AN EXPLICIT FORMULA OF THE SHAPLEY VALUE FOR A COOPERATIVE GAME INDUCED FROM THE CONJUGATE POINT

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AN EXPLICIT FORMULA OF THE SHAPLEY VALUE FOR A COOPERATIVE GAME INDUCED FROM THE CONJUGATE POINT [1]

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Abstract. The conjugate point was introduced by Jacobi to derive a sufficient optimality condition for a variational problem. Recently, the conjugate point was defined for an extremal problem in $\mathbb{R}^n$. The key of the conjugate point is cooperation of variables. Namely, when there exists a conjugate point for a stationary solution $x \in \mathbb{R}^n$, we can improve the solution by suitably changing some of the variables. We call such a set of variables a strict conjugate set. This idea leads us to a cooperative game, which we call a conjugate-set game. The Shapley value is an important value in game theory. It evaluates player's contribution in the cooperative game. However, its calculation is usually very hard. The purpose of this paper is to give an explicit formula of the Shapley value for the conjugate-set game induced from the shortest path problem on an ellipsoid.

1. Introduction

The conjugate point was originally introduced to guarantee local optimality of a stationary solution $x(t)$ for the simplest problem in the calculus of variations

$$\begin{array}{ll}\text{Minimize} & \displaystyle\int_0^T f(t, x(t), \dot{x}(t))dt \\ \text{subject to} & x(0) = A,\ x(T) = B\end{array}$$

where A and B are given points, see e.g. Gelfand and Fomin [2]. Recently, the conjugate point was defined for an extremal problem with n variables

$$(P_0) \qquad \text{Minimize} \quad f(x), \quad x \in \mathbb{R}^n,$$

see Kawasaki [3][4].

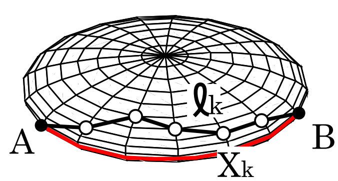

Figure 1. the shortest polygonal path problem

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One can see the typical example of the conjugate point for (P_0) in the shortest polygonal path problem on an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1. \tag{1}$$

This problem is to find the shortest polygonal path

$$A = X_0, X_1, \ldots, X_n, X_{n+1} = B$$

joining two given points $A := (a, 0, 0)$ and $B := (a \cos T, a \sin T, 0)$, where each X_k can move on a longitude ℓ_i that is equally located between A and B. Since each X_k has one-dimensional freedom, this problem is formulated as (P_0), and the equatorial polygonal path is a stationary solution for (P_0).

Further, whether the stationary solution is minimal or not depends on T. It is minimal when $T < a\pi/c$, and not minimal when $T > a\pi/c$ and n is sufficiently large. In the latter case, we call the first number k satisfying $(k+1)T/(n+1) > a\pi/c$ a strict conjugate point, which matches the classical conjugate point, see [4].

When there exists a strict conjugate point $k \leq n$, the Hessian matrix $A := f''(x)$ is not positive semidefinite. So, according to Sylvester's criterion, A has a negative leading principal minor. Then we can improve the stationary solution $x = (x_1, \ldots, x_n)$ by suitably changing some variables $\{x_i\}_{i \in I}$. We call such a subset $\{x_i\}_{i \in I}$ (or I) a strict conjugate set. Namely, when $A_I := (a_{ij})_{i,j \in I}$ has a negative principal minor, we call $\{x_i\}_{i \in I}$ (or I) a strict conjugate set, see Kawasaki [6][7]. In this paper, we consider a cooperative game based on strict conjugate sets, and we present an explicit formula of the Shapley value for this game.

2. Definitions and Notations

In this section, we first define conjugate-set game induced from the shortest path problem on the ellipsoid (1). Next, we introduce tools $I(i; S)$ and $Ker(i; S)$ to compute the Shapley value.

Definition 1. *Let $N = \{1, \ldots, n\}$ be the players set and $1 \leq k \leq n$ a natural number. We call a subset*

$$[j, j+k-1] := \{j, j+1, \ldots, j+k-1\}$$

of N an interval of length k. For any subset S of N, we define a characteristic function $v(S)$ as the maximum number of disjoint intervals of length k contained in S. We call this cooperative game a conjugate-set game induced from the shortest path problem on an ellipsoid and denote it by $G(n, k)$.

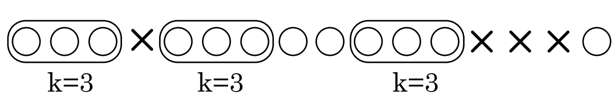

Figure 2. When $k = 3$ and S consists of circles, $v(S) = 3$.

