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Funaki, Tadahisa
Graduate School of Mathematical Sciences, The University of Tokyo

Hariya, Yuu
Departement de Mathematiques, Institut Elie Cartan

Yor, Marc
Laboratoire de Probabilites et Modeles Aleatoires, Universite Pierre et Marie Curie

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T. Funaki, Y. Hariya
M. Yor

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Wiener integrals for centered powers of Bessel processes, I

Tadahisa Funaki*, Yuu Hariya† and Marc Yor‡

Abstract

The stochastic integrals of Wiener's type may be constructed relatively to the centered δ-dimensional Bessel processes (BES(δ)-processes in short) and their variants based on two different approaches. One approach, developed in [3], is via the Brascamp-Lieb inequality which works especially well for the BES(δ)-processes, BES(δ)-bridges with $\delta \geq 3$ or for the Brownian meander. The other approach, which is the subject of the present paper, goes via Hardy's L^2 inequality which is effective for general BES(δ)-processes and their powers. We shall also discuss an interplay of these two methods.

1 Introduction and main results

Consider, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$, a continuous semimartingale $(X_t, t \geq 0)$, with canonical decomposition:

$$X_t = M_t + V_t,$$

where (M_t) denotes an $(\mathcal{F}_t)$-local martingale, and (V_t) an $(\mathcal{F}_t)$-adapted, continuous process of bounded variation.

The theory of stochastic integration with respect to X consists in defining:

$$\int_0^t \phi_s \, dX_s = \int_0^t \phi_s \, dM_s + \int_0^t \phi_s \, dV_s,$$

*Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Tokyo 153-8914, Japan

†Département de Mathématiques, Institut Elie Cartan, BP239, 54506 Vandœuvre-lès-Nancy Cedex, France

‡Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, 4, Place Jussieu, Boîte Courrier 188, F-75252 Paris Cedex 05, France

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for $(\mathcal{F}_t)$-predictable processes (ϕ_t) which satisfy:

$$\int_0^t \phi_s^2 \, d\langle M\rangle_s < \infty \quad \text{and} \quad \int_0^t |\phi_s||dV_s| < \infty \tag{1.1}$$

a.s. for every t. However, in a number of situations, one would like to be able to define $(\int_0^t \phi_s \, dX_s, t \geq 0)$ for predictable processes (ϕ_t) which only satisfy:

$$\int_0^t \phi_s^2 \, d\langle M\rangle_s < \infty \quad \text{a.s. for every } t. \tag{1.1$'$}$$

This question arose very naturally in some discussions of enlargements of the Brownian filtration. More precisely, consider a Brownian motion $(\beta_t, t \geq 0)$ (and its filtration), and assume that, in a larger filtration $(\mathcal{F}_t)$, (β_t) remains a semimartingale, which then decomposes as:

$$\beta_t = B_t + V_t, \quad t \geq 0, \tag{1.2}$$

where (B_t) is an $(\mathcal{F}_t)$-Brownian motion. In [9], examples were given for which, although (1.2) holds, nonetheless, there exist even deterministic functions f in $L^2_{\text{loc}}(ds)$, i.e., $\int_0^t f(s)^2 \, ds \equiv \int_0^t f(s)^2 \, d\langle B\rangle_s < \infty$ such that

$$\int_0^t |f(s)||dV_s| = \infty.$$

To be precise, starting from a Brownian motion (B_t), consider the "second" Brownian motion:

$$\beta_t = B_t - \int_0^t \frac{B_s}{s} \, ds, \tag{1.3}$$

whose proper filtration is strictly smaller than that of B, but trivially (1.2) holds with:

$$V_t = -\int_0^t \frac{B_s}{s} \, ds.$$

It then follows from Jeulin's lemma [7, p.44] (see also [8]) that, if $f \in L^2_{\text{loc}}(ds)$, then $(\int_0^t f(s) \, d\beta_s, t \geq 0)$ is an $\mathcal{F}_t \equiv \sigma\{B_s, s \leq t\}$-semimartingale if and only if

$$\int_{0+} \frac{|f(s)|}{\sqrt{s}} \, ds < \infty.$$

(For several applications of Jeulin's lemma, we also refer to [12, 18].) Nonetheless, for every $f \in L^2_{\text{loc}}(ds)$,

$$\int_0^t f(s) \left(\frac{B_s}{s}\right) ds := \lim_{\varepsilon \downarrow 0} \int_\varepsilon^t f(s) \left(\frac{B_s}{s}\right) ds$$

exists both a.s. and in L^2, as follows easily from (1.3). Thus the identity:

$$\int_0^t f(s)\, d\beta_s = \int_0^t f(s)\, dB_s - \int_0^t f(s) \left(\frac{B_s}{s}\right) ds \tag{1.4}$$

is meaningful for every $f \in L^2_{\mathrm{loc}}(ds)$.

In the present paper, we intend to make a similar discussion when instead of (1.2), we consider the semimartingale:

$$X_t = R_\delta(t), \quad t \geq 0,$$

where $(R_\delta(t), t \geq 0)$ is a δ-dimensional Bessel process (BES(δ)-process in short), $\delta \geq 1$, starting from 0. We recall the canonical decomposition:

$$R_\delta(t) = B_t + V_\delta(t), \quad t \geq 0, \tag{1.5}$$

where

$$V_\delta(t) = \begin{cases} L_t, & \delta = 1, \\ \dfrac{\delta - 1}{2} \displaystyle\int_0^t \frac{ds}{R_\delta(s)}, & \delta > 1. \end{cases} \tag{1.6}$$

(For $\delta = 1$, we may consider: $R_1(t) = |\beta_t|$; $B_t = \int_0^t \mathrm{sgn}(\beta_s)\, d\beta_s$, and (L_t) the local time at 0 of β.) Then, again as a consequence of Jeulin's lemma, we can show that, for $f \in L^2_{\mathrm{loc}}(ds)$, there is the equivalence:

$$\int_0^t |f(s)||dV_\delta(s)| < \infty \quad \Longleftrightarrow \quad \int_0^t \frac{|f(s)|}{\sqrt{s}}\, ds < \infty.$$

Despite this, we shall show the following analogue of (1.4): define the centered Bessel process

$$\widehat{R_\delta}(t) = R_\delta(t) - E[R_\delta(t)] \equiv B_t + \widehat{V}_\delta(t), \quad t \geq 0.$$

Then the integral $\int_0^t f(s)\, d\widehat{R_\delta}(s)$ makes sense for every $f \in L^2_{\mathrm{loc}}(ds)$. In fact, we have obtained the much more precise results. For simplicity, we restrict our argument to the time interval $[0, 1]$: For a bounded, measurable function $f : [0, 1] \to \mathbb{R}$, we set

$$I(f; \widehat{R_\delta}) = \int_0^1 f(u)\, d\widehat{R_\delta}(u).$$

Theorem 1.1. *For every $\delta \geq 1$, it holds that, for all bounded, measurable functions $f : [0, 1] \to \mathbb{R}$,*

$$E[I(f; \widehat{R_\delta})^2] \leq \|f\|_2^2. \tag{1.7}$$

Moreover, the constant 1 implicit in (1.7) *is the optimal constant K such that $E[I(f; \widehat{R_\delta})^2] \leq K\|f\|_2^2$.*

Note that, for each $f^* \in L^2([0,1])$, we may find a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded, measurable functions on $[0,1]$, such that $\|f_n - f^*\|_2 \to 0$ as $n \to \infty$. From (1.7), we see that $\{I(f_n; \widehat{R_\delta})\}_{n\in\mathbb{N}}$ forms a Cauchy sequence in $L^2(P)$. So we may define the Wiener integral $I(f^*; \widehat{R_\delta})$ as its limit. Thus we obtain, as a corollary to Theorem 1.1:

Corollary 1.2. *For all $f \in L^2([0,1])$, the Wiener integral $I(f; \widehat{R_\delta})$ is well-defined relative to* BES(δ), *$\delta \geq 1$, starting from* 0. *Moreover, it enjoys the property* (1.7).

We may extend the above results by replacing $(R_\delta(t), t \geq 0)$ by $(R_\delta(t))^\alpha$, for every α such that $(R_\delta(t))^\alpha$ is a semimartingale; that is,

$$\alpha \geq (2-\delta)_+.$$

Here, for $a \in \mathbb{R}$, $(a)_+ = \max\{a, 0\}$. In this case, we may modify the above as follows: for $\beta \in \mathbb{R}$ satisfying $\delta + \beta > 0$, set

$$\kappa_\beta = E[R_1^\beta] \equiv 2^{\beta/2} \frac{\Gamma(\frac{\delta+\beta}{2})}{\Gamma(\frac{\delta}{2})}.$$

Theorem 1.3. *For $\delta > 0$ and $\alpha \in (0,2)$, suppose that $\alpha \geq (2-\delta)_+$. Then it holds that, for all bounded, measurable functions f on* $[0,1]$,

$$E[I(f; \widehat{R_\delta^\alpha})^2] \leq \alpha^2 \kappa_{2(\alpha-1)} \|f_\alpha\|_2^2, \tag{1.8}$$

where $f_\alpha(u) := f(u)u^{(\alpha-1)/2}$.

Remark 1.1. *The constant $\alpha^2\kappa_{2(\alpha-1)}$ in* (1.8) *is also optimal (see Remark 3.1).*

We denote by $L^2_\alpha([0,1])$ the weighted L^2 space defined by

$$L^2_\alpha([0,1]) = \{f; f_\alpha \in L^2([0,1])\}.$$

As a consequence of Theorem 1.3, we also have the following:

Corollary 1.4. *Suppose that δ and α satisfy the assumption of Theorem 1.3. Then, for all $f \in L^2_\alpha([0,1])$, the Wiener integral $I(f; \widehat{R_\delta^\alpha})$ is well-defined.*

The rest of the paper is organized as follows: in Section 2, after developing some Hardy-like inequalities, we prove Theorem 1.1; Section 3 is devoted to the proof of Theorem 1.3; in Section 4, we show that as a consequence of the previous results, Wiener integral may be defined relatively to the centered BES(δ)-bridges, for $\delta \geq 1$; finally, in Section 5, which we consider as an Appendix, we have gathered several topics we refer to in our discussions.

2 An approach via Hardy's L^2 inequality

2.1 Some Hardy-like inequalities

Proposition 2.1. *For a non-negative, integrable function $\Phi(t), 0 \le t \le 1$, the operator*

$$T_\Phi : f \to \{\int_0^1 dt\, f(ut)\Phi(t), u \le 1\}$$

which is well-defined on bounded Borel measurable functions f, extends as a bounded linear operator on $L^2([0,1])$ if and only if

$$K_\Phi := \int_0^1 dt\, \frac{\Phi(t)}{\sqrt{t}} < \infty. \tag{2.1}$$

Moreover,

$$\|T_\Phi f\|_2 \le K_\Phi \|f\|_2 \quad \textit{for all } f \in L^2([0,1]),$$

and the constant K_Φ is the optimal constant K such that: $\|T_\Phi f\|_2 \le K\|f\|_2$; that is, the operator norm $|||T_\Phi|||$ of T_Φ is equal to K_Φ.

