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Abstract

In [3], Kotani proved analytically that expectations for additive functionals of Brownian motion $\{B_t, t \geq 0\}$ of the form

$$E_0[f(B_t)g(\int_0^t \varphi(B_s)\,ds)]$$

have the asymptotics $t^{-3/2}$ as $t \to \infty$ for some suitable non-negative functions φ, f and g. This generalizes, in the asymptotic form, Yor's explicit formula [9] for exponential Brownian functionals.

In the present paper, we discuss this generalization probabilistically, by using a time-change argument. We may easily see from our argument that this asymptotics $t^{-3/2}$ comes from the transition probability of 3-dimensional Bessel process.

1 Introduction

Let $(B = \{B_t, t \geq 0\}, P_x)$ be a one-dimensional Brownian motion starting from x: $P_x(B_0 = x) = 1$. Yor's formula for exponential additive functionals of Brownian motion states that, for all non-negative Borel-measurable functions f and g,

$$E_0[f(B_t)g(\int_0^t e^{-2B_s}\,ds)] = \int_{\mathbb{R}} dx \int_0^\infty \frac{dy}{y} f(x)g(y)\exp(-\frac{1+e^{2x}}{2y})\theta(\frac{e^x}{y}, t). \tag{1.1}$$

See [9, formula (6.e)]; we also refer to [1]. Here, for fixed $z > 0$, $\theta(z, \cdot)$ denotes the density of the so-called Hartman-Watson distribution, whose integral representation is

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obtained in [8, Théorème (5.4)]. It is noted in [1] that $\lim_{t\to\infty}\sqrt{2\pi t^3}\theta(z,t) = K_0(z)$, the Macdonald function of order 0. From these, we may deduce that, for some suitable functions f and g, the expectation as on the LHS of (1.1) has the asymptotics $t^{-3/2}$ as $t\to\infty$.

Later in [3], Kotani proved the same asymptotics for more general additive functionals, replacing e^{-2x} by $\varphi(x)\ge 0$ satisfying certain conditions. He employed an analytic approach, namely the Krein theory, in doing this.

In this paper, we deal with the same problem. Our approach employed here is a probabilistic one. Although we only discuss here the case where g is given by $g(x) = \exp(-x)$, we think that our approach provides us with a simpler way to understand why such an asymptotics appears even for general additive functionals, and that it is worthwhile to present it; we may easily deduce from our argument that the asymptotics $t^{-3/2}$ comes from the transition probability of 3-dimensional Bessel process:

$$\sqrt{2\pi t^3}P_x^{(3)}(R_t\in dz)\to 2z^2\,dz,\quad t\to\infty.$$

We assume $\varphi(x)\ge 0$ $(x\in\mathbb{R})$ is locally integrable and satisfies:

$$\text{(P1)}\quad \int^{\infty} x\varphi(x)\,dx<\infty,\qquad\qquad \text{(P2)}\quad \liminf_{x\to-\infty}\varphi(x)>0.$$

We denote by f_0 the unique, strictly positive solution to the Sturm-Liouville equation

$$\frac{1}{2}f''(x)=\varphi(x)f(x) \tag{1.2}$$

with boundary conditions

$$f'(x)\to 1\quad (x\to\infty)\quad\text{and}\quad f(x)\to 0\quad (x\to-\infty). \tag{1.3}$$

The existence and uniqueness of such a solution is ensured by the above assumptions on φ.

Remark 1.1. By (P2), there exist constants $a<0$ and $c,c'>0$ such that

$$f_0(x)\le c'e^{-c|x|}\quad\text{for all } x<a.$$

See Remark 2.1.

Let f be a non-negative function on $\mathbb{R}$ satisfying

$$\text{(A)}\quad \int_{\mathbb{R}} f(z)f_0(z)\,dz<\infty.$$

The purpose of this paper is to prove the following limit theorem: for every $x\in\mathbb{R}$,

$$\lim_{t\to\infty}\sqrt{2\pi t^3}E_x[f(B_t)\exp\{-\int_0^t\varphi(B_s)\,ds\}] = 2f_0(x)\int_{\mathbb{R}} f(z)f_0(z)\,dz. \tag{$*$}$$

We shall show that $(*)$ holds under some additional condition on f. Although we only discuss the simple case with $g(x) = \exp(-x)$, an assumption on f imposed in [3] is relaxed somewhat; indeed, in some case, we only need the minimal assumption (A) for $(*)$ to hold.

