{aligned}$$

The first term is estimated by means of the definition of ϕ_1; namely

$$\begin{aligned}d_1\langle z - z^\varepsilon, \phi_1(z) - \phi_1(z^\varepsilon)\rangle &\geq d_1\|\phi_1(z) - \phi_1(z^\varepsilon)\|^2_{L^2(\Omega)} \\ &\geq d_1\left(\frac{1}{2}\|e_u\|^2_{L^2(\Omega)} - \|\phi_1(z^\varepsilon) - u^\varepsilon\|^2_{L^2(\Omega)}\right),\end{aligned}$$

where $e_u := u - u^\varepsilon$. In the same way, we obtain

$$-d_2\langle z - z^\varepsilon, \phi_2(z) - \phi_2(z^\varepsilon)\rangle \geq d_2\left(\frac{1}{2}\|e_v\|^2_{L^2(\Omega)} - \|\phi_2(z^\varepsilon) - v^\varepsilon\|^2_{L^2(\Omega)}\right).$$

Here, $e_v := v - v^\varepsilon$. Using the Cauchy-Schwarz inequality, we obtain

$$\langle e_z, \beta(z^\varepsilon) - \theta^\varepsilon\rangle \leq \|e_z\|_{L^2(\Omega)}\|\beta(z^\varepsilon) - \theta^\varepsilon\|_{L^2(\Omega)}.$$

The inequalities for $u, v \geq 0$ and $w \in [0, \lambda]$ are easily obtained:

$$\begin{cases} |u - \phi_1(u - v + w)|^2 \leq uv + (\lambda - w)u, \\ |v - \phi_2(u - v + w)|^2 \leq uv + vw, \\ |w - \phi_3(u - v + w)|^2 \leq vw + (\lambda - w)u. \end{cases} \tag{12}$$

It follows from Lemma 7 and (12) that

$$\|\phi_1(z^\varepsilon) - u^\varepsilon\|^2_{L^2(Q)} + \|\phi_2(z^\varepsilon) - v^\varepsilon\|^2_{L^2(Q)} + \|\beta(z^\varepsilon) - \theta^\varepsilon\|^2_{L^2(Q)} \leq C\varepsilon,$$

where C is a positive constant that is independent of ε.

From the elementary relation

$$2ab \leq \alpha a^2 + \frac{1}{\alpha}b^2 \tag{13}$$

for $a, b \in \mathbb{R}$ and $\alpha > 0$, we have

$$\begin{aligned}\langle e_{z_0}, \zeta(\cdot, 0)\rangle &= \int_0^{t_0} \langle e_{z_0}, e_\theta\rangle = d_1\int_0^{t_0} \langle e_{z_0}, e_u\rangle - d_2\int_0^{t_0} \langle e_{z_0}, e_v\rangle \\ &\leq (d_1 + d_2)T\|e_{z_0}\|^2_{L^2(\Omega)} + \frac{d_1}{4}\int_0^{t_0} \|e_u\|^2_{L^2(\Omega)} + \frac{d_2}{4}\int_0^{t_0} \|e_v\|^2_{L^2(\Omega)}.\end{aligned}$$

The third term of the left-hand side of (11) can be estimated easily as follows:

$$\int_0^T \langle \nabla e_\theta, \nabla \zeta \rangle = \int_0^{t_0} \langle \nabla e_\theta, \int_t^{t_0} \nabla e_\theta \rangle = \frac{1}{2} \left\| \int_0^{t_0} \nabla e_\theta \right\|_{L^2(\Omega)}^2 .$$

Collecting all of the previous bounds yields

$$\begin{aligned} d_1 \|e_u\|_{L^2(0,t_0;L^2(\Omega))}^2 + d_2 \|e_v\|_{L^2(0,t_0;L^2(\Omega))}^2 + \left\| \int_0^{t_0} \nabla e_\theta \right\|_{L^2(\Omega)}^2 \\ \leq C(\varepsilon^{1/2} + \|e_{z_0}\|_{L^2(\Omega)}^2) \end{aligned} \tag{14}$$

for all $t_0 \in (0, T)$ and some positive constant C independent of ε.