Throughout this paper, we put

$$n = pk + r \ (0 \leq r \leq k-1). \tag{2}$$

We denote by $\phi_i(n,k)$ or ϕ_i the Shapley value of $G(n,k)$. That is,

$$\phi_i(n,k) = \sum_{i\in S\subset N} \frac{(s-1)!(n-s)!}{n!}\{v(S) - v(S-\{i\})\}, \tag{3}$$

where $s := \#S$. The following expression is well-known, see e.g. Aumann et al [1].

$$\phi_i(n,k) = \sum_{\pi\in\Pi} \frac{1}{n!}\left\{v(S_{\pi,i}) - v(S_{\pi,i} - \{i\})\right\}, \tag{4}$$

where Π denotes the set of all permutation on N and $S_{\pi,i}$ denotes the union of $\{i\}$ and the set of all players that precedes player i with respect to π, that is,

$$S_{\pi,i} = \{\pi(j) \mid 1 \le j \le \pi^{-1}(i)\}. \tag{5}$$

We note that $S_{\pi,i}$ plays an important role in this paper.

The following proposition is evident from symmetry of the characteristic function.

Proposition 1. *For any $1 \le i \le n$, it holds that $\phi_i = \phi_{n-i+1}$.*

Definition 2. *Any element of $W_S := \{i \,\big|\, v(S) - v(S-\{i\}) = 1\}$ is called a pivot of S.*

Then the Shapley value (4) is simply written as

$$\phi_i = \frac{1}{n!}\#\{\pi \,\big|\, i \in W_{S_{\pi,i}}\}. \tag{6}$$

So it suffices to test whether $i \in S$ is a pivot of S or not in order to compute ϕ_i. For this aim, we introduce two subsets of S, say $I(i;S)$ and $Ker(i;S)$.

Definition 3. *For any $i \in N$ and $S \subset N$ including i, we denote by $I(i;S) \subset S$ the maximum interval including i. We denote by $Ker(i;S)$ the remainder of $I(i;S)$ after removing intervals of length k from both sides of i as much as possible with keeping i.*

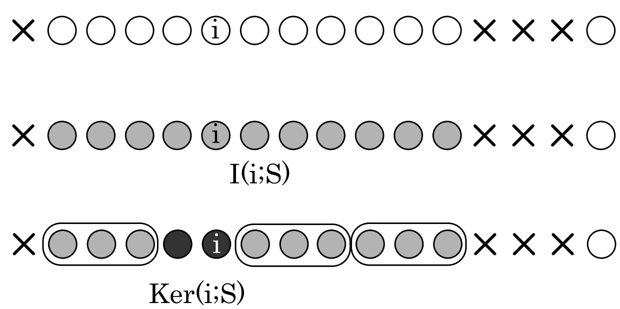

FIGURE 3. When $k = 3$ and S consists of circles, $I(i;S)$ consists of gray circles and $Ker(i;S)$ consists of black cirlces.

Theorem 1. *The following conditions are equivalent to each others.*

(i) $i \in W_S$,
(ii) $i \in W_{I(i;S)}$,
(iii) $i \in W_{Ker(i;S)}$,
(iv) $\#Ker(i;S) \ge k$.

Proof. (i) $\Leftrightarrow$ (ii): Obvious. (ii) $\Leftrightarrow$ (iii): It is enough to consider the case that $[j_1, j_2] := I(i;S) \neq Ker(i;S)$. Because of symmetry, we may assume that $j_1 + k < i$. Then, since $v([j_1, j_2]) = v([j_1 + k, j_2]) + 1$ and $v([j_1, j_2] - \{i\}) = v([j_1 + k, j_2] - \{i\}) + 1$, we have

$$\begin{aligned} i \in W_{I(i;S)} \;&\Leftrightarrow\; v([j_1, j_2]) - v([j_1, j_2] - \{i\}) = 1 \\ &\Leftrightarrow\; v([j_1 + k, j_2]) - v([j_1 + k, j_2] - \{i\}) = 1 \\ &\Leftrightarrow\; i \in W_{I(i;S) - [j_1, j_1 + k - 1]}. \end{aligned}$$

Repeating this procedure, we finally get $Ker(i;S)$ as the remainder and see the equivalence of (ii) and (iii). (iii) $\Leftrightarrow$ (iv): Since we can not remove any interval of length k from $Ker(i;S)$ without deleting i, this assertion is clear. □

3. The Shapley value of player 1

In this section, we compute the Shapley value ϕ_1. It follows from Theorem 1 that

$$1 \in W_S \;\Leftrightarrow\; \#Ker(1;S) \geq k \;\Leftrightarrow\; Ker(1;S) = [1, k]. \tag{7}$$

Since $I(1;S)$ is obtained by adding disjoint intervals of length k to $Ker(1;S)$, we get from (7) that

$$\left\{ I(1;S) \,\middle|\, 1 \in W_S \right\} = \left\{ [1, mk] \,\middle|\, 1 \leq m \leq p \right\}. \tag{8}$$

Lemma 1. *Let $1 \leq m \leq p$. Then π satisfies $I(1; S_{\pi,1}) = [1, mk]$ if and only if*

$$\pi^{-1}(j) < \pi^{-1}(1) \quad \forall j \in [2, mk] \tag{9}$$

and either (a) $\pi^{-1}(mk+1) > \pi^{-1}(1)$ or (b) $mk = n$ holds.