To state the result, we introduce the exponent $\gamma_0 \geq 0$ defined by:

$$\gamma_0 = \inf\{\gamma \geq 0; \liminf_{x\to-\infty} |x|^{-2\gamma}\varphi(x) > 0\}.$$

Theorem 1.1. (i) *The case* $\gamma_0 \leq 1$*: Assume* (A)*. Moreover, we assume*

$$\text{(B)} \qquad \int_{-\infty} |z| f(z) f_0(z)\, dz < \infty.$$

Then $(*)$ *holds.*
(ii) *The case* $\gamma_0 > 1$*: Assume* (A)*. Then* $(*)$ *holds.*

Remark 1.2. In [3], it is assumed that, in the present setting,

$$\int_{-\infty} |z|^{3/2} f(z) f_0(z)\, dz < \infty$$

for both cases (i) and (ii).

Remark 1.3. If, in particular, $\varphi(x) = O(|x|^\gamma)$ as $x \to -\infty$ for some $0 < \gamma \leq 1$, then the condition (B) can be relaxed as:

$$\text{(B}'\text{)} \qquad \begin{cases} \int_{-\infty} |z|^{1-\gamma} f(z) f_0(z)\, dz < \infty & \text{for } \gamma < 1, \\ \int_{-\infty} (\log|z|) f(z) f_0(z)\, dz < \infty & \text{for } \gamma = 1. \end{cases}$$

We may easily deduce this from our argument used in the proof of Theorem 1.1. See, in particular, the proof of Lemma 3.6.

As a corollary to Theorem 1.1, we also see:

Corollary 1.1. *Under the same assumption as in Theorem 1.1, we have, for all* $x \in \mathbb{R}$,

$$\lim_{t\to\infty} \sqrt{t} \int_t^\infty ds\, E_x[f(B_s) \exp\{-\int_0^s \varphi(B_u)\, du\}] = \frac{4}{\sqrt{2\pi}} f_0(x) \int_{\mathbb{R}} f(z) f_0(z)\, dz. \qquad (1.4)$$

Note that the assertion is also a rewriting of Proposition 3.1. We give some remark on this corollary in Section 4.

As an application of Theorem 1.1, we give two examples; in both examples, we take $f(x) = e^{-\mu x}$ ($\mu > 0$), which means, by the Cameron-Martin relation, we may rewrite the assertions using the Brownian motion with drift $B^{(-\mu)} = \{B_t - \mu t, t \geq 0\}$ instead of the Brownian motion.

Example 1.1. For $\alpha > 0$, we take $\varphi(x) = \alpha e^{-2x}$. In this case f_0 is given by

$$f_0(x) = K_0(\sqrt{2\alpha}e^{-x}),$$

where K_0 denotes the Macdonald function of order 0. Using one of its integral representations (see, e.g., [4, formula (5.10.25)]), we may easily see:

$$\int_{\mathbb{R}} e^{-\mu x} K_0(\sqrt{2\alpha}e^{-x})\, dx = 2^{\mu-2} \frac{1}{(\sqrt{2\alpha})^{\mu}} \{\Gamma(\frac{\mu}{2})\}^2.$$

Note that, in this case, we may apply (ii) of Theorem 1.1 and obtain

$$\lim_{t\to\infty} \sqrt{2\pi t^3} e^{\mu^2 t/2} E_x[\exp\{-\alpha \int_0^t e^{-2B_s^{(-\mu)}}\, ds\}] = 2^{\mu-1} \{\Gamma(\frac{\mu}{2})\}^2 e^{\mu x} \frac{K_0(\sqrt{2\alpha}e^{-x})}{(\sqrt{2\alpha})^{\mu}}.$$

This asymptotics has already been discussed in [2, Theorem 2.1], where Yor's formula was used.