For Φ satisfying (2.1), we define the quadratic form J_Φ on $L^2([0,1]) \times L^2([0,1])$ by:

$$J_\Phi(f,g) = \int_0^1 du\, f(u) T_\Phi g(u), \quad f, g \in L^2([0,1]). \tag{2.2}$$

Proof of Proposition 2.1. As is well known, it holds that:

$$|||T_\Phi||| = \sup\{|J_\Phi(f,g)|; f, g \in L^2([0,1]), \|f\|_2 \le 1, \|g\|_2 \le 1\}.$$

By Fubini's theorem and the Cauchy-Schwarz inequality, we have:

$$\begin{aligned} |J_\Phi(f,g)| &\le \int_0^1 dt\, \Phi(t)\|f\|_2 \left(\int_0^1 du\, g(ut)^2\right)^{1/2} \\ &\le \int_0^1 dt\, \frac{\Phi(t)}{\sqrt{t}} \|f\|_2 \|g\|_2, \end{aligned}$$

which shows $|||T_\Phi||| \le K_\Phi$. On the other hand, if we assume that $|J_\Phi(f,g)| \le K\|f\|_2\|g\|_2$, then, by taking $f \equiv g \equiv \psi_\alpha$, where $\psi_\alpha(u) = u^\alpha$ for $\alpha > -1/2$, we get $J_\Phi(\psi_\alpha, \psi_\alpha) \le K(1+2\alpha)^{-1}$, but

$$J_\Phi(\psi_\alpha, \psi_\alpha) = \int_0^1 dt\, \Phi(t) t^\alpha \times \frac{1}{1+2\alpha}.$$

Therefore, $\int_0^1 dt\, \Phi(t) t^\alpha \le K$ for every $\alpha > -1/2$, and thus letting $\alpha \downarrow -1/2$, we get $K_\Phi \le K$, which ends the proof. □

The next two propositions were proven and kindly proposed to us by F. Hirsh [6].

Proposition 2.2. *Suppose* $\Phi \geq 0$ *and satisfies* (2.1). *Moreover, suppose that there exists* $\alpha > -1/2$ *such that* Φ *is expanded as:*

$$\Phi(t) = t^{\alpha} \sum_{n=0}^{\infty} a_n t^n, \quad a_n \geq 0. \tag{2.3}$$

Then the quadratic form J_Φ *is non-negative definite; i.e.,* $J_\Phi(f,f) \geq 0$ *for all* $f \in L^2([0,1])$.

Proof. If we denote $\phi_\beta(t) = t^\beta$, we have:

$$\Phi(t) = \sum_{n=1}^{\infty} a_n \phi_{n+\alpha}(t),$$

and, in order to prove the assertion, it suffices to show: for any $\beta > -1/2$,

$$J_{\phi_\beta}(f,f) \geq 0. \tag{2.4}$$

Moreover, by approximating each $f \in L^2([0,1])$ by functions in $C([0,1])$, we only have to prove (2.4) for $f \in C([0,1])$. By change of variables, and by integration by parts,

$$\begin{aligned} J_{\phi_\beta}(f,f) &= \int_0^1 du\, u^{-2\beta-1} u^\beta f(u) \int_0^u dv\, v^\beta f(v) \\ &= F_\beta(1) + (2\beta+1) \int_0^1 du\, u^{-2\beta-2} F_\beta(u). \end{aligned}$$

Here we set

$$F_\beta(u) = \int_0^u ds\, f(s) s^\beta \int_0^u dv\, f(v) v^\beta \equiv \frac{1}{2} \left(\int_0^u ds\, f(s) s^\beta \right)^2,$$

which is non-negative. So the assertion is proved. □

In fact, the functions Φ such that J_Φ is non-negative definite may be characterized as follows: for $f : [0,1] \to \mathbb{R}$, we denote $\widetilde{f} : \mathbb{R}_+ \to \mathbb{R}$ defined by $\widetilde{f}(x) = e^{-x/2} f(e^{-x})$. The mapping $f \to \widetilde{f}$ is an onto isometry from $L^2([0,1])$ on $L^2(\mathbb{R}_+)$. Let $\phi : [0,1] \to \mathbb{R}$ such that $|\phi|$ satisfies (2.1). Then $\int_0^\infty |\widetilde{\phi}(x)|\, dx < \infty$. Let

$$H(x) = \widetilde{\phi}(|x|) = e^{-|x|/2} \phi(e^{-|x|})$$

Proposition 2.3. *The operator T_ϕ is positive if and only if H is positive definite in the sense that, for every $h \in L^2(\mathbb{R})$,*

$$\iint_{\mathbb{R}^2} h(x)h(y)H(x-y)\,dxdy \geq 0. \tag{2.5}$$

Proof. Note that

$$\widetilde{T_\phi f}(s) = \int_0^\infty \widetilde{f}(t+s)\widetilde{\phi}(t)\,dt.$$

Thus, it holds that

$$\begin{aligned} J_\phi(f,f) &= \int_0^\infty\int_0^\infty dsdt\,\widetilde{f}(s)\widetilde{f}(t+s)\widetilde{\phi}(t) \\ &= \int_0^\infty ds\,\widetilde{f}(s)\int_s^\infty dt\,\widetilde{f}(t)\widetilde{\phi}(t-s) \\ &= \int_0^\infty ds\,\widetilde{f}(s)\int_s^\infty dt\,\widetilde{f}(t)H(t-s). \end{aligned}$$

Hence, from the symmetry of H,

$$J_\phi(f,f) = \frac{1}{2}\iint_{\mathbb{R}^2_+} dsdt\,\widetilde{f}(s)\widetilde{f}(t)H(t-s).$$

Thus, $J_\phi \geq 0$ if and only if

$$\iint_{\mathbb{R}^2_+} dsdt\,g(s)g(t)H(t-s) \geq 0 \quad \text{for all } g \in L^2(\mathbb{R}_+).$$

It follows immediately that, if H is positive definite, then J_ϕ is positive. Conversely, if $J_\phi \geq 0$ and if $h \in L^2(\mathbb{R})$, then, for every $a \leq 0$,

$$\begin{aligned} &\int_a^\infty\int_a^\infty dsdt\,h(s)h(t)H(t-s) \\ &= \iint_{\mathbb{R}^2_+} dsdt\,h(a+s)h(a+t)H(t-s) \geq 0. \end{aligned}$$

Letting a go to $-\infty$, we have:

$$\iint_{\mathbb{R}^2} dsdt\,h(s)h(t)H(t-s) \geq 0.$$

So the proof is complete. □

As a corollary, we have the following (from Bochner's theorem):

Corollary 2.4. *If ϕ is continuous on $(0,1]$, $J_\phi \geq 0$ if and only if there exists a finite positive symmetric measure on $\mathbb{R}$ such that*

$$\phi(t) = \frac{1}{\sqrt{t}} \int_{-\infty}^{+\infty} t^{ix}\, d\mu(x). \tag{2.6}$$

Example 2.1. (i) *When ϕ is given in the form* (2.3)*, the measure μ in the above corollary is given by:*

$$d\mu(x) = \sum_{n=0}^{\infty} a_n \frac{n+\alpha+\frac{1}{2}}{(n+\alpha+\frac{1}{2})^2 + x^2} dx. \tag{2.7}$$

(ii) *An interesting class of functions ϕ which satisfy* (2.6) *are those ϕ's given by:*

$$e^{-|x|/2}\phi(e^{-|x|}) = c \int_0^\infty e^{-x^2 y/2}\, dm(y)$$

for some constant $c > 0$, and for a probability measure m on $\mathbb{R}_+$; the measure μ in the above corollary then appears as the law of B_A, where B is a Brownian motion, and A is a random variable independent of B and distributed as m.

2.2 Proof of Theorem 1.1

We first look for some rahter explicit expression for $E[I(f;\widehat{R_\delta})^2]$:

Proposition 2.5. *Let $f : [0,1] \to \mathbb{R}$ be a bounded measurable function. Then*

$$E[I(f;\widehat{R_\delta})^2] = \|f\|_2^2 - J_{\Phi_\delta}(f,f), \tag{2.8}$$

where

$$\Phi_\delta(t) = \begin{cases} \dfrac{1}{\pi}\left\{\dfrac{1}{\sqrt{t}}(1-\sqrt{1-t}) + \dfrac{\sqrt{t}}{\sqrt{1-t}}\right\}, & \delta = 1, \\ -(\delta-1)\dfrac{d}{dt}E[\widehat{R_\delta(1)^{-1}}\widehat{R_\delta(t)}], & \delta > 1, \end{cases}$$

and J_{Φ_δ} is defined through (2.2)*.*

Example 2.2. *In the case $\delta = 3$, Φ_δ is given by*

$$\Phi_3(t) = \frac{4}{\pi}\frac{1}{\sqrt{t}}(1-\sqrt{1-t}).$$

For some connection between Φ_1 and Φ_3, which may be deduced from Pitman's representation theorem for BES(3)*, see Lemma 5.3 in the appendix.*

Suppose that $(X_t, t \geq 0)$ is an $(\mathcal{F}_t)$-semimartingale with decomposition

$$X_t = M_t + V_t, \quad t \geq 0, \tag{2.9}$$

where (M_t) is an L^2-integrable $(\mathcal{F}_t)$-martingale and (V_t) is an $(\mathcal{F}_t)$-adapted process of bounded variation satisfying $E[(\int_0^t |dV_u|)^2] < \infty$ for every t. For a bounded, measurable function f on $[0,1]$, we set

$$X_t^f = \int_0^t f(u)\, dX_u, \quad 0 \leq t \leq 1.$$

Clearly, (X_t^f) is a semimartingale which decomposes as:

$$X_t^f = \int_0^t f(u)\, dM_u + \int_0^t f(u)\, dV_u, \quad 0 \leq t \leq 1.$$

Lemma 2.6. *It holds that*

$$E[(X_1^f)^2] = E[\int_0^1 f(u)^2\, d\langle M\rangle_u] + 2E[\int_0^1 f(u)\, dV_u \left(\int_0^u f(s)\, dX_s\right)].$$

Proof. By Itô's formula,

$$(X_1^f)^2 = 2\int_0^1 X_u^f\, dX_u^f + \langle X^f\rangle_1.$$

Taking the expectation on both sides, and noting $\langle X^f\rangle_1 = \int_0^1 f(u)^2\, d\langle M\rangle_u$, we obtain the lemma. □

By using Lemma 2.6, we prove Proposition 2.5. We begin with the case $\delta = 1$.

