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Wiener integrals for centered powers of Bessel processes, I

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Abstract

The stochastic integrals of Wiener's type may be constructed relatively to the centered δ-dimensional Bessel processes (BES(δ)-processes in short) and their variants based on two different approaches. One approach, developed in [3], is via the Brascamp-Lieb inequality which works especially well for the BES(δ)-processes, BES(δ)-bridges with $\delta \geq 3$ or for the Brownian meander. The other approach, which is the subject of the present paper, goes via Hardy's L^2 inequality which is effective for general BES(δ)-processes and their powers. We shall also discuss an interplay of these two methods.

1 Introduction and main results

Consider, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$, a continuous semimartingale $(X_t, t \geq 0)$, with canonical decomposition:

$$X_t = M_t + V_t,$$

where (M_t) denotes an $(\mathcal{F}_t)$-local martingale, and (V_t) an $(\mathcal{F}_t)$-adapted, continuous process of bounded variation.

The theory of stochastic integration with respect to X consists in defining:

$$\int_0^t \phi_s \, dX_s = \int_0^t \phi_s \, dM_s + \int_0^t \phi_s \, dV_s,$$

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for $(\mathcal{F}_t)$-predictable processes (ϕ_t) which satisfy:

$$\text{(1.1)} \qquad \int_0^t \phi_s^2 \, d\langle M\rangle_s < \infty \quad \text{and} \quad \int_0^t |\phi_s||dV_s| < \infty$$

a.s. for every t. However, in a number of situations, one would like to be able to define $(\int_0^t \phi_s \, dX_s, t \geq 0)$ for predictable processes (ϕ_t) which only satisfy:

$$\text{(1.1}'\text{)} \qquad \int_0^t \phi_s^2 \, d\langle M\rangle_s < \infty \quad \text{a.s. for every } t.$$

This question arose very naturally in some discussions of enlargements of the Brownian filtration. More precisely, consider a Brownian motion $(\beta_t, t \geq 0)$ (and its filtration), and assume that, in a larger filtration $(\mathcal{F}_t)$, (β_t) remains a semimartingale, which then decomposes as:

$$\text{(1.2)} \qquad \beta_t = B_t + V_t, \quad t \geq 0,$$

where (B_t) is an $(\mathcal{F}_t)$-Brownian motion. In [9], examples were given for which, although (1.2) holds, nonetheless, there exist even deterministic functions f in $L^2_{\text{loc}}(ds)$, i.e., $\int_0^t f(s)^2 \, ds \equiv \int_0^t f(s)^2 \, d\langle B\rangle_s < \infty$ such that

$$\int_0^t |f(s)||dV_s| = \infty.$$

To be precise, starting from a Brownian motion (B_t), consider the "second" Brownian motion:

$$\text{(1.3)} \qquad \beta_t = B_t - \int_0^t \frac{B_s}{s} \, ds,$$

whose proper filtration is strictly smaller than that of B, but trivially (1.2) holds with:

$$V_t = -\int_0^t \frac{B_s}{s} \, ds.$$

It then follows from Jeulin's lemma [7, p.44] (see also [8]) that, if $f \in L^2_{\text{loc}}(ds)$, then $(\int_0^t f(s) \, d\beta_s, t \geq 0)$ is an $\mathcal{F}_t \equiv \sigma\{B_s, s \leq t\}$-semimartingale if and only if

$$\int_{0+} \frac{|f(s)|}{\sqrt{s}} \, ds < \infty.$$

(For several applications of Jeulin's lemma, we also refer to [12, 18].) Nonetheless, for every $f \in L^2_{\text{loc}}(ds)$,

$$\int_0^t f(s) \left(\frac{B_s}{s}\right) ds := \lim_{\varepsilon \downarrow 0} \int_\varepsilon^t f(s) \left(\frac{B_s}{s}\right) ds$$

exists both a.s. and in L^2, as follows easily from (1.3). Thus the identity:

$$\int_0^t f(s)\,d\beta_s = \int_0^t f(s)\,dB_s - \int_0^t f(s)\left(\frac{B_s}{s}\right)ds \tag{1.4}$$

is meaningful for every $f \in L^2_{\mathrm{loc}}(ds)$.

In the present paper, we intend to make a similar discussion when instead of (1.2), we consider the semimartingale:

$$X_t = R_\delta(t), \quad t \geq 0,$$

where $(R_\delta(t), t \geq 0)$ is a δ-dimensional Bessel process (BES(δ)-process in short), $\delta \geq 1$, starting from 0. We recall the canonical decomposition:

$$R_\delta(t) = B_t + V_\delta(t), \quad t \geq 0, \tag{1.5}$$

where

$$V_\delta(t) = \begin{cases} L_t, & \delta = 1, \\ \dfrac{\delta - 1}{2}\displaystyle\int_0^t \frac{ds}{R_\delta(s)}, & \delta > 1. \end{cases} \tag{1.6}$$

(For $\delta = 1$, we may consider: $R_1(t) = |\beta_t|$; $B_t = \int_0^t \mathrm{sgn}(\beta_s)\,d\beta_s$, and (L_t) the local time at 0 of β.) Then, again as a consequence of Jeulin's lemma, we can show that, for $f \in L^2_{\mathrm{loc}}(ds)$, there is the equivalence:

$$\int_0^t |f(s)||dV_\delta(s)| < \infty \qquad \Longleftrightarrow \qquad \int_0^t \frac{|f(s)|}{\sqrt{s}}\,ds < \infty.$$

Despite this, we shall show the following analogue of (1.4): define the centered Bessel process

$$\widehat{R_\delta}(t) = R_\delta(t) - E[R_\delta(t)] \equiv B_t + \widehat{V_\delta}(t), \quad t \geq 0.$$

Then the integral $\int_0^t f(s)\,d\widehat{R_\delta}(s)$ makes sense for every $f \in L^2_{\mathrm{loc}}(ds)$. In fact, we have obtained the much more precise results. For simplicity, we restrict our argument to the time interval $[0,1]$: For a bounded, measurable function $f : [0,1] \to \mathbb{R}$, we set

$$I(f; \widehat{R_\delta}) = \int_0^1 f(u)\,d\widehat{R_\delta}(u).$$

Theorem 1.1. *For every $\delta \geq 1$, it holds that, for all bounded, measurable functions $f : [0,1] \to \mathbb{R}$,*

$$E[I(f; \widehat{R_\delta})^2] \leq \|f\|_2^2. \tag{1.7}$$

Moreover, the constant 1 implicit in (1.7) *is the optimal constant K such that $E[I(f; \widehat{R_\delta})^2] \leq K\|f\|_2^2$.*

Note that, for each $f^* \in L^2([0,1])$, we may find a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded, measurable functions on $[0,1]$, such that $\|f_n - f^*\|_2 \to 0$ as $n \to \infty$. From (1.7), we see that $\{I(f_n; \widehat{R_\delta})\}_{n\in\mathbb{N}}$ forms a Cauchy sequence in $L^2(P)$. So we may define the Wiener integral $I(f^*; \widehat{R_\delta})$ as its limit. Thus we obtain, as a corollary to Theorem 1.1:

Corollary 1.2. *For all $f \in L^2([0,1])$, the Wiener integral $I(f; \widehat{R_\delta})$ is well-defined relative to* BES(δ), $\delta \geq 1$, *starting from* 0. *Moreover, it enjoys the property* (1.7).

We may extend the above results by replacing $(R_\delta(t), t \geq 0)$ by $(R_\delta(t))^\alpha$, for every α such that $(R_\delta(t))^\alpha$ is a semimartingale; that is,

$$\alpha \geq (2-\delta)_+.$$

Here, for $a \in \mathbb{R}$, $(a)_+ = \max\{a, 0\}$. In this case, we may modify the above as follows: for $\beta \in \mathbb{R}$ satisfying $\delta + \beta > 0$, set

$$\kappa_\beta = E[R_1^\beta] \equiv 2^{\beta/2} \frac{\Gamma(\frac{\delta+\beta}{2})}{\Gamma(\frac{\delta}{2})}.$$

Theorem 1.3. *For $\delta > 0$ and $\alpha \in (0,2)$, suppose that $\alpha \geq (2-\delta)_+$. Then it holds that, for all bounded, measurable functions f on $[0,1]$,*

$$E[I(f; \widehat{R_\delta^\alpha})^2] \leq \alpha^2 \kappa_{2(\alpha-1)} \|f_\alpha\|_2^2, \tag{1.8}$$

where $f_\alpha(u) := f(u) u^{(\alpha-1)/2}$.

Remark 1.1. *The constant $\alpha^2 \kappa_{2(\alpha-1)}$ in* (1.8) *is also optimal (see Remark 3.1).*

We denote by $L^2_\alpha([0,1])$ the weighted L^2 space defined by

$$L^2_\alpha([0,1]) = \{f; f_\alpha \in L^2([0,1])\}.$$

As a consequence of Theorem 1.3, we also have the following:

Corollary 1.4. *Suppose that δ and α satisfy the assumption of Theorem 1.3. Then, for all $f \in L^2_\alpha([0,1])$, the Wiener integral $I(f; \widehat{R_\delta^\alpha})$ is well-defined.*

The rest of the paper is organized as follows: in Section 2, after developing some Hardy-like inequalities, we prove Theorem 1.1; Section 3 is devoted to the proof of Theorem 1.3; in Section 4, we show that as a consequence of the previous results, Wiener integral may be defined relatively to the centered BES(δ)-bridges, for $\delta \geq 1$; finally, in Section 5, which we consider as an Appendix, we have gathered several topics we refer to in our discussions.

2 An approach via Hardy's L^2 inequality

2.1 Some Hardy-like inequalities

Proposition 2.1. *For a non-negative, integrable function $\Phi(t), 0 \leq t \leq 1$, the operator*

$$T_\Phi : f \to \{ \int_0^1 dt\, f(ut)\Phi(t), u \leq 1\}$$

which is well-defined on bounded Borel measurable functions f, extends as a bounded linear operator on $L^2([0,1])$ if and only if

$$K_\Phi := \int_0^1 dt\, \frac{\Phi(t)}{\sqrt{t}} < \infty. \tag{2.1}$$

Moreover,

$$\|T_\Phi f\|_2 \leq K_\Phi \|f\|_2 \quad \textit{for all } f \in L^2([0,1]),$$

and the constant K_Φ is the optimal constant K such that: $\|T_\Phi f\|_2 \leq K\|f\|_2$; that is, the operator norm $\|\!|T_\Phi\|\!|$ of T_Φ is equal to K_Φ.