Example 1.2. We take $\varphi(x) = \beta \mathbf{1}_{(-\infty,0)}(x)$ for $\beta > 0$. In this case f_0 is given by

$$f_0(x) = \begin{cases} x + \frac{1}{\sqrt{2\beta}}, & x \geq 0, \\ \frac{1}{\sqrt{2\beta}} e^{-\sqrt{2\beta}|x|}, & x \leq 0. \end{cases}$$

Note that, if $\mu < \sqrt{2\beta}$, then

$$\int_{\mathbb{R}} e^{-\mu x} f_0(x)\, dx = \frac{\sqrt{2\beta}}{\mu^2(\sqrt{2\beta} - \mu)} < \infty,$$

and the assumption (B) is also fulfilled. Therefore, by (i) of Theorem 1.1, we have, for $\mu < \sqrt{2\beta}$,

$$\lim_{t\to\infty} \sqrt{2\pi t^3} e^{\mu^2 t/2} E_x[\exp\{-\beta \int_0^t \mathbf{1}_{(-\infty,0)}(B_s^{(-\mu)})\, ds\}] = \frac{2\sqrt{2\beta}}{\mu^2(\sqrt{2\beta} - \mu)} e^{\mu x} f_0(x).$$

The organization of this paper is as follows: in Section 2, we do some preliminaries; in Subsection 3.a, we prove Theorem 1.1; in Subsections 3.b and 3.c, we prove two propositions that are used in the proof of Theorem 1.1; in Section 4, we give some remark on a connection between our result and a related one in [6].

Throughout this paper, $R = \{R_t, t \geq 0\}$, together with a probability measure $P_x^{(3)}$, denotes a 3-dimensional Bessel process starting from x: $P_x^{(3)}(R_0 = x) = 1$, and $E_x^{(3)}$ denotes the expectation with respect to $P_x^{(3)}$. Other notation will be introduced as needed.

2 Preliminaries

In this section, we prepare several preliminary results.

2.a. *h*-transform with respect to f_0. Let X be the solution to the following SDE:

$$X_t = x + W_t + \int_0^t \frac{f_0'}{f_0}(X_s)\,ds, \quad t \geq 0,\ x \in \mathbb{R}, \tag{2.1}$$

where W is a standard one-dimensional Brownian motion. We denote by $\mathbb{P}_x$ the probability measure on the path space $C([0,\infty);\mathbb{R})$, induced by X. For every $t > 0$ and every non-negative, measurable functional $F(w(s), s \leq t)$ ($w \in C([0,\infty);\mathbb{R})$), it holds that, by the Girsanov theorem,

$$\mathbb{E}_x[F(w(s), s \leq t)] = E_x[F(B_s, s \leq t)\frac{f_0(B_t)}{f_0(x)}\exp\{-\int_0^t \varphi(B_s)\,ds\}].$$

From this relation, we have in particular

$$E_x[f(B_t)\exp\{-\int_0^t \varphi(B_s)\,ds\}] = f_0(x)\mathbb{E}_x[\frac{f}{f_0}(X_t)]. \tag{2.2}$$

Here we made the abuse of notation, letting X denote the canonical path in $C([0,\infty);\mathbb{R})$ under $\mathbb{P}_x$.

2.b. Time-change. Since $f_0'(x) \to 1$ as $x \to \infty$, the drift term $(f_0'/f_0)(x)$ of the SDE (2.1) behaves as $1/x$ when $x \to \infty$. So we may expect the solution X_t to behave asymptotically as 3-dimensional Bessel process as $t \to \infty$. To formulate this intuition mathematically, we shall consider expressing X as a time-change of a 3-dimensional Bessel process. For this purpose, we define the function g_0 by

$$g_0(x) = \left\{\int_x^\infty \frac{dy}{f_0(y)^2}\right\}^{-1}, \quad x \in \mathbb{R}.$$

By using the inverse function g_0^{-1} of g_0, X is expressed as:

$$X_t = g_0^{-1}(R_{a_t(R)}) \tag{2.3}$$

for some 3-dimensional Bessel process R starting from $y = g_0(x) > 0$. Here

$$a_t(R) = \inf\{s \geq 0; A_s(R) > t\},$$
$$A_s(R) = \int_0^s |(g_0^{-1})'(R_u)|^2\,du.$$

Since $(g_0^{-1})'(x) \geq 1$ and converges to 1 as $x \to \infty$ (see Lemma 2.1 below), we see that, $P_y^{(3)}$-a.s.,

$$A_s(R) \geq s \quad \text{for all } s \geq 0 \qquad \text{and} \qquad A_s(R)/s \to 1 \quad \text{as } s \to \infty. \tag{2.4}$$

The latter follows from L'Hospital's rule and the fact that R is transient. Since $a_t(R)$ is the inverse of $A_s(R)$, we also see that, $P_y^{(3)}$-a.s.,

$$a_t(R) \le t \quad \text{for all } t \ge 0 \qquad\qquad \text{and} \qquad\qquad a_t(R)/t \to 1 \quad \text{as } t \to \infty.$$

The latter property, in particular, combined with (2.3) and the fact that $(g_0^{-1})'(x) \to 1$ as $x \to \infty$, does indicate that X_t behaves as R_t as $t \to \infty$.