Proof. Necessity: Since 1 joins $I(1; S_{\pi,1}) = [1, mk]$ last, (9) is clear. If $mk < n$ and $mk + 1$ joins $S_{\pi,1}$ before 1, then the interval $I(1; S_{\pi,1})$ contains $[1, mk+1]$. Sufficiency is evident. □

Theorem 2.

$$\phi_1 = \begin{cases} \displaystyle\sum_{m=1}^{p-1} \frac{1}{mk(mk+1)} + \frac{1}{pk} & \text{if } r = 0, \\ \displaystyle\sum_{m=1}^{p} \frac{1}{mk(mk+1)} & \text{if } r \neq 0, \end{cases} \tag{10}$$

where $n = pk + r$ $(0 \leq r \leq k - 1)$.

Proof. By (6), it suffices to compute $\#\{\pi \mid 1 \in W_{S_{\pi,1}}\}$. Combining (7) and (8), it equals $\#\{\pi \mid 1 \leq \exists m \leq p,\ I(1; S_{\pi,1}) = [1, mk]\}$. Further, it equals

$$\#\{\pi \mid \pi \text{ satisfies (9) and } (a)\} + \#\{\pi \mid \pi \text{ satisfies (9) and } (b)\}. \tag{11}$$

For each m, the first term of (11) is given by

$$\binom{n}{mk+1} (n - mk - 1)!(mk - 1)! = \frac{n!}{mk(mk+1)}. \tag{12}$$

Indeed, such a permutation π satisfies

$$\pi^{-1}(j) < \pi^{-1}(1) < \pi^{-1}(mk+1) \quad \forall j \in [2, mk]. \tag{13}$$

There are ${}_nC_{mk+1}$ ways to choose $P := \pi^{-1}([1, mk+1]) \subset N$. Since $\pi^{-1}([2, mk])$ can freely share the first $mk-1$ places of P, and since the complement of P can be freely shared by other $n-mk-1$ numbers, we get (12). On the other hand, since case (b) occurs only when $m = p$ and $n = pk$ (so that $r = 0$), we similarly see that the second term of (11) is given by

$$(n-1)!. \tag{14}$$

In the case of $r = 0$, ϕ_1 is equal to the total sum of $(12)/n!$ $(m = 1, \dots, p-1)$ and $(14)/n!$. Otherwise, ϕ_1 is equal to the total sum of $(12)/n!$ $(m = 1, \dots, p)$. □

4. A recurrence relation of $\{\phi_i\}$: Case 1

Starting with ϕ_1, we compute ϕ_2, ϕ_3, and so on. For this aim, we compute the difference between ϕ_i and ϕ_{i+1}. Because of symmetry of the game, it suffices to consider $1 \le i \le \left[\frac{n-1}{2}\right]$, where $[\cdot]$ denotes Gauss's symbol. We will deal with this problem in three cases.

Case 1: $n-k+1 \le i \le k-1$ (This is the case that $k \le \frac{n}{2}+1$),
Case 2: $1 \le i \le \min\{n-k,\ k-1\}$,
Case 3: $k \le i \le \left[\frac{n-1}{2}\right]$.

Before dealing with Case 1, we present a lemma that is applicable to any case.

Lemma 2.

$$\phi_{i+1} = \phi_i + \delta_i^+ - \delta_i^-, \tag{15}$$

where

$$\delta_i^+ := \#\{\pi \big| i \notin W_{S_{\pi,i+1}},\ i+1 \in W_{S_{\pi,i+1}}\}/n! \tag{16}$$

and

$$\delta_i^- := \#\{\pi \big| i \in W_{S_{\pi,i}},\ i+1 \notin W_{S_{\pi,i}}\}/n!. \tag{17}$$

Proof. Since

$$\begin{aligned} & n!(\phi_{i+1} - \phi_i) \\ = \ & \#\left\{\pi \big| i+1 \in W_{S_{\pi,i+1}}\right\} - \#\left\{\pi \big|\ i \in W_{S_{\pi,i}}\right\} \\ = \ & \#\left\{\pi \big|\ i, i+1 \in W_{S_{\pi,i+1}}\right\} + \#\left\{\pi \big|\ i \notin W_{S_{\pi,i+1}},\ i+1 \in W_{S_{\pi,i+1}}\right\} \\ & -\#\{\pi \big| i, i+1 \in W_{S_{\pi,i}}\} - \#\{\pi \big|\ i \in W_{S_{\pi,i}},\ i+1 \notin W_{S_{\pi,i}}\}. \end{aligned}$$

it suffices to prove

$$\#\left\{\pi \big| i, i+1 \in W_{S_{\pi,i+1}}\right\} = \#\left\{\pi \big| i, i+1 \in W_{S_{\pi,i}}\right\}. \tag{18}$$

We define a bijection $f : \Pi \to \Pi$ by $f(\pi) := (i, i+1) \circ \pi$, where $(i, i+1)$ is a transposition. Then, for any $\pi \in \Pi$ such that $i, i+1 \in W_{S_{\pi,i}}$, due to definition of W_S,

$$v(S_{\pi,i}) - v(S_{\pi,i} - \{i\}) = v(S_{\pi,i}) - v(S_{\pi,i} - \{i+1\}) = 1. \tag{19}$$