For Φ satisfying (2.1), we define the quadratic form J_Φ on $L^2([0,1]) \times L^2([0,1])$ by:

$$J_\Phi(f,g) = \int_0^1 du\, f(u) T_\Phi g(u), \quad f, g \in L^2([0,1]). \tag{2.2}$$

Proof of Proposition 2.1. As is well known, it holds that:

$$\|\!|T_\Phi\|\!| = \sup\{|J_\Phi(f,g)|; f, g \in L^2([0,1]), \|f\|_2 \leq 1, \|g\|_2 \leq 1\}.$$

By Fubini's theorem and the Cauchy-Schwarz inequality, we have:

$$\begin{aligned} |J_\Phi(f,g)| &\leq \int_0^1 dt\, \Phi(t) \|f\|_2 \left(\int_0^1 du\, g(ut)^2 \right)^{1/2} \\ &\leq \int_0^1 dt\, \frac{\Phi(t)}{\sqrt{t}} \|f\|_2 \|g\|_2, \end{aligned}$$

which shows $\|\!|T_\Phi\|\!| \leq K_\Phi$. On the other hand, if we assume that $|J_\Phi(f,g)| \leq K\|f\|_2\|g\|_2$, then, by taking $f \equiv g \equiv \psi_\alpha$, where $\psi_\alpha(u) = u^\alpha$ for $\alpha > -1/2$, we get $J_\Phi(\psi_\alpha, \psi_\alpha) \leq K(1+2\alpha)^{-1}$, but

$$J_\Phi(\psi_\alpha, \psi_\alpha) = \int_0^1 dt\, \Phi(t) t^\alpha \times \frac{1}{1+2\alpha}.$$

Therefore, $\int_0^1 dt\, \Phi(t) t^\alpha \leq K$ for every $\alpha > -1/2$, and thus letting $\alpha \downarrow -1/2$, we get $K_\Phi \leq K$, which ends the proof. ∎

The next two propositions were proven and kindly proposed to us by F. Hirsh [6].

Proposition 2.2. *Suppose $\Phi \geq 0$ and satisfies (2.1). Moreover, suppose that there exists $\alpha > -1/2$ such that Φ is expanded as:*

$$\Phi(t) = t^{\alpha} \sum_{n=0}^{\infty} a_n t^n, \quad a_n \geq 0. \tag{2.3}$$

Then the quadratic form J_Φ is non-negative definite; i.e., $J_\Phi(f,f) \geq 0$ for all $f \in L^2([0,1])$.

Proof. If we denote $\phi_\beta(t) = t^\beta$, we have:

$$\Phi(t) = \sum_{n=1}^{\infty} a_n \phi_{n+\alpha}(t),$$

and, in order to prove the assertion, it suffices to show: for any $\beta > -1/2$,

$$J_{\phi_\beta}(f,f) \geq 0. \tag{2.4}$$

Moreover, by approximating each $f \in L^2([0,1])$ by functions in $C([0,1])$, we only have to prove (2.4) for $f \in C([0,1])$. By change of variables, and by integration by parts,

$$\begin{aligned} J_{\phi_\beta}(f,f) &= \int_0^1 du\, u^{-2\beta-1} u^\beta f(u) \int_0^u dv\, v^\beta f(v) \\ &= F_\beta(1) + (2\beta+1) \int_0^1 du\, u^{-2\beta-2} F_\beta(u). \end{aligned}$$

Here we set

$$F_\beta(u) = \int_0^u ds\, f(s) s^\beta \int_0^u dv\, f(v) v^\beta \equiv \frac{1}{2} \left(\int_0^u ds\, f(s) s^\beta \right)^2,$$

which is non-negative. So the assertion is proved. □

In fact, the functions Φ such that J_Φ is non-negative definite may be characterized as follows: for $f : [0,1] \to \mathbb{R}$, we denote $\widetilde{f} : \mathbb{R}_+ \to \mathbb{R}$ defined by $\widetilde{f}(x) = e^{-x/2} f(e^{-x})$. The mapping $f \to \widetilde{f}$ is an onto isometry from $L^2([0,1])$ on $L^2(\mathbb{R}_+)$. Let $\phi : [0,1] \to \mathbb{R}$ such that $|\phi|$ satisfies (2.1). Then $\int_0^\infty |\widetilde{\phi}(x)|\, dx < \infty$. Let

$$H(x) = \widetilde{\phi}(|x|) = e^{-|x|/2} \phi(e^{-|x|})$$

Proposition 2.3. *The operator T_ϕ is positive if and only if H is positive definite in the sense that, for every $h \in L^2(\mathbb{R})$,*

$$\iint_{\mathbb{R}^2} h(x)h(y)H(x-y)\,dxdy \geq 0. \tag{2.5}$$

Proof. Note that

$$\widetilde{T_\phi f}(s) = \int_0^\infty \widetilde{f}(t+s)\widetilde{\phi}(t)\,dt.$$

Thus, it holds that

$$\begin{aligned} J_\phi(f,f) &= \int_0^\infty \int_0^\infty dsdt\,\widetilde{f}(s)\widetilde{f}(t+s)\widetilde{\phi}(t) \\ &= \int_0^\infty ds\,\widetilde{f}(s)\int_s^\infty dt\,\widetilde{f}(t)\widetilde{\phi}(t-s) \\ &= \int_0^\infty ds\,\widetilde{f}(s)\int_s^\infty dt\,\widetilde{f}(t)H(t-s). \end{aligned}$$

Hence, from the symmetry of H,

$$J_\phi(f,f) = \frac{1}{2}\iint_{\mathbb{R}_+^2} dsdt\,\widetilde{f}(s)\widetilde{f}(t)H(t-s).$$

Thus, $J_\phi \geq 0$ if and only if

$$\iint_{\mathbb{R}_+^2} dsdt\,g(s)g(t)H(t-s) \geq 0 \quad \text{for all } g \in L^2(\mathbb{R}_+).$$

It follows immediately that, if H is positive definite, then J_ϕ is positive. Conversely, if $J_\phi \geq 0$ and if $h \in L^2(\mathbb{R})$, then, for every $a \leq 0$,

$$\begin{aligned} &\int_a^\infty \int_a^\infty dsdt\,h(s)h(t)H(t-s) \\ &= \iint_{\mathbb{R}_+^2} dsdt\,h(a+s)h(a+t)H(t-s) \geq 0. \end{aligned}$$

Letting a go to $-\infty$, we have:

$$\iint_{\mathbb{R}^2} dsdt\,h(s)h(t)H(t-s) \geq 0.$$

So the proof is complete. □

As a corollary, we have the following (from Bochner's theorem):

Corollary 2.4. *If ϕ is continuous on $(0,1]$, $J_\phi \geq 0$ if and only if there exists a finite positive symmetric measure on $\mathbb{R}$ such that*

$$\phi(t) = \frac{1}{\sqrt{t}} \int_{-\infty}^{+\infty} t^{ix}\, d\mu(x). \tag{2.6}$$

Example 2.1. (i) *When ϕ is given in the form* (2.3)*, the measure μ in the above corollary is given by:*

$$d\mu(x) = \sum_{n=0}^{\infty} a_n \frac{n+\alpha+\frac{1}{2}}{(n+\alpha+\frac{1}{2})^2 + x^2} dx. \tag{2.7}$$

(ii) *An interesting class of functions ϕ which satisfy* (2.6) *are those ϕ's given by:*

$$e^{-|x|/2}\phi(e^{-|x|}) = c\int_0^\infty e^{-x^2 y/2}\, dm(y)$$

for some constant $c > 0$, and for a probability measure m on $\mathbb{R}_+$; the measure μ in the above corollary then appears as the law of B_A, where B is a Brownian motion, and A is a random variable independent of B and distributed as m.

2.2 Proof of Theorem 1.1

We first look for some rahter explicit expression for $E[I(f;\widehat{R_\delta})^2]$:

Proposition 2.5. *Let $f : [0,1] \to \mathbb{R}$ be a bounded measurable function. Then*

$$E[I(f;\widehat{R_\delta})^2] = \|f\|_2^2 - J_{\Phi_\delta}(f,f), \tag{2.8}$$

where

$$\Phi_\delta(t) = \begin{cases} \dfrac{1}{\pi}\left\{\dfrac{1}{\sqrt{t}}(1-\sqrt{1-t}) + \dfrac{\sqrt{t}}{\sqrt{1-t}}\right\}, & \delta = 1, \\ -(\delta-1)\dfrac{d}{dt}E[\widehat{R_\delta(1)^{-1}}\widehat{R_\delta(t)}], & \delta > 1, \end{cases}$$

and J_{Φ_δ} is defined through (2.2)*.*

Example 2.2. *In the case $\delta = 3$, Φ_δ is given by*

$$\Phi_3(t) = \frac{4}{\pi}\frac{1}{\sqrt{t}}(1-\sqrt{1-t}).$$

For some connection between Φ_1 and Φ_3, which may be deduced from Pitman's representation theorem for BES(3)*, see Lemma 5.3 in the appendix.*

Suppose that $(X_t, t \geq 0)$ is an $(\mathcal{F}_t)$-semimartingale with decomposition

$$X_t = M_t + V_t, \quad t \geq 0, \tag{2.9}$$

where (M_t) is an L^2-integrable $(\mathcal{F}_t)$-martingale and (V_t) is an $(\mathcal{F}_t)$-adapted process of bounded variation satisfying $E[(\int_0^t |dV_u|)^2] < \infty$ for every t. For a bounded, measurable function f on $[0,1]$, we set

$$X_t^f = \int_0^t f(u)\, dX_u, \quad 0 \leq t \leq 1.$$

Clearly, (X_t^f) is a semimartingale which decomposes as:

$$X_t^f = \int_0^t f(u)\, dM_u + \int_0^t f(u)\, dV_u, \quad 0 \leq t \leq 1.$$

Lemma 2.6. *It holds that*

$$E[(X_1^f)^2] = E[\int_0^1 f(u)^2\, d\langle M\rangle_u] + 2E[\int_0^1 f(u)\, dV_u \left(\int_0^u f(s)\, dX_s\right)].$$

Proof. By Itô's formula,

$$(X_1^f)^2 = 2\int_0^1 X_u^f\, dX_u^f + \langle X^f\rangle_1.$$

Taking the expectation on both sides, and noting $\langle X^f\rangle_1 = \int_0^1 f(u)^2\, d\langle M\rangle_u$, we obtain the lemma. □

By using Lemma 2.6, we prove Proposition 2.5. We begin with the case $\delta = 1$.