Let φ be a function belonging to $H^1(\Omega)$, and let t_0 be an arbitrary point in $(0, T)$. The function $\chi_\delta = \chi_\delta(t)$ is defined as

$$\chi_\delta(t) = \begin{cases} 1 & t \in [0, t_0 - \delta], \\ (t_0 + \delta - t)/2\delta & t \in (t_0 - \delta, t_0 + \delta), \\ 0 & t \in [t_0 + \delta, T]. \end{cases}$$

The function χ_δ converges in $L^2(0, T)$ to the characteristic function of $(0, t_0)$. Taking $\zeta(x, t) = \phi(x)\chi_\delta(t)$ in (11), we obtain

$$\frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} \langle e_z, \varphi \rangle - \langle e_{z_0}, \varphi \rangle + \int_0^T \chi_\delta \langle \nabla e_\theta, \nabla \varphi \rangle = 0.$$

Using the Lebesgue differentiation theorem and the Cauchy-Schwarz inequality, for a.e. $t_0 \in (0, T)$, we have

$$\begin{aligned} |\langle e_z(t_0), \varphi \rangle| &\leq |\langle e_{z_0}, \varphi \rangle| + \left| \int_0^{t_0} \langle \nabla e_\theta, \nabla \varphi \rangle \right| \\ &\leq \left(\|e_{z_0}\|_{L^2(\Omega)} + \left\| \int_0^{t_0} \nabla e_\theta \right\|_{L^2(\Omega)} \right) \|\varphi\|_{H^1(\Omega)}. \end{aligned}$$

From the last inequality and (14), the desired estimate is obtained. □

4 A proof of Theorem 4

In this section, a proof of Theorem 4 is presented. The strategy is the same as that for the proof of Theorem 3. The following lemmas hold for the functions u_ξ^ε, v_ξ^ε and w_ξ^ε in the same way as in Lemmas 6–8.

Lemma 9 *The following relations hold:*

$$0 \le u_\xi^\varepsilon \le M, \quad 0 \le v_\xi^\varepsilon \le M, \quad 0 \le w_\xi^\varepsilon \le \eta \quad in \quad Q.$$

Lemma 10 *There exists a positive constant C independent of ε and ξ such that*

$$\int_0^T \{\langle u_\xi^\varepsilon, v_\xi^\varepsilon\rangle + \langle \eta - w_\xi^\varepsilon, u_\xi^\varepsilon\rangle + \langle v_\xi^\varepsilon, w_\xi^\varepsilon\rangle\} \le C\varepsilon.$$

Lemma 11 *There exists a positive constant C independent of ε and ξ such that*

$$d_1\|\nabla u_\xi^\varepsilon\|_{L^2(Q)} + d_2\|\nabla v_\xi^\varepsilon\|_{L^2(Q)} + \xi\|\nabla w_\xi^\varepsilon\|_{L^2(Q)} \le C.$$

From $(\mathrm{RD})_\xi$, we have

$$\frac{\partial z_\xi^\varepsilon}{\partial t} = \Delta\theta_\xi^\varepsilon.$$

Multiplying this expression by a test function $\zeta \in \mathcal{K}$ and integrating by parts gives the following identity:

$$\int_0^T \langle z_\xi^\varepsilon, \zeta_t\rangle + \langle z_0^\varepsilon, \zeta(\cdot,0)\rangle = \int_0^T \langle \nabla\theta_\xi^\varepsilon, \nabla\zeta\rangle. \tag{15}$$

After subtraction from (15) to (7), the following relation is obtained:

$$-\int_0^T \langle e_z, \zeta_t\rangle - \langle e_{z_0}, \zeta(\cdot,0)\rangle + \int_0^T \langle \nabla e_\theta, \nabla\zeta\rangle = 0 \tag{16}$$

for all $\zeta \in \mathcal{K}$. Here, the errors are defined as follows:

$$e_z := z - z_\xi^\varepsilon, \quad e_\theta := \theta - \theta_\xi^\varepsilon, \quad e_{z_0} := z_0 - z_0^\varepsilon.$$

Now, we are ready to prove Theorem 4.

