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Quadratic Transformations of the Sixth Painlevé Equation

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Abstract

Concise forms of Kitaev's quadratic transformation between Painleve VI equations with the local monodromy differences $(1/2, 1/2, a, b)$ and (a, a, b, b) are presented. This transformation is related to better known quadratic transformations (due to Manin and Ramani-Grammaticos-Tamizhmani) via Okamoto transformations. Using the new concise formulas, we derive explicit expressions for several algebraic Painleve VI functions.

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1 Introduction

The sixth Painlevé equation is, of course,

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}\\ &+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right),\end{aligned} \tag{1.1}$$

where α, β, γ, $\delta \in \mathbb{C}$ are parameters. As well-known [7], its solutions define isomonodromy deformations (with respect to t) of the 2×2 matrix Fuchsian equation with 4 singular points ($\lambda = 0, 1, t$, and ∞):

$$\frac{d}{d\lambda}\Psi = \left(\frac{A_0}{\lambda}+\frac{A_1}{\lambda-1}+\frac{A_t}{\lambda-t}\right)\Psi, \qquad \frac{d}{d\lambda}A_k = 0, \quad k = 0, 1, t. \tag{1.2}$$

Following Jimbo-Miwa correspondence [7], we assume that the Fuchsian equation is normalized so that the eigenvalues of A_0, A_1, A_t are, respectively, $\pm\theta_0/2$, $\pm\theta_1/2$, $\pm\theta_t/2$, and that the matrix $A_\infty := -A_1 - A_2 - A_3$ is diagonal with the diagonal entries $\pm\theta_\infty/2$. Then the corresponding Painleve equation has the parameters

$$\alpha = \frac{(\theta_\infty-1)^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = \frac{1-\theta_t^2}{2}. \tag{1.3}$$

We refer to the numbers θ_0, θ_1, θ_t and θ_∞ as *local monodromy differences.*

For any numbers $\nu_1, \nu_2, \nu_t, \nu_\infty$, let us denote by $P_{VI}(\nu_0, \nu_1, \nu_t, \nu_\infty; t)$ the Painleve VI equation for the local monodromy differences $\theta_i = \nu_i$ for $i \in \{0, 1, t, \infty\}$, via (1.3). Note that changing the sign of ν_0, ν_1, ν_t or $1 - \nu_\infty$ does not change the Painleve equation. There are fractional-linear transformations for Painleve VI equations, which permute the 4 singular points and the numbers $\nu_0, \nu_1, \nu_t, 1 - \nu_\infty$.

The main subject of this paper is quadratic transformations for the sixth Painleve equation. Their existence was discovered in [8], [9]. In particular [9], a quadratic transformations was found between isomonodromy Fucshian systems (1.2) with the local monodromy differences $(\theta_0, \theta_1, \theta_t, \theta_\infty)$ related as follows:

$$(b, a, b, a) \mapsto \left(\tfrac{1}{2}, a, b, \tfrac{1}{2}\right). \tag{1.4}$$

The corresponding transformation between Painleve VI solutions was implied as a complicated composition of lengthy formulas. The main result of this paper is a compact expression for quadratic transformation (1.4).

Simpler quadratic transformations for Painleve VI equations are obtained in [10] and [12]. Up to fractional-linear transformations, the local monodromy differences are transformed by these quadratic transformations as follows:

$$(0, A, B, 1) \mapsto \left(\tfrac{A}{2}, \tfrac{B}{2}, \tfrac{B}{2}, \tfrac{A}{2}+1\right). \tag{1.5}$$

This quadratic transformation is formulated in the following lemma. It cannot be realized as a quadratic transformation of the Fucshian equation (1.2) via the Jimbo-Miwa correspondence. In [13] quadratic transformations of the Painleve VI equation are referred to as *folding transformations.*

Lemma 1.1 *Suppose that y_0 is a solution of $P_{VI}(0, A, B, 1\,;\, t_0)$. Let us denote*

$$T_0 = \frac{(1-\sqrt{t_0})^2}{(1+\sqrt{t_0})^2}. \tag{1.6}$$

Then

$$\frac{\sqrt{t_0}-1}{\sqrt{t_0}+1}\,\frac{\sqrt{y_0}-1}{\sqrt{y_0}+1} \tag{1.7}$$

is a solution of $P_{VI}\left(\frac{A}{2}, \frac{B}{2}, \frac{B}{2}, \frac{A}{2}+1;\, T_0\right)$.

Proof. See [12]. Alternatively, in [10] this transformation is found as the Landen transformation for the elliptic form of Painleve VI. □

Quadratic transformations (1.4) and (1.5) are related by Okamoto transformations. Recall that Okamoto transformations relate local monodromy differences of Painleve VI equations as follows:

$$(a, b, c, d) \mapsto (a-S, b-S, c-S, d-S)\,, \qquad \text{where} \quad S = \tfrac{a+b+c+d}{2}. \tag{1.8}$$

In particular, Okamoto transformations directly relate

$$(-a, b, b, a) \mapsto (a+b, 0, 0, a-b)\,, \tag{1.9}$$

and

$$\left(-\tfrac{1}{2}, -a, b, \tfrac{1}{2}\right) \mapsto \left(\tfrac{a-b-1}{2}, \tfrac{a+b}{2}, \tfrac{a+b}{2}, \tfrac{a-b+1}{2}\right), \tag{1.10}$$

while a fractional-linear version of quadratic transformation (1.5) relates

$$(a+b, 0, 0, a-b) \mapsto \left(\tfrac{a-b-1}{2}, \tfrac{a+b}{2}, \tfrac{a+b}{2}, \tfrac{a-b+1}{2}\right). \tag{1.11}$$

As shown in [11], Okamoto transformations generate a group isomorphic to the affine Weyl group of type D_4. If we take all fractional-linear transformations into account, the symmetry group is the affine Weyl group of type F_4.

Okamoto transformations imply non-differential (but non-linear in general) relations between any three Painleve VI functions in the same Okamoto orbit. These relations are known as *Backlund transformations*. They are analogous to contiguous relations for Gauss hypergeometric functions. In particular, Backlund transformations relate $P_{VI}(a, b, c, d; t)$ to $P_{VI}(a+1, b, c, d-1; t)$, $P_{VI}(a, b+1, c, d+1; t)$, etc.

We compute quadratic transformation (1.4) by composing transformations (1.9)–(1.11). We strive to use Backlund transformations rather than direct definition of Okamoto transformations, so to avoid differentiation.