Proof of Proposition 2.5 for $\delta = 1$. Noting (1.5), we take $X_t = R_\delta(t) \equiv |\beta_t|$, $M_t = B_t$ and $V_t = V_\delta(t) \equiv L_t$ in (2.9). Then by Lemma 2.6,

$$E[\left(\int_0^1 f(u)\, d|\beta_u|\right)^2] = \|f\|_2^2 + 2E[\int_0^1 f(u)\, dL_u \left(\int_0^u f(s)\, d|\beta_s|\right)]. \tag{2.10}$$

Recall that dL_u is carried by the set of zeros of β. So, if we denote by $(b^u(s), 0 \leq s \leq u)$ a Brownian bridge of length u, the expectation on the RHS of (2.10) may be written as:

$$\int_0^1 f(u)\, dE[L_u] E[\int_0^u f(s)\, d|b^u(s)|]. \tag{2.11}$$

Note that $E[L_u] \equiv E[|\beta_u|] = \sqrt{2u/\pi}$, hence $dE[L_u] = du/\sqrt{2\pi u}$. Moreover, since $b^u(s)$ is Gaussian with mean 0 and variance $s(u-s)/u$,

$$E[|b^u(s)|] = \sqrt{\frac{2}{\pi}}\sqrt{\frac{s(u-s)}{u}},$$

hence

$$dE[|b^u(s)|] = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{s(1-s/u)}} \left(1 - \frac{2s}{u}\right) ds.$$

Combining these, we see that (2.11) is equal to:

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^1 \frac{du}{\sqrt{u}} f(u) \int_0^u ds\, f(s) \frac{1}{\sqrt{s(1-s/u)}} \left(1 - \frac{2s}{u}\right) \\
&= \frac{1}{2\pi} \int_0^1 du\, f(u) \int_0^1 dt\, f(ut) \frac{1-2t}{\sqrt{t(1-t)}}.
\end{aligned}$$

Here, for the equality, we changed variables with $s = ut, 0 \le t \le 1$. From this and (2.10),

$$E\Big[\left(\int_0^1 f(u)\, d|\beta_u|\right)^2\Big] = \|f\|_2^2 + J_{\Psi_1}(f,f), \quad \Psi_1(t) := \frac{1}{\pi} \frac{1-2t}{\sqrt{t(1-t)}}. \tag{2.12}$$

On the other hand, $E[\int_0^1 f(u)\, d|\beta_u|] = \int_0^1 du\, f(u)/\sqrt{2\pi u}$, hence

$$\left(E\Big[\int_0^1 f(u)\, d|\beta_u|\Big]\right)^2 = \frac{1}{\pi} \int_0^1 \frac{du}{\sqrt{u}} f(u) \int_0^u \frac{ds}{\sqrt{s}} f(s) = J_{\Psi_2}(f,f),$$

where $\Psi_2(t) := 1/(\pi\sqrt{t})$. Combining this with (2.12), we see that

$$E[I(f;\widehat{R_1})^2] \equiv \mathrm{var}\left(\int_0^1 f(u)\, d|\beta_u|\right) = \|f\|_2^2 - J_{\Psi_2 - \Psi_1}(f,f).$$

Noting the identity: $\Phi_1 = \Psi_2 - \Psi_1$, we obtain (2.8) for $\delta = 1$. □

Next, we turn to the case $\delta > 1$.

Proof of Proposition 2.5 for $\delta > 1$. By the decomposition (1.5), we take $X_t = \widehat{R_\delta(t)}$, $M_t = B_t$ and $V_t = \widehat{V_\delta(t)}$ in (2.9). Then, noting $dV_u = 2^{-1}(\delta - 1)\widehat{R_\delta(u)^{-1}}\, du$, we see from Lemma 2.6 that

$$\begin{aligned}
&E[(X_1^f)^2] \equiv E[I(f;\widehat{R_\delta})^2] \\
&= \|f\|_2^2 + (\delta - 1)E\Big[\int_0^1 f(u)\widehat{R_\delta(u)^{-1}}\, du \left(\int_0^u f(s)\, d\widehat{R_\delta(s)}\right)\Big] \\
&= \|f\|_2^2 + (\delta - 1) \int_0^1 du\, f(u) \int_0^u ds\, f(s) \frac{d}{ds} E[\widehat{R_\delta(u)^{-1}}\widehat{R_\delta(s)}].
\end{aligned}$$

Now (2.8) follows from this and the change of variables with $s = ut$. □

Lemma 2.7. (i) *For every $\delta \geq 1$, Φ_δ is non-negative and satisfies:*

$$\int_0^1 \frac{dt}{\sqrt{t}} \Phi_\delta(t) < \infty.$$

(ii) *J_{Φ_δ} extends as a quadratic form on $L^2([0,1]) \times L^2([0,1])$. Moreover, it is positive-definite.*

Proof. (i) This assertion for $\delta = 1$ is obvious by definition. For $\delta > 1$, we denote $\varphi_\delta(t) = E[\widehat{R_\delta(1)^{-1}} \widehat{R_\delta(t)}], 0 \leq t \leq 1$, and use the fact that φ_δ admits the following representation:

$$\varphi_\delta(t) = C_\delta t^{1/2} \{{}_2F_1(-\frac{1}{2}, \frac{1}{2}; \frac{\delta}{2}; t) - 1\}, \quad C_\delta = \frac{\Gamma(\frac{\delta+1}{2})\Gamma(\frac{\delta-1}{2})}{\{\Gamma(\frac{\delta}{2})\}^2}.$$

Here ${}_2F_1$ denotes Gauss' hypergeometric function; see Proposition 5.1 in the appendix. Recalling the series expansion (in fact, the definition) of ${}_2F_1$ (e.g., [10, (9.1.1)]), we see that:

$$\text{(a) } \varphi_\delta \leq 0; \qquad \text{(b) } \varphi_\delta' \leq 0; \qquad \text{(c) } \liminf_{t \downarrow 0} \frac{1}{\sqrt{t^3}} \varphi_\delta(t) > -\infty.$$

Indeed,

$$t^{1/2} \{{}_2F_1(-\frac{1}{2}, \frac{1}{2}; \frac{\delta}{2}; t) - 1\} = \sum_{n=1}^{\infty} b_n t^{n+1/2}, \quad b_n = \frac{(-\frac{1}{2})_n (\frac{1}{2})_n}{(\frac{\delta}{2})_n n!} < 0, \tag{2.13}$$

from which the properties (a)–(c) of φ_δ follow. Here, for $\gamma \in \mathbb{R}$, $(\gamma)_n := \gamma(\gamma+1)\cdots(\gamma+n-1)$. (In fact, for (c), the limit itself exists: $\lim_{t\downarrow 0} t^{-3/2} \varphi_\delta(t) = -C_\delta/(2\delta)$.) So, by (b) and by definition, $\Phi_\delta(t) = -(\delta-1)\varphi_\delta'(t) \geq 0$. Moreover, by (c), we may use integration by parts to see:

$$\begin{aligned} \int_0^1 \frac{dt}{\sqrt{t}} \Phi_\delta(t) &\equiv -(\delta-1) \int_0^1 \frac{dt}{\sqrt{t}} \varphi_\delta'(t) \\ &= -(\delta-1) \left\{ \varphi_\delta(1) + \frac{1}{2} \int_0^1 \frac{dt}{\sqrt{t^3}} \varphi_\delta(t) \right\} < \infty. \end{aligned}$$

(ii) The former assertion now follows from (i) and Proposition 2.1. To show the latter, we shall show that Φ_δ can be developed as:

$$\Phi_\delta(t) = t^{1/2} \sum_{n=0}^{\infty} a_n t^n, \quad a_n \geq 0. \tag{2.14}$$

Then the positivity of J_{Φ_δ} follows immediately from Proposition 2.2. For the case $\delta = 1$, note that the function $1 - \sqrt{1-t}, 0 \le t \le 1$, can be developed as:

$$1 - \sqrt{1-t} = \sum_{n=1}^{\infty} c_n t^n, \quad c_n = \frac{(2n)!}{2^{2n}(n!)^2(2n-1)}, \tag{2.15}$$

from which it also follows that, by differentiating both sides,

$$\frac{1}{2\sqrt{1-t}} = \sum_{n=1}^{\infty} n c_n t^{n-1}. \tag{2.16}$$

Combining these, we see that Φ_1 is expanded as (2.14) with

$$a_n = \frac{1}{\pi}\{1 + 2(n+1)\}c_{n+1} > 0, \quad n = 0, 1, 2, \cdots .$$

On the other hand, for $\delta > 1$, we see from (2.13) that

$$\varphi_\delta'(t) = C_\delta \sum_{n=1}^{\infty} \left(n + \frac{1}{2}\right) b_n t^{n-1/2}.$$

Since $\Phi_\delta(t) = -(\delta - 1)\varphi_\delta'(t)$ by definition, $\Phi_\delta, \delta > 1$, is also expanded as (2.14) with

$$a_n = C_\delta(\delta - 1)\left(n + \frac{3}{2}\right) \times (-b_{n+1}) > 0, \quad n = 0, 1, 2, \cdots .$$

So the proof is complete. □

Now we are ready to prove Theorem 1.1:

Proof of Theorem 1.1. The former assertion (1.7) is immediate from Proposition 2.5 and Lemma 2.7. By Proposition 2.5, in order to prove the latter, it suffices to find a sequence (f_n) of functions such that

$$J_{\Phi_\delta}(f_n, f_n) \le \varepsilon_n \|f_n\|_2^2$$

with $\varepsilon_n \to 0$ as $n \to \infty$. This is satisfied by the choice: $f_n(u) \equiv f_{\alpha_n}(u) = u^{\alpha_n}$ for $\alpha_n \to \infty$. Indeed,

$$\begin{aligned} J_{\Phi_\delta}(f_\alpha, f_\alpha) &= \left(\int_0^1 du\, u^{2\alpha}\right) \int_0^1 dt\, t^{\alpha-1/2}\Phi_\delta(t) \\ &= \|f_\alpha\|_2^2 \int_0^1 dt\, t^{\alpha-1/2}\Phi_\delta(t), \end{aligned}$$

in which the integral relative to t decreases to 0 as $\alpha \to \infty$ by the dominated convergence theorem since it is bounded from above by $\int_0^1 dt\, t^{-1/2}\Phi_\delta(t) < \infty$. □

3 Proof of Theorem 1.3

In this section, we denote by the pair $(R = \{R_t, t \geq 0\}, P_x^{(\delta)})$ a BES(δ)-process starting from x: $P_x^{(\delta)}(R_0 = x) = 1$. We denote by $E_x^{(\delta)}$ the expectation with respect to $P_x^{(\delta)}$. When $x = 0$, we often write E for $E_0^{(\delta)}$ and P for $P_0^{(\delta)}$.

3.1 Proof for the case $\delta > 1$ and $\alpha > (2-\delta)_+$

In this subsection, we prove the assertion of Theorem 1.3 in the case $\delta > 1$ and $\alpha > (2-\delta)_+$. Set $\delta_\alpha = \alpha(\delta + \alpha - 2)$.