Proof of Proposition 2.5 for $\delta = 1$. Noting (1.5), we take $X_t = R_\delta(t) \equiv |\beta_t|$, $M_t = B_t$ and $V_t = V_\delta(t) \equiv L_t$ in (2.9). Then by Lemma 2.6,

$$E[\left(\int_0^1 f(u)\, d|\beta_u|\right)^2] = \|f\|_2^2 + 2E[\int_0^1 f(u)\, dL_u \left(\int_0^u f(s)\, d|\beta_s|\right)]. \tag{2.10}$$

Recall that dL_u is carried by the set of zeros of β. So, if we denote by $(b^u(s), 0 \leq s \leq u)$ a Brownian bridge of length u, the expectation on the RHS of (2.10) may be written as:

$$\int_0^1 f(u)\, dE[L_u] E[\int_0^u f(s)\, d|b^u(s)|]. \tag{2.11}$$

Note that $E[L_u] \equiv E[|\beta_u|] = \sqrt{2u/\pi}$, hence $dE[L_u] = du/\sqrt{2\pi u}$. Moreover, since $b^u(s)$ is Gaussian with mean 0 and variance $s(u-s)/u$,

$$E[|b^u(s)|] = \sqrt{\frac{2}{\pi}}\sqrt{\frac{s(u-s)}{u}},$$

hence

$$dE[|b^u(s)|] = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{s(1-s/u)}} \left(1 - \frac{2s}{u}\right) ds.$$

Combining these, we see that (2.11) is equal to:

$$\begin{aligned} &\frac{1}{2\pi} \int_0^1 \frac{du}{\sqrt{u}} f(u) \int_0^u ds\, f(s) \frac{1}{\sqrt{s(1-s/u)}} \left(1 - \frac{2s}{u}\right) \\ &= \frac{1}{2\pi} \int_0^1 du\, f(u) \int_0^1 dt\, f(ut) \frac{1-2t}{\sqrt{t(1-t)}}. \end{aligned}$$

Here, for the equality, we changed variables with $s = ut, 0 \le t \le 1$. From this and (2.10),

$$E[\left(\int_0^1 f(u)\, d|\beta_u|\right)^2] = \|f\|_2^2 + J_{\Psi_1}(f,f), \quad \Psi_1(t) := \frac{1}{\pi} \frac{1-2t}{\sqrt{t(1-t)}}. \tag{2.12}$$

On the other hand, $E[\int_0^1 f(u)\, d|\beta_u|] = \int_0^1 du\, f(u)/\sqrt{2\pi u}$, hence

$$\left(E[\int_0^1 f(u)\, d|\beta_u|]\right)^2 = \frac{1}{\pi} \int_0^1 \frac{du}{\sqrt{u}} f(u) \int_0^u \frac{ds}{\sqrt{s}} f(s) = J_{\Psi_2}(f,f),$$

where $\Psi_2(t) := 1/(\pi\sqrt{t})$. Combining this with (2.12), we see that

$$E[I(f;\widehat{R_1})^2] \equiv \mathrm{var}\left(\int_0^1 f(u)\, d|\beta_u|\right) = \|f\|_2^2 - J_{\Psi_2 - \Psi_1}(f,f).$$

Noting the identity: $\Phi_1 = \Psi_2 - \Psi_1$, we obtain (2.8) for $\delta = 1$. □

Next, we turn to the case $\delta > 1$.

Proof of Proposition 2.5 for $\delta > 1$. By the decomposition (1.5), we take $X_t = \widehat{R_\delta(t)}$, $M_t = B_t$ and $V_t = \widehat{V_\delta(t)}$ in (2.9). Then, noting $dV_u = 2^{-1}(\delta - 1)\widehat{R_\delta(u)^{-1}}\, du$, we see from Lemma 2.6 that

$$\begin{aligned} &E[(X_1^f)^2] \equiv E[I(f;\widehat{R_\delta})^2] \\ &= \|f\|_2^2 + (\delta-1) E[\int_0^1 f(u) \widehat{R_\delta(u)^{-1}}\, du \left(\int_0^u f(s)\, d\widehat{R_\delta(s)}\right)] \\ &= \|f\|_2^2 + (\delta-1) \int_0^1 du\, f(u) \int_0^u ds\, f(s) \frac{d}{ds} E[\widehat{R_\delta(u)^{-1}} \widehat{R_\delta(s)}]. \end{aligned}$$

Now (2.8) follows from this and the change of variables with $s = ut$. □

Lemma 2.7. (i) *For every $\delta \geq 1$, Φ_δ is non-negative and satisfies:*

$$\int_0^1 \frac{dt}{\sqrt{t}} \Phi_\delta(t) < \infty.$$

(ii) *J_{Φ_δ} extends as a quadratic form on $L^2([0,1]) \times L^2([0,1])$. Moreover, it is positive-definite.*

Proof. (i) This assertion for $\delta = 1$ is obvious by definition. For $\delta > 1$, we denote $\varphi_\delta(t) = E[\widehat{R_\delta(1)^{-1}}\widehat{R_\delta(t)}], 0 \leq t \leq 1$, and use the fact that φ_δ admits the following representation:

$$\varphi_\delta(t) = C_\delta t^{1/2}\{{}_2F_1(-\frac{1}{2}, \frac{1}{2}; \frac{\delta}{2}; t) - 1\}, \quad C_\delta = \frac{\Gamma(\frac{\delta+1}{2})\Gamma(\frac{\delta-1}{2})}{\{\Gamma(\frac{\delta}{2})\}^2}.$$

Here ${}_2F_1$ denotes Gauss' hypergeometric function; see Proposition 5.1 in the appendix. Recalling the series expansion (in fact, the definition) of ${}_2F_1$ (e.g., [10, (9.1.1)]), we see that:

$$\text{(a) } \varphi_\delta \leq 0; \qquad \text{(b) } \varphi_\delta' \leq 0; \qquad \text{(c) } \liminf_{t \downarrow 0} \frac{1}{\sqrt{t^3}} \varphi_\delta(t) > -\infty.$$

Indeed,

$$t^{1/2}\{{}_2F_1(-\frac{1}{2}, \frac{1}{2}; \frac{\delta}{2}; t) - 1\} = \sum_{n=1}^{\infty} b_n t^{n+1/2}, \quad b_n = \frac{(-\frac{1}{2})_n (\frac{1}{2})_n}{(\frac{\delta}{2})_n n!} < 0, \tag{2.13}$$

from which the properties (a)–(c) of φ_δ follow. Here, for $\gamma \in \mathbb{R}$, $(\gamma)_n := \gamma(\gamma+1)\cdots(\gamma+n-1)$. (In fact, for (c), the limit itself exists: $\lim_{t \downarrow 0} t^{-3/2}\varphi_\delta(t) = -C_\delta/(2\delta)$.) So, by (b) and by definition, $\Phi_\delta(t) = -(\delta-1)\varphi_\delta'(t) \geq 0$. Moreover, by (c), we may use integration by parts to see:

$$\begin{aligned} \int_0^1 \frac{dt}{\sqrt{t}} \Phi_\delta(t) &\equiv -(\delta-1) \int_0^1 \frac{dt}{\sqrt{t}} \varphi_\delta'(t) \\ &= -(\delta-1)\left\{\varphi_\delta(1) + \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t^3}} \varphi_\delta(t)\right\} < \infty. \end{aligned}$$

(ii) The former assertion now follows from (i) and Proposition 2.1. To show the latter, we shall show that Φ_δ can be developed as:

$$\Phi_\delta(t) = t^{1/2} \sum_{n=0}^{\infty} a_n t^n, \quad a_n \geq 0. \tag{2.14}$$

Then the positivity of J_{Φ_δ} follows immediately from Proposition 2.2. For the case $\delta = 1$, note that the function $1 - \sqrt{1-t}, 0 \le t \le 1$, can be developed as:

$$1 - \sqrt{1-t} = \sum_{n=1}^{\infty} c_n t^n, \quad c_n = \frac{(2n)!}{2^{2n}(n!)^2(2n-1)}, \tag{2.15}$$

from which it also follows that, by differentiating both sides,

$$\frac{1}{2\sqrt{1-t}} = \sum_{n=1}^{\infty} n c_n t^{n-1}. \tag{2.16}$$

Combining these, we see that Φ_1 is expanded as (2.14) with

$$a_n = \frac{1}{\pi}\{1 + 2(n+1)\}c_{n+1} > 0, \quad n = 0, 1, 2, \cdots.$$

On the other hand, for $\delta > 1$, we see from (2.13) that

$$\varphi'_\delta(t) = C_\delta \sum_{n=1}^{\infty} \left(n + \frac{1}{2}\right) b_n t^{n-1/2}.$$

Since $\Phi_\delta(t) = -(\delta - 1)\varphi'_\delta(t)$ by definition, $\Phi_\delta, \delta > 1$, is also expanded as (2.14) with

$$a_n = C_\delta(\delta - 1)\left(n + \frac{3}{2}\right) \times (-b_{n+1}) > 0, \quad n = 0, 1, 2, \cdots.$$

So the proof is complete. □

Now we are ready to prove Theorem 1.1:

Proof of Theorem 1.1. The former assertion (1.7) is immediate from Proposition 2.5 and Lemma 2.7. By Proposition 2.5, in order to prove the latter, it suffices to find a sequence (f_n) of functions such that

$$J_{\Phi_\delta}(f_n, f_n) \le \varepsilon_n \|f_n\|_2^2$$

with $\varepsilon_n \to 0$ as $n \to \infty$. This is satisfied by the choice: $f_n(u) \equiv f_{\alpha_n}(u) = u^{\alpha_n}$ for $\alpha_n \to \infty$. Indeed,

$$\begin{aligned} J_{\Phi_\delta}(f_\alpha, f_\alpha) &= \left(\int_0^1 du\, u^{2\alpha}\right) \int_0^1 dt\, t^{\alpha - 1/2}\Phi_\delta(t) \\ &= \|f_\alpha\|_2^2 \int_0^1 dt\, t^{\alpha-1/2}\Phi_\delta(t), \end{aligned}$$

in which the integral relative to t decreases to 0 as $\alpha \to \infty$ by the dominated convergence theorem since it is bounded from above by $\int_0^1 dt\, t^{-1/2}\Phi_\delta(t) < \infty$. □

3 Proof of Theorem 1.3

In this section, we denote by the pair $(R = \{R_t, t \geq 0\}, P_x^{(\delta)})$ a BES(δ)-process starting from x: $P_x^{(\delta)}(R_0 = x) = 1$. We denote by $E_x^{(\delta)}$ the expectation with respect to $P_x^{(\delta)}$. When $x = 0$, we often write E for $E_0^{(\delta)}$ and P for $P_0^{(\delta)}$.

3.1 Proof for the case $\delta > 1$ and $\alpha > (2-\delta)_+$

In this subsection, we prove the assertion of Theorem 1.3 in the case $\delta > 1$ and $\alpha > (2-\delta)_+$. Set $\delta_\alpha = \alpha(\delta + \alpha - 2)$.

Proposition 3.1. *Suppose $\delta > 1$ and $\alpha > (2-\delta)_+$. Then we have, for every bounded, measurable function f on $[0,1]$,*

$$E[I(f;\widehat{R^\alpha})^2] = \alpha^2 \kappa_{2(\alpha-1)} \|f_\alpha\|_2^2 - J_{\Phi_{\delta,\alpha}}(f_\alpha, f_\alpha), \tag{3.1}$$

where

$$\Phi_{\delta,\alpha}(t) = -\delta_\alpha t^{-(\alpha-1)/2} \varphi'_{\delta,\alpha}(t) \quad \textit{with} \quad \varphi_{\delta,\alpha}(t) = E[\widehat{R_1^{\alpha-2}} \widehat{R_t^\alpha}],\ 0 \leq t \leq 1.$$

Proof. Recall that, for $\delta > 1$, BES(δ) admits a semimartingale decomposition of the following form:

$$R_t = B_t + \frac{\delta - 1}{2} \int_0^t \frac{du}{R_u}, \tag{3.2}$$

where $B = (B_t, t \geq 0)$ is a Brownian motion. By Itô's formula,

$$R_t^\alpha = \alpha \int_0^t R_u^{\alpha-1}\, dB_u + \frac{\delta_\alpha}{2} \int_0^t R_u^{\alpha-2}\, du. \tag{3.3}$$

We take the expectation on both sides to get

$$E[R_t^\alpha] = \frac{\delta_\alpha}{2} \int_0^t E[R_u^{\alpha-2}]\, du. \tag{3.4}$$

From these, we have

$$\widehat{R_t^\alpha} = \alpha \int_0^t R_u^{\alpha-1}\, dB_u + \frac{\delta_\alpha}{2} \int_0^t \widehat{R_u^{\alpha-2}}\, du.$$