Our original motivation was to compute examples of algebraic Painleve VI functions of "icosahedral" type as in [2]. In the sixth version of Boalch's paper, 10 out of 52 icosahedral types were not exemplified yet. Eight out of the 10 missing examples can be obtained from the known cases by quadratic transformations. Very recently [4] Boalch computed those 8 examples by applying quadratic transformations in a similar manner. Compared with Boalch's results, we concentrate on direct quadratic transformation (1.4). Accordingly, we give representative examples for the 8 icosahedral cases with the nicest local monodromy differences $(1/2, 1/2, \ldots)$ within corresponding Okamoto orbits.

To have a convenient notation for Okamoto transformations conveniently, we introduce the following operator on functions. For any $\nu_0, \nu_1, \nu_t, \nu_\infty \in \mathbb{C}$, let

$$K_{(\nu_0,\nu_1,\nu_t,\nu_\infty;\,t)}\,y(t) := y(t) + \frac{\nu_0+\nu_1+\nu_t+\nu_\infty}{Z(t)}, \qquad (1.12)$$

where

$$Z(t) = \frac{(t-1)\,\frac{dy(t)}{dt} - \nu_0}{y(t)} - \frac{t\,\frac{dy(t)}{dt} + \nu_1}{y(t)-1} + \frac{\frac{dy(t)}{dt} - 1 - \nu_t}{y(t)-t}.$$

Okamoto's result in [11] can be formulated as follows.

Lemma 1.2 *Suppose that $y(t)$ is a solution of $P_{VI}(a,b,c,d;t)$, and that*

$$\nu_0 \in \{a,-a\}, \quad \nu_1 = \{b,-b\}, \quad \nu_t = \{c,-c\}, \quad \nu_\infty \in \{d, 2-d\}.$$

Let $\Theta = (\nu_0+\nu_1+\nu_t+\nu_\infty)/2$. Then the function $K_{(\nu_0,\nu_1,\nu_t,\nu_\infty;\,t)}\,y(t)$ is a solution of $P_{VI}(\nu_0-\Theta, \nu_1-\Theta, \nu_t-\Theta, \nu_\infty-\Theta)$. Besides,

$$K_{(\nu_0-\Theta,\,\nu_1-\Theta,\,\nu_t-\Theta,\,\nu_\infty-\Theta;\,t)}K_{(\nu_0,\,\nu_1,\,\nu_t,\,\nu_\infty;\,t)}\,y(t) = y(t).$$

Proof. Straightforward from [11]. The latter claim is equivalent to the Painleve VI equation. □

The main result of this paper is the following Theorem. This is probably the most compact explicit expression for (a fractional-linear version of) quadratic transformation (1.4).

Theorem 1.3 *Suppose that y_0 is a solution of $P_{VI}(a,a,b,b\,;t_1)$. Let us denote*

$$y_1 = K_{(-a,-a,-b,\,b\,;\,t_1)}\,y_0 \qquad (1.13)$$

and

$$T_1 = \frac{1}{2} + \frac{t_1 - \frac{1}{2}}{2\sqrt{t_1^2 - t_1}}. \qquad (1.14)$$

Then the function

$$Y_0 = \frac{1}{2} + \frac{t_1 - y_1 + \sqrt{y_1^2-y_1}}{2\sqrt{t_1^2-t_1}} + \frac{(a-b+1)(y_0-y_1)\left(Y_0 - \sqrt{y_1^2-y_1}\right)}{2\left(a\sqrt{y_1^2-y_1} - (b-1)(y_0-y_1)\right)\sqrt{t_1^2-t_1}} \qquad (1.15)$$

is a solution of $P_{VI}\left(\frac{1}{2},\frac{1}{2},a,b\,;\,T_1\right)$.

We prove this Theorem in Section 3. Essentially, we express solution (1.15) in terms of y_0 and Y_0. An alternative is to express Y_0 in terms of y_0 and dy_0/dt_1, by explicitly spelling out (1.13). But as with Backlund transformations, explicit formulas are much simpler when differentiation is not involved.

In applications to algebraic Painleve VI functions, it is convenient to write t_1, y_0, y_1 of Theorem 1.3 in the form

$$t_1 = \frac{1}{2} + \frac{1}{2}\,\theta, \qquad y_0 = \frac{1}{2} + \frac{1}{2}\,\varphi, \qquad y_1 = \frac{1}{2} + \frac{1}{2}\,\psi. \qquad (1.16)$$

Then formulas (1.14)–(1.15) can be rewritten as follows:

$$T_1 = \frac{1}{2} + \frac{\theta}{2\sqrt{\theta^2-1}}, \tag{1.17}$$

$$Y_0 = \frac{1}{2} + \frac{\theta-\psi+\sqrt{\psi^2-1}}{2\sqrt{\theta^2-1}} + \frac{(a-b+1)(\varphi-\psi)\left(\psi-\sqrt{\psi^2-1}\right)}{2\left(a\sqrt{\psi^2-1}-(b-1)(\varphi-\psi)\right)\sqrt{\theta^2-1}}. \tag{1.18}$$

Similar expression form for Painleve VI functions is used in [4]. It is useful when the local monodromy differences θ_0, θ_1 are equal. We present more expressions for the functions involved in quadratic transformations in Section 5.

At first glance, the function Y_0 is defined over the field $\mathbb{C}(\theta,\varphi,\sqrt{\theta^2-1},\sqrt{\psi^2-1})$, or equivalently, over $\mathbb{C}(t_1,y_0,\sqrt{t_1^2-t_1},\sqrt{y_1^2-y_1})$. It appears that quadratic transformation (1.4) increases algebraic degree by the factor 4. However, the definition field $\mathbb{C}(T_1,Y_0)$ for the transformed function is not an immediate extension of $\mathbb{C}(t_1,y_0)$; see the remark with formula (2.4) below. Particularly, formula (1.18) can be written in the following form:

$$Y_0 = \frac{1}{2} + \widetilde{A}\frac{\theta}{\sqrt{\theta^2-1}} + \widetilde{B}\,\frac{\sqrt{\psi^2-1}}{\sqrt{\theta^2-1}}, \tag{1.19}$$

where

$$\widetilde{A} = \frac{a^2(\psi^2-1)\,(\varphi/\theta-1)+a(b-1)\,(\psi\varphi-1)\,(\psi/\theta-\varphi/\theta)+(b-1)^2(\psi-\varphi)^2}{2\,(a^2(\psi^2-1)-(b-1)^2(\phi-\psi)^2)}$$

$$\widetilde{B} = \frac{a}{2}\cdot\frac{a(\psi\varphi-1)+(b-1)(\psi-\varphi)\varphi}{a^2(\psi^2-1)-(b-1)^2(\varphi-\psi)^2}.$$

After dividing the numerator and denominator of each $\widetilde{A},\widetilde{B}$ by θ^2, one can observe that the definition field for Y_0 is $\mathbb{C}\left(\theta^2,\varphi/\theta,\,\psi/\theta,\,\theta/\sqrt{\theta^2-1},\,\sqrt{\psi^2-1}/\sqrt{\theta^2-1}\right)$. Typically, this is a subfield of index 2 in $\mathbb{C}(\theta,\varphi,\sqrt{\theta^2-1},\sqrt{\psi^2-1})$.