Proposition 3.1. *Suppose $\delta > 1$ and $\alpha > (2-\delta)_+$. Then we have, for every bounded, measurable function f on $[0,1]$,*

$$E[I(f;\widehat{R^\alpha})^2] = \alpha^2 \kappa_{2(\alpha-1)} \|f_\alpha\|_2^2 - J_{\Phi_{\delta,\alpha}}(f_\alpha, f_\alpha), \tag{3.1}$$

where

$$\Phi_{\delta,\alpha}(t) = -\delta_\alpha t^{-(\alpha-1)/2} \varphi'_{\delta,\alpha}(t) \quad \textit{with} \quad \varphi_{\delta,\alpha}(t) = E[\widehat{R_1^{\alpha-2}}\widehat{R_t^\alpha}],\ 0 \leq t \leq 1.$$

Proof. Recall that, for $\delta > 1$, BES(δ) admits a semimartingale decomposition of the following form:

$$R_t = B_t + \frac{\delta - 1}{2} \int_0^t \frac{du}{R_u}, \tag{3.2}$$

where $B = (B_t, t \geq 0)$ is a Brownian motion. By Itô's formula,

$$R_t^\alpha = \alpha \int_0^t R_u^{\alpha-1}\, dB_u + \frac{\delta_\alpha}{2} \int_0^t R_u^{\alpha-2}\, du. \tag{3.3}$$

We take the expectation on both sides to get

$$E[R_t^\alpha] = \frac{\delta_\alpha}{2} \int_0^t E[R_u^{\alpha-2}]\, du. \tag{3.4}$$

From these, we have

$$\widehat{R_t^\alpha} = \alpha \int_0^t R_u^{\alpha-1}\, dB_u + \frac{\delta_\alpha}{2} \int_0^t \widehat{R_u^{\alpha-2}}\, du.$$

Now we take, in (2.9), $X_t = \widehat{R_t^\alpha}$, $M_t = \int_0^t R_u^{\alpha-1}\, dB_u$ and $V_t = (\delta_\alpha/2) \int_0^t \widehat{R_u^{\alpha-2}}\, du$. Then by Lemma 2.6, it holds that, for a bounded, measurable function f on $[0,1]$,

$$\begin{aligned} &E[(X_1^f)^2]\ (\equiv E[I(f;\widehat{R^\alpha})^2]) \\ &= \delta_\alpha E\Big[\int_0^1 \left(\int_0^u f(s)\, d\widehat{R_s^\alpha}\right) f(u) \widehat{R_u^{\alpha-2}}\, du\Big] + \alpha^2 \int_0^1 f(u)^2 E[R_u^{2(\alpha-1)}]\, du \\ &= \delta_\alpha \int_0^1 du\, f(u) \int_0^u ds\, f(s) \left\{\frac{d}{ds} E[\widehat{R_u^{\alpha-2}}\widehat{R_s^\alpha}]\right\} + \alpha^2 E[R_1^{2(\alpha-1)}] \int_0^1 f(u)^2 u^{\alpha-1}\, du. \end{aligned} \tag{3.5}$$

Now the assertion of the proposition is immediate by the change of variables. ☐

Lemma 3.2. *$J_{\Phi_{\delta,\alpha}}$ extends as a quadratic form on $L^2([0,1]) \times L^2([0,1])$. Moreover, it is positive-definite.*

Once this lemma is shown, then, combining this with Proposition 3.1, we may prove the assertion of Theorem 1.3 for $\delta > 1$ and $\alpha > (2-\delta)_+$:

Proof of Theorem 1.3 for $\delta > 1$ and $\alpha > (2-\delta)_+$. By Proposition 3.1 and Lemma 3.2, we have:

$$E_0^{(\delta)}[I(f; \widehat{R^\alpha})^2] \le \alpha^2 \kappa_{2(\alpha-1)} \|f_\alpha\|_2^2,$$

which is (1.8). ☐

Remark 3.1. *Similarly to the proof of Theorem 1.1, we may see that the constant $\alpha^2 \kappa_{2(\alpha-1)}$ is optimal.*

The proof of Lemma 3.2 may be given along the lines of that of Theorem 1.1 for $\delta > 1$, Φ_δ and φ_δ being replaced by $\Phi_{\delta,\alpha}$ and $\varphi_{\delta,\alpha}$, respectively. Instead of giving details (and in order to avoid duplications), we present here a different way, other than relying on series expansions, to show that the function $\varphi_{\delta,\alpha}$ has similar properties to those (a)–(c) of φ_δ, which we shall present in the next proposition:

Proposition 3.3. *Suppose that $\delta > 0$ and $\alpha \in (0,2)$ satisfy $\alpha > 2 - \delta$. Then the function $\varphi_{\delta,\alpha}$ has the following properties:*

$$\text{(a}'\text{)}\ \varphi_{\delta,\alpha} \le 0; \qquad \text{(b}'\text{)}\ \varphi_{\delta,\alpha}' \le 0; \qquad \text{(c}'\text{)}\ \liminf_{t\downarrow 0} t^{-\frac{\alpha}{2}-1}\varphi_{\delta,\alpha}(t) > -\infty.$$

Remark 3.2. *By using the series expansion of $\varphi_{\delta,\alpha}$, we may see that, for the property (c′), the limit itself exists; see Remark 5.1 in the appendix.*

For $x \ge 0$ and $t > 0$, set

$$\begin{aligned}\rho_\alpha(x,t) &\equiv \rho_\alpha^{(\delta)}(x,t) = E_x^{(\delta)}[R_t^\alpha],\\ \sigma_\alpha(x,t) &\equiv \sigma_\alpha^{(\delta)}(x,t) = E_x^{(\delta)}[R_t^{\alpha-2}].\end{aligned}$$

To prove Proposition 3.3, we begin with the following lemma:

Lemma 3.4. *Under the same assumption as in Proposition 3.3, the following assertions hold for every fixed $t > 0$:*
(i) *$\rho_\alpha(x,t)$ is increasing in x.*
(ii) *$\sigma_\alpha(x,t)$ is decreasing in x.*
(iii) *The following lower bound for $\sigma_\alpha(x,t)$ holds:*

$$\sigma_\alpha(x,t) \geq e^{-\frac{x^2}{2t}} \sigma_\alpha(0,t) \quad \textit{for all } x \geq 0.$$

Proof. The assertions (i) and (ii) follow from the additive property of Bessel processes; that is, if $R_x^{(\delta)}(t), t \geq 0$, denotes $\mathrm{BES}(\delta)$ starting from x, then it holds that:

$$(R_x^{(\delta)}(t))^2 \overset{(d)}{=} (R_x^{(0)}(t))^2 + (R_0^{(\delta)}(t))^2, \tag{3.6}$$

where, on the RHS, the two processes are independent. Moreover, considering two SDE's with common Brownian motion B:

$$(R_x^{(0)}(t))^2 = x^2 + \int_0^t 2R_x^{(0)}(u)\, dB_u, \qquad (R_y^{(0)}(t))^2 = y^2 + \int_0^t 2R_y^{(0)}(u)\, dB_u,$$

we see that

$$R_x^{(0)}(t) \leq R_y^{(0)}(t) \quad \text{if} \quad x \leq y,$$

by comparison. The assertion (iii) also follows from (3.6) since we deduce from there that

$$\sigma_\alpha(x,t) \geq E_0^{(\delta)}[R_t^{\alpha-2}] P_x^{(0)}(T_0(R) \leq t),$$

where $T_0(R)$ is the time at which R reaches to 0. It is well-known (e.g., [16, Chapter XI]) that, under $P_x^{(0)}$, $T_0(R)$ is identical in law with $x^2/2\boldsymbol{e}$, $\boldsymbol{e}$ being a standard exponential variable, hence $P_x^{(0)}(T_0(R) \leq t) = e^{-x^2/2t}$. Combining these yields the assertion (iii). □

Remark 3.3. *The assertions* (i) *and* (ii) *may be seen from the FKG inequality; an application of a generalized FKG inequality due to Preston [14], to the laws of* $\mathrm{BES}(\delta)$, *$\delta > 0$, asserts that, for increasing functions F on $C([0,1];\mathbb{R}_+)$, it holds that $E_x^{(\delta)}[F(R)] \leq E_y^{(\delta)}[F(R)]$ if $0 \leq x \leq y$. Here we say that a function F defined on $C([0,1];\mathbb{R}_+)$ is increasing if $F(w_1) \leq F(w_2)$ for all $w_1, w_2 \in C([0,1];\mathbb{R}_+)$ satisfying $w_1(t) \leq w_2(t)$ for all $0 \leq t \leq 1$. So, if we take $F(R) = R_t^\alpha$ (resp. $F(R) = -R_t^{\alpha-2}$), we recover the assertion* (i) *(resp. the assertion* (ii)*). For the applicability of this inequality to Bessel processes, see Proposition 5.6 in the appendix.*

Set

$$\psi_{\delta,\alpha}(t) = E[\widehat{R_1^{\alpha-2}}\widehat{R_t^{\alpha-2}}].$$

The two functions $\varphi_{\delta,\alpha}$ and $\psi_{\delta,\alpha}$ are related via:

Lemma 3.5. *The following identity holds:*

$$\varphi'_{\delta,\alpha}(t) = -\frac{\delta_\alpha}{2}\psi_{\delta,\alpha}(t) + \frac{\alpha}{t}\varphi_{\delta,\alpha}(t). \tag{3.7}$$

Proof. By definition, $\varphi'_{\delta,\alpha}(t)$ is written as:

$$\frac{d}{dt}E[R_1^{\alpha-2}R_t^\alpha] - E[R_1^{\alpha-2}]\frac{d}{dt}E[R_t^\alpha]. \tag{3.8}$$

To work on the first term, we use the time-inversion and the Markov property to see that:

$$\begin{aligned} E[R_1^{\alpha-2}R_t^\alpha] &= t^\alpha E[R_1^{\alpha-2}R_{1/t}^\alpha] \\ &= t^\alpha E[R_1^{\alpha-2}\rho_\alpha(R_1, \frac{1}{t} - 1)]. \end{aligned} \tag{3.9}$$

Noting (3.4) with changing the starting point to $x \geq 0$, we deduce that

$$\rho_\alpha(x,t) = x^\alpha + \frac{\delta_\alpha}{2}\int_0^t \sigma_\alpha(x,u)\,du,$$

hence

$$\frac{\partial}{\partial t}\rho_\alpha(x,t) = \frac{\delta_\alpha}{2}\sigma_\alpha(x,t). \tag{3.10}$$

Therefore, by (3.9) and (3.10),

$$\begin{aligned} \frac{d}{dt}E[R_1^{\alpha-2}R_t^\alpha] &= \alpha t^{\alpha-1}E[R_1^{\alpha-2}R_{1/t}^\alpha] + t^\alpha E[R_1^{\alpha-2} \times \frac{\delta_\alpha}{2}\sigma_\alpha(R_1, \frac{1}{t} - 1) \times (-\frac{1}{t^2})] \\ &= \frac{\alpha}{t}E[R_1^{\alpha-2}R_t^\alpha] - \frac{\delta_\alpha}{2}E[R_1^{\alpha-2}R_t^{\alpha-2}]. \end{aligned} \tag{3.11}$$

Here, for the second equality, we used the Markov property and the time-inversion. On the other hand, for the second term of (3.8), the factor $(d/dt)E[R_t^\alpha]$ may be expressed in two ways:

$$\begin{aligned} \frac{d}{dt}E[R_t^\alpha] &= \frac{\delta_\alpha}{2}E[R_t^{\alpha-2}] \\ &= \frac{\alpha}{2t}E[R_t^\alpha]. \end{aligned}$$

The first expression is nothing but (3.10) with $x = 0$, while the second one is a consequence of the scaling property: $E[R_t^\alpha] = (\sqrt{t})^\alpha E[R_1^\alpha]$. Thus, we may write (3.8) as:

$$\begin{aligned} & -\frac{\delta_\alpha}{2} E[R_1^{\alpha-2} R_t^{\alpha-2}] + \frac{\alpha}{t} E[R_1^{\alpha-2} R_t^\alpha] - E[R_1^{\alpha-2}] \left(-\frac{\delta_\alpha}{2} E[R_t^{\alpha-2}] + \frac{\alpha}{t} E[R_t^\alpha] \right) \\ & = -\frac{\delta_\alpha}{2} \left\{ E[R_1^{\alpha-2} R_t^{\alpha-2}] - E[R_1^{\alpha-2}] E[R_t^{\alpha-2}] \right\} + \frac{\alpha}{t} \left\{ E[R_1^{\alpha-2} R_t^\alpha] - E[R_1^{\alpha-2}] E[R_t^\alpha] \right\}, \end{aligned}$$

which is the RHS of (3.7). $\square$

To prove Proposition 3.3, we also use the following:

Lemma 3.6. *The function $\psi_{\delta,\alpha}$ is non-negative.*

Proof. Note that $\widehat{R_t^{\alpha-2}}, t > 0$, admits the following representation (Itô's representation):

$$\widehat{R_t^{\alpha-2}} = \int_0^t \frac{\partial \sigma_\alpha}{\partial x}(R_u, t-u)\, dB_u, \tag{3.12}$$

where B is the Brownian motion appearing in (3.2). Therefore, by definition,

$$\psi_{\delta,\alpha}(t) \, (= E[\widehat{R_1^{\alpha-2}} \widehat{R_t^{\alpha-2}}]) = \int_0^t du\, E[\frac{\partial \sigma_\alpha}{\partial x}(R_u, 1-u) \frac{\partial \sigma_\alpha}{\partial x}(R_u, t-u)].$$

Note that, by Lemma 3.4 (ii), $(\partial/\partial x)\sigma_\alpha \le 0$, which implies $\psi_{\delta,\alpha} \ge 0$. $\square$

Remark 3.4. *This lemma may also be seen from the FKG inequality; from the (classical) formulation of the FKG inequality, we may also see that, for each fixed $x \ge 0$, and for increasing functions F, G on $C([0,1]; \mathbb{R}_+)$, it holds that $E_x^{(\delta)}[F(R)G(R)] \ge E_x^{(\delta)}[F(R)] E_x^{(\delta)}[G(R)]$. So we may recover this lemma by taking $F(R) = -R_1^{\alpha-2}$ and $G(R) = -R_t^{\alpha-2}$.*

Now we are prepared to prove Proposition 3.3:

Proof of Proposition 3.3. Property (a′): By Itô's representation:

$$\widehat{R_t^\alpha} = \int_0^t \frac{\partial \rho_\alpha}{\partial x}(R_u, t-u)\, dB_u,$$

we have, together with (3.12) for $t = 1$,

$$\varphi_{\delta,\alpha}(t) = \int_0^t du\, E[\frac{\partial \sigma_\alpha}{\partial x}(R_u, 1-u) \frac{\partial \rho_\alpha}{\partial x}(R_u, t-u)].$$

By Lemma 3.4 (i) and (ii), we see that the inside of the expectation is negative. This shows (a′).

Property (b′): This property is now immediate from Lemmas 3.5 and 3.6, and (a′).
Property (c′): To prove this property, note that, by definition,

$$\varphi_{\delta,\alpha}(t) = E[R_1^{\alpha-2}R_t^{\alpha}] - E[R_1^{\alpha-2}]E[R_t^{\alpha}].$$

For the second term, we easily see that, by scaling,

$$E[R_1^{\alpha-2}]E[R_t^{\alpha}] = t^{\alpha/2}\kappa_{\alpha-2}\kappa_{\alpha}. \tag{3.13}$$

On the other hand, to estimate the first term, we rewrite it as, by using the Markov property,

$$E[R_1^{\alpha-2}R_t^{\alpha}] = E[R_t^{\alpha}\sigma_{\alpha}(R_t, 1-t)]. \tag{3.14}$$

By Lemma 3.4 (iii), and by scaling, (3.14) is bounded from below by

$$\begin{aligned}
&E[R_t^{\alpha}\exp\{-\frac{R_t^2}{2(1-t)}\}\sigma_{\alpha}(0,1-t)] \\
&= \kappa_{\alpha-2}t^{\alpha/2}(1-t)^{(\alpha-2)/2}E[R_1^{\alpha}\exp\{-\frac{t}{2(1-t)}R_1^2\}].
\end{aligned}$$

By using the fact that $R_1^2/2 \overset{(d)}{=} \gamma_{\delta/2}$, a gamma variable of index $\delta/2$, the expectation on the RHS is computed explicitly as $\kappa_{\alpha}(1-t)^{(\delta+\alpha)/2}$. So (3.14) is bounded from below by

$$\kappa_{\alpha-2}\kappa_{\alpha}t^{\alpha/2}(1-t)^{(\delta+2\alpha-2)/2}. \tag{3.15}$$

Combining (3.13) and (3.15) yields

$$\varphi_{\delta,\alpha}(t) \geq -\kappa_{\alpha-2}\kappa_{\alpha}t^{\frac{\alpha}{2}}\{1-(1-t)^{\frac{\delta}{2}+\alpha-1}\},$$

from which the property (c′) follows. □

Remark 3.5. *We may also see the property* (a′) *directly from the FKG inequality.*

3.2 Proof for the case $0 < \delta < 2$ and $\alpha = 2 - \delta$

For $0 < \delta < 2$, define $\mu \in (0,1)$ via: $\delta = 2(1-\mu)$. Then α and μ are related as $\alpha = 2\mu$.

Proposition 3.7. *For every bounded, measurable function f on $[0,1]$, it holds that*

$$E[I(f;\widehat{R^{\alpha}})^2] = \alpha^2\kappa_{2(\alpha-1)}\|f_{\alpha}\|_2^2 - \frac{\alpha^2}{2}(\kappa_{\alpha})^2 J^{(\alpha)}(f_{\alpha}, f_{\alpha}),$$

where $J^{(\alpha)} \equiv J_{\psi^{(\mu)}}$ with

$$\psi^{(\mu)}(t) = \frac{1}{\sqrt{t}}\{1-(1-t)^{\mu}\} + \frac{\sqrt{t}}{(1-t)^{1-\mu}}.$$

Remark 3.6. *Taking* $\alpha = 1$ ($\iff$ $\mu = 1/2$) *above, we recover* (2.8) *for* $\delta = 1$.

The key fact to Proposition 3.7 is that $R_t^{2\mu}$ may be decomposed as:

$$R_t^{2\mu} = M_t^{(\mu)} + L_t, \tag{3.16}$$

where $(M_t^{(\mu)}, t \geq 0)$ is a martingale with $\langle M^{(\mu)} \rangle_t = 4\mu^2 \int_0^t R_u^{2(2\mu-1)} \, du$, and (L_t) is a possible choice of the local time at 0 of (R_t) as a diffusion.

Proof of Proposition 3.7. We take $X_t = R_t^{2\mu}$, $M_t = M_t^{(\mu)}$ and $V_t = L_t$ in (2.6). Then by Lemma 2.6, we have, for a bounded, measurable function f on $[0, 1]$,

$$\begin{aligned} E[\Big(\int_0^1 f(u) \, dR_u^{2\mu}\Big)^2] &= E[\int_0^1 f(u) \, d\langle M^{(\mu)} \rangle_u] + 2E[\int_0^1 f(u) \, dL_u \Big(\int_0^u f(s) \, dR_s^{2\mu}\Big)] \\ &=: I_1^f + 2I_2^f. \end{aligned}$$

Since $(d/du)E[\langle M^{(\mu)} \rangle_u] = 4\mu^2 E[R_u^{2(2\mu-1)}] \equiv 4\mu^2 \kappa_{2(2\mu-1)} u^{2\mu-1}$, we have

$$I_1^f = 4\mu^2 \kappa_{2(2\mu-1)} \|f_{2\mu}\|_2^2.$$

On the other hand, for I_2^f, recall that dL_u is carried by the set of zeros of R. So, denoting by $(r_s^u, 0 \leq s \leq u)$ a Bessel bridge of length u, we have:

$$\begin{aligned} I_2^f &= \int_0^1 f(u) \, dE[L_u] E[\int_0^u f(s) \, dR_s^{2\mu} \big| R_u = 0] \\ &= \int_0^1 f(u) \, dE[L_u] \int_0^u f(s) \, dE[(r_s^u)^{2\mu}] \\ &= \int_0^1 f(u) \, dE[L_u] \int_0^1 f(ut) u^\mu \, dE[(r_t^1)^{2\mu}], \end{aligned}$$

where we changed variables with $s = ut, 0 \leq t \leq 1$, for the third line. Note that $E[L_u] = E[R_u^{2\mu}] = \kappa_{2\mu} u^\mu$, hence $dE[L_u] = \mu \kappa_{2\mu} u^{\mu-1} \, du$. Moreover, noting $r_t^1 \overset{(d)}{=} (1-t) R_{t/(1-t)}$, we see $E[(r_t^1)^{2\mu}] = \kappa_{2\mu} t^\mu (1-t)^\mu$. Combining these yields:

$$\begin{aligned} I_2^f &= \mu(\kappa_{2\mu})^2 \int_0^1 du \, f(u) u^{2\mu-1} \int_0^1 dt \, f(ut) \frac{d}{dt}\{t^\mu (1-t)^\mu\} \\ &= \int_0^1 du \, f_{2\mu}(u) \int_0^1 dt \, f_{2\mu}(ut) \Psi_1^{(\mu)}(t), \end{aligned}$$

where $\Psi_1^{(\mu)}(t) := \mu(\kappa_{2\mu})^2 t^{-\mu+1/2} (d/dt)\{t^\mu (1-t)^\mu\}$. Besides these, setting

$$I_3^f = E[\int_0^1 f(u) \, dR_u^{2\mu}] \equiv \mu \kappa_{2\mu} \int_0^1 f(u) u^{\mu-1} \, du,$$

we easily see that

$$(I_3^f)^2 = \int_0^1 du\, f_{2\mu}(u) \int_0^1 dt\, f_{2\mu}(ut)\Psi_2^{(\mu)}(t), \quad \Psi_2^{(\mu)}(t) := 2\mu^2(\kappa_{2\mu})^2 t^{-1/2}.$$

Consequently, we obtain

$$\begin{aligned} E[I(f;\widehat{R^{2\mu}})^2] &\equiv I_1^f + 2I_2^f - (I_3^f)^2 \\ &= (2\mu)^2\kappa_{2(2\mu-1)}\|f_{2\mu}\|_2^2 - J_{\Psi_2^{(\mu)}-2\Psi_1^{(\mu)}}(f_{2\mu}, f_{2\mu}) \end{aligned}$$

with the identity

$$\Psi_2^{(\mu)}(t) - 2\Psi_1^{(\mu)}(t) = 2\mu^2(\kappa_{2\mu})^2\psi^{(\mu)}(t).$$

Recalling $\alpha = 2\mu$, we obtain the proposition. □

Lemma 3.8. *$J^{(\alpha)}$ extends as a quadratic form on $L^2([0,1]) \times L^2([0,1])$. Moreover, it is positive-definite.*

Proof. Clearly $\psi^{(\mu)}$ is non-negative and satisfies $\int_0^1 dt\, t^{-1/2}\psi^{(\mu)}(t) < \infty$. So the first assertion follows from Proposition 2.1. So, by Proposition 2.2, it suffices to show that $\psi^{(\mu)}$ is expanded as (2.3). To this end, first note that

$$\frac{1}{(1-t)^{1-\mu}} = \frac{1}{\Gamma(1-\mu)} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-\mu)}{n!} t^n. \tag{3.17}$$