Now we take, in (2.9), $X_t = \widehat{R_t^\alpha}$, $M_t = \int_0^t R_u^{\alpha-1}\, dB_u$ and $V_t = (\delta_\alpha/2) \int_0^t \widehat{R_u^{\alpha-2}}\, du$. Then by Lemma 2.6, it holds that, for a bounded, measurable function f on $[0,1]$,

$$\begin{aligned} &E[(X_1^f)^2]\ (\equiv E[I(f;\widehat{R^\alpha})^2]) \\ &= \delta_\alpha E\Big[\int_0^1 \Big(\int_0^u f(s)\, d\widehat{R_s^\alpha}\Big) f(u) \widehat{R_u^{\alpha-2}}\, du\Big] + \alpha^2 \int_0^1 f(u)^2 E[R_u^{2(\alpha-1)}]\, du \\ &= \delta_\alpha \int_0^1 du\, f(u) \int_0^u ds\, f(s) \left\{ \frac{d}{ds} E[\widehat{R_u^{\alpha-2}} \widehat{R_s^\alpha}] \right\} + \alpha^2 E[R_1^{2(\alpha-1)}] \int_0^1 f(u)^2 u^{\alpha-1}\, du. \end{aligned} \tag{3.5}$$

Now the assertion of the proposition is immediate by the change of variables. □

Lemma 3.2. *$J_{\Phi_{\delta,\alpha}}$ extends as a quadratic form on $L^2([0,1]) \times L^2([0,1])$. Moreover, it is positive-definite.*

Once this lemma is shown, then, combining this with Proposition 3.1, we may prove the assertion of Theorem 1.3 for $\delta > 1$ and $\alpha > (2-\delta)_+$:

Proof of Theorem 1.3 for $\delta > 1$ and $\alpha > (2-\delta)_+$. By Proposition 3.1 and Lemma 3.2, we have:

$$E_0^{(\delta)}[I(f;\widehat{R^\alpha})^2] \le \alpha^2 \kappa_{2(\alpha-1)} \|f_\alpha\|_2^2,$$

which is (1.8). □

Remark 3.1. *Similarly to the proof of Theorem 1.1, we may see that the constant $\alpha^2\kappa_{2(\alpha-1)}$ is optimal.*

The proof of Lemma 3.2 may be given along the lines of that of Theorem 1.1 for $\delta > 1$, Φ_δ and φ_δ being replaced by $\Phi_{\delta,\alpha}$ and $\varphi_{\delta,\alpha}$, respectively. Instead of giving details (and in order to avoid duplications), we present here a different way, other than relying on series expansions, to show that the function $\varphi_{\delta,\alpha}$ has similar properties to those (a)–(c) of φ_δ, which we shall present in the next proposition:

Proposition 3.3. *Suppose that $\delta > 0$ and $\alpha \in (0,2)$ satisfy $\alpha > 2-\delta$. Then the function $\varphi_{\delta,\alpha}$ has the following properties:*

$$\text{(a}') \ \varphi_{\delta,\alpha} \le 0; \qquad \text{(b}') \ \varphi'_{\delta,\alpha} \le 0; \qquad \text{(c}') \ \liminf_{t\downarrow 0} t^{-\frac{\alpha}{2}-1}\varphi_{\delta,\alpha}(t) > -\infty.$$

Remark 3.2. *By using the series expansion of $\varphi_{\delta,\alpha}$, we may see that, for the property (c′), the limit itself exists; see Remark 5.1 in the appendix.*

For $x \ge 0$ and $t > 0$, set

$$\rho_\alpha(x,t) \equiv \rho_\alpha^{(\delta)}(x,t) = E_x^{(\delta)}[R_t^\alpha],$$
$$\sigma_\alpha(x,t) \equiv \sigma_\alpha^{(\delta)}(x,t) = E_x^{(\delta)}[R_t^{\alpha-2}].$$

To prove Proposition 3.3, we begin with the following lemma:

Lemma 3.4. *Under the same assumption as in Proposition 3.3, the following assertions hold for every fixed $t > 0$:*
(i) *$\rho_\alpha(x,t)$ is increasing in x.*
(ii) *$\sigma_\alpha(x,t)$ is decreasing in x.*
(iii) *The following lower bound for $\sigma_\alpha(x,t)$ holds:*

$$\sigma_\alpha(x,t) \geq e^{-\frac{x^2}{2t}}\sigma_\alpha(0,t) \quad \textit{for all } x \geq 0.$$

Proof. The assertions (i) and (ii) follow from the additive property of Bessel processes; that is, if $R_x^{(\delta)}(t), t \geq 0$, denotes BES($\delta$) starting from x, then it holds that:

$$(R_x^{(\delta)}(t))^2 \overset{(d)}{=} (R_x^{(0)}(t))^2 + (R_0^{(\delta)}(t))^2, \tag{3.6}$$

where, on the RHS, the two processes are independent. Moreover, considering two SDE's with common Brownian motion B:

$$(R_x^{(0)}(t))^2 = x^2 + \int_0^t 2R_x^{(0)}(u)\,dB_u, \qquad (R_y^{(0)}(t))^2 = y^2 + \int_0^t 2R_y^{(0)}(u)\,dB_u,$$

we see that

$$R_x^{(0)}(t) \leq R_y^{(0)}(t) \quad \text{if} \quad x \leq y,$$

by comparison. The assertion (iii) also follows from (3.6) since we deduce from there that

$$\sigma_\alpha(x,t) \geq E_0^{(\delta)}[R_t^{\alpha-2}]P_x^{(0)}(T_0(R) \leq t),$$

where $T_0(R)$ is the time at which R reaches to 0. It is well-known (e.g., [16, Chapter XI]) that, under $P_x^{(0)}$, $T_0(R)$ is identical in law with $x^2/2\boldsymbol{e}$, $\boldsymbol{e}$ being a standard exponential variable, hence $P_x^{(0)}(T_0(R) \leq t) = e^{-x^2/2t}$. Combining these yields the assertion (iii). □

Remark 3.3. *The assertions* (i) *and* (ii) *may be seen from the FKG inequality; an application of a generalized FKG inequality due to Preston [14], to the laws of* BES(δ), $\delta > 0$, *asserts that, for increasing functions F on $C([0,1];\mathbb{R}_+)$, it holds that $E_x^{(\delta)}[F(R)] \leq E_y^{(\delta)}[F(R)]$ if $0 \leq x \leq y$. Here we say that a function F defined on $C([0,1];\mathbb{R}_+)$ is increasing if $F(w_1) \leq F(w_2)$ for all $w_1, w_2 \in C([0,1];\mathbb{R}_+)$ satisfying $w_1(t) \leq w_2(t)$ for all $0 \leq t \leq 1$. So, if we take $F(R) = R_t^\alpha$ (resp. $F(R) = -R_t^{\alpha-2}$), we recover the assertion* (i) *(resp. the assertion* (ii)*). For the applicability of this inequality to Bessel processes, see Proposition 5.6 in the appendix.*

Set

$$\psi_{\delta,\alpha}(t) = E[\widehat{R_1^{\alpha-2}}\widehat{R_t^{\alpha-2}}].$$

The two functions $\varphi_{\delta,\alpha}$ and $\psi_{\delta,\alpha}$ are related via:

Lemma 3.5. *The following identity holds:*

$$\varphi'_{\delta,\alpha}(t) = -\frac{\delta_\alpha}{2}\psi_{\delta,\alpha}(t) + \frac{\alpha}{t}\varphi_{\delta,\alpha}(t). \tag{3.7}$$

Proof. By definition, $\varphi'_{\delta,\alpha}(t)$ is written as:

$$\frac{d}{dt}E[R_1^{\alpha-2}R_t^\alpha] - E[R_1^{\alpha-2}]\frac{d}{dt}E[R_t^\alpha]. \tag{3.8}$$

To work on the first term, we use the time-inversion and the Markov property to see that:

$$\begin{aligned} E[R_1^{\alpha-2}R_t^\alpha] &= t^\alpha E[R_1^{\alpha-2}R_{1/t}^\alpha] \\ &= t^\alpha E[R_1^{\alpha-2}\rho_\alpha(R_1, \frac{1}{t}-1)]. \end{aligned} \tag{3.9}$$

Noting (3.4) with changing the starting point to $x \geq 0$, we deduce that

$$\rho_\alpha(x,t) = x^\alpha + \frac{\delta_\alpha}{2}\int_0^t \sigma_\alpha(x,u)\,du,$$

hence

$$\frac{\partial}{\partial t}\rho_\alpha(x,t) = \frac{\delta_\alpha}{2}\sigma_\alpha(x,t). \tag{3.10}$$

Therefore, by (3.9) and (3.10),

$$\begin{aligned} \frac{d}{dt}E[R_1^{\alpha-2}R_t^\alpha] &= \alpha t^{\alpha-1}E[R_1^{\alpha-2}R_{1/t}^\alpha] + t^\alpha E[R_1^{\alpha-2}\times\frac{\delta_\alpha}{2}\sigma_\alpha(R_1,\frac{1}{t}-1)\times(-\frac{1}{t^2})] \\ &= \frac{\alpha}{t}E[R_1^{\alpha-2}R_t^\alpha] - \frac{\delta_\alpha}{2}E[R_1^{\alpha-2}R_t^{\alpha-2}]. \end{aligned} \tag{3.11}$$

Here, for the second equality, we used the Markov property and the time-inversion. On the other hand, for the second term of (3.8), the factor $(d/dt)E[R_t^\alpha]$ may be expressed in two ways:

$$\begin{aligned} \frac{d}{dt}E[R_t^\alpha] &= \frac{\delta_\alpha}{2}E[R_t^{\alpha-2}] \\ &= \frac{\alpha}{2t}E[R_t^\alpha]. \end{aligned}$$

The first expression is nothing but (3.10) with $x = 0$, while the second one is a consequence of the scaling property: $E[R_t^\alpha] = (\sqrt{t})^\alpha E[R_1^\alpha]$. Thus, we may write (3.8) as:

$$
\begin{aligned}
&-\frac{\delta_\alpha}{2}E[R_1^{\alpha-2}R_t^{\alpha-2}] + \frac{\alpha}{t}E[R_1^{\alpha-2}R_t^\alpha] - E[R_1^{\alpha-2}]\left(-\frac{\delta_\alpha}{2}E[R_t^{\alpha-2}] + \frac{\alpha}{t}E[R_t^\alpha]\right) \\
&= -\frac{\delta_\alpha}{2}\left\{E[R_1^{\alpha-2}R_t^{\alpha-2}] - E[R_1^{\alpha-2}]E[R_t^{\alpha-2}]\right\} + \frac{\alpha}{t}\left\{E[R_1^{\alpha-2}R_t^\alpha] - E[R_1^{\alpha-2}]E[R_t^\alpha]\right\},
\end{aligned}
$$

which is the RHS of (3.7). □

To prove Proposition 3.3, we also use the following:

Lemma 3.6. *The function $\psi_{\delta,\alpha}$ is non-negative.*

Proof. Note that $\widehat{R_t^{\alpha-2}}, t > 0$, admits the following representation (Itô's representation):

$$
\widehat{R_t^{\alpha-2}} = \int_0^t \frac{\partial \sigma_\alpha}{\partial x}(R_u, t-u)\, dB_u, \tag{3.12}
$$

where B is the Brownian motion appearing in (3.2). Therefore, by definition,

$$
\psi_{\delta,\alpha}(t)\,(= E[\widehat{R_1^{\alpha-2}}\widehat{R_t^{\alpha-2}}]) = \int_0^t du\, E[\frac{\partial \sigma_\alpha}{\partial x}(R_u, 1-u)\frac{\partial \sigma_\alpha}{\partial x}(R_u, t-u)].
$$

Note that, by Lemma 3.4 (ii), $(\partial/\partial x)\sigma_\alpha \le 0$, which implies $\psi_{\delta,\alpha} \ge 0$. □

Remark 3.4. *This lemma may also be seen from the FKG inequality; from the (classical) formulation of the FKG inequality, we may also see that, for each fixed $x \ge 0$, and for increasing functions F, G on $C([0,1];\mathbb{R}_+)$, it holds that $E_x^{(\delta)}[F(R)G(R)] \ge E_x^{(\delta)}[F(R)]E_x^{(\delta)}[G(R)]$. So we may recover this lemma by taking $F(R) = -R_1^{\alpha-2}$ and $G(R) = -R_t^{\alpha-2}$.*

Now we are prepared to prove Proposition 3.3:

Proof of Proposition 3.3. Property (a′): By Itô's representation:

$$
\widehat{R_t^\alpha} = \int_0^t \frac{\partial \rho_\alpha}{\partial x}(R_u, t-u)\, dB_u,
$$

we have, together with (3.12) for $t = 1$,

$$
\varphi_{\delta,\alpha}(t) = \int_0^t du\, E[\frac{\partial \sigma_\alpha}{\partial x}(R_u, 1-u)\frac{\partial \rho_\alpha}{\partial x}(R_u, t-u)].
$$

By Lemma 3.4 (i) and (ii), we see that the inside of the expectation is negative. This shows (a′).

