Wiener integrals for centered powers of Bessel processes, I

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Abstract

The stochastic integrals of Wiener’s type may be constructed relatively to the centered $\delta$-dimensional Bessel processes (BES(\(\delta\))-processes in short) and their variants based on two different approaches. One approach, developed in [3], is via the Brascamp-Lieb inequality which works especially well for the BES(\(\delta\))-processes, BES(\(\delta\))-bridges with \(\delta \geq 3\) or for the Brownian meander. The other approach, which is the subject of the present paper, goes via Hardy’s \(L^2\) inequality which is effective for general BES(\(\delta\))-processes and their powers. We shall also discuss an interplay of these two methods.

1 Introduction and main results

Consider, on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)\), a continuous semimartingale \((X_t, t \geq 0)\), with canonical decomposition:

\[ X_t = M_t + V_t, \]

where \((M_t)\) denotes an \((\mathcal{F}_t)\)-local martingale, and \((V_t)\) an \((\mathcal{F}_t)\)-adapted, continuous process of bounded variation.

The theory of stochastic integration with respect to \(X\) consists in defining:

\[
\int_0^t \phi_s \, dX_s = \int_0^t \phi_s \, dM_s + \int_0^t \phi_s \, dV_s,
\]

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for \((\mathcal{F}_t)\)-predictable processes \((\phi_t)\) which satisfy:

\[(1.1) \quad \int_0^t \phi_s^2 \, d\langle M \rangle_s < \infty \quad \text{and} \quad \int_0^t |\phi_s| \, |dV_s| < \infty \]

a.s. for every \(t\). However, in a number of situations, one would like to be able to define \((\int_0^t \phi_s \, dX_s, \, t \geq 0)\) for predictable processes \((\phi_t)\) which only satisfy:

\[(1.1') \quad \int_0^t \phi_s^2 \, d\langle M \rangle_s < \infty \quad \text{a.s. for every } t.\]

This question arose very naturally in some discussions of enlargements of the Brownian filtration. More precisely, consider a Brownian motion \((\beta_t, \, t \geq 0)\) (and its filtration), and assume that, in a larger filtration \((\mathcal{F}_t)\), \((\beta_t)\) remains a semimartingale, which then decomposes as:

\[(1.2) \quad \beta_t = B_t + V_t, \quad t \geq 0,\]

where \((B_t)\) is an \((\mathcal{F}_t)\)-Brownian motion. In [9], examples were given for which, although \((1.2)\) holds, nonetheless, there exist even deterministic functions \(f\) in \(L^2_{\text{loc}}(ds)\), i.e., \(\int_0^t f(s)^2 \, ds \equiv \int_0^t f(s)^2 \, d\langle B \rangle_s < \infty\) such that

\[\int_0^t |f(s)| \, |dV_s| = \infty.\]

To be precise, starting from a Brownian motion \((B_t)\), consider the “second” Brownian motion:

\[(1.3) \quad \beta_t = B_t - \int_0^t \frac{B_s}{s} \, ds,\]

whose proper filtration is strictly smaller than that of \(B\), but trivially \((1.2)\) holds with:

\[V_t = -\int_0^t \frac{B_s}{s} \, ds.\]

It then follows from Jeulin’s lemma [7, p.44] (see also [8]) that, if \(f \in L^2_{\text{loc}}(ds)\), then \((\int_0^t f(s) \, d\beta_s, \, t \geq 0)\) is an \(\mathcal{F}_t \equiv \sigma\{B_s, \, s \leq t\}\)-semimartingale if and only if

\[\int_{0^+} |f(s)| / \sqrt{s} \, ds < \infty.\]

(For several applications of Jeulin’s lemma, we also refer to [12, 18].) Nonetheless, for every \(f \in L^2_{\text{loc}}(ds)\),

\[\int_0^t f(s) \left( \frac{B_s}{s} \right) \, ds := \lim_{\varepsilon \downarrow 0} \int_0^t f(s) \left( \frac{B_s}{s} \right) \, ds\]
exists both a.s. and in $L^2$, as follows easily from (1.3). Thus the identity:

\[ \int_0^t f(s) \, d\beta_s = \int_0^t f(s) \, dB_s - \int_0^t f(s) \left( \frac{B_s}{s} \right) \, ds \]

is meaningful for every $f \in L^2_{\text{loc}}(ds)$.

In the present paper, we intend to make a similar discussion when instead of (1.2), we consider the semimartingale:

\[ X_t = R_\delta(t), \quad t \geq 0, \]

where $(R_\delta(t), t \geq 0)$ is a $\delta$-dimensional Bessel process (BES($\delta$)-process in short), $\delta \geq 1$, starting from 0. We recall the canonical decomposition:

\[ R_\delta(t) = B_t + V_\delta(t), \quad t \geq 0, \]

where

\[ V_\delta(t) = \begin{cases} L_t, & \delta = 1, \\ \frac{\delta - 1}{2} \int_0^t \frac{ds}{R_\delta(s)}, & \delta > 1. \end{cases} \]

(For $\delta = 1$, we may consider: $R_1(t) = |\beta_t|$; $B_t = \int_0^t \text{sgn}(\beta_s) \, d\beta_s$, and $(L_t)$ the local time at 0 of $\beta$.) Then, again as a consequence of Jeulin’s lemma, we can show that, for $f \in L^2_{\text{loc}}(ds)$, there is the equivalence:

\[ \int_0^t |f(s)||dV_\delta(s)| < \infty \iff \int_0^t \frac{|f(s)|}{\sqrt{s}} \, ds < \infty. \]

Despite this, we shall show the following analogue of (1.4): define the centered Bessel process

\[ \hat{R}_\delta(t) = R_\delta(t) - E[R_\delta(t)] = B_t + \hat{V}_\delta(t), \quad t \geq 0. \]

Then the integral $\int_0^t f(s) \, d\hat{R}_\delta(s)$ makes sense for every $f \in L^2_{\text{loc}}(ds)$. In fact, we have obtained the much more precise results. For simplicity, we restrict our argument to the time interval $[0, 1]$: For a bounded, measurable function $f : [0, 1] \to \mathbb{R}$, we set

\[ I(f; \hat{R}_\delta) = \int_0^1 f(u) \, d\hat{R}_\delta(u). \]

**Theorem 1.1.** For every $\delta \geq 1$, it holds that, for all bounded, measurable functions $f : [0, 1] \to \mathbb{R}$,

\[ E[I(f; \hat{R}_\delta)^2] \leq \|f\|_2^2. \]

Moreover, the constant 1 implicit in (1.7) is the optimal constant $K$ such that $E[I(f; \hat{R}_\delta)^2] \leq K\|f\|_2^2$. 

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Note that, for each \( f^* \in L^2([0, 1]) \), we may find a sequence \( (f_n)_{n \in \mathbb{N}} \) of bounded, measurable functions on \([0, 1]\), such that \( \|f_n - f^*\|_2 \to 0 \) as \( n \to \infty \). From (1.7), we see that \( \{I(f_n; \hat{R}_\delta)\}_{n \in \mathbb{N}} \) forms a Cauchy sequence in \( L^2(P) \). So we may define the Wiener integral \( I(f^*; \hat{R}_\delta) \) as its limit. Thus we obtain, as a corollary to Theorem 1.1:

**Corollary 1.2.** For all \( f \in L^2([0, 1]) \), the Wiener integral \( I(f; \hat{R}_\delta) \) is well-defined relative to \( \text{BES}(\delta), \delta \geq 1 \), starting from 0. Moreover, it enjoys the property (1.7).

We may extend the above results by replacing \((R_\delta(t), t \geq 0)\) by \((R_\delta(t))^\alpha\), for every \( \alpha \) such that \((R_\delta(t))^\alpha\) is a semimartingale; that is,

\[
\alpha \geq (2 - \delta)_+.
\]

Here, for \( a \in \mathbb{R} \), \((a)_+ = \max\{a, 0\}\). In this case, we may modify the above as follows: for \( \beta \in \mathbb{R} \) satisfying \( \delta + \beta > 0 \), set

\[
\kappa_{\beta} = E[R_1^\beta] \equiv 2^{\beta/2} \frac{\Gamma(\frac{\delta + \beta}{2})}{\Gamma(\frac{\delta}{2})}.
\]

**Theorem 1.3.** For \( \delta > 0 \) and \( \alpha \in (0, 2) \), suppose that \( \alpha \geq (2 - \delta)_+ \). Then it holds that, for all bounded, measurable functions \( f \) on \([0, 1]\),

\[
E[I(f; \hat{R}_\delta)]^2 \leq \alpha^2 \kappa_{2(\alpha - 1)} \|f\|_2^2,
\]

where \( f_\alpha(u) := f(u)u^{(\alpha - 1)/2} \).

**Remark 1.1.** The constant \( \alpha^2 \kappa_{2(\alpha - 1)} \) in (1.8) is also optimal (see Remark 3.1).

We denote by \( L^2_\alpha([0, 1]) \) the weighted \( L^2 \) space defined by

\[
L^2_\alpha([0, 1]) = \{f; f_\alpha \in L^2([0, 1])\}.
\]

As a consequence of Theorem 1.3, we also have the following:

**Corollary 1.4.** Suppose that \( \delta \) and \( \alpha \) satisfy the assumption of Theorem 1.3. Then, for all \( f \in L^2_\alpha([0, 1]) \), the Wiener integral \( I(f; \hat{R}_\delta) \) is well-defined.

The rest of the paper is organized as follows: in Section 2, after developing some Hardy-like inequalities, we prove Theorem 1.1; Section 3 is devoted to the proof of Theorem 1.3; in Section 4, we show that as a consequence of the previous results, Wiener integral may be defined relatively to the centered \( \text{BES}(\delta) \)-bridges, for \( \delta \geq 1 \); finally, in Section 5, which we consider as an Appendix, we have gathered several topics we refer to in our discussions.
2 An approach via Hardy’s $L^2$ inequality

2.1 Some Hardy-like inequalities

**Proposition 2.1.** For a non-negative, integrable function $\Phi(t), 0 \leq t \leq 1$, the operator

$$T_\Phi : f \to \{ \int_0^1 dt f(ut)\Phi(t), u \leq 1 \}$$

which is well-defined on bounded Borel measurable functions $f$, extends as a bounded linear operator on $L^2([0,1])$ if and only if

$$K_\Phi := \int_0^1 dt \frac{\Phi(t)}{\sqrt{t}} < \infty.$$  \quad (2.1)

Moreover,

$$\|T_\Phi f\|_2 \leq K_\Phi \|f\|_2 \quad \text{for all } f \in L^2([0,1]),$$

and the constant $K_\Phi$ is the optimal constant $K$ such that: $\|T_\Phi f\|_2 \leq K \|f\|_2$; that is, the operator norm $\|T_\Phi\|$ of $T_\Phi$ is equal to $K_\Phi$.