Since $S_{\pi,i} = S_{f(\pi),i+1}$, (19) implies that $i, i+1 \in W_{S_{f(\pi),i+1}}$. That is, $f(\left\{\pi \big| i, i+1 \in W_{S_{\pi,i}}\right\}) \subset \left\{\pi \big| i, i+1 \in W_{S_{\pi,i+1}}\right\}$. Since f is an injection, we have

$$\#\left\{\pi \big| i, i+1 \in W_{S_{\pi,i}}\right\} \le \#\left\{\pi \big| i, i+1 \in W_{S_{\pi,i+1}}\right\}. \tag{20}$$

The converse inequality is similarly obtained. □

Let us now consider Case 1.

Theorem 3. *For any i such that $n-k+1 \le i \le k-1$, it holds that*

$$\delta_i^+ = \delta_i^- = 0. \tag{21}$$

Therefore, $\phi_{n-k+1} = \phi_{n-k+2} = \cdots = \phi_{k-1}$.

Proof. By Theorem 1, $i \in W_{S_{\pi,i}}$ if and only if $\#Ker(i; S_{\pi,i}) \ge k$. For such a π, since $n-k+1 \le i \le k-1$ and since we can not remove any intervals of length k from $Ker(i; S_{\pi,i})$ without removing i, we have

$$\left\{ Ker(i; S_{\pi,i}) \middle| \ i \in W_{S_{\pi,i}} \right\} = \left\{ \begin{matrix} [1,k], & [1,k+1], & \dots & [1,n], \\ & [2,k+1], & \dots & [2,n], \\ & & \ddots & \vdots \\ & & & [n-k+1,n] \end{matrix} \right\}. \tag{22}$$

We similarly see that $\left\{ Ker(i+1; S_{\pi,i+1}) \middle| \ i+1 \in W_{S_{\pi,i+1}} \right\}$ coincides with set (22). So $\{\pi | \ i \in W_{S_{\pi,i}}, \ i+1 \notin W_{S_{\pi,i}}\}$ is empty. (Remark that not $i+1 \notin W_{S_{\pi,i+1}}$ but $i+1 \notin W_{S_{\pi,i}}$.) Indeed, if π is an element of this set, then $Ker(i; S_{\pi,i})$ is one of the intervals in (22) and i is its element. Since $i+1 \le k$, $i+1$ belongs to the interval, which implies that $i+1$ is also an element of $S_{\pi,i}$. Since the length of the interval is greater than or equal to k, we see from Theorem 1 and (22) that $i+1 \in W_{S_{\pi,i}}$. Therefore $\delta_i^- = 0$. Similarly, we have $\delta_i^+ = 0$. □

5. A recurrence relation of $\{\phi_i\}$: Case 2

In this section, we consider the case that $1 \le i \le \min\{n-k, \ k-1\}$.

Since $i \le k$ and $k+i-1 \le n$, we get

$$\{ Ker(i; S_{\pi,i}) | \ i \in W_{S_{\pi,i}} \} = \left\{ \begin{matrix} [1,k], & [1,k+1], & \dots & [1,k+i-1], \\ & [2,k+1], & \dots & [2,k+i-1], \\ & & \ddots & \vdots \\ & & & [i,k+i-1] \end{matrix} \right\} \tag{23}$$

as well as (22), where the difference between (22) and (23) is caused from $k+i-1 \le n$. Since $i+1 \le k$ and $k+i \le n$, we similarly see that $\{Ker(i+1; S_{\pi,i+1}) | \ i+1 \in W_{S_{\pi,i+1}}\}$ equals

$$\left\{ \begin{matrix} [1,k], & [1,k+1], & \dots & [1,k+i-1], & [1,k+i], \\ & [2,k+1], & \dots & [2,k+i-1], & \vdots \\ & & \ddots & \vdots & \vdots \\ & & & [i,k+i-1], & [i,k+i], \\ & & & & [i+1,k+i] \end{matrix} \right\}. \tag{24}$$

Comparing (23) and (24), we get $\delta_i^- = 0$ as well as Theorem 3. On the other hand,

$$\begin{aligned} & \{ Ker(i+1; S_{\pi,i+1}) | \ i \notin W_{S_{\pi,i+1}}, \ i+1 \in W_{S_{\pi,i+1}} \} \\ = \ & \{ [j_1, k+i] | \ 1 \le j_1 \le i+1 \}. \end{aligned} \tag{25}$$

Indeed, since i belongs to any interval in (24) except $[i+1, k+i]$, Theorem 1 asserts that $i \notin W_{S_{\pi,i+1}}$ implies $\#Ker(i; S_{\pi,i+1}) < k$. The intervals in (24) that satisfy this condition are those in the last column of (24). So we get (25).