Property (b′): This property is now immediate from Lemmas 3.5 and 3.6, and (a′).
Property (c′): To prove this property, note that, by definition,

$$\varphi_{\delta,\alpha}(t) = E[R_1^{\alpha-2}R_t^\alpha] - E[R_1^{\alpha-2}]E[R_t^\alpha].$$

For the second term, we easily see that, by scaling,

$$E[R_1^{\alpha-2}]E[R_t^\alpha] = t^{\alpha/2}\kappa_{\alpha-2}\kappa_\alpha. \tag{3.13}$$

On the other hand, to estimate the first term, we rewrite it as, by using the Markov property,

$$E[R_1^{\alpha-2}R_t^\alpha] = E[R_t^\alpha \sigma_\alpha(R_t, 1-t)]. \tag{3.14}$$

By Lemma 3.4 (iii), and by scaling, (3.14) is bounded from below by

$$\begin{aligned} &E[R_t^\alpha \exp\{-\frac{R_t^2}{2(1-t)}\}\sigma_\alpha(0,1-t)] \\ &= \kappa_{\alpha-2}t^{\alpha/2}(1-t)^{(\alpha-2)/2}E[R_1^\alpha \exp\{-\frac{t}{2(1-t)}R_1^2\}]. \end{aligned}$$

By using the fact that $R_1^2/2 \stackrel{(d)}{=} \gamma_{\delta/2}$, a gamma variable of index $\delta/2$, the expectation on the RHS is computed explicitly as $\kappa_\alpha(1-t)^{(\delta+\alpha)/2}$. So (3.14) is bounded from below by

$$\kappa_{\alpha-2}\kappa_\alpha t^{\alpha/2}(1-t)^{(\delta+2\alpha-2)/2}. \tag{3.15}$$

Combining (3.13) and (3.15) yields

$$\varphi_{\delta,\alpha}(t) \geq -\kappa_{\alpha-2}\kappa_\alpha t^{\frac{\alpha}{2}}\{1-(1-t)^{\frac{\delta}{2}+\alpha-1}\},$$

from which the property (c′) follows. □

Remark 3.5. *We may also see the property* (a′) *directly from the FKG inequality.*

3.2 Proof for the case $0 < \delta < 2$ and $\alpha = 2 - \delta$

For $0 < \delta < 2$, define $\mu \in (0,1)$ via: $\delta = 2(1-\mu)$. Then α and μ are related as $\alpha = 2\mu$.

Proposition 3.7. *For every bounded, measurable function f on $[0,1]$, it holds that*

$$E[I(f;\widehat{R^\alpha})^2] = \alpha^2\kappa_{2(\alpha-1)}\|f_\alpha\|_2^2 - \frac{\alpha^2}{2}(\kappa_\alpha)^2 J^{(\alpha)}(f_\alpha, f_\alpha),$$

where $J^{(\alpha)} \equiv J_{\psi^{(\mu)}}$ with

$$\psi^{(\mu)}(t) = \frac{1}{\sqrt{t}}\{1-(1-t)^\mu\} + \frac{\sqrt{t}}{(1-t)^{1-\mu}}.$$

Remark 3.6. *Taking $\alpha = 1$ ($\Longleftrightarrow$ $\mu = 1/2$) above, we recover* (2.8) *for $\delta = 1$.*

The key fact to Proposition 3.7 is that $R_t^{2\mu}$ may be decomposed as:

$$R_t^{2\mu} = M_t^{(\mu)} + L_t, \tag{3.16}$$

where $(M_t^{(\mu)}, t \geq 0)$ is a martingale with $\langle M^{(\mu)} \rangle_t = 4\mu^2 \int_0^t R_u^{2(2\mu-1)}\, du$, and (L_t) is a possible choice of the local time at 0 of (R_t) as a diffusion.

Proof of Proposition 3.7. We take $X_t = R_t^{2\mu}$, $M_t = M_t^{(\mu)}$ and $V_t = L_t$ in (2.6). Then by Lemma 2.6, we have, for a bounded, measurable function f on $[0,1]$,

$$\begin{aligned} E[\left(\int_0^1 f(u)\, dR_u^{2\mu}\right)^2] &= E[\int_0^1 f(u)\, d\langle M^{(\mu)} \rangle_u] + 2E[\int_0^1 f(u)\, dL_u \left(\int_0^u f(s)\, dR_s^{2\mu}\right)] \\ &=: I_1^f + 2I_2^f. \end{aligned}$$

Since $(d/du)E[\langle M^{(\mu)} \rangle_u] = 4\mu^2 E[R_u^{2(2\mu-1)}] \equiv 4\mu^2 \kappa_{2(2\mu-1)} u^{2\mu-1}$, we have

$$I_1^f = 4\mu^2 \kappa_{2(2\mu-1)} \|f_{2\mu}\|_2^2.$$

On the other hand, for I_2^f, recall that dL_u is carried by the set of zeros of R. So, denoting by $(r_s^u, 0 \leq s \leq u)$ a Bessel bridge of length u, we have:

$$\begin{aligned} I_2^f &= \int_0^1 f(u)\, dE[L_u] E[\int_0^u f(s)\, dR_s^{2\mu} \big| R_u = 0] \\ &= \int_0^1 f(u)\, dE[L_u] \int_0^u f(s)\, dE[(r_s^u)^{2\mu}] \\ &= \int_0^1 f(u)\, dE[L_u] \int_0^1 f(ut) u^\mu\, dE[(r_t^1)^{2\mu}], \end{aligned}$$

where we changed variables with $s = ut, 0 \leq t \leq 1$, for the third line. Note that $E[L_u] = E[R_u^{2\mu}] = \kappa_{2\mu} u^\mu$, hence $dE[L_u] = \mu \kappa_{2\mu} u^{\mu-1}\, du$. Moreover, noting $r_t^1 \stackrel{(d)}{=} (1-t) R_{t/(1-t)}$, we see $E[(r_t^1)^{2\mu}] = \kappa_{2\mu} t^\mu (1-t)^\mu$. Combining these yields:

$$\begin{aligned} I_2^f &= \mu(\kappa_{2\mu})^2 \int_0^1 du\, f(u) u^{2\mu-1} \int_0^1 dt\, f(ut) \frac{d}{dt}\{t^\mu (1-t)^\mu\} \\ &= \int_0^1 du\, f_{2\mu}(u) \int_0^1 dt\, f_{2\mu}(ut) \Psi_1^{(\mu)}(t), \end{aligned}$$

where $\Psi_1^{(\mu)}(t) := \mu(\kappa_{2\mu})^2 t^{-\mu+1/2} (d/dt)\{t^\mu (1-t)^\mu\}$. Besides these, setting

$$I_3^f = E[\int_0^1 f(u)\, dR_u^{2\mu}] \equiv \mu \kappa_{2\mu} \int_0^1 f(u) u^{\mu-1}\, du,$$

we easily see that

$$(I_3^f)^2 = \int_0^1 du\, f_{2\mu}(u) \int_0^1 dt\, f_{2\mu}(ut)\Psi_2^{(\mu)}(t), \quad \Psi_2^{(\mu)}(t) := 2\mu^2(\kappa_{2\mu})^2 t^{-1/2}.$$

Consequently, we obtain

$$\begin{aligned} E[I(f;\widehat{R^{2\mu}})^2] &\equiv I_1^f + 2I_2^f - (I_3^f)^2 \\ &= (2\mu)^2\kappa_{2(2\mu-1)}\|f_{2\mu}\|_2^2 - J_{\Psi_2^{(\mu)}-2\Psi_1^{(\mu)}}(f_{2\mu}, f_{2\mu}) \end{aligned}$$

with the identity

$$\Psi_2^{(\mu)}(t) - 2\Psi_1^{(\mu)}(t) = 2\mu^2(\kappa_{2\mu})^2\psi^{(\mu)}(t).$$

Recalling $\alpha = 2\mu$, we obtain the proposition. □

Lemma 3.8. *$J^{(\alpha)}$ extends as a quadratic form on $L^2([0,1]) \times L^2([0,1])$. Moreover, it is positive-definite.*

Proof. Clearly $\psi^{(\mu)}$ is non-negative and satisfies $\int_0^1 dt\, t^{-1/2}\psi^{(\mu)}(t) < \infty$. So the first assertion follows from Proposition 2.1. So, by Proposition 2.2, it suffices to show that $\psi^{(\mu)}$ is expanded as (2.3). To this end, first note that

$$\frac{1}{(1-t)^{1-\mu}} = \frac{1}{\Gamma(1-\mu)} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-\mu)}{n!} t^n. \tag{3.17}$$

Indeed, using the elementary formula: $a^{-\gamma} = \Gamma(\gamma)^{-1}\int_0^\infty dx\, x^{\gamma-1}e^{-ax}$ for $a, \gamma > 0$, we have:

$$\begin{aligned} \frac{1}{(1-t)^{1-\mu}} &= \frac{1}{\Gamma(1-\mu)} \int_0^\infty dx\, x^{-\mu} e^{-(1-t)x} \\ &= \frac{1}{\Gamma(1-\mu)} \int_0^\infty dx\, x^{-\mu} e^{-x} \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} \\ &= \frac{1}{\Gamma(1-\mu)} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^\infty dx\, x^{n-\mu} e^{-x}, \end{aligned}$$

and arrive at (3.17). Moreover, since $1-(1-t)^\mu = \int_0^t ds\,(1-s)^{-(1-\mu)}$, we also have:

$$1 - (1-t)^\mu = \frac{1}{\Gamma(1-\mu)} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-\mu)}{(n+1)!} t^{n+1}. \tag{3.18}$$

Combining (3.17) and (3.18), we see that $\psi^{(\mu)}$ is expanded as (2.3) with $\alpha = 1/2$ and

$$a_n = \frac{\Gamma(n+1-\mu)}{\Gamma(1-\mu)} \left(\frac{1}{n!} + \frac{1}{(n+1)!} \right) > 0, \quad n = 0, 1, 2, \cdots.$$

This concludes the assertion. □

The proof of Theorem 1.3 for the present case is now immediate from Proposition 3.7 and Lemma 3.8:

Proof of Theorem 1.3 for $0 < \delta < 2$ *and* $\alpha = 2 - \delta$. By Proposition 3.7 and Lemma 3.8,

$$E[I(f; \widehat{R^\alpha})^2] \le \alpha^2 \kappa_{2(\alpha-1)} \|f_\alpha\|_2^2.$$

So the assertion for the present case is proved. □

3.3 Proof for the case $0 < \delta \le 1$ and $\alpha > 2 - \delta$

For $0 < \delta \le 1$, let μ be as given in the previous subsection: $\delta = 2(1-\mu)$. By the decomposition (3.16), and by Itô's formula, it is not hard to see that, for $\alpha > 2\mu$, (R_t^α) is a semimartingale which decomposes as

$$R_t^\alpha \equiv (R_t^{2\mu})^{\alpha/2\mu} = M_t^{(\alpha)} + V_t^{(\alpha)},$$

where $(M_t^{(\alpha)})$ is a martingale with quadratic variation $\langle M^{(\alpha)} \rangle_t = \alpha^2 \int_0^t R_u^{2(\alpha-1)}\, du$, and $V_t^{(\alpha)} = (\delta_\alpha/2) \int_0^t R_u^{\alpha-2}\, du$. Now the proof in the present case can be done in the same way as that of the proof in the case $\delta > 1$ and $\alpha > (2-\delta)_+$ (Subsection 3.1). So we omit the proof.

