A duality theorem based on triangles separating three convex sets

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Abstract. Separation theorems play the central role in duality theory. Recently, the author proposed a duality theorem for a three-phase partition problem in [4]. It is based on triangles separating three convex sets. However, the dual problem in [4] includes a variable of the primal problem. The aim of this paper is to remove the variable from the dual problem.

Key words. Duality theorem, Separation theorem, Convex set, Partition problem, Triangle

1. Introduction

The three-phase partition problem is to divide a given domain $\Omega \subset \mathbb{R}^2$ into three subdomains with a triple junction having least interfacial area (Fig.1). Sternberg and Zeimer [6] established the existence of local minimizers to the problem. Ikota and Yanagida [1] investigated not only stability but also instability for stationary curves in terms of the curvature of the boundary $\partial \Omega$.

They formulated the problem as a variational problem. However, since the shortest curve joining two points is the line segment, it can be formulated as an extremal problems in $\mathbb{R}^n$. From this point of view, the author discussed stability and instability of the three-phase partition problem and studied its game-theoretic aspect in [2][3]. Further, he gave a duality theorem for the following problem in [4].

\[(P_0) \quad \text{Minimize} \quad f(X_0, \ldots, X_3) := \sum_{i=1}^{3} ||X_i - X_0|| \]

subject to $X_0 \in \Omega, \ X_i \in C_i \ (i = 1, 2, 3),$

where $|| \cdot ||$ denotes the Euclidean norm and $C_i \ (i = 1, 2, 3)$ are closed convex sets with non-empty interior in $\mathbb{R}^2$ such that $\Omega := \text{cl}(\bigcap_{i=1}^{3} C_i)$ is non-empty (Fig. 2).

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Figure 2. Primal problem ($P_0$)

Let ($X_0, \ldots, X_3$) be a non-degenerate minimum solution for ($P_0$), that is, $X_0$ does not coincide with any $X_i$ ($i = 1, 2, 3$). Assume that $\Omega$ is bounded. Then, for the following dual problem ($D_0^*$), we have $\min(P_0) = \max(D_0^*)$ ([4]).

$$
\begin{align*}
(D_0^*) & \quad \text{Maximize} \quad \text{the smallest height of a triangle } \Delta \\
& \quad \text{subject to} \quad X_0 \in \Delta \subset \Omega.
\end{align*}
$$

The main aim of this paper is to remove $X_0$ from the dual problem ($D_0^*$). For this aim, we slightly change the primal problem as follows.

$$
\begin{align*}
(P) & \quad \text{Minimize} \quad \sum_{i=1}^{3} ||X_i - X_0|| \\
& \quad \text{subject to} \quad X_0 \in \mathbb{R}^2, \ X_i \in C_i \ (i = 1, 2, 3).
\end{align*}
$$

The only difference between ($P$) and ($P_0$) is the domain of $X_0$. When we emphasize the domain $\Omega$, we denote ($P$) by ($P_\Omega$).

This paper is organized as follows. In Section 2, we briefly review classical duality theorems and introduce the concept of separation of three convex sets by a triangle. In Section 3, we characterize minimum solutions for ($P$). In Section 4, we define the dual problem ($D$) and show duality.

We close this section with our notations. For any closed convex sets $C_1$ and $C_2$, we define $d(C_1, C_2) := \min\{||X_1 - X_2|| \mid X_i \in C_i \ (i = 1, 2)\}$. We denote by $N(X_i; C_i)$ the normal cone of $C_i$ at $X_i$. When $X_i \neq X_0$, we denote by $e_i$ the unit vector $(X_i - X_0)/||X_i - X_0||$.

2. Separation by a triangle

In this section, we first review classical duality theorems in brief. Next, we introduce separation of three convex sets by a triangle.