2.c. Key identity. By (2.2), we are led to study the asymptotics of $\mathbb{E}_x[\frac{f}{f_0}(X_t)]$ instead of that of $E_x[f(B_t)\exp\{-\int_0^t \varphi(B_s)\,ds\}]$ itself. The key to doing this is the following identity:

$$\int_0^t \frac{f}{f_0}(X_s)\,ds = \int_0^{a_t(R)} \frac{f}{f_0}(g_0^{-1}(R_s))|(g_0^{-1})'(R_s)|^2\,ds. \tag{2.5}$$

To see that this relation holds, we differentiate the RHS with respect to t, noting $\frac{d}{dt}a_t(R) = |(g_0^{-1})'(R_{a_t(R)})|^{-2}$:

$$\begin{aligned}\frac{d}{dt}(\text{RHS of (2.5)}) &= \frac{f}{f_0}((g_0^{-1})'(R_{a_t(R)}))|(g_0^{-1})'(R_{a_t(R)})|^2\frac{d}{dt}a_t(R)\\ &= \frac{f}{f_0}(g_0^{-1}(R_{a_t(R)}))\\ &= \frac{f}{f_0}(X_t), \qquad\qquad \text{(by (2.3))}\end{aligned}$$

which implies (2.5).

2.d. Properties of g_0. We summarize here several properties of g_0 in a lemma. Some of them were already referred to above.

Lemma 2.1. (i) $\lim_{x\to\infty} g_0'(x) = 1$, $\lim_{x\to-\infty} g_0(x) = 0$.
(ii) g_0 *is convex.*
(iii) $(g_0^{-1})'(x) \ge 1$, *is non-increasing, and converges to 1 as* $x \to \infty$.
(iv) $g_0 \ge f_0 f_0'$.
(v) $\limsup_{x\downarrow 0} x(g_0^{-1})'(x) < \infty$.

Before giving a proof, we give an example:

Example 2.1 *(recall Example 1.2).* In the case $\varphi(x) = \beta\mathbf{1}_{(-\infty,0)}(x)$ for $\beta > 0$, g_0 and $(g_0^{-1})'$ are given respectively by:

$$g_0(x) = \begin{cases} x + \frac{1}{\sqrt{2\beta}}, & x \ge 0,\\ \frac{2}{\sqrt{2\beta}}\frac{1}{1+\exp(-2\sqrt{2\beta}x)}, & x \le 0;\end{cases}$$

$$(g_0^{-1})'(x) = \begin{cases} \frac{1}{\sqrt{2\beta}}\frac{1}{x(2-\sqrt{2\beta}x)}, & 0 < x \le \frac{1}{\sqrt{2\beta}},\\ 1, & x \ge \frac{1}{\sqrt{2\beta}}.\end{cases}$$

Note that $x(g_0^{-1})'(x) \to 1/(2\sqrt{2\beta})$ as $x \downarrow 0$.

Proof of Lemma 2.1. The latter assertion of (i) is obvious. For the former, note that $g_0' = (g_0/f_0)^2$. So it suffices to check $f_0(x)/g_0(x) \to 1$ as $x \to \infty$, which is immediate from L'Hospital's rule:

$$\lim_{x\to\infty} \frac{f_0(x)}{g_0(x)} = \lim_{x\to\infty} \frac{\left(\int_x^\infty \frac{dy}{f_0(y)^2}\right)'}{\left(\frac{1}{f_0(x)}\right)'} = \lim_{x\to\infty} \frac{1}{f_0'(x)} = 1.$$