4 Defining Wiener integrals relative to the centered BES(δ)-bridges for $\delta \ge 1$

In this section, we denote by $(r_\delta(t), 0 \le t \le 1)$ a BES(δ)-bridge, $\delta \ge 1$, with length 1 such that $r_\delta(0) = r_\delta(1) = 0$. With the help of the results obtained in Section 2, we may also prove that the Wiener integral $I(f; \widehat{r_\delta})$ is well-defined for all $f \in L^2([0,1])$; in fact, we obtain the following a priori estimate:

Proposition 4.1. *Let* $\delta \ge 1$. *Then, for all* $f \in C([0,1])$, *it holds that*

$$E[I(f; \widehat{r_\delta})^2] \le \|f\|_2^2. \tag{4.1}$$

Remark 4.1. *As we see in the proof below, we have, more precisely that*

$$E[I(f; \widehat{r_\delta})^2] \le \|f\|_2^2 - \left(\int_0^1 du\, f(u) \right)^2. \tag{4.2}$$

This estimate coincides with that obtained by the approach via the Brascamp-Lieb inequality for $\delta \ge 3$*; in fact, if we denote by* $(b(u), 0 \le u \le 1)$ *a Brownian bridge of length* 1 *such that* $b(0) = b(1) = 0$, *then the RHS of* (4.2) *is nothing but* $E[I(f; b)^2]$.

Proof of Proposition 4.1. It is well-known that $(r_\delta(t), 0 \le t \le 1)$ is identical in law with the process $(X_\delta(t), 0 \le t \le 1)$ defined by

$$X_\delta(t) = \begin{cases} (1-t)R_\delta(\dfrac{t}{1-t}), & 0 \le t < 1, \\ 0, & t = 1, \end{cases}$$

where R_δ is a BES(δ)-process starting from 0. Therefore

$$\begin{aligned} I(f; \widehat{r_\delta}) &\stackrel{(d)}{=} \int_0^1 f(u)\, d\widehat{X_\delta}(u) \\ &= -\int_0^1 f(u)\widehat{R_\delta}(\frac{u}{1-u})\, du + \int_0^1 f(u)(1-u)\, d_u\widehat{R_\delta}(\frac{u}{1-u}). \end{aligned}$$

We use integration by parts to see that the first term on the RHS is equal to:

$$-\int_0^1 \left(\int_u^1 dv\, f(v)\right) d_u\widehat{R_\delta}(\frac{u}{1-u}).$$

Thus, setting $G(u) = f(u) - (1-u)^{-1}\int_u^1 dv\, f(v)$, we have

$$\begin{aligned} I(f; \widehat{r_\delta}) &\stackrel{(d)}{=} \int_0^1 (1-u)G(u)\, d_u\widehat{R_\delta}(\frac{u}{1-u}) \\ &= \int_0^\infty \frac{1}{1+s} G(\frac{s}{1+s})\, d\widehat{R_\delta}(s) \end{aligned} \tag{4.3}$$

So, denoting $\widetilde{G}(s) = (1+s)^{-1}G(s/(1+s))$ and using the similar argument to that in the proof of Proposition 2.5, we deduce that

$$\begin{aligned} E[I(f; \widehat{r_\delta})^2] &= E[\left(\int_0^\infty \widetilde{G}(u)\, d\widehat{R_\delta}(u)\right)^2] \\ &= \|\widetilde{G}\|_{L^2(\mathbb{R}_+)} - \widetilde{J}_{\Phi_\delta}(\widetilde{G}, \widetilde{G}), \end{aligned} \tag{4.4}$$

where, for $F \in L^2(\mathbb{R}_+)$,

$$\widetilde{J}_{\Phi_\delta}(F, F) = \int_0^\infty du\, F(u) \int_0^1 dt\, F(ut)\Phi_\delta(t)$$

and Φ_δ is as given in Proposition 2.5. With the help of the argument used in the proof of Lemma 2.7, we may also deduce that $\widetilde{J}_{\Phi_\delta}$ is non-negative definite. So, by (4.4),

$$\begin{aligned} E[I(f; \widehat{r_\delta})^2] &\le \|\widetilde{G}\|^2_{L^2(\mathbb{R}_+)} \\ &= \|G\|^2_{L^2([0,1])} \\ &= \|f\|^2_{L^2([0,1])} - \left(\int_0^1 du\, f(u)\right)^2, \end{aligned}$$

which implies (4.1). □

5 Appendix

5.1 An explicit representation for $\varphi_{\delta,\alpha}$ in terms of hypergeometric functions

Following [17], for complex numbers a, b, c, we denote by ${}_2F_1(a, b; c; w)$, $|w| < 1$, Gauss's hypergeometric functions. Recall that, when $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$, they admit the following integral representation:

$$ {}_2F_1(a, b; c; w) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dz\, z^{b-1}(1-z)^{c-b-1}(1-wz)^{-a}. \tag{5.1}$$

See, e.g, [10, (9.1.4)].

In this subsection, we discuss an explicit representation for the function $\varphi_{\delta,\alpha}(t) \equiv E[\widehat{R_1^{\alpha-2}}\widehat{R_t^{\alpha}}]$ in terms of ${}_2F_1$. We do this in a slightly general situation: For dimension $\delta > 0$, and for two exponents $\alpha_i \in \mathbb{R}, i = 1, 2$, satisfying $\delta + \alpha_i > 0$, set

$$\phi_{\alpha_1;\alpha_2}(t) = E[\widehat{R_t^{\alpha_1}}\widehat{R_1^{\alpha_2}}], \quad 0 < t < 1.$$

Proposition 5.1. *Suppose $\alpha \in \mathbb{R}$ and $\beta > 0$ satisfy $\delta + \alpha > 0$ and $\delta > \beta$, respectively. Then the function $\phi_{\alpha;-\beta}(t)$ admits the following representation:*

$$\phi_{\alpha;-\beta}(t) = 2^{\frac{\alpha-\beta}{2}} \frac{\Gamma(\frac{\delta+\alpha}{2})\Gamma(\frac{\delta-\beta}{2})}{\{\Gamma(\frac{\delta}{2})\}^2} t^{\frac{\alpha}{2}} \left\{ {}_2F_1(-\frac{\alpha}{2}, \frac{\beta}{2}; \frac{\delta}{2}; t) - 1 \right\}.$$

Proof. Since $\beta > 0$, we may rewrite $R_1^{-\beta}$ into:

$$R_1^{-\beta} = 2^{-\beta/2} \left(R_1^2/2\right)^{-\beta/2} = 2^{-\beta/2}\Gamma(\beta/2)^{-1} \int_0^\infty \frac{dx}{x} x^{\beta/2} \exp\left(-\frac{x}{2}R_1^2\right).$$

Then by Fubini's theorem,

$$E[R_t^\alpha R_1^{-\beta}] = 2^{-\beta/2}\Gamma(\beta/2)^{-1} \int_0^\infty \frac{dx}{x} x^{\beta/2} E[R_t^\alpha \exp\left(-\frac{x}{2}R_1^2\right)]. \tag{5.2}$$

By the Markov property, we have, conditionally on $\mathcal{F}_t^R$,

$$\begin{aligned} E_0^{(\delta)}[\exp\left(-\frac{x}{2}R_1^2\right)|\mathcal{F}_t^R] &= E_{R_t}^{(\delta)}[\exp\left(-\frac{x}{2}R_{1-t}^2\right)] \\ &= \{1 + x(1-t)\}^{-\delta/2} \exp\left\{-\frac{x}{2(1+x(1-t))}R_t^2\right\}. \end{aligned} \tag{5.3}$$

Here the second equality follows from the well-known fact that: for $a \geq 0$,

$$E_a^{(\delta)}[\exp\left(-\frac{x}{2}R_s^2\right)] = (1+xs)^{-\delta/2} \exp\left(-\frac{x}{1+xs}a^2\right).$$

See, e.g., [16, p.441]. By (5.3), we then have:

$$E[R_t^\alpha \exp\left(-\frac{x}{2}R_1^2\right)] = \{1+x(1-t)\}^{-\delta/2} E[R_t^\alpha \exp\left\{-\frac{x}{2(1+x(1-t))}R_t^2\right\}]$$
$$= (2t)^{\alpha/2}\Gamma(\delta/2)^{-1}\Gamma((\delta+\alpha)/2)\frac{1}{(1+x)^{\delta/2}}\left(1-\frac{x}{1+x}t\right)^{\alpha/2},$$

where, for the second line, we used the fact that $R_t \overset{(d)}{=} \sqrt{2t\gamma_{\delta/2}}$. Plugging this into (5.2), and changing variables with $x/(1+x)=z$, we arrive at

$$E[R_t^\alpha R_1^{-\beta}] = 2^{(\alpha-\beta)/2}\frac{\Gamma(\frac{\delta+\alpha}{2})}{\Gamma(\frac{\beta}{2})\Gamma(\frac{\delta}{2})}t^{\alpha/2}\int_0^1 dz\, z^{\frac{\beta}{2}-1}(1-z)^{\frac{\delta-\beta}{2}-1}(1-tz)^{\frac{\alpha}{2}}.$$

By the integral representation (5.1), the above integral in z is expressed as:

$$\frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\delta-\beta}{2})}{\Gamma(\frac{\delta}{2})}{}_2F_1(-\frac{\alpha}{2},\frac{\beta}{2};\frac{\delta}{2};t).$$

Now we see

$$E[R_t^\alpha R_1^{-\beta}] = 2^{(\alpha-\beta)/2}\frac{\Gamma(\frac{\delta+\alpha}{2})\Gamma(\frac{\delta-\beta}{2})}{\{\Gamma(\frac{\delta}{2})\}^2}t^{\alpha/2}{}_2F_1(-\frac{\alpha}{2},\frac{\beta}{2};\frac{\delta}{2};t). \tag{5.4}$$

On the other hand, by scaling,

$$E[R_t^\alpha]E[R_1^{-\beta}] = t^{\alpha/2}E[R_1^\alpha]E[R_1^{-\beta}] \equiv t^{\alpha/2}\times 2^{(\alpha-\beta)/2}\frac{\Gamma(\frac{\delta+\alpha}{2})\Gamma(\frac{\delta-\beta}{2})}{\{\Gamma(\frac{\delta}{2})\}^2}.$$

Therefore, combining this with (5.4), and noting $\phi_{\alpha;-\beta}(t) = E[R_t^\alpha R_1^{-\beta}] - E[R_t^\alpha]E[R_1^{-\beta}]$, we have the lemma. □