One of the simplest duality theorems is the following. Let $C_1$ be a non-empty convex set in $\mathbb{R}^2$ and $A \notin C_1$ a point. Then the primal problem is

$$
\begin{align*}
(P_1) & \quad \text{Minimize} \quad ||X_1 - A|| \\
& \quad \text{subject to} \quad X_1 \in C_1.
\end{align*}
$$

Its dual problem ($D_1$) is to maximize the distance from $A$ to a hyperplane $H$ that separates $A$ and $C_1$. We can rephrase it as maximizing the width of a strip that separates $A$ and $C_1$ (Fig. 3), where a strip stands for the area sandwiched between two parallel lines.
If we replace $A$ with a convex set $C_2$ such that $C_1 \cap C_2 = \emptyset$, then the primal problem is as follows.

\[(P_2) \quad \text{Minimize } ||X_1 - X_2|| \]
\[\text{subject to } X_i \in C_i \ (i = 1, 2).\]

Its dual problem $(D_2)$ is to minimize the width of a strip that separates $C_1$ and $C_2$ (Fig. 4).

If we take the epigraph of a convex function $f$ and the hypograph of a concave function $g$ as $C_1$ and $C_2$, respectively, and measure the width of the strip in the vertical direction, duality between $(P_2)$ and $(D_2)$ reduces to Fenchel’s duality, see e.g. [5, Theorem 31.1].

Therefore, classical dual problems can be described in terms of strips or hyperplanes separating two convex sets. In this paper, we need a concept of triangles separating three convex sets in order to deal with $(P)$.

**Definition 2.1.** ([4]) Let $C_i \ (i = 1, 2, 3)$ be convex sets in $\mathbb{R}^2$ such that $\Omega = \text{cl}(\cap_{i=1}^3 C_i^c)$ is not empty, and let $\Delta \subset \Omega$ a triangle. Then, we say that $\Delta$ separates $\{C_i\}_{i=1}^3$ if there are three closed half spaces $\{H_i^-\}_{i=1}^3$ such that $C_i \subset H_i^-$ for every $i$ and $\Delta = \cap_{i=1}^3 H_i^+$, where $H_i^+$ denotes the closed half space opposite to $H_i^-$ (Fig. 5).

The following lemma is useful in this paper.

**Lemma 2.1.** Let $(X_0, \ldots, X_3)$ be a feasible solution for $(P)$ and let a triangle $\Delta$ separate $\{C_i\}_{i=1}^3$. Then $\min(P_\Delta) \leq \sum_{i=1}^3 ||X_i - X_0||$.

**Proof.** Since $X_i \in C_i \subset H_i^-$, we have $\sum_{i=1}^3 ||X_i - X_0|| \geq \sum_{i=1}^3 d(X_0, H_i^-) \geq \min(P_\Delta)$. \(\square\)
3. Characterization of minimum solutions

In this section, we first give a characterization theorem of minimum solutions for (P). Next, we consider the special case that $C_i$'s are closed half spaces.

Although $(P_0)$ is not a convex program, the present primal problem $(P)$ is a convex program. So optimal solutions are characterized by the first-order optimality condition below. Since the proof is almost same with [4, Theorem 3.1], we omit the proof.

**Theorem 3.1.** Let $(X_0, \ldots, X_3)$ be a non-degenerate feasible solution for $(P)$. Then it is a minimum solution if and only if it satisfies Young’s law

$$\angle X_i X_0 X_j = 120^\circ \text{ for any } i \neq j$$

(3.1)

and the transversality condition

$$X_0 - X_i \in N(X_i; C_i) \quad (i = 1, 2, 3).$$

(3.2)

Next, we consider the special case that $\Omega$ is a triangle determined by closed half spaces $C_i \ (i = 1, 2, 3)$. Then it is clear that the minimum is attained by $(X_0, \ldots, X_3)$ satisfying $X_0 \in \Omega$. So, $(P)$ reduces to $(P_0)$. Hence, Corollary 1 in [4] is available to $(P)$.