Indeed, using the elementary formula: $a^{-\gamma} = \Gamma(\gamma)^{-1}\int_0^\infty dx\, x^{\gamma-1}e^{-ax}$ for $a, \gamma > 0$, we have:

$$\begin{aligned} \frac{1}{(1-t)^{1-\mu}} &= \frac{1}{\Gamma(1-\mu)} \int_0^\infty dx\, x^{-\mu} e^{-(1-t)x} \\ &= \frac{1}{\Gamma(1-\mu)} \int_0^\infty dx\, x^{-\mu} e^{-x} \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} \\ &= \frac{1}{\Gamma(1-\mu)} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^\infty dx\, x^{n-\mu} e^{-x}, \end{aligned}$$

and arrive at (3.17). Moreover, since $1-(1-t)^\mu = \int_0^t ds\,(1-s)^{-(1-\mu)}$, we also have:

$$1-(1-t)^\mu = \frac{1}{\Gamma(1-\mu)} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-\mu)}{(n+1)!} t^{n+1}. \tag{3.18}$$

Combining (3.17) and (3.18), we see that $\psi^{(\mu)}$ is expanded as (2.3) with $\alpha = 1/2$ and

$$a_n = \frac{\Gamma(n+1-\mu)}{\Gamma(1-\mu)} \left(\frac{1}{n!} + \frac{1}{(n+1)!} \right) > 0, \quad n = 0, 1, 2, \cdots.$$

This concludes the assertion. □

The proof of Theorem 1.3 for the present case is now immediate from Proposition 3.7 and Lemma 3.8:

Proof of Theorem 1.3 for $0 < \delta < 2$ *and* $\alpha = 2 - \delta$. By Proposition 3.7 and Lemma 3.8,

$$E[I(f; \widehat{R^\alpha})^2] \leq \alpha^2 \kappa_{2(\alpha-1)} \|f_\alpha\|_2^2.$$

So the assertion for the present case is proved. □

3.3 Proof for the case $0 < \delta \leq 1$ and $\alpha > 2 - \delta$

For $0 < \delta \leq 1$, let μ be as given in the previous subsection: $\delta = 2(1 - \mu)$. By the decomposition (3.16), and by Itô's formula, it is not hard to see that, for $\alpha > 2\mu$, (R_t^α) is a semimartingale which decomposes as

$$R_t^\alpha \equiv (R_t^{2\mu})^{\alpha/2\mu} = M_t^{(\alpha)} + V_t^{(\alpha)},$$

where $(M_t^{(\alpha)})$ is a martingale with quadratic variation $\langle M^{(\alpha)} \rangle_t = \alpha^2 \int_0^t R_u^{2(\alpha-1)}\, du$, and $V_t^{(\alpha)} = (\delta_\alpha/2) \int_0^t R_u^{\alpha-2}\, du$. Now the proof in the present case can be done in the same way as that of the proof in the case $\delta > 1$ and $\alpha > (2 - \delta)_+$ (Subsection 3.1). So we omit the proof.

4 Defining Wiener integrals relative to the centered BES(δ)-bridges for $\delta \geq 1$

In this section, we denote by $(r_\delta(t), 0 \leq t \leq 1)$ a BES(δ)-bridge, $\delta \geq 1$, with length 1 such that $r_\delta(0) = r_\delta(1) = 0$. With the help of the results obtained in Section 2, we may also prove that the Wiener integral $I(f; \widehat{r_\delta})$ is well-defined for all $f \in L^2([0, 1])$; in fact, we obtain the following a priori estimate:

Proposition 4.1. *Let* $\delta \geq 1$. *Then, for all* $f \in C([0, 1])$, *it holds that*

$$E[I(f; \widehat{r_\delta})^2] \leq \|f\|_2^2. \tag{4.1}$$

Remark 4.1. *As we see in the proof below, we have, more precisely that*

$$E[I(f; \widehat{r_\delta})^2] \leq \|f\|_2^2 - \left(\int_0^1 du\, f(u) \right)^2. \tag{4.2}$$

This estimate coincides with that obtained by the approach via the Brascamp-Lieb inequality for $\delta \geq 3$; *in fact, if we denote by* $(b(u), 0 \leq u \leq 1)$ *a Brownian bridge of length 1 such that* $b(0) = b(1) = 0$, *then the RHS of* (4.2) *is nothing but* $E[I(f; b)^2]$.

Proof of Proposition 4.1. It is well-known that $(r_\delta(t), 0 \le t \le 1)$ is identical in law with the process $(X_\delta(t), 0 \le t \le 1)$ defined by

$$X_\delta(t) = \begin{cases} (1-t)R_\delta(\frac{t}{1-t}), & 0 \le t < 1, \\ 0, & t = 1, \end{cases}$$

where R_δ is a BES(δ)-process starting from 0. Therefore

$$\begin{aligned} I(f; \widehat{r_\delta}) &\overset{(d)}{=} \int_0^1 f(u)\, d\widehat{X_\delta}(u) \\ &= -\int_0^1 f(u)\widehat{R_\delta}(\frac{u}{1-u})\, du + \int_0^1 f(u)(1-u)\, d_u\widehat{R_\delta}(\frac{u}{1-u}). \end{aligned}$$

We use integration by parts to see that the first term on the RHS is equal to:

$$-\int_0^1 \left(\int_u^1 dv\, f(v)\right) d_u\widehat{R_\delta}(\frac{u}{1-u}).$$

Thus, setting $G(u) = f(u) - (1-u)^{-1}\int_u^1 dv\, f(v)$, we have

$$\begin{aligned} I(f; \widehat{r_\delta}) &\overset{(d)}{=} \int_0^1 (1-u)G(u)\, d_u\widehat{R_\delta}(\frac{u}{1-u}) \\ &= \int_0^\infty \frac{1}{1+s}G(\frac{s}{1+s})\, d\widehat{R_\delta}(s) \end{aligned} \tag{4.3}$$

So, denoting $\widetilde{G}(s) = (1+s)^{-1}G(s/(1+s))$ and using the similar argument to that in the proof of Proposition 2.5, we deduce that

$$\begin{aligned} E[I(f; \widehat{r_\delta})^2] &= E[\left(\int_0^\infty \widetilde{G}(u)\, d\widehat{R_\delta}(u)\right)^2] \\ &= \|\widetilde{G}\|_{L^2(\mathbb{R}_+)} - \widetilde{J}_{\Phi_\delta}(\widetilde{G}, \widetilde{G}), \end{aligned} \tag{4.4}$$

where, for $F \in L^2(\mathbb{R}_+)$,

$$\widetilde{J}_{\Phi_\delta}(F, F) = \int_0^\infty du\, F(u) \int_0^1 dt\, F(ut)\Phi_\delta(t)$$

and Φ_δ is as given in Proposition 2.5. With the help of the argument used in the proof of Lemma 2.7, we may also deduce that $\widetilde{J}_{\Phi_\delta}$ is non-negative definite. So, by (4.4),

$$\begin{aligned} E[I(f; \widehat{r_\delta})^2] &\le \|\widetilde{G}\|^2_{L^2(\mathbb{R}_+)} \\ &= \|G\|^2_{L^2([0,1])} \\ &= \|f\|^2_{L^2([0,1])} - \left(\int_0^1 du\, f(u)\right)^2, \end{aligned}$$

which implies (4.1). $\square$

5 Appendix

5.1 An explicit representation for $\varphi_{\delta,\alpha}$ in terms of hypergeometric functions

Following [17], for complex numbers a, b, c, we denote by ${}_2F_1(a, b; c; w)$, $|w| < 1$, Gauss's hypergeometric functions. Recall that, when $\operatorname{Re} c > \operatorname{Re} b > 0$, they admit the following integral representation:

$$
{}_2F_1(a, b; c; w) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dz\, z^{b-1}(1-z)^{c-b-1}(1-wz)^{-a}. \tag{5.1}
$$

See, e.g, [10, (9.1.4)].

In this subsection, we discuss an explicit representation for the function $\varphi_{\delta,\alpha}(t) \equiv E[\widehat{R_1^{\alpha-2}}\widehat{R_t^{\alpha}}]$ in terms of ${}_2F_1$. We do this in a slightly general situation: For dimension $\delta > 0$, and for two exponents $\alpha_i \in \mathbb{R}, i = 1, 2$, satisfying $\delta + \alpha_i > 0$, set

$$
\phi_{\alpha_1;\alpha_2}(t) = E[\widehat{R_t^{\alpha_1}}\widehat{R_1^{\alpha_2}}], \quad 0 < t < 1.
$$

Proposition 5.1. *Suppose $\alpha \in \mathbb{R}$ and $\beta > 0$ satisfy $\delta + \alpha > 0$ and $\delta > \beta$, respectively. Then the function $\phi_{\alpha;-\beta}(t)$ admits the following representation:*

$$
\phi_{\alpha;-\beta}(t) = 2^{\frac{\alpha-\beta}{2}} \frac{\Gamma(\frac{\delta+\alpha}{2})\Gamma(\frac{\delta-\beta}{2})}{\{\Gamma(\frac{\delta}{2})\}^2} t^{\frac{\alpha}{2}} \left\{ {}_2F_1(-\frac{\alpha}{2}, \frac{\beta}{2}; \frac{\delta}{2}; t) - 1 \right\}.
$$

Proof. Since $\beta > 0$, we may rewrite $R_1^{-\beta}$ into:

$$
R_1^{-\beta} = 2^{-\beta/2}\left(R_1^2/2\right)^{-\beta/2} = 2^{-\beta/2}\Gamma(\beta/2)^{-1} \int_0^\infty \frac{dx}{x} x^{\beta/2} \exp\left(-\frac{x}{2}R_1^2\right).
$$

Then by Fubini's theorem,

$$
E[R_t^\alpha R_1^{-\beta}] = 2^{-\beta/2}\Gamma(\beta/2)^{-1} \int_0^\infty \frac{dx}{x} x^{\beta/2} E[R_t^\alpha \exp\left(-\frac{x}{2}R_1^2\right)]. \tag{5.2}
$$

By the Markov property, we have, conditionally on $\mathcal{F}_t^R$,

$$
\begin{aligned}
E_0^{(\delta)}[\exp\left(-\frac{x}{2}R_1^2\right)|\mathcal{F}_t^R] &= E_{R_t}^{(\delta)}[\exp\left(-\frac{x}{2}R_{1-t}^2\right)] \\
&= \{1 + x(1-t)\}^{-\delta/2} \exp\left\{-\frac{x}{2(1+x(1-t))}R_t^2\right\}.
\end{aligned} \tag{5.3}
$$

Here the second equality follows from the well-known fact that: for $a \geq 0$,

$$
E_a^{(\delta)}[\exp\left(-\frac{x}{2}R_s^2\right)] = (1+xs)^{-\delta/2} \exp\left(-\frac{x}{1+xs}a^2\right).
$$