Since $I(i+1; S_{\pi,i+1})$ is an interval obtained by adding disjoint intervals of length k to $Ker(i+1; S_{\pi,i+1})$, we get from (25) that

$$\begin{aligned} & \{I(i+1; S_{\pi,i+1}) \big| \ i \notin W_{S_{\pi,i+1}}, \ i+1 \in W_{S_{\pi,i+1}}\} \\ = & \ \{[j_1, mk+i] \big| \ 1 \leq j_1 \leq i+1, \ m \geq 1, \ mk+i \leq n\}. \end{aligned} \tag{26}$$

Lemma 3. *Let $m \geq 1$ satisfy $mk+i \leq n$. Then there exists $1 \leq j_1 \leq i+1$ such that $I(i+1; S_{\pi,i+1}) = [j_1, mk+i]$ if and only if*

$$\pi^{-1}(j) < \pi^{-1}(i+1) \quad \forall j \in [i+2, mk+i] \tag{27}$$

and either (c) $\pi^{-1}(mk+i+1) > \pi^{-1}(i+1)$ or (d) $mk+i = n$ holds.

Proof. Necessity: Since $i+1$ joins $S_{\pi,i+1}$ last, $I(i+1; S_{\pi,i+1}) = [j_1, mk+i]$ implies that (27) and $mk+i+1$ dose not joint $S_{\pi,i+1}$ before $i+1$ if $mk+i < n$. Conversely, it follows from (c) or (d) that any number greater than $mk+i$ does not join $S_{\pi,i+1}$ before $i+1$. Hence $mk+i$ is the maximum number of $I(i+1; S_{\pi,i+1})$. Since $I(i+1; S_{\pi,i+1})$ is an interval, it has a form of $[j_1, mk+i]$ for some $1 \leq j_1 \leq i+1$. □

Theorem 4. *In the case of $1 \leq i \leq \min\{n-k, \ k-1\}$, it holds that*

$$\delta_i^+ = \begin{cases} \displaystyle\sum_{m=1}^{p} \frac{1}{mk(mk+1)} & 1 \leq i \leq r-1, \\ \displaystyle\sum_{m=1}^{p-1} \frac{1}{mk(mk+1)} + \frac{1}{pk} & i = r, \\ \displaystyle\sum_{m=1}^{p-1} \frac{1}{mk(mk+1)} & r+1 \leq i \leq k-1, \end{cases} \tag{28}$$

$$\delta_i^- = 0, \tag{29}$$

where $n = pk + r$ $(0 \leq r \leq k-1)$.

Proof. Assume that π satisfies that $i \notin W_{S_{\pi,i+1}}$ and $i+1 \in W_{S_{\pi,i+1}}$. Then it is easily seen from (26) and Lemma 3 that (27) and either (c) or (d) hold. The number of π's satisfying (27) and (c) is given by

$$\binom{n}{mk+1}(mk-1)!(n-mk-1)! = \frac{n!}{mk(mk+1)}. \tag{30}$$

Indeed, such a permutation π satisfies

$$\pi^{-1}(j) < \pi^{-1}(i+1) < \pi^{-1}(mk+i+1) \quad \forall j \in [i+2, mk+i]. \tag{31}$$

There are ${}_nC_{mk+1}$ ways to choose $P := \pi^{-1}([i+1, mk+i+1])$. Since $\pi^{-1}([i+2, mk+i])$ can freely share the first $mk-1$ places of P, and since the complement of P can be freely shared by other $n-mk-1$ numbers, we get (30).

Since case (d) occurs only when $m = p$, we similarly see that the number of π's satisfying (27) and (d) is given by

$$\binom{n}{pk}(pk-1)!(n-pk)! = \frac{n!}{pk}. \tag{32}$$

In the cases of $0 \le i < r$, since $mk + i < n$ for any $1 \le m \le p$, δ_i^+ equals the total sum of $(30)/n!$ $(m = 1, \ldots, p)$. In the cases of $i = r$, since $mk + i$ equals n only when $m = p$, δ_i^+ equals the total sum of $(30)/n!$ $(m = 1, \ldots, p-1)$ and $(32)/n!$. In the cases of $i > r$, since $mk + i < n$ for any $1 \le m \le p-1$, δ_i^+ equals the total sum of $(30)/n!$ $(m = 1, \ldots, p-1)$. □

6. A recurrence relation of $\{\phi_i\}$: Case 3

In this section, we consider the case of $k \le i \le [\frac{n-1}{2}]$. Then i is expressed as

$$i = qk + s \tag{33}$$

for some $q \ge 1$ and $0 \le s \le k-1$. Since $i + k \le n$, we get from Theorem 1 that $\{Ker(i; S_{\pi,i}) | i \in W_{S_{\pi,i}}\}$ is given by

$$\left\{\begin{matrix} [i-k+1, i] & [i-k+1, i+1] & \ldots & [i-k+1, i+k-1] \\ & [i-k+2, i+1] & \ldots & [i-k+2, i+k-1] \\ & & \ddots & \vdots \\ & & & [i, i+k-1] \end{matrix}\right\} \tag{34}$$

and $\{Ker(i+1; S_{\pi,i+1}) | \; i+1 \in W_{S_{\pi,i+1}}\}$ is given by

$$\left\{\begin{matrix} [i-k+2, i+1] & \ldots & [i-k+2, i+k-1] & [i-k+2, i+k] \\ & \ddots & \vdots & \vdots \\ & & [i, i+k-1] & [i, i+k] \\ & & & [i+1, i+k] \end{matrix}\right\}. \tag{35}$$