**Proposition 3.1.** When $\Omega$ is a triangle, $\min(P)$ equals to the smallest height of $\Omega$.  

4. Duality theorem

The dual problem $(D)$ is defined as follows.

\[
\text{Maximize} \quad \text{the smallest height of a triangle } \Delta \\
\text{subject to} \quad \text{there exists a triangle } \Delta' \text{ such that } \Delta \subset \Delta' \subset \Omega, \Delta' \text{ separates } \{C_i\}_{i=1}^3.
\]
When \( \Omega \) is bounded, it has a simplified form \((D^*)\) defined in Theorem 4.2 below.

**Theorem 4.1.** Let \((X_0, X_1, X_2, X_3)\) and \(\Delta\) be feasible solutions for \((P)\) and \((D)\), respectively, then it holds that \(\min(P\Delta) \leq \sum_{i=1}^{3} ||X_i - X_0||\), so that

\[
\sup(D) \leq \inf(P). \tag{4.1}
\]

Furthermore, if \((P)\) has a non-degenerate minimum, then

\[
\min(P_0) = \min(P) = \max(D). \tag{4.2}
\]

**Proof.** Let \(\Delta\) be a feasible solution for \((D)\). Then there exists a triangle \(\Delta'\) such that \(\Delta \subset \Delta' \subset \Omega\) and \(\Delta'\) separates \(\{C_i\}_{i=1}^{3}\). Let \((X_0, X_1, X_2, X_3)\) be a feasible solution for \((P)\). Then, combining Lemma 2.1 and Proposition 3.1, we have

\[
\min(P_\Delta) \leq \min(P_{\Delta'}) \leq \sum_{i=1}^{3} ||X_i - X_0||, \tag{4.3}
\]

which implies the weak duality (4.1). By Theorem 3.1, the non-degenerate minimum solution forms a regular triangle \(\Delta^*\) such that

\[
\min(P_{\Delta}) = \text{the height of } \Delta^* = \min(P_{\Delta^*}). \tag{4.4}
\]

It follows from definition of the normal cone that \(\Delta^*\) itself separates \(\{C_i\}_{i=1}^{3}\). Therefore, \(\Delta^*\) attains the maximum of \((D)\). So we get the strong duality \(\min(P) = \max(D)\). On the other hand, since \(X_0\) is in the interior of \(\Omega\), there exists a convex neighborhood \(C_0\) of \(X_0\) such that \(C_0 \subset \Omega\). Since the primal problem \((P_0)\) restricted on \(C_0\) is a convex program, \((X_0, X_1, X_2, X_3)\) is a minimum solution for \((P)\).

**Theorem 4.2.** When \(\Omega\) is bounded, the dual problem \((D)\) is simplified as follows.

\[ (D^*) \quad \text{Maximize the smallest height of a triangle } \Delta \subset \Omega. \]

**Proof.** Assume that \(\Delta \subset \Omega\). Then, by separation theorem, there are closed half spaces \(H_i^-\) \((i = 1, 2, 3)\) such that \(C_i \subset H_i^-\) and \(\Delta \subset \cap_{i=1}^{3} (H_i^+) =: \Delta'\). Since \(\Omega\) is bounded and since \(\Delta' \subset \cap_{i=1}^{3} C_i = \Omega\), \(\Delta'\) is a triangle separating \(\{C_i\}_{i=1}^{3}\). \(\square\)

![Figure 7. Separating hyperplanes form an unbounded polygon \(\Delta'\)](image)
5. Concluding remarks

When $\Omega$ is not bounded, separating hyperplanes do not necessarily form a triangle, see Fig. 7. So duality relationship $\min(P) = \max(D^*)$ does not always hold. Indeed, since we can enlarge $\Delta$ rightward within the dark gray area as we like, $\sup(D^*)$ equals $+\infty$.

We can replace a triangle by a regular triangle in our dual problems $(D)$ and $(D^*)$, because the maximum is attained by a regular triangle. However, it is clear that regular triangles are not enough when $\Omega$ is a (general) triangle. That’s why we defined the dual problem with (general) triangles.

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