For $\Phi$ satisfying (2.1), we define the quadratic form $J_\Phi$ on $L^2([0,1]) \times L^2([0,1])$ by:

$$J_\Phi(f,g) = \int_0^1 du f(u)T_\Phi g(u), \quad f, g \in L^2([0,1]).$$ \quad (2.2)

**Proof of Proposition 2.1.** As is well known, it holds that:

$$\|T_\Phi\| = \sup\{ |J_\Phi(f,g)|; f, g \in L^2([0,1]), \|f\|_2 \leq 1, \|g\|_2 \leq 1 \}.$$  

By Fubini’s theorem and the Cauchy-Schwarz inequality, we have:

$$|J_\Phi(f,g)| \leq \int_0^1 dt \Phi(t)\|f\|_2 \left( \int_0^1 du g(ut)^2 \right)^{1/2}$$

$$\leq \int_0^1 dt \frac{\Phi(t)}{\sqrt{t}} \|f\|_2 \|g\|_2,$$

which shows $\|T_\Phi\| \leq K_\Phi$. On the other hand, if we assume that $|J_\Phi(f,g)| \leq K \|f\|_2 \|g\|_2$, then, by taking $f \equiv g \equiv \psi_\alpha$, where $\psi_\alpha(u) = u^\alpha$ for $\alpha > -1/2$, we get $J_\Phi(\psi_\alpha, \psi_\alpha) \leq K(1 + 2\alpha)^{-1}$, but

$$J_\Phi(\psi_\alpha, \psi_\alpha) = \int_0^1 dt t^\alpha \frac{1}{1 + 2\alpha}.$$  

Therefore, $\int_0^1 dt \Phi(t)t^\alpha \leq K$ for every $\alpha > -1/2$, and thus letting $\alpha \downarrow -1/2$, we get $K_\Phi \leq K$, which ends the proof.  \qed
The next two propositions were proven and kindly proposed to us by F. Hirsh [6].

**Proposition 2.2.** Suppose $\Phi \geq 0$ and satisfies (2.1). Moreover, suppose that there exists $\alpha > -1/2$ such that $\Phi$ is expanded as:

$$\Phi(t) = t^\alpha \sum_{n=0}^{\infty} a_n t^n, \quad a_n \geq 0.$$  \hspace{1cm} (2.3)

Then the quadratic form $J_\Phi$ is non-negative definite; i.e., $J_\Phi(f, f) \geq 0$ for all $f \in L^2([0,1])$.

**Proof.** If we denote $\phi_\beta(t) = t^\beta$, we have:

$$\Phi(t) = \sum_{n=1}^{\infty} a_n \phi_{n+\alpha}(t),$$

and, in order to prove the assertion, it suffices to show: for any $\beta > -1/2$,

$$J_{\phi_\beta}(f, f) \geq 0.$$  \hspace{1cm} (2.4)

Moreover, by approximating each $f \in L^2([0,1])$ by functions in $C([0,1])$, we only have to prove (2.4) for $f \in C([0,1])$. By change of variables, and by integration by parts,

$$J_{\phi_\beta}(f, f) = \int_0^1 du u^{-2\beta-1} u^\beta f(u) \int_0^u dv v^\beta f(v) = \int_0^1 du u^{-2\beta-2} F_\beta(u).$$

Here we set

$$F_\beta(u) = \int_0^u ds f(s)s^\beta \int_0^u dv f(v)v^\beta \equiv \frac{1}{2} \left( \int_0^u ds f(s)s^\beta \right)^2,$$

which is non-negative. So the assertion is proved. \hfill \Box

In fact, the functions $\Phi$ such that $J_\Phi$ is non-negative definite may be characterized as follows: for $f : [0,1] \to \mathbb{R}$, we denote $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}$ defined by $\tilde{f}(x) = e^{-x/2} f(e^{-x})$. The mapping $f \to \tilde{f}$ is an onto isometry from $L^2([0,1])$ on $L^2(\mathbb{R}_+)$. Let $\phi : [0,1] \to \mathbb{R}$ such that $|\phi|$ satisfies (2.1). Then $\int_0^\infty |\tilde{\phi}(x)| \, dx < \infty$. Let

$$H(x) = \tilde{\phi}(|x|) = e^{-|x|/2}\phi(e^{-|x|})$$
Proposition 2.3. The operator $T_\phi$ is positive if and only if $H$ is positive definite in the sense that, for every $h \in L^2(\mathbb{R})$,

\[(2.5) \quad \iint_{\mathbb{R}^2} h(x)h(y)H(x - y)\,dx\,dy \geq 0.\]

Proof. Note that

$$\tilde{T}_\phi f(s) = \int_0^\infty \tilde{f}(t + s)\tilde{\phi}(t)\,dt.$$ 

Thus, it holds that

$$J_\phi(f, f) = \int_0^\infty \int_0^\infty ds dt \tilde{f}(s)\tilde{f}(t + s)\tilde{\phi}(t)$$

$$= \int_0^\infty ds \tilde{f}(s) \int_0^\infty dt \tilde{f}(t)\tilde{\phi}(t - s)$$

$$= \int_0^\infty ds \tilde{f}(s) \int_0^\infty dt \tilde{f}(t)H(t - s).$$

Hence, from the symmetry of $H$,

$$J_\phi(f, f) = \frac{1}{2} \iint_{\mathbb{R}^2_+} ds dt \tilde{f}(s)\tilde{f}(t)H(t - s).$$

Thus, $J_\phi \geq 0$ if and only if

$$\iint_{\mathbb{R}^2_+} ds dt g(s)g(t)H(t - s) \geq 0 \quad \text{for all } g \in L^2(\mathbb{R}_+).$$

It follows immediately that, if $H$ is positive definite, then $J_\phi$ is positive. Conversely, if $J_\phi \geq 0$ and if $h \in L^2(\mathbb{R})$, then, for every $a \leq 0$,

$$\int_a^\infty \int_a^\infty ds dt h(s)h(t)H(t - s)$$

$$= \iint_{\mathbb{R}^2} ds dt h(a + s)h(a + t)H(t - s) \geq 0.$$ 

Letting $a$ go to $-\infty$, we have:

$$\iint_{\mathbb{R}^2} ds dt h(s)h(t)H(t - s) \geq 0.$$ 

So the proof is complete. \qed

As a corollary, we have the following (from Bochner’s theorem):

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Corollary 2.4. If \( \phi \) is continuous on \((0, 1]\), \( J_\phi \geq 0 \) if and only if there exists a finite positive symmetric measure on \( \mathbb{R} \) such that

\[
(2.6) \quad \phi(t) = \frac{1}{\sqrt{t}} \int_{-\infty}^{+\infty} t^{i \alpha} \, d\mu(x).
\]

Example 2.1. (i) When \( \phi \) is given in the form (2.3), the measure \( \mu \) in the above corollary is given by:

\[
(2.7) \quad d\mu(x) = \sum_{n=0}^{\infty} a_n \frac{n + \alpha + \frac{1}{2}}{(n + \alpha + \frac{1}{2})^2 + x^2} \, dx.
\]

(ii) An interesting class of functions \( \phi \) which satisfy (2.6) are those \( \phi \)'s given by:

\[
e^{-|x|/2} \phi(e^{-|x|}) = c \int_{0}^{\infty} e^{-x^2 y/2} \, dm(y)
\]

for some constant \( c > 0 \), and for a probability measure \( m \) on \( \mathbb{R}_+ \); the measure \( \mu \) in the above corollary then appears as the law of \( BA \), where \( B \) is a Brownian motion, and \( A \) is a random variable independent of \( B \) and distributed as \( m \).

2.2 Proof of Theorem 1.1

We first look for some rahter explicit expression for \( E[I(f; \hat{R}_\delta)]^2 \):

Proposition 2.5. Let \( f : [0, 1] \to \mathbb{R} \) be a bounded measurable function. Then

\[
(2.8) \quad E[I(f; \hat{R}_\delta)^2] = \|f\|_2^2 - J_{\Phi_\delta}(f, f),
\]

where

\[
\Phi_\delta(t) = \begin{cases} 
\frac{1}{\pi} \left\{ \frac{1}{\sqrt{t}} \left(1 - \sqrt{1 - t} \right) + \frac{\sqrt{t}}{\sqrt{1 - t}} \right\}, & \delta = 1, \\
-(\delta - 1) \frac{d}{dt} E[R_{\delta}(1)^{-1} R_{\delta}(t)], & \delta > 1,
\end{cases}
\]

and \( J_{\Phi_\delta} \) is defined through (2.2).

Example 2.2. In the case \( \delta = 3 \), \( \Phi_\delta \) is given by

\[
\Phi_3(t) = \frac{4}{\pi} \frac{1}{\sqrt{t}} (1 - \sqrt{1 - t}).
\]

For some connection between \( \Phi_1 \) and \( \Phi_3 \), which may be deduced from Pitman's representation theorem for BES(3), see Lemma 5.3 in the appendix.
Suppose that \((X_t, t \geq 0)\) is an \((\mathcal{F}_t)\)-semimartingale with decomposition
\[
X_t = M_t + V_t, \quad t \geq 0,
\]
where \((M_t)\) is an \(L^2\)-integrable \((\mathcal{F}_t)\)-martingale and \((V_t)\) is an \((\mathcal{F}_t)\)-adapted process of bounded variation satisfying \(E[\int_0^t |dV_u|^2] < \infty\) for every \(t\). For a bounded, measurable function \(f\) on \([0, 1]\), we set
\[
X^f_t = \int_0^t f(u) \, dX_u, \quad 0 \leq t \leq 1.
\]
Clearly, \((X^f_t)\) is a semimartingale which decomposes as:
\[
X^f_t = \int_0^t f(u) \, dM_u + \int_0^t f(u) \, dV_u, \quad 0 \leq t \leq 1.
\]

**Lemma 2.6.** It holds that
\[
E[(X^f_1)^2] = E[\int_0^1 f(u)^2 \, d\langle M \rangle_u] + 2E[\int_0^1 f(u) \, dV_u \left(\int_0^u f(s) \, dX_s\right)].
\]

**Proof.** By Itô’s formula,
\[
(X^f_1)^2 = 2 \int_0^1 X^f_u \, dX^f_u + \langle X^f \rangle_1.
\]
Taking the expectation on both sides, and noting \(\langle X^f \rangle_1 = \int_0^1 f(u)^2 \, d\langle M \rangle_u\), we obtain the lemma.

By using Lemma 2.6, we prove Proposition 2.5. We begin with the case \(\delta = 1\).