So, as well as (25), we have

$$\{Ker(i+1; S_{\pi,i+1}) | \; i \notin W_{S_{\pi,i+1}}, \; i+1 \in W_{S_{\pi,i+1}}\} = \{[j_1, i+k] | \; i-k+2 \le j_1 \le i+1\} \tag{36}$$

and

$$\{Ker(i; S_{\pi,i}) | \; i \in W_{S_{\pi,i}}, \; i+1 \notin W_{S_{\pi,i}}\} = \{[i-k+1, j_2] | \; i \le j_2 \le i+k-1\}. \tag{37}$$

A well as (26), we get from (36) and (37) that

$$\{I(i+1; S_{\pi,i+1}) | \; i \notin W_{S_{\pi,i+1}}, \; i+1 \in W_{S_{\pi,i+1}}\} = \{[j_1, mk+i] | \; 1 \le j_1 \le i+1, \; m \ge 1, \; mk+i \le n\} \tag{38}$$

and

$$\{I(i;S_{\pi,i})\big|\ i\in W_{S_{\pi,i}},\ i+1\notin W_{S_{\pi,i}}\}$$
$$=\{[i-mk+1,j_2]\big|\ i\le j_2\le n,\ m\ge 1,\ 1\le i-mk+1\}. \quad (39)$$

Lemma 4. *Let $m\ge 1$ satisfy $mk+i\le n$. Then there exists $i\le j_2\le n$ such that $I(i;S_{\pi,i})=[i-mk+1,j_2]$ if and only if*

$$\pi^{-1}(j)<\pi^{-1}(i)\quad \forall j\in[i-mk+1,i-1] \quad (40)$$

and either (e) $\pi^{-1}(i-mk)>\pi^{-1}(i)$ or (f) $i-mk+1=1$ holds.

Proof. Almost same with Lemma 3. The only difference is that we make $I(i;S_{\pi,i})$ by attaching intervals of length k to $Ker(i;S_{\pi,i})$ from not the right side of i but the left side of i. □

Theorem 5. *In the case of $k\le i\le[\frac{n-1}{2}]$, it holds that*

$$\delta_i^+=\begin{cases}\displaystyle\sum_{m=1}^{p-q}\frac{1}{mk(mk+1)} & 0\le s\le r-1,\\ \displaystyle\sum_{m=1}^{p-q-1}\frac{1}{mk(mk+1)}+\frac{1}{(p-q)k} & s=r,\\ \displaystyle\sum_{m=1}^{p-q-1}\frac{1}{mk(mk+1)} & r+1\le s\le k-1,\end{cases} \quad (41)$$

and

$$\delta_i^-=\begin{cases}\displaystyle\sum_{m=1}^{q-1}\frac{1}{mk(mk+1)}+\frac{1}{qk} & s=0,\\ \displaystyle\sum_{m=1}^{q}\frac{1}{mk(mk+1)} & s\ne 0,\end{cases} \quad (42)$$

where q and j are defined by (33).

Proof. One can easily prove (41) as well as (28). The only difference is that p is replaced by $p-q$. The difference comes from that i is expressed as $i=qk+s$. So, the condition $m\ge 1$ and $mk+i\le n$ in (38) is equivalent to $m\ge 1$ and $(m+q)k+s\le n$. When $s>r$, the latter implies that $1\le m\le p-q-1$. When $s\le r$, it implies that $1\le m\le p-q$. In particular, when $s=r$, $m=p-q$ corresponds to (d) in Lemma 3.

We use (39) and Lemma 4 to prove (42). By Lemma 4, π satisfies $i\in W_{S_{\pi,i}}$ and $i+1\notin W_{S_{\pi,i}}$ if and only if π satisfies (40) and either (e) or (f). The condition $m\ge 1$ and $1\le i-mk+1$ in (39) is equivalent to $1\le m\le q$. In particular, when $s=0$, $m=q$ corresponds to (f). So we get (42). □

Following is the graphs of the Shapley values of 28 players.

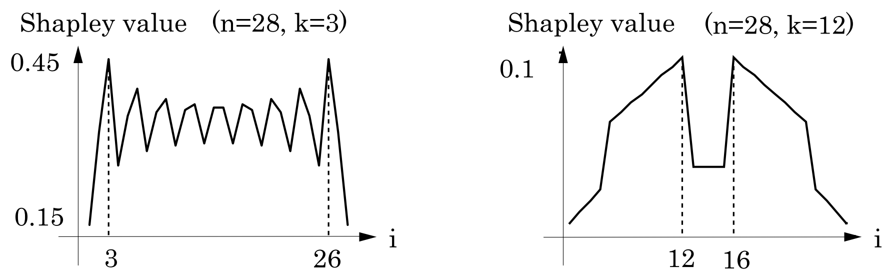

FIGURE 4. Shapley values of 28 players. Left: $k = 3$, Right: $k = 12$

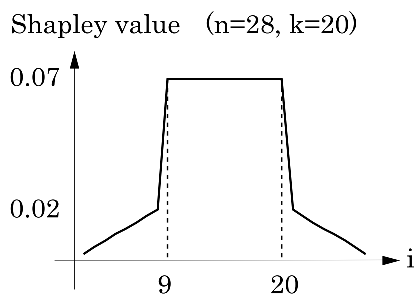

FIGURE 5. Shapley values of 28 players when $k = 20$.