Remark 5.1. *By Proposition 5.1, and by recalling the definition (series expansion) of hypergeometric functions (cf. [10, (9.1.1)]), we may easily see that*

$$\lim_{t\downarrow 0} t^{-\frac{\alpha}{2}-1}\phi_{\alpha;-\beta}(t) = 2^{(\alpha-\beta)/2}\frac{\Gamma(\frac{\delta+\alpha}{2})\Gamma(\frac{\delta-\beta}{2})}{\{\Gamma(\frac{\delta}{2})\}^2}\times\left(-\frac{\alpha\beta}{2\delta}\right).$$

We may compare this with the assertion (c′) *of Proposition 3.3.*

5.2 On a connection between Φ_1 and Φ_3 via perturbations

In this part, we shall consider the family of processes $(X_t^{(\gamma)}, 0\le t\le 1), \gamma\in\mathbb{R}$, defined by

$$X_t^{(\gamma)} = |\beta_t| + \gamma L_t, \tag{5.5}$$

and discuss the Wiener integrals $I(f;\widehat{X^{(\gamma)}})$ relative to the centered processes $\widehat{X^{(\gamma)}}$. Here, as in the proof of Proposition 2.5 for $\delta = 1$, (β_t) is a Brownian motion and (L_t) the local time of β at 0. Note that, in particular,

$$\text{(5.6)} \qquad \begin{cases} \gamma = -1 & : \ X^{(-1)} \text{ is a Brownian motion (Tanaka's formula);} \\ \gamma = 0 & : \ X^{(0)} \text{ is a reflecting Brownian motion;} \\ \gamma = +1 & : \ X^{(+1)} \text{ is BES(3) (Pitman's theorem).} \end{cases}$$

We shall prove that, for every $\gamma \in \mathbb{R}$, $I(f;\widehat{X^{(\gamma)}})$ is well-defined for all $f \in L^2([0,1])$. In fact, we obtain the following a priori estimate:

Proposition 5.2. *There exists a positive constant K_γ such that, for all bounded, measurable functions f on $[0,1]$,*

$$E[I(f;\widehat{X^{(\gamma)}})^2] \leq K_\gamma \|f\|_2^2. \tag{5.7}$$

In particular, if $|\gamma| \leq 1$, then (5.7) holds with $K_\gamma = 1$.

To prove this proposition, we prepare the following:

Lemma 5.3. *For every bounded, measurable function f on $[0,1]$, it holds that*

$$E[I(f;\widehat{X^{(\gamma)}})^2] = \|f\|_2^2 - J_{\Phi^{(\gamma)}}(f,f),$$

where

$$\Phi^{(\gamma)}(t) = \frac{(1+\gamma)^2}{\pi\sqrt{t}}(1-\sqrt{1-t}) + \frac{1-\gamma^2}{\pi}\frac{\sqrt{t}}{\sqrt{1-t}}$$

Note that, in the three particular cases $\gamma = -1, 0, +1$, we recover the previous results; indeed,

$$\begin{aligned} &\Phi^{(-1)}(t) \equiv 0 \text{ (since } X^{(\gamma)} \text{ is the Brownian motion);} \\ &\Phi^{(0)}(t) = \frac{1}{\pi}\left\{\frac{1}{\sqrt{t}}(1-\sqrt{1-t}) + \frac{\sqrt{t}}{\sqrt{1-t}}\right\} \equiv \Phi_1(t); \\ &\Phi^{(+1)}(t) = \frac{4}{\pi\sqrt{t}}(1-\sqrt{1-t}) \equiv \Phi_3(t). \end{aligned}$$

The proof of this lemma is given in the same manner as that of Proposition 2.5 for $\delta = 1$:

Proof of Lemma 5.3. Since $|\beta_t|$ decomposes as $|\beta_t| = B_t + L_t$, $X_t^{(\gamma)}$ admits the following decomposition:

$$X_t^{(\gamma)} = B_t + (\gamma+1)L_t. \tag{5.8}$$

We take $X_t = X_t^{(\gamma)}$, $M_t = B_t$ and $V_t = (\gamma+1)L_t$ in (2.9). Then by Lemma 2.6,

$$\begin{aligned} E[\left(\int_0^1 f(u)\,dX_u^{(\gamma)}\right)^2] &= \|f\|_2^2 + 2(\gamma+1)E[\int_0^1 f(u)\,dL_u\left(\int_0^u f(s)\,dX_s^{(\gamma)}\right)] \\ &= \|f\|_2^2 + 2(\gamma+1)\int_0^1 f(u)\,dE[L_u]E[\int_0^u f(s)\,dX_s^{(\gamma)}\big|\beta_u = 0]. \end{aligned}$$

We have already seen that $dE[L_u] = du/\sqrt{2\pi u}$ just below (2.11). Moreover, by the definition of $X^{(\gamma)}$,

$$\begin{aligned} &\frac{d}{ds}E[X_s^{(\gamma)}|\beta_u = 0] = \frac{d}{ds}E[|\beta_s||\beta_u = 0] + \gamma\frac{d}{ds}E[L_s|\beta_u = 0] \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{s(1-s/u)}}\left(1 - \frac{2s}{u}\right) + \gamma\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{s(1-s/u)}}. \end{aligned}$$

We thus obtain

$$E[\left(\int_0^1 f(u)\,dX_u^{(\gamma)}\right)^2] = \|f\|_2^2 + J_{\varphi_1^{(\gamma)}}(f,f), \quad \varphi_1^{(\gamma)}(t) := \frac{(\gamma+1)\{(1-2t)+\gamma\}}{\pi\sqrt{t(1-t)}}.$$

On the other hand, since $E[\int_0^1 f(u)\,dX_u^{(\gamma)}] = (\gamma+1)\int_0^1 du\, f(u)/\sqrt{2\pi u}$,

$$\left(E[\int_0^1 f(u)\,dX_u^{(\gamma)}]\right)^2 = J_{\varphi_2^{(\gamma)}}(f,f), \quad \varphi_2^{(\gamma)}(t) := \frac{(\gamma+1)^2}{\pi\sqrt{t}}.$$

Combining these, we obtain

$$E[I(f;\widehat{X^{(\gamma)}})^2] = \|f\|_2^2 - J_{\varphi_2^{(\gamma)} - \varphi_1^{(\gamma)}}(f,f)$$

with the identity $\varphi_2^{(\gamma)} - \varphi_1^{(\gamma)} = \Phi^{(\gamma)}$. So the proposition is proved. □

With Lemma 5.3 at disposal, we prove Proposition 5.2:

Proof of Proposition 5.2. The case $|\gamma| > 1$: Since it is clear that $K_{\Phi^{(\gamma)}} := \int_0^1 dt\, t^{-1/2}|\Phi^{(\gamma)}(t)|$ is finite, we may use Proposition 2.1 to have that

$$|J_{\Phi^{(\gamma)}}(f,f)| \le K_{\Phi^{(\gamma)}}\|f\|_2^2.$$

Therefore, by Lemma 5.3, (5.7) holds with $K_\gamma = 1 + K_{\Phi^{(\gamma)}}$.
The case $|\gamma| \le 1$: Using the same argument as in the proof of Lemma 2.7 for $\delta = 1$, we

easily see that $J_{\Phi^{(\gamma)}}$ extends as a positive-definite quadratic form on $L^2([0,1]) \times L^2([0,1])$ if $|\gamma| \leq 1$; indeed, $\Phi^{(\gamma)}$ is expanded as (2.14) with

$$a_n = \frac{1}{\pi} \left\{ (1+\gamma)^2 + 2(n+1)(1-\gamma^2) \right\} c_{n+1},$$

which is positive for every $n = 0, 1, 2, \cdots$, as long as $|\gamma| \leq 1$. Here (c_n) is the positive sequence given in (2.15). Thus, if $|\gamma| \leq 1$, then by Lemma 5.3, the a priori estimate (5.7) holds with $K_\gamma = 1$. □

The latter assertion of this proposition may also be explained by the following perturbation argument: Let X be a semimartingale which decomposes as $X_t = B_t + V_t$ with V of finite variation satisfying $E[(\int_0^1 |dV_u|)^2] < \infty$. As we have seen in Lemma 2.6, it holds that, for every bounded, measurable function f on $[0,1]$,

$$E[I(f;X)^2] = \|f\|_2^2 - \mathfrak{J}_X(f), \quad \mathfrak{J}_X(f) := -2E[\int_0^1 f(u)\, dV_u \int_0^u f(s)\, dX_s].$$

Now we suppose $\mathfrak{J}_X(f) \geq 0$. As we have already seen, there are a number of such situations. For $\eta \in \mathbb{R}$, define

$$X_t^\eta = B_t + \eta V_t.$$

Then it is easily seen that $E[I(f;X^\eta)^2] = \|f\|_2^2 - \mathfrak{J}_{X^\eta}(f)$ with

$$J_{X^\eta}(f) = \eta \mathfrak{J}_X(f) - \eta(\eta-1) E[\left(\int_0^1 f(u)\, dV_u \right)^2]. \tag{5.9}$$

Indeed, by Lemma 2.6,

$$\begin{aligned}
\mathfrak{J}_{X^\eta}(f) &= -2E[\int_0^1 f(u)\, dV_u \int_0^u f(s)\, dX_s^\eta] \\
&= -2\eta E[\int_0^1 f(u)\, dV_u \int_0^u f(s)\, (dX_u + (\eta-1) dV_u)] \\
&= \eta \mathfrak{J}_X(f) - 2\eta(\eta-1) E[\int_0^1 f(u)\, dV_u \int_0^u f(s)\, dV_s],
\end{aligned}$$

hence (5.9) follows. From this, it is clear that,

$$\text{if } 0 \leq \eta \leq 1, \text{ then } \mathfrak{J}_{X^\eta}(f) \geq 0. \tag{5.10}$$

Now we turn to the case that we have just discussed above; we may recover the latter assertion of Proposition 5.2 by using (5.10):

Alternative proof of the latter assertion of Proposition 5.2. We rewrite the decomposition (5.8) above as

$$X_t^{(\gamma)} = B_t + \eta \times (2L_t), \quad \text{with } \eta = \frac{1+\gamma}{2}.$$

In the case $\eta = 1$ ($\Longleftrightarrow$ $\gamma = 1$), it is already seen that $\mathfrak{J}_{\widehat{X^{(1)}}}(f) \geq 0$ since $X^{(1)}$ is BES(3). So, by the above perturbation result (5.10), we may conclude that $\mathfrak{J}_{\widehat{X^{(\gamma)}}}(f) \geq 0$ when

$$0 \leq \frac{1+\gamma}{2} \leq 1 \ (\Longleftrightarrow \ -1 \leq \gamma \leq 1),$$

which is nothing but the latter assertion of Proposition 5.2. □

5.3 On the applicability of the FKG inequality to BES(δ), $\delta > 0$

In this part, we prove that the FKG inequality is applicable to the laws of BES(δ), $\delta > 0$, (or, more precisely, to their finite-dimensional marginals). For the formulation of the FKG inequality, we refer to [14, 15].