Proof of Theorem 4. The strategy is the same as that in the proof of Theorem 3. To this end, we take

$$\zeta(x,t) = \begin{cases} \int_t^{t_0} e_\theta(x,s)ds & \text{if } \ 0 \le t < t_0, \\ 0 & \text{if } \ t_0 \le t \le T \end{cases}$$

for fixed $t_0 \in (0,T]$ in (16).

The term $\langle e_z, e_\theta\rangle$ is then estimated. For a.e. $(x,t) \in Q$,

$$\langle e_z, e_\theta\rangle = \langle e_z, \theta - \beta_\xi(z)\rangle + \langle e_z, \beta_\xi(z) - \beta_\xi(z_\xi^\varepsilon)\rangle + \langle e_z, \beta_\xi(z_\xi^\varepsilon) - \theta_\xi^\varepsilon\rangle. \tag{17}$$

Using the elementary relation (13), the first term is estimated as follows:

$$\begin{aligned}\langle e_z, \theta - \beta_\xi(z)\rangle =&\langle e_u - e_v + e_w, \beta(z) - \beta_\xi(z)\rangle \\ \geq& -\frac{d_1}{8}\|e_u\|^2_{L^2(\Omega)} - \frac{d_2}{8}\|e_v\|^2_{L^2(\Omega)} - \frac{\xi}{8}\|e_w\|^2_{L^2(\Omega)} \\ &-\left(\frac{2}{d_1} + \frac{2}{d_2} + \frac{2}{\xi}\right)\|\beta(z) - \beta_\xi(z)\|^2_{L^2(\Omega)},\end{aligned}$$

where $e_u := u - u^\varepsilon_\xi$, $e_v := v - v^\varepsilon_\xi$ and $e_w := w - w^\varepsilon_\xi$.

The last term of (17) can be estimated in a similar manner:

$$\begin{aligned}\langle e_z, \beta_\xi(z^\varepsilon_\xi) - \theta^\varepsilon_\xi\rangle \geq& -\frac{d_1}{16}\|e_u\|^2_{L^2(\Omega)} - \frac{d_2}{16}\|e_v\|^2_{L^2(\Omega)} - \frac{\xi}{16}\|e_w\|^2_{L^2(\Omega)} \\ &-\left(\frac{4}{d_1} + \frac{4}{d_2} + \frac{4}{\xi}\right)\|\beta_\xi(z^\varepsilon_\xi) - \theta^\varepsilon_\xi\|^2_{L^2(\Omega)}.\end{aligned}$$

It follows from the property of ϕ^ξ_1 and the triangle inequality that

$$\begin{aligned}\langle e_z, &\phi^\xi_1(z) - \phi^\xi_1(z^\varepsilon_\xi)\rangle \geq \|\phi^\xi_1(z) - \phi^\xi_1(z^\varepsilon_\xi)\|^2_{L^2(\Omega)} \\ &\geq \frac{1}{2}\|\phi_1(z) - \phi^\xi_1(z^\varepsilon_\xi)\|^2_{L^2(\Omega)} - \|\phi_1(z) - \phi^\xi_1(z)\|^2_{L^2(\Omega)} \\ &\geq \frac{1}{4}\|e_u\|^2_{L^2(\Omega)} - \frac{1}{2}\|\phi^\xi_1(z^\varepsilon_\xi) - u^\varepsilon_\xi\|^2_{L^2(\Omega)} - \|\phi_1(z) - \phi^\xi_1(z)\|^2_{L^2(\Omega)}.\end{aligned}$$

Similarly, we obtain

$$-\langle e_z, \phi_2(z) - \phi_2(z^\varepsilon_\xi)\rangle \geq \frac{1}{2}\|e_v\|^2_{L^2(\Omega)} - \|\phi_2(z^\varepsilon_\xi) - v^\varepsilon_\xi\|^2_{L^2(\Omega)}$$

and

$$\begin{aligned}\langle e_w, &\phi^\xi_3(z) - \phi^\xi_3(z^\varepsilon_\xi)\rangle \\ &\geq \frac{1}{4}\|e_w\|^2_{L^2(\Omega)} - \frac{1}{2}\|\phi^\xi_3(z^\varepsilon_\xi) - w^\varepsilon_\xi\|^2_{L^2(\Omega)} - \|\phi_3(z) - \phi^\xi_3(z)\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore, the second term of (17) can be estimated as follows:

$$\begin{aligned}\langle e_z, &\beta_\xi(z) - \beta_\xi(z^\varepsilon_\xi)\rangle \geq \frac{d_1}{4}\|e_u\|^2_{L^2(\Omega)} + \frac{d_2}{2}\|e_v\|^2_{L^2(\Omega)} + \frac{\xi}{4}\|e_w\|^2_{L^2(\Omega)} \\ &- \frac{d_1}{2}\|\phi^\xi_1(z^\varepsilon_\xi) - u^\varepsilon_\xi\|^2_{L^2(\Omega)} - d_2\|\phi_2(z^\varepsilon_\xi) - v^\varepsilon_\xi\|^2_{L^2(\Omega)} - \frac{\xi}{2}\|\phi^\xi_3(z^\varepsilon_\xi) - w^\varepsilon_\xi\|^2_{L^2(\Omega)} \\ &- d_1\|\phi_1(z) - \phi^\xi_1(z)\|^2_{L^2(\Omega)} - \xi\|\phi_3(z) - \phi^\xi_3(z)\|^2_{L^2(\Omega)}.\end{aligned}$$

It follows from the property of ϕ_i^ξ $(i = 1, 3)$ and the property $\beta(z) = \beta_\xi(z)$ if $(x, t) \notin A_\xi(z)$ that

$$\|\phi_1(z) - \phi_1^\xi(z)\|_{L^2(\Omega)}^2 + \|\phi_3(z) - \phi_3^\xi(z)\|_{L^2(\Omega)}^2 \leq |\Omega|\xi^2,$$
$$\|\beta(z) - \beta_\xi(z)\|_{L^2(\Omega)}^2 \leq \xi^2 \left| A_\xi(z) \right|.$$

Using Lemma 10 and the inequalities for $u, v \geq 0$ and $w \in [0, \eta]$:

$$\begin{cases} |u - \phi_1^\xi(u - v + w)|^2 \leq uv + (\eta - w)u, \\ |v - \phi_2(u - v + w)|^2 \leq uv + vw, \\ |w - \phi_3^\xi(u - v + w)|^2 \leq vw + (\eta - w)u \end{cases}$$

we obtain

$$\|\phi_1^\xi(z_\xi^\varepsilon) - u_\xi^\varepsilon\|_{L^2(Q)}^2 + \|\phi_2(z_\xi^\varepsilon) - v_\xi^\varepsilon\|_{L^2(Q)}^2 + \|\phi_3^\xi(z_\xi^\varepsilon) - w_\xi^\varepsilon\|_{L^2(Q)}^2 \\ + \|\beta_\xi(z_\xi^\varepsilon) - \theta_\xi^\varepsilon\|_{L^2(Q)}^2 \leq C\varepsilon$$

for some positive constant C independent of ε and ξ.

The other terms of (16) can be estimated in the same manner as in Theorem 3. Collecting these bounds yields

$$d_1\|e_u\|_{L^2(0,t_0;L^2(\Omega))}^2 + d_2\|e_v\|_{L^2(0,t_0;L^2(\Omega))}^2 + \xi\|e_w\|_{L^2(0,t_0;L^2(\Omega))}^2 + \left\| \int_0^{t_0} \nabla e_\theta \right\|_{L^2(\Omega)}^2 \\ \leq C \left(\xi \left| A_\xi(z) \right| + \xi^2 + \frac{\varepsilon}{\xi} + \sigma(\varepsilon, \xi) \right)$$

for all $t_0 \in (0, T)$.

The assertion can be proven analogously as Theorem 3. □

5 Concluding remarks

A new reaction-diffusion system approximation to the classical Stefan problem with sufficiently small parameters ε and ξ was proposed from a regularization point of view. The rates of convergence with respect to ε and ξ have been investigated theoretically. Numerical experiments indicate that the present results for the convergence rates can be improved [13]. The optimal rate of convergence, including the convergence of the interface, should be investigated.

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