7. The maximal values of the Shapley value

In Figure 4, the maximum value of the Shapley value is attained by $i = k$ and $i = n + 1 - k$. The aim of this section is to show that this is true for any $2 \le k \le n/2$. Otherwise, the graph of the Shapley value has a shape in Figure 5. We first cosider the case of $k \ge n/2 + 1$.

Theorem 6. *When $k \ge n/2 + 1$, it holds that*

$$\phi_1 < \phi_2 < \cdots < \phi_{n-k+1} = \cdots = \phi_k > \phi_{k+1} > \cdots > \phi_n.$$

Proof. The assertion is a direct consequence of Theorems 3 and 4. □

Next, we consider the case of $2 \le k \le n/2$. We list up the maximal values of the Shapley value. By virtue of symmetry of the Shapley value, it suffices to consider $i \le [(n-1)/2]$, so that $i + 1 \le [(n+1)/2]$. Since n and i are expressed as $n = pk + r$ and $i = qk + s$, $i \le [(n-1)/2]$ implies that

$$(p - 2q)k + r - 2s - 1 \ge 0. \tag{43}$$

Since $r \le k$, we see from (43) that $p \ge 2q$.

Theorem 7. *The maximal points of $\{\phi_i\}_i$ on the interval $[1, [(n+1)/2]]$ are $\{k, 2k, \dots\}$.*

Proof. Figure 6 below shows the outline of the proof. Step 1. It follows from Theorem 4 that $\phi_1 < \phi_2 < \cdots < \phi_k$. Step 2. We show

$$\phi_{qk+1} - \phi_{qk} < 0 \quad (q = 1, 2, \dots). \tag{44}$$

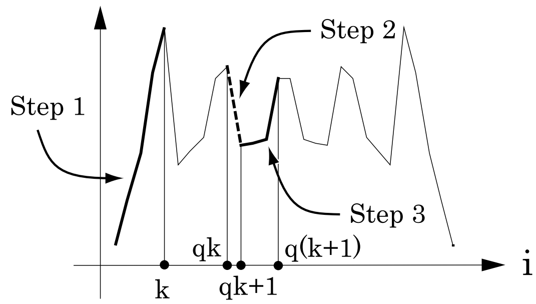

FIGURE 6. $\phi_{qk} > \phi_{qk+1} \le \phi_{qk+2} \le \cdots \le \phi_{q(k+1)}$.

We get from Theorem 5 that, for any $q = 1, 2, \ldots,$

$$\phi_{qk+1} - \phi_{qk} = \begin{cases} \displaystyle\sum_{m=q}^{p-q} \frac{1}{mk(mk+1)} - \frac{1}{qk} & r > 0 \\ \displaystyle\sum_{m=q}^{p-q-1} \frac{1}{mk(mk+1)} + \frac{1}{(p-q)k} - \frac{1}{qk} & r = 0. \end{cases} \tag{45}$$

Indeed, since $s = 0$ for $i = qk$, the first two cases in (41) and the first case in (42) are applicable, and we easily get (45). Further, we get (44) from (45). Indeed, for any $r > 0$ and $p = 2q$, we have

$$\phi_{qk+1} - \phi_{qk} = \frac{1}{qk}\left(\frac{1}{qk+1} - 1\right) < 0. \tag{46}$$

For any $r > 0$ and $p > 2q$, we have

$$\begin{aligned} \phi_{qk+1} - \phi_{qk} &= \sum_{m=q}^{p-q-1} \frac{1}{mk(mk+1)} + \frac{1}{(p-q)k\{(p-q)k+1\}} - \frac{1}{qk} \\ &< \sum_{m=q}^{p-q-1} \frac{1}{mkmk} + \frac{1}{(p-q)k} - \frac{1}{qk} \\ &= \frac{1}{k}\left(\frac{1}{k}\sum_{m=q}^{p-q-1} \frac{1}{m^2} + \frac{1}{p-q} - \frac{1}{q}\right). \end{aligned}$$

For $r = 0$, we have from (45) that

$$\phi_{qk+1} - \phi_{qk} < \frac{1}{k}\left(\frac{1}{k}\sum_{m=q}^{p-q-1} \frac{1}{m^2} + \frac{1}{p-q} - \frac{1}{q}\right).$$