**Proof of Proposition 2.5 for \(\delta = 1\).** Noting (1.5), we take \(X_t = R_\delta(t) \equiv |\beta_t|, \: M_t = B_t\) and \(V_t = V_\delta(t) \equiv L_t\) in (2.9). Then by Lemma 2.6,
\[
E[\left(\int_0^1 f(u) \, d|\beta_u|\right)^2] = \|f\|^2_2 + 2E[\int_0^1 f(u) \, dL_u \left(\int_0^u f(s) \, d|\beta_s|\right)].
\]
Recall that \(dL_u\) is carried by the set of zeros of \(\beta\). So, if we denote by \((b^u(s), 0 \leq s \leq u)\) a Brownian bridge of length \(u\), the expectation on the RHS of (2.10) may be written as:
\[
\int_0^1 f(u) \, dE[L_u]E[\int_0^u f(s) \, db^u(s)].
\]
Note that \(E[L_u] \equiv E[|\beta_u|] = \sqrt{2u/\pi}\), hence \(E[L_u] = du/\sqrt{2\pi u}\). Moreover, since \(b^u(s)\) is Gaussian with mean 0 and variance \(s(u - s)/u\),
\[
E[|b^u(s)|] = \sqrt{\frac{2}{\pi}} \sqrt{\frac{s(u - s)}{u}}.
\]
On the other hand, (2.12) follows from this and the change of variables with \( s = ut \). Combining these, we see that (2.11) is equal to:

\[
\frac{1}{2\pi} \int_0^1 \frac{du}{\sqrt{u}} f(u) \int_0^u ds \frac{1}{s(1-s/u)} \left( 1 - \frac{2s}{u} \right) ds.
\]

Here, for the equality, we changed variables with \( s = ut, 0 \leq t \leq 1 \). From this and (2.10),

\[
E\left[ \left( \int_0^1 f(u) d|\beta_u| \right)^2 \right] = ||f||_2^2 + J_{\Phi_1}(f, f), \quad \Psi_1(t) := \frac{1}{\pi} \frac{1-2t}{\sqrt{t(1-t)}}.
\]

On the other hand, \( E[\int_0^1 f(u) d|\beta_u|] = \int_0^1 du f(u)/\sqrt{2\pi u} \), hence

\[
E[\int_0^1 f(u) d|\beta_u|] = \frac{1}{\pi} \int_0^1 \frac{du}{\sqrt{u}} f(u) \int_0^u ds \frac{1}{\sqrt{s}} f(s) = J_{\Psi_2}(f, f),
\]

where \( \Psi_2(t) := 1/(\pi \sqrt{t}) \). Combining this with (2.12), we see that

\[
E[I(f; \hat{R}_1)^2] \equiv \text{var} \left( \int_0^1 f(u) d|\beta_u| \right) = ||f||_2^2 - J_{\Psi_2 - \Psi_1}(f, f).
\]

Noting the identity: \( \Phi_1 = \Psi_2 - \Psi_1 \), we obtain (2.8) for \( \delta = 1 \).

Next, we turn to the case \( \delta > 1 \).

**Proof of Proposition 2.5 for \( \delta > 1 \)**. By the decomposition (1.5), we take \( X_t = \hat{R}_\delta(t) \), \( M_t = B_t \) and \( V_t = \hat{V}_\delta(t) \) in (2.9). Then, noting \( dV_u = 2^{-1}(\delta - 1)\hat{R}_\delta(u)^{-1} du \), we see from Lemma 2.6 that

\[
E[(X_t^f)^2] = E[I(f; \hat{R}_\delta)^2]
= ||f||_2^2 + (\delta - 1) \int_0^1 f(u) R\hat{\delta}(u)^{-1} du \left( \int_0^u f(s) d\hat{R}_\delta(s) \right).
\]

Now (2.8) follows from this and the change of variables with \( s = ut \).
Lemma 2.7. (i) For every $\delta \geq 1$, $\Phi_\delta$ is non-negative and satisfies:

$$\int_{0}^{1} \frac{dt}{\sqrt{t}} \Phi_\delta(t) < \infty.$$  

(ii) $J_{\Phi_\delta}$ extends as a quadratic form on $L^2([0, 1]) \times L^2([0, 1])$. Moreover, it is positive-definite.

Proof. (i) This assertion for $\delta = 1$ is obvious by definition. For $\delta > 1$, we denote $\varphi_\delta(t) = E[\widehat{R_\delta(1)}^{-1} \widehat{R_\delta(t)}], 0 \leq t \leq 1$, and use the fact that $\varphi_\delta$ admits the following representation:

$$\varphi_\delta(t) = C_\delta t^{1/2}\{\, _2F_1\left(-\frac{1}{2}, \frac{1}{2}; \delta; t\right) - 1\}, \quad C_\delta = \frac{\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{\delta-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}.$$  

Here $\, _2F_1$ denotes Gauss’ hypergeometric function; see Proposition 5.1 in the appendix. Recalling the series expansion (in fact, the definition) of $\, _2F_1$ (e.g., [10, (9.1.1)]), we see that:

(a) $\varphi_\delta \leq 0$;  (b) $\varphi'_\delta \leq 0$;  (c) $\lim \inf_{t \downarrow 0} \frac{1}{\sqrt{t^3}} \varphi_\delta(t) > -\infty$.

Indeed,

$$t^{1/2}\{\, _2F_1\left(-\frac{1}{2}, \frac{1}{2}; \delta; t\right) - 1\} = \sum_{n=1}^{\infty} b_n t^{n+1/2}, \quad b_n = \frac{(-\frac{1}{2})_n (\frac{1}{2})_n}{(\frac{1}{2})_n n!} < 0,$$  

from which the properties (a)-(c) of $\varphi_\delta$ follow. Here, for $\gamma \in \mathbb{R}$, $(\gamma)_n := \gamma(\gamma+1) \cdots (\gamma+n-1)$. (In fact, for (c), the limit itself exists: $\lim_{t \downarrow 0} t^{-3/2} \varphi_\delta(t) = -C_\delta/(2\delta)$.) So, by (b) and by definition, $\Phi_\delta(t) = -(\delta - 1)\varphi'_\delta(t) \geq 0$. Moreover, by (c), we may use integration by parts to see:

$$\int_{0}^{1} \frac{dt}{\sqrt{t}} \Phi_\delta(t) = -(\delta - 1) \int_{0}^{1} \frac{dt}{\sqrt{t}} \varphi'_\delta(t)$$

$$= -(\delta - 1) \left\{ \varphi_\delta(1) + \frac{1}{2} \int_{0}^{1} \frac{dt}{\sqrt{t^3}} \varphi_\delta(t) \right\} < \infty.$$  

(ii) The former assertion now follows from (i) and Proposition 2.1. To show the latter, we shall show that $\Phi_\delta$ can be developed as:

$$\Phi_\delta(t) = t^{1/2} \sum_{n=0}^{\infty} a_n t^n, \quad a_n \geq 0.$$  

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Then the positivity of \( J_{\Phi_{\delta}} \) follows immediately from Proposition 2.2. For the case \( \delta = 1 \), note that the function \( 1 - \sqrt{1 - t}, 0 \leq t \leq 1 \), can be developed as:

\[
1 - \sqrt{1 - t} = \sum_{n=1}^{\infty} c_n t^n, \quad c_n = \frac{(2n)!}{2^{2n} (n!)^2 (2n - 1)},
\]

from which it also follows that, by differentiating both sides,

\[
\frac{1}{2\sqrt{1 - t}} = \sum_{n=1}^{\infty} n c_n t^{n-1}.
\]

Combining these, we see that \( \Phi_1 \) is expanded as (2.14) with

\[
a_n = \frac{1}{\pi} \{1 + 2(n + 1)\} c_{n+1} > 0, \quad n = 0, 1, 2, \ldots.
\]

On the other hand, for \( \delta > 1 \), we see from (2.13) that

\[
\varphi'_{\delta}(t) = C_{\delta} \sum_{n=1}^{\infty} \left(n + \frac{1}{2}\right) b_n t^{n-1/2}.
\]

Since \( \Phi_{\delta}(t) = - (\delta - 1) \varphi'_{\delta}(t) \) by definition, \( \Phi_{\delta}, \delta > 1 \), is also expanded as (2.14) with

\[
a_n = C_{\delta} (\delta - 1) \left(n + \frac{3}{2}\right) \times (-b_{n+1}) > 0, \quad n = 0, 1, 2, \ldots.
\]

So the proof is complete.

Now we are ready to prove Theorem 1.1:

**Proof of Theorem 1.1.** The former assertion (1.7) is immediate from Proposition 2.5 and Lemma 2.7. By Proposition 2.5, in order to prove the latter, it suffices to find a sequence \((f_n)\) of functions such that

\[
J_{\Phi_{\delta}}(f_n, f_n) \leq \varepsilon_n \| f_n \|_2^2
\]

with \( \varepsilon_n \to 0 \) as \( n \to \infty \). This is satisfied by the choice: \( f_{\alpha_n}(u) \equiv f_{\alpha_n}(u) = u^{\alpha_n} \) for \( \alpha_n \to \infty \). Indeed,

\[
J_{\Phi_{\delta}}(f_{\alpha}, f_{\alpha}) = \left( \int_0^1 du u^{2\alpha} \right) \int_0^1 dt t^{\alpha - 1/2} \Phi_{\delta}(t)
\]

\[
= \| f_{\alpha} \|_2^2 \int_0^1 dt t^{\alpha - 1/2} \Phi_{\delta}(t),
\]

in which the integral relative to \( t \) decreases to 0 as \( \alpha \to \infty \) by the dominated convergence theorem since it is bounded from above by \( \int_0^1 dt t^{-1/2} \Phi_{\delta}(t) < \infty \).  

\( \square \)
3 Proof of Theorem 1.3

In this section, we denote by the pair \( (R_t, t \geq 0), P_x^{(\delta)} \) a BES(\( \delta \))-process starting from \( x \): \( P_x^{(\delta)}(R_0 = x) = 1 \). We denote by \( E_x^{(\delta)} \) the expectation with respect to \( P_x^{(\delta)} \). When \( x = 0 \), we often write \( E \) for \( E_0^{(\delta)} \) and \( P \) for \( P_0^{(\delta)} \).

3.1 Proof for the case \( \delta > 1 \) and \( \alpha > (2 - \delta)_+ \)

In this subsection, we prove the assertion of Theorem 1.3 in the case \( \delta > 1 \) and \( \alpha > (2 - \delta)_+ \). Set \( \delta_\alpha = \alpha(\delta + \alpha - 2) \).