2 Preliminaries

For convenience, in Appendix Section 5 we give a list of fractional-linear transformations for Painleve VI equations or functions; see Table 5 there. It is useful to note that Okamoto and fractional-linear transformations commute.

Lemma 2.1 *Let (A,B,C,D) be a permutation of $(0,1,t,\infty)$, and let $L:(y,t)\mapsto(Y,T)$ with $T\in\{t,1-t,t/(t-1),1/t,1/(1-t),(t-1)/t\}$ denote the corresponding fractional linear transformation from Table 5. Then for any numbers $\nu_0,\nu_1,\nu_t,\nu_\infty\in\mathbb{C}$ we have*

$$L\,K_{(\nu_0,\nu_1,\nu_t,1+\nu_\infty;\,t)} = K_{(\nu_A,\nu_B,\nu_C,1+\nu_D;\,T)}\,L.$$

Proof. It is enough to check the statement explicitly for a generating set of the permutations. One can take, for example, the three transpositions corresponding to the 5th, 8th and 9th rows of Table 5. □

In the same Appendix, we give various expressions for different Painleve VI functions related to the quadratic transformations. Here are useful fractional-linear variations of quadratic transformations of Lemma 1.1. Formulas (2.1)–(2.2) are reminiscent to Corollary 3 in [4].

Lemma 2.2 *Suppose that* y_1 *is a solution of* $P_{VI}(0,0,B,C;t_1)$*. Let us denote*

$$T_1 = \frac{1}{2} + \frac{t_1 - \frac{1}{2}}{2\sqrt{t_1^2 - t_1}}. \tag{2.1}$$

Then the function

$$\frac{1}{2} + \frac{\sqrt{y_1^2 - y_1} - \sqrt{t_1^2 - t_1}}{2\,(y_1 - t_1)} \tag{2.2}$$

is a solution of $P_{VI}\left(\frac{C-1}{2}, \frac{C-1}{2}, \frac{B}{2}, \frac{B}{2}+1;\, T_1\right)$*, and the function*

$$\frac{1}{2} + \frac{t_1 - y_1 + \sqrt{y_1^2 - y_1}}{2\sqrt{t_1^2 - t_1}} \tag{2.3}$$

is a solution of $P_{VI}\left(\frac{B}{2}, \frac{B}{2}, \frac{C-1}{2}, \frac{C+1}{2};\, T_1\right)$*.*

Proof. Set $T_0 = T_1/(T_1 - 1)$. The function $y_2 := y_1/(y_1 - 1)$ is a solution of $P_{VI}(0, C-1, B, 1; t_1/(t_1 - 1))$ by fractional-linear transformations. We can apply Lemma 1.1 to y_2 and get a solution Y_2 of $P_{VI}(\frac{C-1}{2}, \frac{B}{2}, \frac{B}{2}, \frac{C+1}{2}; T_0)$. The functions in (2.2) and (2.3) are the fractional-linear transformations $Y/(Y-1)$ and $(Y - T_0)/(1 - T_0)$ of Y_2. □

Let Y_1 denote the function (2.3). One can observe the following inclusions of fields:

$$\begin{array}{ccccc} \mathbb{C}(t_1, y_1) & \subset & \mathbb{C}\,(t_1, T_1, y_1) & \subset & \mathbb{C}\,(t_1, T_1, Y_1) \\ & & \cup & & \cup \\ & & \mathbb{C}\,(T_1, y_1) & \subset & \mathbb{C}\,(T_1, Y_1) \end{array} \tag{2.4}$$

All immediate extensions have degree 2. As we see, the definition field $\mathbb{C}(T_1, Y_1)$ for Y_1 is not an immediate extension of $\mathbb{C}(t_1, y_0)$, but a subfield of index 2 of the degree 4 extension $\mathbb{C}\,(t_1, T_1, Y_1) \supset \mathbb{C}(t_1, y_0)$. Since fractional-linear and Okamoto transformations do not change function fields, we can replace y_1, Y_1 in (2.4) with y_0, Y_0 of Theorem 1.3.

In encountered examples of algebraic Painleve VI functions with the local monodromy differences θ_0, θ_1 equal, the algebraic remain invariant under the fractional-linear transformation $(y, t) \mapsto (1 - y, 1 - t)$. Then the subfields $\mathbb{C}\,(T_1, y_0) \subset \mathbb{C}\,(t_1, T_1, y_0)$ and $\mathbb{C}\,(T_1, Y) \subset \mathbb{C}\,(t_1, T_1, Y)$ are defined by this symmetry. In terms of (1.16), the symmetry flips the sign of θ, ϕ, etc. In the examples, y_0 is defined on a (hyper)elliptic curve, and the symmetry $(y, t) \mapsto (1 - y, 1 - t)$ is easily realized by interchanging the 2 branches of a (hyper)elliptic covering of $\mathbb{P}^1$.

The following Lemma shows some "commutativity" of quadratic and Okamoto transformations under proper conditions.