For $t > 0$ and $x_1, x_2 > 0$, let $p^\delta(t; x_1, x_2)$ denote the transition density function of BES(δ):

$$p^\delta(t; x_1, x_2) = t^{-1} x_1^{-\nu} x_2^{\nu+1} \exp(-\frac{x_1^2 + x_2^2}{2t}) I_\nu(\frac{x_1 x_2}{t}).$$

Here $\nu = \delta/2 - 1$ (> -1), the index of BES(δ). Set

$$\mathfrak{I}_\nu(x) = \frac{x I_{\nu+1}(x)}{I_\nu(x)}, \quad x > 0.$$

Lemma 5.4. *If $\nu > -1$, then $\mathfrak{I}_\nu$ is non-decreasing.*

Proof. We compute the derivative:

$$\begin{aligned}
\frac{d}{dx}\mathfrak{I}_\nu(x) &\equiv \frac{d}{dx}\frac{x^{\nu+1} I_{\nu+1}(x)}{x^\nu I_\nu(x)} \\
&= \frac{1}{\{x^\nu I_\nu(x)\}^2} \left\{ x^{\nu+1} I_\nu(x) x^\nu I_\nu(x) - x^{\nu+1} I_{\nu+1}(x) x^\nu I_{\nu-1}(x) \right\} \\
&= \frac{x^{2\nu+1}}{\{x^\nu I_\nu(x)\}^2} \left\{ I_\nu(x)^2 - I_{\nu+1}(x) I_{\nu-1}(x) \right\},
\end{aligned}$$

where we used the recurrence relation: $\{x^\mu I_\mu(x)\}' = x^\mu I_{\mu-1}(x)$. By [11, (9.t5)], the last quantity is non-negative for $\nu > -1$. This shows the lemma. □

Remark 5.2. *It is known (e.g., [20, (11.27)]) that, for $\nu > -1$,*

$$\frac{1}{x^2}\mathfrak{I}_\nu(x) = 2\sum_{n=1}^{\infty}\frac{1}{x^2 + j_{\nu,n}^2},$$

where $(j_{\nu,n})$ is the simple, positive zeros of J_ν, the Bessel function of the first kind of order ν. From here, we may also deduce Lemma 5.4.

Lemma 5.5. *For each fixed $t > 0$, it holds that*

$$p^\delta(t; x_1 \vee y_1, x_2 \vee y_2)p^\delta(t; x_1 \wedge y_1, x_2 \wedge y_2) \geq p^\delta(t; x_1, x_2)p^\delta(t; y_1, y_2) \tag{5.11}$$

for all $(x_1, x_2), (y_1, y_2) \in (0, \infty) \times (0, \infty)$. Here $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$.

Proof. We divide the case into four cases: (i) $x_1 \geq y_1$, $x_2 \geq y_2$; (ii) $x_1 \leq y_1$, $x_2 \leq y_2$; (iii) $x_1 \geq y_1$, $x_2 \leq y_2$; (iv) $x_1 \leq y_1$, $x_2 \geq y_2$. In both cases (i) and (ii), (5.11) holds as an equality. So, by symmetry, we only need to consider either (iii) or (iv). Here we give a proof in the case (iii). By the definition of $p^\delta(t; x, y)$, the proof is reduced to showing the following: for $x_1 \geq y_1$ and $x_2 \leq y_2$,

$$I_\nu(\frac{x_1y_2}{t})I_\nu(\frac{x_2y_1}{t}) \geq I_\nu(\frac{x_1x_2}{t})I_\nu(\frac{y_1y_2}{t}). \tag{5.12}$$

Rewriting (5.12) as

$$\frac{I_\nu(\frac{y_2}{t}x_1)}{I_\nu(\frac{x_2}{t}x_1)} \geq \frac{I_\nu(\frac{y_2}{t}y_1)}{I_\nu(\frac{x_2}{t}y_1)},$$

we see that it suffices to prove, for $\beta > \alpha > 0$,

$$\frac{I_\nu(\beta x)}{I_\nu(\alpha x)} \text{ is non-decreasing in } x > 0. \tag{5.13}$$

To this end, we compute:

$$\begin{aligned}\frac{d}{dx}\left\{\frac{I_\nu(\beta x)}{I_\nu(\alpha x)}\right\} \times \frac{\beta^{-\nu}}{\alpha^{-\nu}} &\equiv \frac{d}{dx}\left\{\frac{(\beta x)^{-\nu}I_\nu(\beta x)}{(\alpha x)^{-\nu}I_\nu(\alpha x)}\right\} \\ &= \frac{(\alpha\beta x^2)^{-\nu}}{\{(\alpha x)^{-\nu}I_\nu(\alpha x)\}^2}\left\{\beta I_{\nu+1}(\beta x)I_\nu(\alpha x) - \alpha I_\nu(\beta x)I_{\nu+1}(\alpha x)\right\},\end{aligned}$$

where, for the equality, we used the recurrence relation: $\{x^{-\mu}I_\mu(x)\}' = x^{-\mu}I_{\mu+1}(x)$. We rewrite the last factor as:

$$\beta I_{\nu+1}(\beta x)I_\nu(\alpha x) - \alpha I_\nu(\beta x)I_{\nu+1}(\alpha x) = \frac{I_\nu(\alpha x)I_\nu(\beta x)}{x}\{\mathfrak{I}_\nu(\beta x) - \mathfrak{I}_\nu(\alpha x)\}$$

to see that this is non-negative by Lemma 5.4. Hence

$$\frac{d}{dx}\left\{\frac{I_\nu(\beta x)}{I_\nu(\alpha x)}\right\} \geq 0,$$

which shows (5.13). So the lemma is proved. □

Let $\Delta = \{0 < t_1 < \cdots < t_n\}$ be an increasing sequence in $[0, \infty)$. For $a > 0$, we denote by $\Phi^\delta_\Delta(x; a)$ $(x = (x_i)_{1 \le i \le n})$ the finite-dimensional distribution function of BES(δ) starting from a taken at the time sequence $(t_i)_{1 \le i \le n}$:

$$\Phi^\delta_\Delta(x; a) = p^\delta(t_1; a, x_1) p^\delta(t_2 - t_1; x_1, x_2) \cdots p^\delta(t_n - t_{n-1}; x_{n-1}, x_n) \tag{5.14}$$

The next proposition shows $\Phi^\delta_\Delta(\cdot; a)$ fulfills the assumption of [14, Theorem 3]:

Proposition 5.6. *For $a \ge a' > 0$, it holds that*

$$\Phi^\delta_\Delta(x \vee y; a) \Phi^\delta_\Delta(x \wedge y; a') \ge \Phi^\delta_\Delta(x; a) \Phi^\delta_\Delta(y; a')$$

for all $x = (x_i)_{1 \le i \le n} \in (0, \infty)^n$ and $y = (y_i)_{1 \le i \le n} \in (0, \infty)^n$. Here $x \vee y = (x_i \vee y_i)_{1 \le i \le n}$ and $x \wedge y = (x_i \wedge y_i)_{1 \le i \le n}$.

Proof. The assertion is immediate from Lemma 5.5 and (5.14). □

Remark 5.3. *It is easily checked that the assertion of this lemma still holds even if a' is equal to 0; in that case, $\Phi^\delta_\Delta(x; a')$ should be replaced by*

$$\Phi^\delta_\Delta(x; 0) = \widetilde{p^\delta}(t_1; x_1) p^\delta(t_2 - t_1; x_1, x_2) \cdots p^\delta(t_n - t_{n-1}; x_{n-1}, x_n),$$

where

$$\widetilde{p^\delta}(t; x) = 2^{-\nu} t^{-(\nu+1)} \Gamma(\nu + 1)^{-1} x^{2\nu+1} \exp(-\frac{x^2}{2t}).$$

Remark 5.4. *There is a probabilistic understanding of* (5.13)*: For $x > 0$, we denote $T_x(R) = \inf\{t > 0; R_t = x\}$. It is well-known (e.g., [20,* (11.31)*]) that, for every dimension $\delta > 0$, and for $\gamma > 0$,*

$$E^{(\delta)}_0[\exp(-\frac{\gamma^2}{2} T_x(R))] = \frac{(x\gamma)^\nu}{2^\nu \Gamma(1 + \nu) I_\nu(x\gamma)}.$$

Therefore, for $0 < y < x$, we get, from the strong Markov property,

$$E^{(\delta)}_0[\exp(-\frac{\gamma^2}{2} T_x(R))] = E^{(\delta)}_0[\exp(-\frac{\gamma^2}{2} T_y(R))] E^{(\delta)}_y[\exp(-\frac{\gamma^2}{2} T_x(R))]$$

hence

$$\frac{I_\nu(y\gamma)}{I_\nu(x\gamma)} = \Big(\frac{y}{x}\Big)^\nu E^{(\delta)}_y[\exp(-\frac{\gamma^2}{2} T_x(R))],$$

which indicates that, for $0 < y < x$, the function $\gamma \mapsto I_\nu(y\gamma)/I_\nu(x\gamma)$ is strictly decreasing.

References

[1] H.J. Brascamp and E.H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Funct. Anal., **22** (1976), pp. 366–389.

[2] C. Donati-Martin and M. Yor, *Mouvement brownien et inégalité de Hardy dans L^2*, Sém. Prob., XXIII, pp. 315–323, Lect. Notes Math., **1372**, Springer, Berlin, 1989.

[3] T. Funaki, Y. Hariya and M. Yor, *Wiener integrals for centered powers of Bessel processes, II*, preprint, 2005.

[4] T. Funaki and K. Ishitani, *Integration by parts formulae for Wiener measures on a path space between two curves*, preprint, 2005, available at http://www.ms.u-tokyo.ac.jp/~funaki/publ/publ.htm

[5] G. Giacomin, *On stochastic domination in the Brascamp-Lieb framework*, Math. Proc. Cambridge Philos. Soc., **134** (2003), pp. 507–514.

[6] F. Hirsch, *personal communication*.

[7] T. Jeulin, *Semi-Martingales et Grossissement d'une Filtration*, Lect. Notes Math., **833**, Springer, Berlin, 1980.

[8] T. Jeulin, *Sur la convergence absolue de certaines intégrales*, Sém. Prob., XVI, pp. 248–256, Lect. Notes Math., **920**, Springer, Berlin, 1982.

[9] T. Jeulin and M. Yor, *Inégalité de Hardy, semimartingales, et faux-amis*, Sém. Prob., XIII, pp. 332–359, Lect. Notes Math., **721**, Springer, Berlin, 1979.

[10] N.N. Lebedev, *Special Functions and their Applications*, Dover, New York, 1972.

[11] J.W. Pitman and M. Yor, *Bessel processes and infinitely divisible laws*, In: "Stochastic Integrals" (Proc. Sympos., Univ. Durham, Durham, 1980), pp. 285–370, Lect. Notes Math., **851**, Springer, Berlin, 1981.

[12] J.W. Pitman and M. Yor, *A decomposition of Bessel bridges*, Zeit. Wahrsch. Geb., **59** (1982), pp. 425–457.

[13] J.W. Pitman and M. Yor, *Some divergent integrals of Brownian motion*, In: "Analytic and Geometric Stochastics" (ed. D.G. Kendall), Adv. in Appl. Probab., 1986, suppl., pp. 109–116.

[14] C.J. Preston, *A generalization of the FKG inequality*, Comm. Math. Phys., bf 36 (1974), pp. 233–241.

[15] B. Simon, *Functional Integration and Quantum Physics*, Academic Press, New York, 1979.

[16] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, 3rd ed., Springer, Berlin, 1999.

[17] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, London, 1962.

[18] M. Yor, *Application d'un lemme de T. Jeulin au grossissement de la filtration brownienne*, Sém. Prob., XIV, pp. 189–199, Lect. Notes Math., **784**, Springer, Berlin, 1980.

[19] M. Yor, *Some Aspects of Brownian Motion. Part I: Some special functionals*, Birkhäuser, Basel, 1992.

[20] M. Yor, *Some Aspects of Brownian Motion. Part II: Some recent martingale problems*, Birkhäuser, Basel, 1997.

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