**Proposition 3.1.** Suppose \( \delta > 1 \) and \( \alpha > (2 - \delta)_+ \). Then we have, for every bounded, measurable function \( f \) on \([0, 1]\),

\[
E[I(f; \hat{R}_{\alpha}^2)] = \alpha^2 \kappa_{2(\alpha-1)} \|f_\alpha\|^2_2 - J_{\delta_\alpha}(f_\alpha, f_\alpha),
\]

where

\[
\Phi_{\delta_\alpha}(t) = -\delta_\alpha t^{-(\alpha-1)/2} \varphi_\delta'(t) \quad \text{with} \quad \varphi_\delta(t) = E[\hat{R}_{\alpha}^2 \hat{R}_{\alpha}^2], \quad 0 \leq t \leq 1.
\]

**Proof.** Recall that, for \( \delta > 1 \), BES(\( \delta \)) admits a semimartingale decomposition of the following form:

\[
R_t = B_t + \frac{\delta - 1}{2} \int_0^t \frac{du}{R_u},
\]

where \( B = (B_t, t \geq 0) \) is a Brownian motion. By Itô’s formula,

\[
R_{\alpha}^2 = \alpha \int_0^t R_{u-1}^\alpha dB_u + \frac{\delta_\alpha}{2} \int_0^t R_{u-2}^\alpha du.
\]

We take the expectation on both sides to get

\[
E[R_{\alpha}^2] = \frac{\delta_\alpha}{2} \int_0^t E[R_{u-2}^\alpha] du.
\]

From these, we have

\[
\hat{R}_{\alpha}^2 = \alpha \int_0^t R_{u-1}^\alpha dB_u + \frac{\delta_\alpha}{2} \int_0^t \hat{R}_{u-2}^\alpha du.
\]

Now we take, in (2.9), \( X_t = \hat{R}_{\alpha}^2, M_t = \int_0^t R_{u-1}^\alpha dB_u \) and \( V_t = (\delta_\alpha/2) \int_0^t \hat{R}_{u-2}^\alpha du \). Then by Lemma 2.6, it holds that, for a bounded, measurable function \( f \) on \([0, 1]\),

\[
E[(X_t^2)] (\equiv E[I(f; \hat{R}_{\alpha}^2)])
= \delta_\alpha E\left[\int_0^1 \left( \int_0^u f(s) dB_s \right) f(u) \hat{R}_{u-2}^\alpha du \right] + \alpha^2 \int_0^1 f(u)^2 E[R_{u-2}^{(\alpha-1)}] du
= \delta_\alpha \int_0^1 du f(u) \int_0^u ds f(s) \left\{ \frac{d}{ds} E[\hat{R}_{u-2}^\alpha \hat{R}_s^\alpha] \right\} + \alpha^2 E[R_{1}^{(\alpha-1)}] \int_0^1 f(u)^2 u^{\alpha-1} du.
\]
Now the assertion of the proposition is immediate by the change of variables. \qed

**Lemma 3.2.** $J_{\Phi,\alpha}$ extends as a quadratic form on $L^2([0,1]) \times L^2([0,1])$. Moreover, it is positive-definite.

Once this lemma is shown, then, combining this with Proposition 3.1, we may prove the assertion of Theorem 1.3 for $\delta > 1$ and $\alpha > (2 - \delta)_+$:

**Proof of Theorem 1.3 for $\delta > 1$ and $\alpha > (2 - \delta)_+$.** By Proposition 3.1 and Lemma 3.2, we have:

$$E_0^{(\delta)}[I(f; \hat{R}^\alpha)^2] \leq \alpha^2 \kappa_{2(\alpha-1)} \|f\|^2_2,$$

which is (1.8). \qed

**Remark 3.1.** Similarly to the proof of Theorem 1.1, we may see that the constant $\alpha^2 \kappa_{2(\alpha-1)}$ is optimal.

The proof of Lemma 3.2 may be given along the lines of that of Theorem 1.1 for $\delta > 1$, $\Phi_\delta$ and $\varphi_\delta$ being replaced by $\Phi_{\delta,\alpha}$ and $\varphi_{\delta,\alpha}$, respectively. Instead of giving details (and in order to avoid duplications), we present here a different way, other than relying on series expansions, to show that the function $\varphi_{\delta,\alpha}$ has similar properties to those (a)–(c) of $\varphi_\delta$, which we shall present in the next proposition:

**Proposition 3.3.** Suppose that $\delta > 0$ and $\alpha \in (0,2)$ satisfy $\alpha > 2 - \delta$. Then the function $\varphi_{\delta,\alpha}$ has the following properties:

(a') $\varphi_{\delta,\alpha} \leq 0$; (b') $\varphi'_{\delta,\alpha} \leq 0$; (c') $\lim\inf_{t \downarrow 0} t^{-\frac{\alpha}{2} - 1} \varphi_{\delta,\alpha}(t) > -\infty$.

**Remark 3.2.** By using the series expansion of $\varphi_{\delta,\alpha}$, we may see that, for the property (c'), the limit itself exists; see Remark 5.1 in the appendix.

For $x \geq 0$ and $t > 0$, set

$$\rho_\alpha(x,t) \equiv \rho_{\alpha}^{(\delta)}(x,t) = E_x^{(\delta)}[R_t^\alpha],$$

$$\sigma_\alpha(x,t) \equiv \sigma_{\alpha}^{(\delta)}(x,t) = E_x^{(\delta)}[R_t^{\alpha-2}].$$

To prove Proposition 3.3, we begin with the following lemma:
Lemma 3.4. Under the same assumption as in Proposition 3.3, the following assertions hold for every fixed $t > 0$:

(i) $\rho_\alpha(x, t)$ is increasing in $x$.
(ii) $\sigma_\alpha(x, t)$ is decreasing in $x$.
(iii) The following lower bound for $\sigma_\alpha(x, t)$ holds:
\[
\sigma_\alpha(x, t) \geq e^{-x^2/2t} \sigma_\alpha(0, t) \quad \text{for all } x \geq 0.
\]

Proof. The assertions (i) and (ii) follow from the additive property of Bessel processes; that is, if $R_{\delta_x}(\delta, x, t)$, $t \geq 0$, denotes BES($\delta$) starting from $x$, then it holds that:
\[
(R_{\delta_x}(\delta, x, t))^2 \overset{(d)}{=} (R_{\delta_x}(0, x, t))^2 + (R_{\delta_0}(0, t))^2,
\]
where, on the RHS, the two processes are independent. Moreover, considering two SDE’s with common Brownian motion $B$:
\[
(R_{\delta_x}(0, x, t))^2 = x^2 + \int_0^t 2R_{\delta_x}(0, u) dB_u, \quad (R_{\delta_y}(0, y, t))^2 = y^2 + \int_0^t 2R_{\delta_y}(0, u) dB_u,
\]
we see that
\[
R_{\delta_x}(0, x, t) \leq R_{\delta_y}(0, y, t) \quad \text{if } x \leq y,
\]
by comparison. The assertion (iii) also follows from (3.6) since we deduce from there that
\[
\sigma_\alpha(x, t) \geq E_{\delta_x}[R_{\delta_t}^{\alpha-2} | P_x(0)(T_0(R) \leq t)],
\]
where $T_0(R)$ is the time at which $R$ reaches to 0. It is well-known (e.g., [16, Chapter XI]) that, under $P_x^{(0)}$, $T_0(R)$ is identical in law with $x^2/2e$, $e$ being a standard exponential variable, hence $P_x^{(0)}(T_0(R) \leq t) = e^{-x^2/2t}$. Combining these yields the assertion (iii).

Remark 3.3. The assertions (i) and (ii) may be seen from the FKG inequality; an application of a generalized FKG inequality due to Preston [14], to the laws of BES($\delta$), $\delta > 0$, asserts that, for increasing functions $F$ on $C([0, 1]; \mathbb{R}_+)$, it holds that $E_{\delta_x}^{(\delta)}[F(R)] \leq E_{\delta_y}^{(\delta)}[F(R)]$ if $0 \leq x \leq y$. Here we say that a function $F$ defined on $C([0, 1]; \mathbb{R}_+)$ is increasing if $F(w_1) \leq F(w_2)$ for all $w_1, w_2 \in C([0, 1]; \mathbb{R}_+)$ satisfying $w_1(t) \leq w_2(t)$ for all $0 \leq t \leq 1$. So, if we take $F(R) = R_{\delta_t}^\alpha$ (resp. $F(R) = -R_{\delta_t}^{\alpha-2}$), we recover the assertion (i) (resp. the assertion (ii)). For the applicability of this inequality to Bessel processes, see Proposition 5.6 in the appendix.
Set
\[ \psi_{\delta,\alpha}(t) = E[R_1^{\alpha-2}R_t^{\alpha-2}] . \]

The two functions \( \varphi_{\delta,\alpha} \) and \( \psi_{\delta,\alpha} \) are related via:

**Lemma 3.5.** The following identity holds:

(3.7) \[ \varphi'_{\delta,\alpha}(t) = -\frac{\delta}{2} \psi_{\delta,\alpha}(t) + \frac{\alpha}{t} \varphi_{\delta,\alpha}(t) . \]

**Proof.** By definition, \( \varphi'_{\delta,\alpha}(t) \) is written as:

(3.8) \[ \frac{d}{dt} E[R_1^{\alpha-2}R_t^\alpha] - E[R_1^{\alpha-2}] \frac{d}{dt} E[R_t^\alpha] . \]

To work on the first term, we use the time-inversion and the Markov property to see that:

(3.9) \[ E[R_1^{\alpha-2}R_t^\alpha] = t^\alpha E[R_1^{\alpha-2}R_{1/t}^\alpha] \]
\[ = t^\alpha E[R_1^{\alpha-2} \rho_\alpha(R_1, \frac{1}{t} - 1)] . \]

Noting (3.4) with changing the starting point to \( x \geq 0 \), we deduce that

\[ \rho_\alpha(x, t) = x^\alpha + \frac{\delta}{2} \int_0^t \sigma_\alpha(x, u) \, du , \]

hence

(3.10) \[ \frac{\partial}{\partial t} \rho_\alpha(x, t) = \frac{\delta}{2} \sigma_\alpha(x, t) . \]

Therefore, by (3.9) and (3.10),

(3.11) \[ \frac{d}{dt} E[R_1^{\alpha-2}R_t^\alpha] = \alpha t^{\alpha-1} E[R_1^{\alpha-2}R_{1/t}^\alpha] + t^\alpha E[R_1^{\alpha-2}] \frac{\delta}{2} \sigma_\alpha(R_1, \frac{1}{t} - 1) \times (-\frac{1}{t^2}) \]
\[ = \frac{\alpha}{t} E[R_1^{\alpha-2}R_t^\alpha] - \frac{\delta}{2} E[R_1^{\alpha-2}R_t^{\alpha-2}] . \]

Here, for the second equality, we used the Markov property and the time-inversion. On the other hand, for the second term of (3.8), the factor \( (d/dt)E[R_t^\alpha] \) may be expressed in two ways:

\[ \frac{d}{dt} E[R_t^\alpha] = \frac{\delta}{2} E[R_t^{\alpha-2}] \]
\[ = \frac{\alpha}{2t} E[R_t^\alpha] . \]

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The first expression is nothing but (3.10) with \( x = 0 \), while the second one is a consequence of the scaling property: \( E[R_1^{\alpha}] = (\sqrt{t})^{\alpha} E[R_t^{\alpha}] \). Thus, we may write (3.8) as:

\[
- \frac{\delta_{\alpha}}{2} E[R_1^{\alpha-2} R_t^{\alpha-2}] + \frac{\alpha}{t} E[R_1^{\alpha-2} R_t^{\alpha-2}] - E[R_1^{\alpha-2}] \left( - \frac{\delta_{\alpha}}{2} E[R_1^{\alpha-2}] + \frac{\alpha}{t} E[R_t^{\alpha}] \right)
\]

\[
= - \frac{\delta_{\alpha}}{2} \left[ E[R_1^{\alpha-2} R_t^{\alpha-2}] - E[R_1^{\alpha-2}] E[R_t^{\alpha-2}] \right] + \frac{\alpha}{t} \left[ E[R_1^{\alpha-2} R_t^{\alpha}] - E[R_1^{\alpha-2}] E[R_t^{\alpha}] \right],
\]

which is the RHS of (3.7).