Lemma 2.3 *Suppose that* y_1 *is a solution of* $P_{VI}(0,0,B,C;t_1)$*, and that* y_2 *is a solution of* $P_{VI}(0,0,C,B;t_1)$*. Suppose that*

$$K_{(0,0,-B,C;t_1)}\, y_1 = K_{(0,0,-C,B;t_1)}\, y_2. \tag{2.5}$$

Let y_0 *denote the evaluation of any side of this equality. Let* S_1 *denote a branch of* $\sqrt{y_1^2 - y_1}$*. We set*

$$S_2 = \frac{y_2(y_0 - 1)}{(1 - y_1)y_0}\, S_1. \tag{2.6}$$

Then S_2 is a branch of $\sqrt{y_2^2 - y_2}$. Further, define T_1 as in (2.1), and let

$$Y_1 = \frac{1}{2} + \frac{t_1 - y_1 + S_1}{2\sqrt{t_1^2 - t_1}}, \qquad Y_2 = \frac{1}{2} + \frac{t_1 - y_2 + S_2}{2\sqrt{t_1^2 - t_1}}. \tag{2.7}$$

Then Y_1 and Y_2 are solutions of, respectively,

$$P_{VI}\left(\tfrac{B}{2}, \tfrac{B}{2}, \tfrac{C-1}{2}, \tfrac{C+1}{2}; T_1\right) \qquad \textit{and} \qquad P_{VI}\left(\tfrac{C}{2}, \tfrac{C}{2}, \tfrac{B-1}{2}, \tfrac{B+1}{2}; T_1\right), \tag{2.8}$$

and

$$Y_2 = K_{\left(-\frac{B}{2}, -\frac{B}{2}, \frac{C-1}{2}, \frac{C+1}{2}; T_1\right)} Y_1. \tag{2.9}$$

Proof. The Backlund relation between y_1, y_2 and y_0 can be written as follows:

$$\frac{(y_1 - 1)(y_2 - 1)}{y_1\, y_2} = \frac{(y_0 - 1)^2}{y_0^2}. \tag{2.10}$$

It can be derived by expressing y_1, y_2 as Okamoto transformations of y_0, and eliminating the derivative of y_0 from two identities. If we square both sides of (2.6) and identify $S_1^2 = y_1^2 - y_1$, $S_2^2 = y_2^2 - y_2$, we get an equivalent equality to (2.10). This implies that S_2 is a branch of $\sqrt{y_2^2 - y_2}$. The functions Y_1 and Y_2 satisfy indicated Painleve equations by Lemma 2.2. Relation (2.9) can be checked explicitly. □

Recall that a *Painleve curve* is the normalization of an algebraic curve defined by the minimal equation for an algebraic Painleve VI solution $y(t)$. The minimal equation is a polynomial in y and t. The indeterminant t defines an algebraic map from the Painleve curve to $\mathbb{P}^1$, and in [6] it is mentioned that this map is a Belyi map. The reason is that the corresponding field extension $\mathbb{C}(y, t) \supset \mathbb{C}(t)$ ramifies only above $t = 0, 1, \infty$ due to the Painleve property (i.e., solutions of Painleve equations do not have “moving” essential singularities). Accordingly, in [3] an algebraic Painleve VI solution is defined as a triple (Π, y, t), where Π is a compact curve, y and t are rational functions on Π such that t is a Belyi function and $y(t)$ is a Painleve VI solution. However, this definition does not look precise, because it apparently allows non-minimal Painleve curves (since Belyi maps can be appropriately composed).

In Section 4, we will follow change of t-Belyi maps under quadratic transformations. For icosahedral Painleve VI solutions, branching patterns for t-Belyi maps are given b the ”Partition” column of Table 1 in [2], since these partitions characterize conjugacy classes for local monodromies around $t = 0, 1, \infty$.

3 Proof of Theorem 1.3

The function y_1 is a solution of $P_{VI}(0, 0, b - a, a + b; t_1)$. Let us introduce

$$y_2 = K_{(a,a,-b,b;\, t_1)}\, y_0. \tag{3.1}$$

This is a solution of $P_{VI}\,(0, 0, a + b, b - a; t_1)$. The Backlund relation between y_0, y_1 and y_2 is the same as in (2.10); we rewrite it as

$$y_2 = \frac{y_0^2(1 - y_1)}{y_0^2 - 2y_1 y_0 + y_1}. \tag{3.2}$$

We choose a branch of $\sqrt{y_1^2-y_1}$, and identify $\sqrt{y_2^2-y_2}$ as follows:

$$\sqrt{y_2^2-y_2} = \frac{(y_0^2-y_0)\sqrt{y_1^2-y_1}}{y_0^2-2y_1y_0+y_1}. \tag{3.3}$$

Let us define

$$T_1 = \frac{1}{2} + \frac{2t_1-1}{4\sqrt{t_1^2-t_1}}, \tag{3.4}$$

$$Y_1 = \frac{1}{2} + \frac{t_1-y_1+\sqrt{y_1^2-y_1}}{2\sqrt{t_1^2-t_1}}, \tag{3.5}$$

$$Y_2 = \frac{1}{2} + \frac{t_1-y_2+\sqrt{y_2^2-y_2}}{2\sqrt{t_1^2-t_1}}. \tag{3.6}$$

By Lemma 2.3, the functions Y_1, Y_2 are solutions of, respectively,

$$P_{VI}\left(\tfrac{b-a}{2}, \tfrac{b-a}{2}, \tfrac{a+b-1}{2}, \tfrac{a+b+1}{2}; T_1\right) \quad \text{and} \quad P_{VI}\left(\tfrac{a+b}{2}, \tfrac{a+b}{2}, \tfrac{b-a-1}{2}, \tfrac{b-a+1}{2}; T_1\right), \tag{3.7}$$

and

$$Y_2 = K_{\left(\frac{a-b}{2}, \frac{a-b}{2}, \frac{a+b-1}{2}, \frac{a+b+1}{2}; T_1\right)} Y_1. \tag{3.8}$$

By combining (3.2), (3.3) and (3.6), we obtain the expression

$$Y_2 = \frac{1}{2} + \frac{t_1-y_1+\sqrt{y_1^2-y_1}}{2\sqrt{t_1^2-t_1}} + \frac{(y_0-y_1)\left(y_1-\frac{1}{2}-\sqrt{y_1^2-y_1}\right)}{\left(y_0-y_1+\sqrt{y_1^2-y_1}\right)\sqrt{t_1^2-t_1}}. \tag{3.9}$$

We define now

$$Y_0 = K_{\left(\frac{a-b}{2}, \frac{a-b}{2}, \frac{1-a-b}{2}, \frac{a+b+1}{2}; T_1\right)} Y_1. \tag{3.10}$$

This is a solution of $P_{VI}\left(\frac{1}{2}, \frac{1}{2}, a, b\right)$. The Backlund relation between Y_0, Y_1, Y_2 can be computed can be deduced by writing down (3.8) and (3.10) explicitly and eliminating the derivative of Y_1. We obtain

$$Y_0 = \frac{2aY_1Y_2-(a+b-1)T_1Y_1-(a-b-1)T_1Y_2}{(a-b+1)Y_1+(a+b-1)Y_2-2aT_1}. \tag{3.11}$$

After substitution of (3.4), (3.5) and (3.9), we obtain expression (1.15). Q.E.D.