See, e.g., [16, p.441]. By (5.3), we then have:

$$
\begin{aligned}
E[R_t^\alpha \exp\left(-\frac{x}{2}R_1^2\right)] &= \{1+x(1-t)\}^{-\delta/2} E[R_t^\alpha \exp\left\{-\frac{x}{2(1+x(1-t))}R_t^2\right\}] \\
&= (2t)^{\alpha/2}\Gamma(\delta/2)^{-1}\Gamma((\delta+\alpha)/2)\frac{1}{(1+x)^{\delta/2}}\left(1-\frac{x}{1+x}t\right)^{\alpha/2},
\end{aligned}
$$

where, for the second line, we used the fact that $R_t \overset{(d)}{=} \sqrt{2t\gamma_{\delta/2}}$. Plugging this into (5.2), and changing variables with $x/(1+x) = z$, we arrive at

$$
E[R_t^\alpha R_1^{-\beta}] = 2^{(\alpha-\beta)/2}\frac{\Gamma(\frac{\delta+\alpha}{2})}{\Gamma(\frac{\beta}{2})\Gamma(\frac{\delta}{2})}t^{\alpha/2}\int_0^1 dz\, z^{\frac{\beta}{2}-1}(1-z)^{\frac{\delta-\beta}{2}-1}(1-tz)^{\frac{\alpha}{2}}.
$$

By the integral representation (5.1), the above integral in z is expressed as:

$$
\frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\delta-\beta}{2})}{\Gamma(\frac{\delta}{2})}\,{}_2F_1(-\frac{\alpha}{2},\frac{\beta}{2};\frac{\delta}{2};t).
$$

Now we see

$$
E[R_t^\alpha R_1^{-\beta}] = 2^{(\alpha-\beta)/2}\frac{\Gamma(\frac{\delta+\alpha}{2})\Gamma(\frac{\delta-\beta}{2})}{\{\Gamma(\frac{\delta}{2})\}^2}t^{\alpha/2}\,{}_2F_1(-\frac{\alpha}{2},\frac{\beta}{2};\frac{\delta}{2};t). \tag{5.4}
$$

On the other hand, by scaling,

$$
E[R_t^\alpha]E[R_1^{-\beta}] = t^{\alpha/2}E[R_1^\alpha]E[R_1^{-\beta}] \equiv t^{\alpha/2}\times 2^{(\alpha-\beta)/2}\frac{\Gamma(\frac{\delta+\alpha}{2})\Gamma(\frac{\delta-\beta}{2})}{\{\Gamma(\frac{\delta}{2})\}^2}.
$$

Therefore, combining this with (5.4), and noting $\phi_{\alpha;-\beta}(t) = E[R_t^\alpha R_1^{-\beta}] - E[R_t^\alpha]E[R_1^{-\beta}]$, we have the lemma. □

Remark 5.1. *By Proposition 5.1, and by recalling the definition (series expansion) of hypergeometric functions (cf. [10, (9.1.1)]), we may easily see that*

$$
\lim_{t\downarrow 0} t^{-\frac{\alpha}{2}-1}\phi_{\alpha;-\beta}(t) = 2^{(\alpha-\beta)/2}\frac{\Gamma(\frac{\delta+\alpha}{2})\Gamma(\frac{\delta-\beta}{2})}{\{\Gamma(\frac{\delta}{2})\}^2}\times\left(-\frac{\alpha\beta}{2\delta}\right).
$$

We may compare this with the assertion (c′) *of Proposition 3.3.*

5.2 On a connection between Φ_1 and Φ_3 via perturbations

In this part, we shall consider the family of processes $(X_t^{(\gamma)}, 0 \le t \le 1), \gamma \in \mathbb{R}$, defined by

$$
X_t^{(\gamma)} = |\beta_t| + \gamma L_t, \tag{5.5}
$$

and discuss the Wiener integrals $I(f;\widehat{X^{(\gamma)}})$ relative to the centered processes $\widehat{X^{(\gamma)}}$. Here, as in the proof of Proposition 2.5 for $\delta = 1$, (β_t) is a Brownian motion and (L_t) the local time of β at 0. Note that, in particular,

$$\begin{cases} \gamma = -1 & : \ X^{(-1)} \text{ is a Brownian motion (Tanaka's formula);} \\ \gamma = 0 & : \ X^{(0)} \text{ is a reflecting Brownian motion;} \\ \gamma = +1 & : \ X^{(+1)} \text{ is BES(3) (Pitman's theorem).} \end{cases} \tag{5.6}$$

We shall prove that, for every $\gamma \in \mathbb{R}$, $I(f;\widehat{X^{(\gamma)}})$ is well-defined for all $f \in L^2([0,1])$. In fact, we obtain the following a priori estimate:

Proposition 5.2. *There exists a positive constant K_γ such that, for all bounded, measurable functions f on $[0,1]$,*

$$E[I(f;\widehat{X^{(\gamma)}})^2] \le K_\gamma \|f\|_2^2. \tag{5.7}$$

In particular, if $|\gamma| \le 1$, then (5.7) holds with $K_\gamma = 1$.

To prove this proposition, we prepare the following:

Lemma 5.3. *For every bounded, measurable function f on $[0,1]$, it holds that*

$$E[I(f;\widehat{X^{(\gamma)}})^2] = \|f\|_2^2 - J_{\Phi^{(\gamma)}}(f,f),$$

where

$$\Phi^{(\gamma)}(t) = \frac{(1+\gamma)^2}{\pi\sqrt{t}}(1-\sqrt{1-t}) + \frac{1-\gamma^2}{\pi}\frac{\sqrt{t}}{\sqrt{1-t}}$$

Note that, in the three particular cases $\gamma = -1, 0, +1$, we recover the previous results; indeed,

$$\begin{aligned} &\Phi^{(-1)}(t) \equiv 0 \text{ (since } X^{(\gamma)} \text{ is the Brownian motion);} \\ &\Phi^{(0)}(t) = \frac{1}{\pi}\left\{\frac{1}{\sqrt{t}}(1-\sqrt{1-t}) + \frac{\sqrt{t}}{\sqrt{1-t}}\right\} \equiv \Phi_1(t); \\ &\Phi^{(+1)}(t) = \frac{4}{\pi\sqrt{t}}(1-\sqrt{1-t}) \equiv \Phi_3(t). \end{aligned}$$

The proof of this lemma is given in the same manner as that of Proposition 2.5 for $\delta = 1$:

Proof of Lemma 5.3. Since $|\beta_t|$ decomposes as $|\beta_t| = B_t + L_t$, $X_t^{(\gamma)}$ admits the following decomposition:

$$X_t^{(\gamma)} = B_t + (\gamma+1)L_t. \tag{5.8}$$

We take $X_t = X_t^{(\gamma)}$, $M_t = B_t$ and $V_t = (\gamma+1)L_t$ in (2.9). Then by Lemma 2.6,

$$\begin{aligned} E[\left(\int_0^1 f(u)\,dX_u^{(\gamma)}\right)^2] &= \|f\|_2^2 + 2(\gamma+1)E[\int_0^1 f(u)\,dL_u\left(\int_0^u f(s)\,dX_s^{(\gamma)}\right)] \\ &= \|f\|_2^2 + 2(\gamma+1)\int_0^1 f(u)\,dE[L_u]E[\int_0^u f(s)\,dX_s^{(\gamma)}\big|\beta_u = 0]. \end{aligned}$$

We have already seen that $dE[L_u] = du/\sqrt{2\pi u}$ just below (2.11). Moreover, by the definition of $X^{(\gamma)}$,

$$\begin{aligned} &\frac{d}{ds}E[X_s^{(\gamma)}|\beta_u = 0] = \frac{d}{ds}E[|\beta_s||\beta_u = 0] + \gamma\frac{d}{ds}E[L_s|\beta_u = 0] \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{s(1-s/u)}}\left(1 - \frac{2s}{u}\right) + \gamma\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{s(1-s/u)}}. \end{aligned}$$

We thus obtain

$$E[\left(\int_0^1 f(u)\,dX_u^{(\gamma)}\right)^2] = \|f\|_2^2 + J_{\varphi_1^{(\gamma)}}(f,f), \quad \varphi_1^{(\gamma)}(t) := \frac{(\gamma+1)\{(1-2t)+\gamma\}}{\pi\sqrt{t(1-t)}}.$$

On the other hand, since $E[\int_0^1 f(u)\,dX_u^{(\gamma)}] = (\gamma+1)\int_0^1 du\, f(u)/\sqrt{2\pi u}$,

$$\left(E[\int_0^1 f(u)\,dX_u^{(\gamma)}]\right)^2 = J_{\varphi_2^{(\gamma)}}(f,f), \quad \varphi_2^{(\gamma)}(t) := \frac{(\gamma+1)^2}{\pi\sqrt{t}}.$$

Combining these, we obtain

$$E[I(f;\widehat{X^{(\gamma)}})^2] = \|f\|_2^2 - J_{\varphi_2^{(\gamma)}-\varphi_1^{(\gamma)}}(f,f)$$

with the identity $\varphi_2^{(\gamma)} - \varphi_1^{(\gamma)} = \Phi^{(\gamma)}$. So the proposition is proved. $\square$

With Lemma 5.3 at disposal, we prove Proposition 5.2:

Proof of Proposition 5.2. The case $|\gamma| > 1$: Since it is clear that $K_{\Phi^{(\gamma)}} := \int_0^1 dt\, t^{-1/2}|\Phi^{(\gamma)}(t)|$ is finite, we may use Proposition 2.1 to have that

$$|J_{\Phi^{(\gamma)}}(f,f)| \le K_{\Phi^{(\gamma)}}\|f\|_2^2.$$

Therefore, by Lemma 5.3, (5.7) holds with $K_\gamma = 1 + K_{\Phi^{(\gamma)}}$.
The case $|\gamma| \le 1$: Using the same argument as in the proof of Lemma 2.7 for $\delta = 1$, we

easily see that $J_{\Phi^{(\gamma)}}$ extends as a positive-definite quadratic form on $L^2([0,1]) \times L^2([0,1])$ if $|\gamma| \le 1$; indeed, $\Phi^{(\gamma)}$ is expanded as (2.14) with

$$a_n = \frac{1}{\pi}\left\{(1+\gamma)^2 + 2(n+1)(1-\gamma^2)\right\} c_{n+1},$$

which is positive for every $n = 0, 1, 2, \cdots$, as long as $|\gamma| \le 1$. Here (c_n) is the positive sequence given in (2.15). Thus, if $|\gamma| \le 1$, then by Lemma 5.3, the a priori estimate (5.7) holds with $K_\gamma = 1$. □

The latter assertion of this proposition may also be explained by the following perturbation argument: Let X be a semimartingale which decomposes as $X_t = B_t + V_t$ with V of finite variation satisfying $E[(\int_0^1 |dV_u|)^2] < \infty$. As we have seen in Lemma 2.6, it holds that, for every bounded, measurable function f on $[0,1]$,

$$E[I(f;X)^2] = \|f\|_2^2 - \mathfrak{J}_X(f), \quad \mathfrak{J}_X(f) := -2E[\int_0^1 f(u)\, dV_u \int_0^u f(s)\, dX_s].$$

Now we suppose $\mathfrak{J}_X(f) \ge 0$. As we have already seen, there are a number of such situations. For $\eta \in \mathbb{R}$, define

$$X_t^\eta = B_t + \eta V_t.$$

Then it is easily seen that $E[I(f;X^\eta)^2] = \|f\|_2^2 - \mathfrak{J}_{X^\eta}(f)$ with

$$J_{X^\eta}(f) = \eta \mathfrak{J}_X(f) - \eta(\eta-1)E[\left(\int_0^1 f(u)\, dV_u\right)^2]. \tag{5.9}$$