To prove Proposition 3.3, we also use the following:

**Lemma 3.6.** The function \( \psi_{\delta,\alpha} \) is non-negative.

*Proof.* Note that \( \widehat{R}_{t}^{\alpha-2} \), \( t > 0 \), admits the following representation (Itô’s representation):

\[(3.12) \quad \widehat{R}_{t}^{\alpha-2} = \int_{0}^{t} \frac{\partial \sigma_{\alpha}}{\partial x}(R_{u}, t-u) dB_{u},\]

where \( B \) is the Brownian motion appearing in (3.2). Therefore, by definition,

\[
\psi_{\delta,\alpha}(t) = E[\widehat{R}_{1}^{\alpha-2} \widehat{R}_{t}^{\alpha-2}] = \int_{0}^{t} du E[\frac{\partial \sigma_{\alpha}}{\partial x}(R_{u}, 1-u) \frac{\partial \sigma_{\alpha}}{\partial x}(R_{u}, t-u)].
\]

Note that, by Lemma 3.4 (ii), \((\partial/\partial x) \sigma_{\alpha} \leq 0\), which implies \( \psi_{\delta,\alpha} \geq 0.\)

**Remark 3.4.** This lemma may also be seen from the FKG inequality; from the (classical) formulation of the FKG inequality, we may also see that, for each fixed \( x \geq 0 \), and for increasing functions \( F, G \) on \( C([0,1]; \mathbb{R}_+) \), it holds that \( E_{x}^{(\delta)}[F(R)G(R)] \geq E_{x}^{(\delta)}[F(R)] E_{x}^{(\delta)}[G(R)]. \) So we may recover this lemma by taking \( F(R) = -R_1^{\alpha-2} \) and \( G(R) = -R_t^{\alpha-2}. \)

Now we are prepared to prove Proposition 3.3:

**Proof of Proposition 3.3.** Property (a’): By Itô’s representation:

\[
\widehat{R}_{t}^{\alpha} = \int_{0}^{t} \frac{\partial \rho_{\alpha}}{\partial x}(R_{u}, t-u) dB_{u},
\]

we have, together with (3.12) for \( t = 1 \),

\[
\varphi_{\delta,\alpha}(t) = \int_{0}^{t} du E[\frac{\partial \rho_{\alpha}}{\partial x}(R_{u}, 1-u) \frac{\partial \rho_{\alpha}}{\partial x}(R_{u}, t-u)].
\]

By Lemma 3.4 (i) and (ii), we see that the inside of the expectation is negative. This shows (a’).
Property (b'): This property is now immediate from Lemmas 3.5 and 3.6, and (a').

Property (c'): To prove this property, note that, by definition,

$$\varphi_{\delta,\alpha}(t) = E[R_1^{\alpha-2}R_t^\alpha] - E[R_1^{\alpha-2}E[R_t^\alpha]].$$

For the second term, we easily see that, by scaling,

$$E[R_1^{\alpha-2}E[R_t^\alpha]] = t^{\alpha/2}\kappa_{\alpha-2}\kappa_\alpha. \quad (3.13)$$

On the other hand, to estimate the first term, we rewrite it as, by using the Markov property,

$$E[R_1^{\alpha-2}R_t^\alpha] = E[R_1^\alpha \sigma_\alpha(R_t,1-t)]. \quad (3.14)$$

By Lemma 3.4 (iii), and by scaling, (3.14) is bounded from below by

$$E[R_1^\alpha \sigma_\alpha(0,1-t)] \geq \kappa_{\alpha-2}\kappa_\alpha t^{\alpha/2}(1-t)^{(\alpha-2)/2}E[R_1^\alpha \sigma_\alpha(0,1-t)].$$

By using the fact that $R_1^2/2 \overset{(d)}{=} \gamma_{\delta/2}$, a gamma variable of index $\delta/2$, the expectation on the RHS is computed explicitly as $\kappa_\alpha(1-t)^{(\delta+\alpha)/2}$. So (3.14) is bounded from below by

$$\kappa_{\alpha-2}\kappa_\alpha t^{\alpha/2}(1-t)^{(\delta+2\alpha-2)/2}. \quad (3.15)$$

Combining (3.13) and (3.15) yields

$$\varphi_{\delta,\alpha}(t) \geq -\kappa_{\alpha-2}\kappa_\alpha t^{\alpha/2} \{1 - (1-t)^{\delta+\alpha-1}\},$$

from which the property (c') follows.

Remark 3.5. We may also see the property (a') directly from the FKG inequality.

3.2 Proof for the case $0 < \delta < 2$ and $\alpha = 2 - \delta$

For $0 < \delta < 2$, define $\mu \in (0,1)$ via: $\delta = 2(1-\mu)$. Then $\alpha$ and $\mu$ are related as $\alpha = 2\mu$.

Proposition 3.7. For every bounded, measurable function $f$ on $[0,1]$, it holds that

$$E[I(f;\hat{R}^\alpha)^2] = \alpha^2\kappa_2(\alpha-1)\|f_\alpha\|^2 - \frac{\alpha^2}{2}(\kappa_\alpha)^2J^{(\alpha)}(f_\alpha, f_\alpha),$$

where $J^{(\alpha)} \equiv J_{\psi(\mu)}$ with

$$\psi(\mu)(t) = \frac{1}{\sqrt{t}} \{1 - (1-t)^\mu\} + \frac{\sqrt{t}}{(1-t)^{1-\mu}}.$$
Lemma 2.6, we have, for a bounded, measurable function \( f \),

\[
R_t^{2\mu} = M_t^{(\mu)} + L_t,
\]

where \( (M_t^{(\mu)}, t \geq 0) \) is a martingale with \( \langle M^{(\mu)} \rangle_t = 4\mu^2 \int_0^t R_u^{2(2\mu-1)} \, du \), and \( (L_t) \) is a possible choice of the local time at 0 of \( (R_t) \) as a diffusion.

**Proof of Proposition 3.7.** We take \( X_t = R_t^{2\mu}, M_t = M_t^{(\mu)} \) and \( V_t = L_t \) in (2.6). Then by Lemma 2.6, we have, for a bounded, measurable function \( f \) on \([0, 1],\)

\[
E\left[ \left( \int_0^1 f(u) \, dR_u^{2\mu} \right)^2 \right] = E\left[ \int_0^1 f(u) \, d\langle M^{(\mu)} \rangle_u \right] + 2E\left[ \int_0^1 f(u) \, dL_u \left( \int_0^u f(s) \, dR_s^{2\mu} \right) \right]
\]

\[=: I_1^f + 2I_2^f.\]

Since \((d/du)E[\langle M^{(\mu)} \rangle_u] = 4\mu^2 E[R_u^{2(2\mu-1)}] \equiv 4\mu^2 \kappa_2(2\mu-1)u^{2\mu-1},\) we have

\[I_1^f = 4\mu^2 \kappa_2(2\mu-1)\|f_{2\mu}\|_2^2.\]

On the other hand, for \(I_2^f\), recall that \(dL_u\) is carried by the set of zeros of \(R\). So, denoting by \((r_s^u, 0 \leq s \leq u)\) a Bessel bridge of length \(u\), we have:

\[
I_2^f = \int_0^1 f(u) \, dE[L_u]E[\int_0^u f(s) \, dR_s^{2\mu} | R_u = 0]
\]

\[= \int_0^1 f(u) \, dE[L_u] \int_0^u f(s) \, dE[(r_s^u)^{2\mu}]\]

\[= \int_0^1 f(u) \, dE[L_u] \int_0^u (ut)^\mu \, dE[(r_t^1)^{2\mu}],\]

where we changed variables with \(s = ut, 0 \leq t \leq 1,\) for the third line. Note that \(E[L_u] = E[R_u^{2\mu}] = \kappa_2 u^\mu,\) hence \(dE[L_u] = \mu \kappa_2 u^{\mu-1} \, du.\) Moreover, noting \(r_t^1 \overset{(d)}{=} (1 - t)R_{t/(1-t)},\) we see \(E[(r_t^1)^{2\mu}] = \kappa_2 t^{\mu}(1 - t)^\mu.\) Combining these yields:

\[
I_2^f = \mu(\kappa_2 t)^2 \int_0^1 du \, f(u) u^{2\mu-1} \int_0^1 dt \, f(ut) \frac{d}{dt} \{t^{\mu}(1 - t)^\mu\}
\]

\[= \int_0^1 du \, f_2(u) \int_0^1 dt \, f_2(ut) \Psi_1^{(\mu)}(t),\]

where \(\Psi_1^{(\mu)}(t) := \mu(\kappa_2 t)^2 t^{-\mu+1/2}(d/dt)\{t^{\mu}(1 - t)^\mu\}.\) Besides these, setting

\[
I_3^f = E\left[ \int_0^1 f(u) \, dR_u^{2\mu} \right] \equiv \mu \kappa_2 \int_0^1 f(u) u^{\mu-1} \, du,
\]

\[\Rightarrow \]

**Remark 3.6.** Taking \(\alpha = 1 (\iff \mu = 1/2)\) above, we recover (2.8) for \(\delta = 1.\)
we easily see that

\[
(I_3^f)^2 = \int_0^1 du \int_0^1 dt f_{2\mu}(ut) \Psi_2^{(\mu)}(t), \quad \Psi_2^{(\mu)}(t) := 2\mu^2(\kappa_{2\mu})^2 t^{-1/2}.
\]

Consequently, we obtain

\[
E[I(f; \hat{R}^{2\mu})^2] \equiv I_1^f + 2I_2^f - (I_3^f)^2
\]

\[
= (2\mu)^2 \kappa_{2(2\mu - 1)} \|f_{2\mu}\|_2^2 - J_{\Psi_2^{(\mu)}}(f_{2\mu}, f_{2\mu})
\]

with the identity

\[
\Psi_2^{(\mu)}(t) - 2\Psi_1^{(\mu)}(t) = 2\mu^2(\kappa_{2\mu})^2 \psi_1^{(\mu)}(t).
\]

Recalling \(\alpha = 2\mu\), we obtain the proposition.

**Lemma 3.8.** \(J^{(\alpha)}\) extends as a quadratic form on \(L^2([0,1]) \times L^2([0,1])\). Moreover, it is positive-definite.