4 Algebraic Painleve VI functions

Here we apply our compact formulas to compute algebraic Painleve VI functions. In [4] eight icosahedral Painleve VI solutions are computed; these explicit results complement the examples in [2] so that all 52 icosahedral classes are eventually exemplified. The eight solutions are precisely the cases which can be obtained from earlier know examples by quadratic transformations. Here is the transformation scheme, where numbers indicate the classes in Boalch's [2] classification:

$$31, 32 \Rightarrow 44, 45 \Rightarrow 50, 51; \qquad 39, 40 \Rightarrow 47, 48; \qquad 41 \Rightarrow 49 \Rightarrow 52. \tag{4.1}$$

(There is also quadratic transformation $21 \Rightarrow 28$, but Case 28 is not hard.) Possibility of using these quadratic transformations in computation of algebraic Painleve VI functions was mentioned in correspondence between the first author and Boalch a few months ago. Boalch was first to express the transformed solutions in a concise form.

Nevertheless, the eight representative examples in [4] do not have the most attractive local monodromy differences $(1/2, 1/2, \ldots)$. Here we use Theorem 1.3 to find algebraic Painleve VI solutions within the same Boalch classes with the attractive local monodromy differences. (In this version, we present only the transformation $39, 40 \Rightarrow 47, 48$.)

First we compute an example for Case 47 in [2]. We start with type-39 Boalch's example (reparametrized by the substitution $s \mapsto s-2$):

$$t_{39} = \frac{1}{2} - \frac{(2s^7 - 18s^6 + 48s^5 - 50s^4 + 105s^3 + 3s^2 - 7s - 3)u}{18(s^2-4s-1)(4s^2-s+1)^2}, \tag{4.2}$$

$$y_{39} = \frac{1}{2} + \frac{14s^5 - 79s^4 + 6s^3 + 80s^2 + 116s - 9}{6(s-1)(s^2-4s-1)\,u}, \tag{4.3}$$

where $u = \sqrt{3(s+3)(4s^2-s+1)}$, so the function y_{39} is defined on a genus 1 curve. It a solution for $P_{VI}\left(\frac{1}{3}, \frac{1}{3}, \frac{4}{5}, \frac{4}{5}; t_{39}\right)$. Following Theorem 1.3, we compute

$$\widetilde{y}_{39} = K_{\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{4}{5}, \frac{4}{5}; t_{39}\right)}\, y_{39} = \frac{1}{2} - \frac{(s^2-5s-1)\,u}{6\,(s-1)\,(4s^2-s-1)}. \tag{4.4}$$

This is a solution of $P_{VI}\left(0, 0, \frac{7}{15}, \frac{13}{15}; t_{39}\right)$. We compute:

$$\sqrt{t_{39}^2 - t_{39}} = \frac{s\,(s+1)^2(s-2)^2(s-5)\sqrt{s\,(s+1)(s-2)(s+3)}}{3\,(s^2-4s-1)(4s^2-s+1)\,u}, \tag{4.5}$$

$$\sqrt{\widetilde{y}_{39}^2 - \widetilde{y}_{39}} = \frac{(s+1)\sqrt{s\,(s-2)(s+3)(s-5)}}{(s-1)\,u}. \tag{4.6}$$

We keep the factor u in these expressions because we expect it will disappear after simplifications. Eventually, the new square roots define the field extension $\mathbb{C}(T, Y)$ in (2.4). The Painleve curve is the fiber product of the elliptic curves

$$v^2 = s\,(s+1)(s-2)(s+3) \qquad \text{and} \qquad w^2 = s\,(s-2)(s+3)(s-5). \tag{4.7}$$

The Painleve curve is hyperelliptic of genus 2. Its Weierstrass form can be obtained by introducing the parameter $q = \sqrt{(s-5)/(s+1)}$, so that $s = (q^2+5)/(1-q^2)$. Then the hyperelliptic curve can be represented as

$$V^2 = -(q^2+1)(q^2+5)(q^2-4), \tag{4.8}$$

and we can identify $v = 6V/(q^2-1)^2$, $w = qv$. The connection with Boalch's hyperelliptic form in [4] is $q = (j-3)/(j+3)$. The field inclusions in (2.4) are the following:

$$\begin{array}{ccccc} \mathbb{C}(s,u) & \subset & \mathbb{C}\,(s,u,v) & \subset & \mathbb{C}\,(s,u,v,w) \\ & & \cup & & \cup \\ & & \mathbb{C}\,(s,v) & \subset & \mathbb{C}\,(s,v,w) = \mathbb{C}\,(q,V) \end{array} \tag{4.9}$$