Indeed, by Lemma 2.6,

$$\begin{aligned}
\mathfrak{J}_{X^\eta}(f) &= -2E[\int_0^1 f(u)\, dV_u \int_0^u f(s)\, dX_s^\eta] \\
&= -2\eta E[\int_0^1 f(u)\, dV_u \int_0^u f(s)\, (dX_u + (\eta-1)dV_u)] \\
&= \eta \mathfrak{J}_X(f) - 2\eta(\eta-1)E[\int_0^1 f(u)\, dV_u \int_0^u f(s)\, dV_s],
\end{aligned}$$

hence (5.9) follows. From this, it is clear that,

$$\text{if } 0 \le \eta \le 1, \text{ then } \mathfrak{J}_{X^\eta}(f) \ge 0. \tag{5.10}$$

Now we turn to the case that we have just discussed above; we may recover the latter assertion of Proposition 5.2 by using (5.10):

Alternative proof of the latter assertion of Proposition 5.2. We rewrite the decomposition (5.8) above as

$$X_t^{(\gamma)} = B_t + \eta \times (2L_t), \quad \text{with } \eta = \frac{1+\gamma}{2}.$$

In the case $\eta = 1$ ($\iff \gamma = 1$), it is already seen that $\mathfrak{I}_{\widehat{X^{(1)}}}(f) \geq 0$ since $X^{(1)}$ is BES(3). So, by the above perturbation result (5.10), we may conclude that $\mathfrak{I}_{\widehat{X^{(\gamma)}}}(f) \geq 0$ when

$$0 \leq \frac{1+\gamma}{2} \leq 1 \ (\iff -1 \leq \gamma \leq 1),$$

which is nothing but the latter assertion of Proposition 5.2. $\square$

5.3 On the applicability of the FKG inequality to $\mathrm{BES}(\delta), \delta > 0$

In this part, we prove that the FKG inequality is applicable to the laws of $\mathrm{BES}(\delta), \delta > 0$, (or, more precisely, to their finite-dimensional marginals). For the formulation of the FKG inequality, we refer to [14, 15].

For $t > 0$ and $x_1, x_2 > 0$, let $p^\delta(t; x_1, x_2)$ denote the transition density function of $\mathrm{BES}(\delta)$:

$$p^\delta(t; x_1, x_2) = t^{-1} x_1^{-\nu} x_2^{\nu+1} \exp(-\frac{x_1^2 + x_2^2}{2t}) I_\nu(\frac{x_1 x_2}{t}).$$

Here $\nu = \delta/2 - 1$ (> -1), the index of $\mathrm{BES}(\delta)$. Set

$$\mathfrak{I}_\nu(x) = \frac{x I_{\nu+1}(x)}{I_\nu(x)}, \quad x > 0.$$

Lemma 5.4. *If $\nu > -1$, then $\mathfrak{I}_\nu$ is non-decreasing.*

Proof. We compute the derivative:

$$\begin{aligned}
\frac{d}{dx}\mathfrak{I}_\nu(x) &\equiv \frac{d}{dx}\frac{x^{\nu+1} I_{\nu+1}(x)}{x^\nu I_\nu(x)} \\
&= \frac{1}{\{x^\nu I_\nu(x)\}^2}\left\{x^{\nu+1} I_\nu(x) x^\nu I_\nu(x) - x^{\nu+1} I_{\nu+1}(x) x^\nu I_{\nu-1}(x)\right\} \\
&= \frac{x^{2\nu+1}}{\{x^\nu I_\nu(x)\}^2}\left\{I_\nu(x)^2 - I_{\nu+1}(x) I_{\nu-1}(x)\right\},
\end{aligned}$$

where we used the recurrence relation: $\{x^\mu I_\mu(x)\}' = x^\mu I_{\mu-1}(x)$. By [11, (9.t5)], the last quantity is non-negative for $\nu > -1$. This shows the lemma. $\square$

Remark 5.2. *It is known (e.g., [20, (11.27)]) that, for $\nu > -1$,*

$$\frac{1}{x^2}\mathfrak{I}_\nu(x) = 2\sum_{n=1}^{\infty}\frac{1}{x^2 + j_{\nu,n}^2},$$

where $(j_{\nu,n})$ is the simple, positive zeros of J_ν, the Bessel function of the first kind of order ν. From here, we may also deduce Lemma 5.4.

Lemma 5.5. *For each fixed $t > 0$, it holds that*

$$p^\delta(t; x_1 \vee y_1, x_2 \vee y_2)p^\delta(t; x_1 \wedge y_1, x_2 \wedge y_2) \geq p^\delta(t; x_1, x_2)p^\delta(t; y_1, y_2) \tag{5.11}$$

for all $(x_1, x_2), (y_1, y_2) \in (0, \infty) \times (0, \infty)$. Here $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$.

Proof. We divide the case into four cases: (i) $x_1 \geq y_1$, $x_2 \geq y_2$; (ii) $x_1 \leq y_1$, $x_2 \leq y_2$; (iii) $x_1 \geq y_1$, $x_2 \leq y_2$; (iv) $x_1 \leq y_1$, $x_2 \geq y_2$. In both cases (i) and (ii), (5.11) holds as an equality. So, by symmetry, we only need to consider either (iii) or (iv). Here we give a proof in the case (iii). By the definition of $p^\delta(t; x, y)$, the proof is reduced to showing the following: for $x_1 \geq y_1$ and $x_2 \leq y_2$,

$$I_\nu(\frac{x_1y_2}{t})I_\nu(\frac{x_2y_1}{t}) \geq I_\nu(\frac{x_1x_2}{t})I_\nu(\frac{y_1y_2}{t}). \tag{5.12}$$

Rewriting (5.12) as

$$\frac{I_\nu(\frac{y_2}{t}x_1)}{I_\nu(\frac{x_2}{t}x_1)} \geq \frac{I_\nu(\frac{y_2}{t}y_1)}{I_\nu(\frac{x_2}{t}y_1)},$$

we see that it suffices to prove, for $\beta > \alpha > 0$,

$$\frac{I_\nu(\beta x)}{I_\nu(\alpha x)} \text{ is non-decreasing in } x > 0. \tag{5.13}$$

To this end, we compute:

$$\begin{aligned}
\frac{d}{dx}\left\{\frac{I_\nu(\beta x)}{I_\nu(\alpha x)}\right\} \times \frac{\beta^{-\nu}}{\alpha^{-\nu}} &\equiv \frac{d}{dx}\left\{\frac{(\beta x)^{-\nu}I_\nu(\beta x)}{(\alpha x)^{-\nu}I_\nu(\alpha x)}\right\} \\
&= \frac{(\alpha\beta x^2)^{-\nu}}{\{(\alpha x)^{-\nu}I_\nu(\alpha x)\}^2}\left\{\beta I_{\nu+1}(\beta x)I_\nu(\alpha x) - \alpha I_\nu(\beta x)I_{\nu+1}(\alpha x)\right\},
\end{aligned}$$

where, for the equality, we used the recurrence relation: $\{x^{-\mu}I_\mu(x)\}' = x^{-\mu}I_{\mu+1}(x)$. We rewrite the last factor as:

$$\beta I_{\nu+1}(\beta x)I_\nu(\alpha x) - \alpha I_\nu(\beta x)I_{\nu+1}(\alpha x) = \frac{I_\nu(\alpha x)I_\nu(\beta x)}{x}\{\mathfrak{I}_\nu(\beta x) - \mathfrak{I}_\nu(\alpha x)\}$$

to see that this is non-negative by Lemma 5.4. Hence

$$\frac{d}{dx}\left\{\frac{I_\nu(\beta x)}{I_\nu(\alpha x)}\right\} \geq 0,$$

which shows (5.13). So the lemma is proved. $\square$

Let $\Delta = \{0 < t_1 < \cdots < t_n\}$ be an increasing sequence in $[0, \infty)$. For $a > 0$, we denote by $\Phi^\delta_\Delta(x; a)$ $(x = (x_i)_{1 \le i \le n})$ the finite-dimensional distribution function of BES(δ) starting from a taken at the time sequence $(t_i)_{1 \le i \le n}$:

$$\Phi^\delta_\Delta(x; a) = p^\delta(t_1; a, x_1) p^\delta(t_2 - t_1; x_1, x_2) \cdots p^\delta(t_n - t_{n-1}; x_{n-1}, x_n) \tag{5.14}$$

The next proposition shows $\Phi^\delta_\Delta(\cdot; a)$ fulfills the assumption of [14, Theorem 3]:

Proposition 5.6. *For $a \ge a' > 0$, it holds that*

$$\Phi^\delta_\Delta(x \vee y; a) \Phi^\delta_\Delta(x \wedge y; a') \ge \Phi^\delta_\Delta(x; a) \Phi^\delta_\Delta(y; a')$$

for all $x = (x_i)_{1 \le i \le n} \in (0, \infty)^n$ and $y = (y_i)_{1 \le i \le n} \in (0, \infty)^n$. Here $x \vee y = (x_i \vee y_i)_{1 \le i \le n}$ and $x \wedge y = (x_i \wedge y_i)_{1 \le i \le n}$.

Proof. The assertion is immediate from Lemma 5.5 and (5.14). ∎

Remark 5.3. *It is easily checked that the assertion of this lemma still holds even if a' is equal to 0; in that case, $\Phi^\delta_\Delta(x; a')$ should be replaced by*

$$\Phi^\delta_\Delta(x; 0) = \widetilde{p^\delta}(t_1; x_1) p^\delta(t_2 - t_1; x_1, x_2) \cdots p^\delta(t_n - t_{n-1}; x_{n-1}, x_n),$$

where

$$\widetilde{p^\delta}(t; x) = 2^{-\nu} t^{-(\nu+1)} \Gamma(\nu + 1)^{-1} x^{2\nu+1} \exp(-\frac{x^2}{2t}).$$

Remark 5.4. *There is a probabilistic understanding of (5.13): For $x > 0$, we denote $T_x(R) = \inf\{t > 0; R_t = x\}$. It is well-known (e.g., [20, (11.31)]) that, for every dimension $\delta > 0$, and for $\gamma > 0$,*

$$E^{(\delta)}_0[\exp(-\frac{\gamma^2}{2} T_x(R))] = \frac{(x\gamma)^\nu}{2^\nu \Gamma(1 + \nu) I_\nu(x\gamma)}.$$

Therefore, for $0 < y < x$, we get, from the strong Markov property,

$$E^{(\delta)}_0[\exp(-\frac{\gamma^2}{2} T_x(R))] = E^{(\delta)}_0[\exp(-\frac{\gamma^2}{2} T_y(R))] E^{(\delta)}_y[\exp(-\frac{\gamma^2}{2} T_x(R))]$$

hence

$$\frac{I_\nu(y\gamma)}{I_\nu(x\gamma)} = \left(\frac{y}{x}\right)^\nu E^{(\delta)}_y[\exp(-\frac{\gamma^2}{2} T_x(R))],$$

which indicates that, for $0 < y < x$, the function $\gamma \mapsto I_\nu(y\gamma)/I_\nu(x\gamma)$ is strictly decreasing.

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