**Proof.** Clearly \(\psi^{(\mu)}\) is non-negative and satisfies \(\int_0^1 dt t^{-1/2} \psi^{(\mu)}(t) < \infty\). So the first assertion follows from Proposition 2.1. So, by Proposition 2.2, it suffices to show that \(\psi^{(\mu)}\) is expanded as (2.3). To this end, first note that

\[
(3.17) \quad \frac{1}{(1-t)^{1-\mu}} = \frac{1}{\Gamma(1-\mu)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \mu)}{n!} t^n.
\]

Indeed, using the elementary formula:

\[
a^{-\gamma} = \Gamma(\gamma)^{-1} \int_0^\infty dx x^{\gamma-1} e^{-ax} \quad \text{for } a, \gamma > 0,
\]

we have:

\[
\frac{1}{(1-t)^{1-\mu}} = \frac{1}{\Gamma(1-\mu)} \int_0^\infty dx x^{-\mu} e^{-(1-t)x}
\]

\[
= \frac{1}{\Gamma(1-\mu)} \int_0^\infty dx x^{-\mu} e^{-x} \sum_{n=0}^{\infty} \frac{(tx)^n}{n!}
\]

\[
= \frac{1}{\Gamma(1-\mu)} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^\infty dx x^{n-\mu} e^{-x},
\]

and arrive at (3.17). Moreover, since \(1 - (1-t)^{\mu} = \int_0^t ds (1-s)^{-(1-\mu)}\), we also have:

\[
(3.18) \quad 1 - (1-t)^{\mu} = \frac{1}{\Gamma(1-\mu)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \mu)}{(n+1)!} t^{n+1}.
\]

Combining (3.17) and (3.18), we see that \(\psi^{(\mu)}\) is expanded as (2.3) with \(\alpha = 1/2\) and

\[
a_n = \frac{\Gamma(n + 1 - \mu)}{\Gamma(1-\mu)} \left( \frac{1}{n!} + \frac{1}{(n+1)!} \right) > 0, \quad n = 0, 1, 2, \cdots.
\]

This concludes the assertion. \(\square\)
The proof of Theorem 1.3 for the present case is now immediate from Proposition 3.7 and Lemma 3.8:

**Proof of Theorem 1.3 for** \(0 < \delta < 2\) **and** \(\alpha = 2 - \delta\). By Proposition 3.7 and Lemma 3.8,

\[
E[I(f; \hat{R}^\alpha)^2] \leq \alpha^2 \kappa_2(\alpha-1) \|f\|_2^2.
\]

So the assertion for the present case is proved. \(\square\)

### 3.3 Proof for the case \(0 < \delta \leq 1\) **and** \(\alpha > 2 - \delta\)

For \(0 < \delta \leq 1\), let \(\mu\) be as given in the previous subsection: \(\delta = 2(1 - \mu)\). By the decomposition (3.16), and by Itô’s formula, it is not hard to see that, for \(\alpha > 2\mu\), \((R^\alpha_t)\) is a semimartingale which decomposes as

\[
R^\alpha_t \equiv (R_t^{2\mu})^{\alpha/2\mu} = M_t^{(\alpha)} + V_t^{(\alpha)},
\]

where \((M_t^{(\alpha)})\) is a martingale with quadratic variation \((M^{(\alpha)})_t = \alpha^2 \int_0^t R_u^{2(\alpha-1)} du\), and \(V_t^{(\alpha)} = (\delta / 2) \int_0^t R_u^{\alpha-2} du\). Now the proof in the present case can be done in the same way as that of the proof in the case \(\delta > 1\) and \(\alpha > (2 - \delta)_+\) (Subsection 3.1). So we omit the proof.

### 4 Defining Wiener integrals relative to the centered BES(\(\delta\))-bridges for \(\delta \geq 1\)

In this section, we denote by \((r_\delta(t), 0 \leq t \leq 1)\) a BES(\(\delta\))-bridge, \(\delta \geq 1\), with length 1 such that \(r_\delta(0) = r_\delta(1) = 0\). With the help of the results obtained in Section 2, we may also prove that the Wiener integral \(I(f; \hat{r}_\delta)\) is well-defined for all \(f \in L^2([0, 1])\); in fact, we obtain the following a priori estimate:

**Proposition 4.1.** Let \(\delta \geq 1\). Then, for all \(f \in C([0, 1])\), it holds that

\[
E[I(f; \hat{r}_\delta)^2] \leq \|f\|_2^2.
\]

**Remark 4.1.** As we see in the proof below, we have, more precisely that

\[
E[I(f; \hat{r}_\delta)^2] \leq \|f\|_2^2 - \left(\int_0^1 du f(u)\right)^2.
\]

This estimate coincides with that obtained by the approach via the Brascamp-Lieb inequality for \(\delta \geq 3\); in fact, if we denote by \((b(u), 0 \leq u \leq 1)\) a Brownian bridge of length 1 such that \(b(0) = b(1) = 0\), then the RHS of (4.2) is nothing but \(E[I(f; b)^2]\).
Proof of Proposition 4.1. It is well-known that \((r_\delta(t), 0 \leq t \leq 1)\) is identical in law with the process \((X_\delta(t), 0 \leq t \leq 1)\) defined by
\[
X_\delta(t) = \begin{cases} 
(1-t)R_\delta\left(\frac{t}{1-t}\right), & 0 \leq t < 1, \\
0, & t = 1,
\end{cases}
\]
where \(R_\delta\) is a BES(\(\delta\))-process starting from 0. Therefore
\[
I(f; \hat{r}_\delta) \overset{(d)}{=} \int_0^1 f(u) \, d\hat{X}_\delta(u) 
\]
\[
= -\int_0^1 f(u) \hat{R}_\delta(u) \, du + \int_0^1 f(u)(1-u) \, d_u \hat{R}_\delta(u).
\]
We use integration by parts to see that the first term on the RHS is equal to:
\[
-\int_0^1 \left( \int_u^1 dv f(v) \right) d_u \hat{R}_\delta(u).
\]
Thus, setting \(G(u) = f(u) - (1-u)^{-1} \int_u^1 dv f(v)\), we have
\[
(4.3) \quad I(f; \hat{r}_\delta) \overset{(d)}{=} \int_0^1 (1-u)G(u) \, d_u \hat{R}_\delta(u)
\]
\[
= \int_0^\infty \frac{1}{1+s}G\left(\frac{s}{1+s}\right) \, d_R_\delta(s).
\]
So, denoting \(\tilde{G}(s) = (1+s)^{-1}G(s/(1+s))\) and using the similar argument to that in the proof of Proposition 2.5, we deduce that
\[
(4.4) \quad E[I(f; \hat{r}_\delta)^2] = E[\left( \int_0^\infty \tilde{G}(u) \, d\hat{R}_\delta(u) \right)^2]
\]
\[
= \|\tilde{G}\|_{L^2(\mathbb{R}_+)}^2 - \tilde{J}_{\Phi_\delta}(\tilde{G}, \tilde{G}),
\]
where, for \(F \in L^2(\mathbb{R}_+)\),
\[
\tilde{J}_{\Phi_\delta}(F, F) = \int_0^\infty du F(u) \int_0^1 dt F(ut)\Phi_\delta(t)
\]
and \(\Phi_\delta\) is as given in Proposition 2.5. With the help of the argument used in the proof of Lemma 2.7, we may also deduce that \(\tilde{J}_{\Phi_\delta}\) is non-negative definite. So, by (4.4),
\[
E[I(f; \hat{r}_\delta)^2] \leq \|\tilde{G}\|_{L^2(\mathbb{R}_+)}^2
\]
\[
= \|G\|_{L^2([0,1])}^2
\]
\[
= \|f\|_{L^2([0,1])}^2 - \left( \int_0^1 du f(u) \right)^2,
\]
which implies (4.1). \(\square\)
5 Appendix

5.1 An explicit representation for $\varphi_{\delta,\alpha}$ in terms of hypergeometric functions

Following [17], for complex numbers $a, b, c$, we denote by $\,_{2}F_{1}(a, b; c; w), |w| < 1$, Gauss’s hypergeometric functions. Recall that, when $\Re c > \Re b > 0$, they admit the following integral representation:

\[
\,_{2}F_{1}(a, b; c; w) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} dz \, z^{b-1}(1-z)^{c-b-1}(1-wz)^{-a}.
\]

(5.1)

See, e.g, [10, (9.1.4)].

In this subsection, we discuss an explicit representation for the function $\varphi_{\delta,\alpha}(t) \equiv E[R_{1}^{\alpha-2}R_{1}^{\delta}]$ in terms of $\,_{2}F_{1}$. We do this in a slightly general situation: For dimension $\delta > 0$, and for two exponents $\alpha_{i} \in \mathbb{R}$, $i = 1, 2$, satisfying $\delta + \alpha_{i} > 0$, set

$\phi_{\alpha_{1};\alpha_{2}}(t) = E[R_{1}^{\alpha_{1}}R_{1}^{\alpha_{2}}], \quad 0 < t < 1.$

**Proposition 5.1.** Suppose $\alpha \in \mathbb{R}$ and $\beta > 0$ satisfy $\delta + \alpha > 0$ and $\delta > \beta$, respectively. Then the function $\phi_{\alpha;\beta}(t)$ admits the following representation:

$\phi_{\alpha;\beta}(t) = 2^{\alpha - \beta} \frac{\Gamma(\frac{\delta + \alpha}{2})\Gamma(\frac{\delta - \beta}{2})}{\Gamma(\frac{\delta}{2})^{2}} \left\{ \,_{2}F_{1}\left(-\frac{\alpha}{2}, \frac{\beta}{2}; \frac{\delta}{2}; t\right) - 1 \right\}.$

**Proof.** Since $\beta > 0$, we may rewrite $R_{1}^{-\beta}$ into:

$R_{1}^{-\beta} = 2^{-\beta/2} \left( R_{1}^{2}/2 \right)^{-\beta/2} = 2^{-\beta/2} \Gamma(\beta/2)^{-1} \int_{0}^{\infty} \frac{dx}{x} x^{\beta/2} \exp \left(-\frac{x}{2} R_{1}^{2}\right).$

Then by Fubini’s theorem,

\[
E[R_{1}^{\alpha}R_{1}^{-\beta}] = 2^{-\beta/2} \Gamma(\beta/2)^{-1} \int_{0}^{\infty} \frac{dx}{x} x^{\beta/2} E[R_{1}^{\alpha} \exp \left(-\frac{x}{2} R_{1}^{2}\right)].
\]

(5.2)

By the Markov property, we have, conditionally on $\mathcal{F}^{R}_{t}$,

\[
E_{0}^{(\delta)}[\exp \left(-\frac{x}{2} R_{1}^{2}\right) | \mathcal{F}^{R}_{t}] = E_{R_{1}^{\alpha}}^{(\delta)}[\exp \left(-\frac{x}{2} R_{1-t}^{2}\right)]
\]

(5.3)

$= \left\{ 1 + x(1-t) \right\}^{-\delta/2} \exp \left\{ -\frac{x}{2(1+x(1-t))} R_{1-t}^{2} \right\}.$

Here the second equality follows from the well-known fact that: for $a \geq 0$,

$E_{a}^{(\delta)}[\exp \left(-\frac{x}{2} R_{s}^{2}\right)] = (1+xs)^{-\delta/2} \exp \left(-\frac{x}{1+xs} a^{2}\right).$