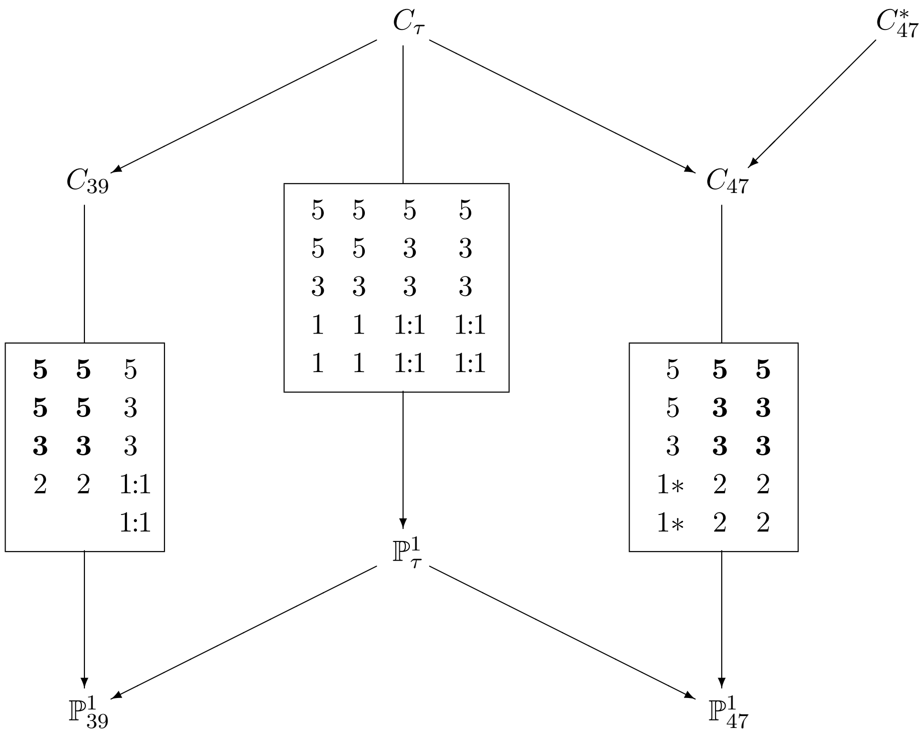

Figure 1: Quadratic transformations for the 47th Boalch solution

We may use formulas (1.14)–(1.15), or equivalently, we may express t_{39}, y_{39}, $\widetilde{y}_{39}$ like in (1.16) and use formulas (1.17)–(1.18). Either way, we obtain the following solution of $P_{VI}\left(\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{4}{5};t_{47}\right)$:

$$t_{47} = \frac{1}{2} - \frac{(2s^7-18s^6+48s^5-50s^4+105s^3+3s^2-7s-3)v}{4s^2(s+1)^3(s-2)^3(s-5)}, \tag{4.10}$$

$$\begin{aligned} y_{47} = \; & \frac{1}{2} - \frac{(3s^7-27s^6+107s^5-205s^4+105s^3-37s^2-7s-3)v}{2s(s+1)(s-2)^2(s-1)(3s^4-12s^3-14s^2-12s+3)} \\ & + \frac{(4s^2-s+1)(s^2-4s-1)(7s^4-52s^3+34s^2-36s+15)}{2s(s+1)(s-2)^2(s-1)(3s^4-12s^3-14s^2-12s+3)\sqrt{(s+1)(s-5)}}. \end{aligned} \tag{4.11}$$

Of course, we can parametrize everything in terms of q and V. The expressions are then somewhat longer, but they may be more convenient for manipulation with a computer algebra package.

As mentioned in Section 2, the variables t_{39} and t_{47} define Belyi maps. Figure 1 depicts change of branching of these Belyi maps. The map t_{39} is from C_{39} to $\mathbb{P}^1_{39}$. The map t_{47} is from C^*_{47} to $\mathbb{P}^1_{47}$. Non-vertical arrows represent degree 2 coverings. The function fields of C_{47}, C_τ, $\mathbb{P}^1_\tau$ are, respectively, $\mathbb{C}(T,y_{39})=\mathbb{C}(s,v)$, $\mathbb{C}(t,T,y_{39})=\mathbb{C}(s,u,v)$ and $\mathbb{C}(t,T)$. In boxes we represent the branching patterns of the morphisms. Each column gives branching orders of one fiber. In the middle box, the first two columns represent points with $s\in\{-1,2,0,5\}$; the last two columns represent the points with $s=\infty$, $4s^2=s+1$ or $s^2=4s+1$. The bold numbers represent points where the upper degree 2 coverings above parallelograms ramify. The star represents ramification points of the upper-right degree 2 covering. Each parallelogram is a commutative fiber product diagram. The genus of C_τ is 4. The two composite coverings have the following ramification pattern, respectively:

$$\begin{array}{ccc} 10 & 10 & 5{:}5 \\ 10 & 10 & 3{:}3 \\ 6 & 6 & 3{:}3 \\ 2 & 2 & 1{:}1{:}1{:}1 \\ 2 & 2 & 1{:}1{:}1{:}1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} 5{:}5 & 10 & 10 \\ 5{:}5 & 6 & 6 \\ 3{:}3 & 6 & 6 \\ 1{:}1 & 2{:}2 & 2{:}2 \\ 1{:}1 & 2{:}2 & 2{:}2 \end{array}.$$

The quadratic covering $\mathbb{P}^1_\tau \to \mathbb{P}^1_{39}$ ramifies above the two points represented by the first two columns of the first box. The covering $\mathbb{P}^1_\tau \to \mathbb{P}^1_{47}$ ramifies above the two points represented by the last two columns of the third box.

Similarly, we can compute a type-48 example. With the same elliptic curve and t_{39} as for y_{39}, we have the following type-40 icosahedral function (obtained by Okamoto transformations from the corresponding example in [2]):

$$y_{40} = \frac{1}{2} + \frac{2s^6 - 14s^5 + 17s^4 + 16s^3 - 112s^2 - 2s - 3}{6u\,(3s-1)(s^2-4s-1)} \tag{4.12}$$

This is a solution of $P_{VI}\left(\frac{2}{5},\frac{2}{5},\frac{2}{3},\frac{2}{3};t_{39}\right)$. Similar application of Theorem 1.3 gives the following solution of $P_{VI}\left(\frac{1}{2},\frac{1}{2},\frac{2}{5},\frac{2}{3};t_{47}\right)$.

$$\begin{aligned} y_{48} \;=\; & \frac{1}{2} - \frac{(19s^6 + 138s^5 - 195s^4 + 380s^3 - 195s^2 - 138s - 89)\,v}{2(s+1)^2(s-2)(19s^5 - 155s^4 + 390s^3 - 590s^2 - 5s - 3)} \\ & + \frac{9(4s^2 - s + 1)(s^2 - 4s - 1)(s^5 - 7s^4 + 13s^3 - 115s^2 - 2s - 10)}{2\,(s+1)^2(19s^5 - 155s^4 + 390s^3 - 590s^2 - 5s - 3)\sqrt{s(s-2)^3(s+3)(s-5)}}. \end{aligned} \tag{4.13}$$

We note also that some (perhaps "half") algebraic solutions of $P_{VI}(1/2, 1/2, 1/2, 1/2; t)$ in the Hitchin's series [6] can be computed by iteratively applying Theorem 1.3.

5 Appendix

For convenience, we give general fractional-linear transformations for Painleve VI functions in Table 5 below. If one starts with a solution $y(t)$ of $P_{VI}(a, b, c, d+1)$, in each row we give a solution of a Painleve VI equation with permuted singular points in terms of $y(t)$ and t. We mostly use the four transformations which do not change the argument t.