So, letting $f(p) := \dfrac{1}{k}\displaystyle\sum_{m=q}^{p-q-1} \frac{1}{m^2} + \frac{1}{p-q} - \frac{1}{q}$, we see that

$$0 \ge f(2q+1) > f(2q+2) > \cdots > f(p). \tag{47}$$

In fact,

$$f(2q+1) = \frac{1}{kq^2} + \frac{1}{q+1} - \frac{1}{q} = \frac{1-q(k-1)}{kq^2(q+1)} \leq 0$$

and

$$f(2q+j+1) - f(2q+j) = \frac{1}{k(q+j)^2} + \frac{1}{q+j+1} - \frac{1}{q+j} = \frac{1-(q+j)(k-1)}{k(q+j)^2(q+j+1)} < 0.$$

Hence f is nonincreasing, so that (44) has been proved. Step 3. We show $\phi_{qk+s} \leq \phi_{qk+s+1}$ for any $s \geq 1$ and $q \neq 0$. (i) When $1 \leq s \leq r-1$, we see from the first case of (41) and the second case of (42) that

$$\phi_{qk+s+1} - \phi_{qk+s} = \sum_{m=q+1}^{p-q} \frac{1}{mk(mk+1)} \geq 0, \tag{48}$$

where the summation equals 0 when $p-q < q+1$. (ii) When $s = r$, it follows from the second case of (41) and the second case of (42) that

$$\phi_{qk+s+1} - \phi_{qk+s} = \sum_{m=q+1}^{p-q-1} \frac{1}{mk(mk+1)} + \frac{1}{(p-q)k} \geq 0. \tag{49}$$

(iii) When $s > r$, it follows from the third case of (41) and the second case of (42) that

$$\phi_{qk+s+1} - \phi_{qk+s} = \sum_{m=q+1}^{p-q-1} \frac{1}{mk(mk+1)} \geq 0. \tag{50}$$

Therefore $\phi_{qk+s+1} \geq \phi_{qk+s}$. □

Theorem 8. *When $2 \leq k \leq n/2$, the maximum points of $\{\phi_i\}$ are $i = k$ and $i = n-k+1$.*

Proof. By Theorem 7, the maximum value is attained by either $i = qk$. In the case of $r = 0$, it follows from the second case of (45) and (50) that

$$\phi_{(q+1)k} - \phi_{qk} = \sum_{s=0}^{k-1}(\phi_{qk+s+1} - \phi_{qk+s}) = k\sum_{m=q+1}^{p-q-1} \frac{1}{mk(mk+1)} + \frac{1}{(p-q)k} - \frac{1}{qk+1}. \tag{51}$$

Here, remark that the summations in (45) and (48) are taken from $m = q+1$ to not $p-q-1$ but $p-q$. So, in the case of $r > 0$, it follows from the first case of (45), (48), (49), and (50) that

$$\begin{aligned}\phi_{(q+1)k} - \phi_{qk} &= \sum_{s=0}^{k-1}(\phi_{qk+s+1} - \phi_{qk+s}) \\ &= k\sum_{m=q+1}^{p-q-1} \frac{1}{mk(mk+1)} + \frac{r}{(p-q)k\{(p-q)k+1\}} + \frac{1}{(p-q)k} - \frac{1}{qk+1}.\end{aligned} \tag{52}$$

Since (52) reduces to (51) when $r = 0$, (52) is correct for $r = 0$.

For $1 \le q$, $2q+2 \le p$, $0 \le r \le k-1$, and $2 \le k$, let $f_1(p,q,r,k) := \phi_{(q+1)k} - \phi_{qk}$. Then, it is obvious that $f_1(p,q,r,k) \le f_1(p,q,k-1,k) =: f_2(p,q,k)$ for any $0 \le r \le k-1$. Further,

$$f_2(p+1,q,k) - f_2(p,q,k) = \frac{-(k-1)}{(p-q+1)\{(p-q+1)k+1\}\{(p-q)k+1\}} < 0.$$

Hence $f_2(p,q,k)$ is the strict decreasing w.r.t. p. So, let $f_3(q,k) := f_2(2q+2,q,k)$. Then

$$\begin{aligned} f_3(q,k) &= \frac{1}{(q+1)\{(q+1)k+1\}} + \frac{k-1}{(q+2)k\{(q+2)k+1\}} + \frac{1}{(q+2)k} - \frac{1}{qk+1} \\ &= \frac{-2q^2k^2 + 2qk^2 - 5qk^2 + 3qk - 4k^2 + 2q + k + 3}{(q+1)(q+2)(qk+1)(qk+k+1)(qk+2k+1)}. \end{aligned}$$

Since the numerator of the right-hand side is expressed as

$$-(k-1)\left\{2k\left(q + \frac{5k+2}{4k}\right)^2 - \frac{(5k+2)^2}{8k} + 4k + 3\right\},$$

the maximum value of $f_3(q,k)$ on $q \ge 1$ is attained by $q = 1$. Then the numerator of $f_3(1,k)$ is $-(k-1)(11k+5) < 0$. So, $f_1(p,q,k,r) \le f_2(p,q,k) \le f_3(q,k) < 0$ as desired. □

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