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See, e.g., [16, p.441]. By (5.3), we then have:

\[
E[R_t^\alpha \exp \left(-\frac{x}{2}R_t^2\right)] = \{1 + x(1-t)\}^{-\delta/2} E[R_t^\alpha \exp \left(-\frac{x}{2(1+x(1-t))}R_t^2\right)] \\
= (2t)^{\alpha/2} \Gamma(\delta/2)^{-1} \Gamma((\delta + \alpha)/2) \frac{1}{(1+x)^{\delta/2}} \left(1 - \frac{x}{1+x}\right)^{\alpha/2},
\]

where, for the second line, we used the fact that \( R_t \equiv \sqrt{2t\gamma \beta/2} \). Plugging this into (5.2), and changing variables with \( x/(1+x) = z \), we arrive at

\[
E[R_t^\alpha R_1^{-\beta}] = 2^{(\alpha-\beta)/2} \frac{\Gamma(\frac{\delta+\alpha}{2}) \Gamma(\frac{\delta-\beta}{2})}{\Gamma(\frac{\delta}{2})} \int_0^1 dz \, z^{\beta-1}(1-z)^{\frac{\delta-\beta}{2}-1}(1-tz)^{\frac{\alpha}{2}}.
\]

By the integral representation (5.1), the above integral in \( z \) is expressed as:

\[
\frac{\Gamma(\frac{\delta}{2}) \Gamma(\frac{\delta-\beta}{2})}{\Gamma(\frac{\delta}{2})} \binom{\alpha}{\frac{\beta}{2}} \binom{\frac{\beta}{2}}{\frac{\delta}{2}} F_1(-\frac{\alpha}{2}, \frac{\beta}{2}; \frac{\delta}{2}; t).
\]

Now we see

\[
(5.4) \quad E[R_t^\alpha R_1^{-\beta}] = 2^{(\alpha-\beta)/2} \frac{\Gamma(\frac{\delta+\alpha}{2}) \Gamma(\frac{\delta-\beta}{2})}{\Gamma(\frac{\delta}{2})^2} t^{\alpha/2} \binom{\alpha}{\frac{\beta}{2}} \binom{\frac{\beta}{2}}{\frac{\delta}{2}} F_1(-\frac{\alpha}{2}, \frac{\beta}{2}; \frac{\delta}{2}; t).
\]

On the other hand, by scaling,

\[
E[R_t^\alpha] E[R_1^{-\beta}] = t^{\alpha/2} E[R_t^\alpha] E[R_1^{-\beta}] \equiv t^{\alpha/2} \frac{2^{(\alpha-\beta)/2} \Gamma(\frac{\delta+\alpha}{2}) \Gamma(\frac{\delta-\beta}{2})}{\Gamma(\frac{\delta}{2})^2}.
\]

Therefore, combining this with (5.4), and noting \( \phi_{\alpha-\beta}(t) = E[R_t^\alpha R_1^{-\beta}] - E[R_t^\alpha] E[R_1^{-\beta}] \), we have the lemma. \( \Box \)

Remark 5.1. By Proposition 5.1, and by recalling the definition (series expansion) of hypergeometric functions (cf. [10, (9.1.1)]), we may easily see that

\[
\lim_{t \downarrow 0} t^{-\frac{\beta}{2}-1} \phi_{\alpha-\beta}(t) = 2^{(\alpha-\beta)/2} \frac{\Gamma(\frac{\delta+\alpha}{2}) \Gamma(\frac{\delta-\beta}{2})}{\Gamma(\frac{\delta}{2})^2} \times \left( -\frac{\alpha \beta}{2\delta} \right).
\]

We may compare this with the assertion \( (c') \) of Proposition 3.3.

5.2 On a connection between \( \Phi_1 \) and \( \Phi_3 \) via perturbations

In this part, we shall consider the family of processes \((X_t^{(\gamma)}, 0 \leq t \leq 1), \gamma \in \mathbb{R}, \) defined by

\[
(5.5) \quad X_t^{(\gamma)} = |\beta_t| + \gamma L_t,
\]

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and discuss the Wiener integrals $I(f; X^{(\gamma)})$ relative to the centered processes $\widehat{X}^{(\gamma)}$. Here, as in the proof of Proposition 2.5 for $\delta = 1$, $(\beta_t)$ is a Brownian motion and $(L_t)$ the local time of $\beta$ at 0. Note that, in particular,

$$
\begin{cases}
\gamma = -1 & : X^{(-1)} \text{ is a Brownian motion (Tanaka’s formula)};
\gamma = 0 & : X^{(0)} \text{ is a reflecting Brownian motion};
\gamma = +1 & : X^{(+1)} \text{ is BES(3) (Pitman’s theorem)}.
\end{cases}
$$

(5.6)

We shall prove that, for every $\gamma \in \mathbb{R}$, $I(f; \widehat{X}^{(\gamma)})$ is well-defined for all $f \in L^2([0,1])$. In fact, we obtain the following a priori estimate:

**Proposition 5.2.** There exists a positive constant $K_\gamma$ such that, for all bounded, measurable functions $f$ on $[0,1]$,

$$
E[I(f; \widehat{X}^{(\gamma)})^2] \leq K_\gamma \|f\|_2^2.
$$

(5.7)

In particular, if $|\gamma| \leq 1$, then (5.7) holds with $K_\gamma = 1$.

To prove this proposition, we prepare the following:

**Lemma 5.3.** For every bounded, measurable function $f$ on $[0,1]$, it holds that

$$
E[I(f; \widehat{X}^{(\gamma)})^2] = \|f\|_2^2 - J_{\Phi^{(\gamma)}}(f, f),
$$

where

$$
\Phi^{(\gamma)}(t) = \frac{(1 + \gamma)^2}{\pi \sqrt{t}}(1 - \sqrt{1-t}) + \frac{1-\gamma^2}{\pi} \frac{\sqrt{t}}{\sqrt{1-t}}
$$

Note that, in the three particular cases $\gamma = -1, 0, +1$, we recover the previous results; indeed,

$$
\Phi^{(-1)}(t) \equiv 0 \text{ (since } X^{(\gamma)} \text{ is the Brownian motion)};
\Phi^{(0)}(t) = \frac{1}{\pi} \left\{ \frac{1}{\sqrt{t}}(1 - \sqrt{1-t}) + \frac{\sqrt{t}}{\sqrt{1-t}} \right\} \equiv \Phi_1(t);
\Phi^{(+1)}(t) = \frac{4}{\pi \sqrt{t}}(1 - \sqrt{1-t}) \equiv \Phi_3(t).
$$

The proof of this lemma is given in the same manner as that of Proposition 2.5 for $\delta = 1$: 25
Proof of Lemma 5.3. Since $|\beta_t|$ decomposes as $|\beta_t| = B_t + L_t$, $X_t^{(\gamma)}$ admits the following decomposition:

\begin{equation}
X_t^{(\gamma)} = B_t + (\gamma + 1)L_t.
\end{equation}

We take $X_t = X_t^{(\gamma)}$, $M_t = B_t$ and $V_t = (\gamma + 1)L_t$ in (2.9). Then by Lemma 2.6,

$$E[\left(\int_0^1 f(u)\,dX_u^{(\gamma)}\right)^2] = \|f\|^2_2 + 2(\gamma + 1)E[\int_0^1 f(u)\,dL_u\left(\int_0^u f(s)\,dX_s^{(\gamma)}\right)]$$

$$= \|f\|^2_2 + 2(\gamma + 1)\int_0^1 f(u)\,dE[L_u]\,E[\int_0^u f(s)\,dX_s^{(\gamma)}|\beta_u = 0].$$

We have already seen that $dE[L_u] = du/\sqrt{2\pi u}$ just below (2.11). Moreover, by the definition of $X^{(\gamma)}$,

$$\frac{d}{ds}E[X_s^{(\gamma)}|\beta_u = 0] = \frac{d}{ds}E[|\beta_u||\beta_u = 0] + \gamma \frac{d}{ds}E[L_s|\beta_u = 0]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{s}} \frac{1}{(1-s/u)} \left(1 - \frac{2s}{u}\right) + \gamma \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{s}} \frac{1}{(1-s/u)}.$$

We thus obtain

$$E[\left(\int_0^1 f(u)\,dX_u^{(\gamma)}\right)^2] = \|f\|^2_2 + J_{\varphi_1^{(\gamma)}}(f,f), \quad \varphi_1^{(\gamma)}(t) := \frac{(\gamma + 1)\{(1 - \gamma) + t\}}{\pi \sqrt{t(1-t)}}.$$

On the other hand, since $E[\int_0^1 f(u)\,dX_u^{(\gamma)}] = (\gamma + 1)\int_0^1 du f(u)/\sqrt{2\pi u},$

$$E[\left(\int_0^1 f(u)\,dX_u^{(\gamma)}\right)^2] = J_{\varphi_2^{(\gamma)}}(f,f), \quad \varphi_2^{(\gamma)}(t) := \frac{(\gamma + 1)^2}{\pi \sqrt{t}}.$$

Combining these, we obtain

$$E[I(f;\hat{X^{(\gamma)}})^2] = \|f\|^2_2 - J_{\varphi_2^{(\gamma)} - \varphi_1^{(\gamma)}}(f,f)$$

with the identity $\varphi_2^{(\gamma)} - \varphi_1^{(\gamma)} = \Phi^{(\gamma)}$. So the proposition is proved. \hfill \Box

With Lemma 5.3 at disposal, we prove Proposition 5.2:

Proof of Proposition 5.2. The case $|\gamma| > 1$: Since it is clear that $K_{\Phi^{(\gamma)}} := \int_0^1 dt t^{-1/2} |\Phi^{(\gamma)}(t)|$ is finite, we may use Proposition 2.1 to have that

$$|J_{\Phi^{(\gamma)}}(f,f)| \leq K_{\Phi^{(\gamma)}} \|f\|^2_2.$$

Therefore, by Lemma 5.3, (5.7) holds with $K_\gamma = 1 + K_{\Phi^{(\gamma)}}$.