Now we wish to present various forms of quadratic transformations. In the context of Lemma 1.1, its y_0 should be identified with y_8 here. In the context of other results, we keep the same notation.

Accordingly, as the starting point we assume that y_8 is a solution of $P_{VI}(0, A, B, 1; t_1)$.

λ	$(\theta_0, \theta_1, \theta_t, 1-\theta_\infty)$	y	t
λ	(a, b, c, d)	y	t
$t\lambda/(\lambda+t-1)$	(a, b, d, c)	$(1-t)y/(y-t)$	$1-t$
$t\lambda$	(a, c, b, d)	y/t	$1/t$
$t\lambda/(t\lambda+1-t)$	(a, c, d, b)	$(t-1)y/t(y-1)$	$(t-1)/t$
$t\lambda/(\lambda-1)$	(a, d, b, c)	$y/(y-t)$	$1/(1-t)$
$\lambda/(\lambda-1)$	(a, d, c, b)	$y/(y-1)$	$t/(t-1)$
$1-\lambda$	(b, a, c, d)	$1-y$	$1-t$
$t(\lambda-1)/(\lambda-t)$	(b, a, d, c)	$t(y-1)/(y-t)$	t
$t\lambda-\lambda+1$	(b, c, a, d)	$(y-1)/(t-1)$	$1/(1-t)$
$t/(\lambda-t\lambda+t)$	(b, c, d, a)	$t(y-1)/(t-1)y$	$t/(t-1)$
$(t\lambda-1)/(\lambda-1)$	(b, d, a, c)	$(y-1)/(y-t)$	$1/t$
$1/(1-\lambda)$	(b, d, c, a)	$(y-1)/y$	$(t-1)/t$
$t(1-\lambda)$	(c, a, b, d)	$(t-y)/t$	$(t-1)/t$
$t(\lambda-1)/(t\lambda-1)$	(c, a, d, b)	$(y-t)/t(y-1)$	$1/t$
$\lambda-t\lambda+t$	(c, b, a, d)	$(y-t)/(1-t)$	$t/(t-1)$
$t/(t\lambda-\lambda+1)$	(c, b, d, a)	$(y-1)/(1-t)y$	$1/(1-t)$
$(\lambda-t)/(\lambda-1)$	(c, d, a, b)	$(y-t)/(y-1)$	t
$t/(1-\lambda)$	(c, d, b, a)	$(y-t)/y$	$1-t$
$t(\lambda-1)/\lambda$	(d, a, b, c)	$t/(t-y)$	$t/(t-1)$
$(\lambda-1)/\lambda$	(d, a, c, b)	$1/(1-y)$	$1/(1-t)$
$(t\lambda-t+1)/\lambda$	(d, b, a, c)	$(1-t)/(y-t)$	$(t-1)/t$
$1/\lambda$	(d, b, c, a)	$1/y$	$1/t$
$(\lambda+t-1)/\lambda$	(d, c, a, b)	$(t-1)/(y-1)$	$1-t$
t/λ	(d, c, b, a)	t/y	t

Figure 2: Fractional-linear transformations

We set

$$y_1 = \frac{1}{1-y_8}, \qquad t_1 = \frac{1}{1-t_0}, \qquad \psi = 2y_1 - 1, \qquad \theta = 2t_1 - 1, \tag{5.1}$$

$$\eta = \sqrt{y_8}, \quad \tau = \sqrt{t_0}, \qquad T_0 = \frac{(\tau-1)^2}{(\tau+1)^2}, \quad Y_8 = \frac{(\tau-1)(\eta-1)}{(\tau+1)(\eta+1)}, \tag{5.2}$$

$$\widetilde{Y}_1 = \frac{1}{1-Y_8}, \qquad T_1 = \frac{1}{1-t_0}, \qquad Y_1 = \frac{\widetilde{Y}_1}{T_1}, \tag{5.3}$$

$$\sigma = \frac{1}{2}\left(\tau + \frac{1}{\tau}\right), \quad \varrho = \frac{1}{2}\left(\eta + \frac{1}{\eta}\right), \quad \omega = \frac{1}{2}\left(\frac{\eta}{\tau} + \frac{\tau}{\eta}\right). \tag{5.4}$$

By fractional-linear transformations, the function y_1 is a solution of $(0, 0, B, A{+}1; t_1)$.
By Lemma 1.1, the function Y_8 is a solution of $P_{VI}\left(\frac{A}{2}, \frac{B}{2}, \frac{B}{2}, \frac{A}{2}{+}1; T_0\right)$.
By Lemma 2.2, the function Y_1 is a solution of $P_{VI}\left(\frac{B}{2}, \frac{B}{2}, \frac{A}{2}, \frac{A}{2}{+}1; T_0\right)$.
Observe the following relations:

$$t_0 = \frac{\theta-1}{\theta+1}, \qquad \theta = \frac{1+\tau^2}{1-\tau^2}, \qquad y_2 = \frac{\psi-1}{\psi+1}, \qquad \psi = \frac{1+\eta^2}{1-\eta^2}, \tag{5.5}$$

$$T_0 = \frac{\sigma-1}{\sigma+1}, \qquad \eta = \frac{\omega - \varrho\,\tau}{\sigma - \tau}, \qquad Y_8 = \frac{\sigma\,\varrho - \omega}{(\sigma+1)(\varrho+1)}, \tag{5.6}$$

$$2\,\sigma\,\varrho\,\omega = \sigma^2 + \varrho^2 + \omega^2 - 1. \tag{5.7}$$

We identify the following roots:

$$\sqrt{T_0} = \frac{\tau-1}{\tau+1}, \qquad \sqrt{\theta^2-1} = \frac{2\tau}{1-\tau^2}, \qquad \sqrt{\psi^2-1} = \frac{2\eta}{1-\eta^2}. \tag{5.8}$$

The symmetry which reduces the field extension by 2 is realized by the involutions $(y_1, t_1) \mapsto (1-y_1, 1-t_1)$, or $(y_8, t_0) \mapsto (1/y_8, 1/t_0)$. The functions σ, ϱ, ω are invariants of this symmetry. So they express the solution (Y, T) minimally. The invariants can be computed directly by

$$\sigma = \frac{1}{2}\sqrt{t_0 + \frac{1}{t_0} + 2}, \qquad \varrho = \frac{1}{2}\sqrt{y_8 + \frac{1}{y_8} + 2}, \qquad \omega = \frac{1}{2}\sqrt{\frac{y_8}{t_0} + \frac{t_0}{y_8} + 2}. \tag{5.9}$$

These square roots are rather nice explicitly. The root signs should be chosen so that relation (5.7) holds.

Now we list various expressions for the functions which are related to Lemaa 1.1 by fractional-linear transformations. In each case, we express t and y in some of the defined variables. To compute a quadratic transformation, one may start with one of the first 6 Painleve VI equations, compute (for example) τ and η from the corresponding expressions, pick one the last three list entries and read off local monodromy differences and an expression in terms of τ and v of other Painleve VI equation.

$\boxed{0, A, B, 1}$

$$t_0 = \tau^2 = \frac{\theta-1}{\theta+1}, \qquad y_8 = \eta^2 = \frac{\psi-1}{\psi+1}. \tag{5.10}$$

$\boxed{A, 0, B, 1}$

$$\frac{1}{t_1} = 1 - \tau^2 = \frac{2}{\theta+1}, \qquad \frac{1}{y_1} = 1 - \eta^2 = \frac{2}{\psi+1}. \tag{5.11}$$

$\boxed{0,0,B,A+1}$

$$t_1 = \frac{1}{1-\tau^2} = \frac{1+\theta}{2}, \qquad y_1 = \frac{1}{1-\eta^2} = \frac{1+\psi}{2}. \tag{5.12}$$

$\boxed{B,0,0,A+1}$

$$\frac{t_1-1}{t_1} = \tau^2 = \frac{\theta-1}{\theta+1}, \qquad \frac{t_1-y_1}{t_1} = \frac{\eta^2-\tau^2}{\eta^2-1} = \frac{\theta-\psi}{\theta+1}. \tag{5.13}$$

$\boxed{0,B,0,A+1}$

$$\frac{1}{t_1} = 1-\tau^2 = \frac{2}{\theta+1}, \qquad \frac{y_1}{t_1} = \frac{\tau^2-1}{\eta^2-1} = \frac{\psi+1}{\theta+1}. \tag{5.14}$$

$\boxed{A,B,0,1}$

$$t_1 = \frac{1}{1-\tau^2} = \frac{1+\theta}{2}, \qquad \frac{t_1}{y_1} = \frac{\eta^2-1}{\tau^2-1} = \frac{\theta+1}{\psi+1}. \tag{5.15}$$

$\boxed{\frac{A}{2},\frac{B}{2},\frac{B}{2},\frac{A}{2}+1}$

$$T_0 = \frac{(\tau-1)^2}{(\tau+1)^2} = (\theta-\sqrt{\theta^2-1})^2, \tag{5.16}$$

$$Y_8 = \frac{(\tau-1)(\eta-1)}{(\tau+1)(\eta+1)} = (\psi-\sqrt{\psi^2-1})(\theta-\sqrt{\theta^2-1}). \tag{5.17}$$

$\boxed{\frac{A}{2},\frac{A}{2},\frac{B}{2},\frac{B}{2}+1}$

$$T_1 = \frac{(\tau+1)^2}{4\tau} = \frac{1}{2} + \frac{\theta}{2\sqrt{\theta^2-1}}. \tag{5.18}$$

$$Y_1 = \frac{(\tau+1)(\eta+1)}{2\,(\eta+\tau)} = \frac{1}{2} + \frac{\sqrt{\psi^2-1}-\sqrt{\theta^2-1}}{2\,(\psi-\theta)}. \tag{5.19}$$

$\boxed{\frac{B}{2},\frac{A}{2},\frac{B}{2},\frac{A}{2}+1}$

$$1-T_0 = \frac{4\tau}{(\tau+1)^2}, \qquad 1-Y_8 = \frac{2\,(\eta+\tau)}{(\tau+1)(\eta+1)}. \tag{5.20}$$

Note that some fractional-linear transformations act identically on Painleve VI equations; these transformations may act as $\tau \to -\tau$, $\eta \to -\eta$ or simultaneous $\tau \to 1/\tau$, $\eta \to 1/\eta$. Interchanging A and B is equivalent to $\eta \to \tau/\eta$.

Now we wish to present some expressions for the "long" quadratic transformation. To comply with Theorem 1.3 and other results, we set $A = a+b-1$, $B = b-a$, and

$$y_0 = K_{(0,0,a-b,a+b;t_1)}\, y_1, \qquad \varphi = 2y_0 - 1, \qquad y_2 = K_{(a,a,-b,b;t_1)}\, y_0,$$
$$Y_2 = K_{\left(\frac{a-b}{2},\frac{a-b}{2},\frac{a+b-1}{2},\frac{a+b+1}{2};T_1\right)}\, Y_1, \qquad Y_0 = K_{\left(\frac{a-b}{2},\frac{a-b}{2},\frac{1-a-b}{2},\frac{a+b+1}{2};T_1\right)}\, Y_1.$$

Then

$$y_2 = \frac{1}{2} - \frac{\psi\varphi^2 - 2\varphi + \psi}{2\left(\varphi^2 - 2\psi\varphi + 1\right)}, \tag{5.21}$$

$$Y_2 = \frac{1}{2} + \frac{\theta - \psi + \sqrt{\psi^2 - 1}}{2\sqrt{\theta^2 - 1}} + \frac{(\varphi - \psi)\left(\psi - \sqrt{\psi^2 - 1}\right)}{\left(\varphi - \psi + \sqrt{\psi^2 - 1}\right)\sqrt{\theta^2 - 1}}. \tag{5.22}$$

Finally, let $y_7 = y_0/(y_0 - 1)$. This is a solution of $P_{VI}(a, b-1, b, a+1; t_0)$. Then the function

$$\frac{1-\tau}{1+\tau}\,\frac{a\,(\eta - \tau)\,(\tau\eta - y_7)}{b\,\tau(y_7 - \eta^2) + a(\tau^2 - y_7)\eta} \tag{5.23}$$

is a solution of $P_{VI}(a, 1/2, 1/2, b+1; T_0)$, and the function

$$\frac{1-\tau}{1+\tau}\,\frac{(a+b)(\eta + \tau y_7) + (a-b)(y_7 + \tau\eta) - 2a(y_7 + \tau)\eta}{(a+b)(\eta - \tau y_7) + (a-b)(y_7 - \tau\eta) - 2a(y_7 - \tau)\eta} \tag{5.24}$$

is a solution of $P_{VI}(1/2, b, a, 1/2; T_0)$.

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