The case $|\gamma| \leq 1$: Using the same argument as in the proof of Lemma 2.7 for $\delta = 1$, we
easily see that \( J_{\Phi(\gamma)} \) extends as a positive-definite quadratic form on \( L^2([0, 1]) \times L^2([0, 1]) \) if \( |\gamma| \leq 1 \); indeed, \( \Phi(\gamma) \) is expanded as (2.14) with

\[
a_n = \frac{1}{\pi} \left\{ (1 + \gamma)^2 + 2(n + 1)(1 - \gamma^2) \right\} c_{n+1},
\]

which is positive for every \( n = 0, 1, 2, \cdots \), as long as \( |\gamma| \leq 1 \). Here \( (c_n) \) is the positive sequence given in (2.15). Thus, if \( |\gamma| \leq 1 \), then by Lemma 5.3, the a priori estimate (5.7) holds with \( K_\gamma = 1 \). \( \square \)

The latter assertion of this proposition may also be explained by the following perturbation argument: Let \( X \) be a semimartingale which decomposes as \( X_t = B_t + V_t \) with \( V \) of finite variation satisfying \( E[(\int_0^1 |dV_u|)^2] < \infty \). As we have seen in Lemma 2.6, it holds that, for every bounded, measurable function \( f \) on \([0, 1]\),

\[
E[I(f; X)^2] = \|f\|_2^2 - 3_X(f), \quad 3_X(f) := -2E[\int_0^1 f(u) dV_u \int_0^u f(s) dX_s].
\]

Now we suppose \( 3_X(f) \geq 0 \). As we have already seen, there are a number of such situations. For \( \eta \in \mathbb{R} \), define

\[
X^\eta_t = B_t + \eta V_t.
\]

Then it is easily seen that \( E[I(f; X^\eta)^2] = \|f\|_2^2 - 3_{X^\eta}(f) \) with

\[
J_{X^\eta}(f) = \eta 3_X(f) - \eta(\eta - 1)E[\left( \int_0^1 f(u) dV_u \right)^2]. \tag{5.9}
\]

Indeed, by Lemma 2.6,

\[
3_{X^\eta}(f) = -2E[\int_0^1 f(u) dV_u \int_0^u f(s) dX_s^\eta] \\
= -2\eta E[\int_0^1 f(u) dV_u \int_0^u f(s) (dX_u + (\eta - 1)dV_u)] \\
= \eta 3_X(f) - 2\eta(\eta - 1)E[\int_0^1 f(u) dV_u \int_0^u f(s) dV_s],
\]

hence (5.9) follows. From this, it is clear that,

\[
J_{X^\eta}(f) \geq 0 \quad \text{if} \ 0 \leq \eta \leq 1.
\]

Now we turn to the case that we have just discussed above; we may recover the latter assertion of Proposition 5.2 by using (5.10):
Alternative proof of the latter assertion of Proposition 5.2. We rewrite the decomposition (5.8) above as
\[ X_t^{(\gamma)} = B_t + \eta \times (2L_t), \quad \text{with } \eta = \frac{1 + \gamma}{2}. \]
In the case \( \eta = 1 \) (\( \iff \gamma = 1 \)), it is already seen that \( J_{X_t^{(\gamma)}}(f) \geq 0 \) since \( X^{(1)} \) is BES(3). So, by the above perturbation result (5.10), we may conclude that \( J_{X_t^{(\gamma)}}(f) \geq 0 \) when
\[ 0 \leq \frac{1 + \gamma}{2} \leq 1 \quad (\iff -1 \leq \gamma \leq 1), \]
which is nothing but the latter assertion of Proposition 5.2.

5.3 On the applicability of the FKG inequality to BES(\( \delta \)), \( \delta > 0 \)
In this part, we prove that the FKG inequality is applicable to the laws of BES(\( \delta \)), \( \delta > 0 \), (or, more precisely, to their finite-dimensional marginals). For the formulation of the FKG inequality, we refer to \([14, 15]\).

For \( t > 0 \) and \( x_1, x_2 > 0 \), let \( p^\delta(t; x_1, x_2) \) denote the transition density function of BES(\( \delta \)):
\[ p^\delta(t; x_1, x_2) = t^{-1} x_1^{-\nu} x_2^{\nu+1} \exp\left(-\frac{x_1^2 + x_2^2}{2t}\right) I_\nu\left(\frac{x_1 x_2}{t}\right). \]
Here \( \nu = \delta/2 - 1 \) (> -1), the index of BES(\( \delta \)). Set
\[ J_\nu(x) = \frac{x I_{\nu+1}(x)}{I_\nu(x)}, \quad x > 0. \]

**Lemma 5.4.** If \( \nu > -1 \), then \( J_\nu \) is non-decreasing.

**Proof.** We compute the derivative:
\[
\frac{d}{dx} J_\nu(x) = \frac{d}{dx} \frac{x^{\nu+1} I_{\nu+1}(x)}{x^\nu I_\nu(x)}
= \frac{1}{\{x^\nu I_\nu(x)\}^2} \left\{ x^{\nu+1} I_\nu(x) x' I_\nu(x) - x^{\nu+1} I_{\nu+1}(x) x' I_{\nu-1}(x) \right\}
= \frac{x^{2\nu+1}}{\{x^\nu I_\nu(x)\}^2} \left\{ I_\nu(x)^2 - I_{\nu+1}(x) I_{\nu-1}(x) \right\},
\]
where we used the recurrence relation: \( \{x^\mu I_\mu(x)\}' = x^\mu I_{\mu-1}(x) \). By \([11, (9.t5)]\), the last quantity is non-negative for \( \nu > -1 \). This shows the lemma.


\[ \square \]
Remark 5.2. It is known (e.g., [20, (11.27)]) that, for $\nu > -1$,
\[
\frac{1}{x^2}J_\nu(x) = 2\sum_{n=1}^{\infty} \frac{1}{x^2 + j_{\nu,n}^2},
\]
where $(j_{\nu,n})$ is the simple, positive zeros of $J_\nu$, the Bessel function of the first kind of order $\nu$. From here, we may also deduce Lemma 5.4.

Lemma 5.5. For each fixed $t > 0$, it holds that
\[
(5.11) \quad p^\delta(t; x_1 \vee y_1, x_2 \vee y_2)p^\delta(t; x_1 \wedge y_1, x_2 \wedge y_2) \geq p^\delta(t; x_1, x_2)p^\delta(t; y_1, y_2)
\]
for all $(x_1, x_2), (y_1, y_2) \in (0, \infty) \times (0, \infty)$. Here $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$.

Proof. We divide the case into four cases: (i) $x_1 \geq y_1, x_2 \geq y_2$; (ii) $x_1 \leq y_1, x_2 \leq y_2$; (iii) $x_1 \geq y_1, x_2 \leq y_2$; (iv) $x_1 \leq y_1, x_2 \geq y_2$. In both cases (i) and (ii), (5.11) holds as an equality. So, by symmetry, we only need to consider either (iii) or (iv). Here we give a proof in the case (iii). By the definition of $p^\delta(t; x, y)$, the proof is reduced to showing the following: for $x_1 \geq y_1$ and $x_2 \leq y_2$,
\[
(5.12) \quad I_\nu(x_1/x_2)I_\nu(y_2/y_1) \geq I_\nu(x_1/x_2)I_\nu(y_1/y_2).
\]
Rewriting (5.12) as
\[
\frac{I_\nu(y_2/x_1)}{I_\nu(y_2/x_1)} \geq \frac{I_\nu(y_2/y_1)}{I_\nu(y_2/y_1)},
\]
we see that it suffices to prove, for $\beta > \alpha > 0$,
\[
(5.13) \quad \frac{I_\nu(\beta x)}{I_\nu(\alpha x)} \text{ is non-decreasing in } x > 0.
\]
To this end, we compute:
\[
\frac{d}{dx} \left\{ \frac{I_\nu(\beta x)}{I_\nu(\alpha x)} \right\} \times \frac{\beta^{-\nu}}{\alpha^{-\nu}} = \frac{d}{dx} \left\{ \frac{(\beta x)^{-\nu}I_\nu(\beta x)}{(\alpha x)^{-\nu}I_\nu(\alpha x)} \right\} = \frac{(\alpha^2 x^{-\nu})}{((\alpha x)^{-\nu}I_\nu(\alpha x))^2} \left\{ \beta I_{\nu+1}(\beta x)I_\nu(\alpha x) - \alpha I_\nu(\beta x)I_{\nu+1}(\alpha x) \right\},
\]
where, for the equality, we used the recurrence relation: $\{x^{-\nu}I_\mu(x)\}' = x^{-\nu}I_{\mu+1}(x)$. We rewrite the last factor as:
\[
\beta I_{\nu+1}(\beta x)I_\nu(\alpha x) - \alpha I_\nu(\beta x)I_{\nu+1}(\alpha x) = \frac{I_\nu(\alpha x)I_\nu(\beta x)}{x} \{J_\nu(\beta x) - J_\nu(\alpha x)\}
\]
to see that this is non-negative by Lemma 5.4. Hence
\[
\frac{d}{dx} \left\{ \frac{I_\nu(\beta x)}{I_\nu(\alpha x)} \right\} \geq 0,
\]
which shows (5.13). So the lemma is proved. \qed
Let $\Delta = \{0 < t_1 < \cdots < t_n\}$ be an increasing sequence in $[0, \infty)$. For $a > 0$, we denote by $\Phi^\delta_\Delta(x; a)$ \((x = (x_i)_{1 \leq i \leq n})\) the finite-dimensional distribution function of $\text{BES}(\delta)$ starting from $a$ taken at the time sequence $(t_i)_{1 \leq i \leq n}$:

\[ \Phi^\delta_\Delta(x; a) = p^\delta(t_1; a, x_1)p^\delta(t_2 - t_1; x_1, x_2) \cdots p^\delta(t_n - t_{n-1}; x_{n-1}, x_n) \tag{5.14} \]

The next proposition shows $\Phi^\delta_\Delta(\cdot; a)$ fulfills the assumption of [14, Theorem 3]:

**Proposition 5.6.** For $a \geq a' > 0$, it holds that

\[ \Phi^\delta_\Delta(x \lor y; a) \Phi^\delta_\Delta(x \land y; a') \geq \Phi^\delta_\Delta(x; a) \Phi^\delta_\Delta(y; a') \]

for all $x = (x_i)_{1 \leq i \leq n} \in (0, \infty)^n$ and $y = (y_i)_{1 \leq i \leq n} \in (0, \infty)^n$. Here $x \lor y = (x_i \lor y_i)_{1 \leq i \leq n}$ and $x \land y = (x_i \land y_i)_{1 \leq i \leq n}$.

**Proof.** The assertion is immediate from Lemma 5.5 and (5.14).

**Remark 5.3.** It is easily checked that the assertion of this lemma still holds even if $a'$ is equal to 0; in that case, $\Phi^\delta_\Delta(x; a')$ should be replaced by

\[ \Phi^\delta_\Delta(x; 0) = \tilde{p}^\delta(t_1; x_1)p^\delta(t_2 - t_1; x_1, x_2) \cdots p^\delta(t_n - t_{n-1}; x_{n-1}, x_n), \]

where

\[ \tilde{p}^\delta(t; x) = 2^{-\nu} t^{-(\nu+1)} \Gamma(\nu + 1)^{-1} x^{2\nu+1} \exp\left(-\frac{x^2}{2t}\right). \]

**Remark 5.4.** There is a probabilistic understanding of (5.13): For $x > 0$, we denote $T_x(R) = \inf\{t > 0; R_t = x\}$. It is well-known (e.g., [20, (11.31)]) that, for every dimension $\delta > 0$, and for $\gamma > 0$,

\[ E_\nu^{(\delta)}[\exp(-\frac{\gamma^2}{2} T_x(R))] = \frac{(x\gamma)^\nu}{2^\nu \Gamma(1+\nu) I_\nu(x\gamma)}. \]

Therefore, for $0 < y < x$, we get, from the strong Markov property,

\[ E_\nu^{(\delta)}[\exp(-\frac{\gamma^2}{2} T_x(R))] = E_\nu^{(\delta)}[\exp(-\frac{\gamma^2}{2} T_y(R))] E_y^{(\delta)}[\exp(-\frac{\gamma^2}{2} T_x(R))], \]

hence

\[ \frac{I_\nu(y\gamma)}{I_\nu(x\gamma)} = \left(\frac{y}{x}\right)^\nu E_y^{(\delta)}[\exp(-\frac{\gamma^2}{2} T_x(R))], \]

which indicates that, for $0 < y < x$, the function $\gamma \mapsto I_\nu(y\gamma)/I_\nu(x\gamma)$ is strictly decreasing.
References


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