Now we set $h_0 = f_0/g_0$. We have just seen $h_0(x) \to 1$ as $x \to \infty$. Note that h_0 also satisfies $(1/2)h_0'' = \varphi h_0$ (in fact, h_0 gives a solution to (1.2) linearly independent of f_0). This indicates, in particular, that h_0 is convex. Combining these, we see that $h_0 \geq 1$ and is non-increasing. Properties (ii)–(iv) are variants of this fact on h_0, so we omit the proof. For (v), first note that, by the condition (P2) on φ, there exist $a < 0, c > 0$ such that $\varphi \geq c$ on $(-\infty, a)$. Therefore $f_0'' = 2\varphi f_0 \geq 2cf_0$ on $(-\infty, a)$. Multiplying both sides by $f_0' > 0$ and integrating over $(-\infty, x)$ for $x < a$, we get $f_0'(x)^2 \geq 2cf_0(x)^2$, hence

$$\frac{f_0'(x)}{f_0(x)} \geq \sqrt{2c} \quad \text{for all } x < a. \tag{2.6}$$

Noting $(g_0^{-1})'(x) = 1/g_0'(g_0^{-1}(x)) = f_0(g_0^{-1}(x))^2/x^2$, we see that

$$\limsup_{x\downarrow 0} x(g_0^{-1})'(x) = \limsup_{y\to-\infty} \frac{f_0(y)^2}{g_0(y)} \leq \limsup_{y\to-\infty} \frac{f_0(y)}{f_0'(y)} \leq \frac{1}{\sqrt{2c}},$$

where we used the property (iv) for the first inequality and (2.6) for the second. This shows (v). □

Remark 2.1. From (2.6), we may see that, as $x \to -\infty$, f_0 decays exponentially or faster; indeed, by (2.6),

$$\log \frac{f_0(a)}{f_0(x)} = \int_x^a \frac{f_0'(y)}{f_0(y)}\, dy \geq \sqrt{2c}(a - x), \quad x < a,$$

which is rewritten as

$$f_0(x) \leq f_0(a)e^{\sqrt{2c}(x-a)}, \quad x < a.$$

2.e. Proof of (2.3)**.** Before closing this section, we prove the time-change relation (2.3) for the sake of completeness of the paper.

By definition, it is easily checked that

$$\frac{1}{2}g_0''(x) + \frac{f_0'}{f_0}(x)g_0'(x) = \frac{g_0'(x)^2}{g_0(x)}.$$

So, by Itô's formula,

$$g_0(X_t) = y + \int_0^t g_0'(X_s)\,dW_s + \int_0^t \frac{g_0'(X_s)^2}{g_0(X_s)}\,ds, \tag{2.7}$$

where, as before, we write $y = g_0(x)$. Since the second term on the RHS is a martingale, there exists a Brownian motion $\widetilde{W}$ such that

$$\int_0^t g_0'(X_s)\,dW_s = \widetilde{W}_{G_t(X)}, \quad G_t(X) = \int_0^t g_0'(X_s)^2\,ds.$$

Now we prepare the 3-dimensional Bessel process R that is given as the strong solution to the following SDE driven by $\widetilde{W}$:

$$R_t = y + \widetilde{W}_t + \int_0^t \frac{ds}{R_s}.$$

Note that $R_{G_t(X)}$ satisfies:

$$\begin{aligned} R_{G_t(X)} &= y + \widetilde{W}_{G_t(X)} + \int_0^{G_t(X)} \frac{ds}{R_s} \\ &= y + \int_0^t g_0'(X_s)\,dW_s + \int_0^t \frac{g_0'(X_s)^2}{R_{G_s(X)}}\,ds. \end{aligned}$$

Comparing this with (2.7), we conclude the following relation:

$$g_0(X_t) = R_{G_t(X)}. \tag{2.8}$$

It remains to prove $G_t(X) = a_t(R)$. Since $a_t(R)$ is the inverse of $A_s(R)$, it suffices to check $A_{G_t(X)}(R) = t$. To this end, we compute:

$$\begin{aligned} \frac{d}{dt} A_{G_t(X)}(R) &= |(g_0^{-1})'(R_{G_t(R)})|^2 \frac{d}{dt} G_t(X) && \text{(by definition)} \\ &= |g_0'(g_0^{-1}(R_{G_t(R)}))|^{-2} g_0'(X_t)^2 \\ &= g_0'(X_t)^{-2} g_0'(X_t)^2 && \text{(by (2.8))} \\ &= 1, \end{aligned}$$

which implies $A_{G_t(X)}(R) = t$. Here, for the second line, we used the relation $(g_0^{-1})' = 1/g_0'(g_0^{-1})$. Now (2.3) is proved.

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

3.a. Proof of Theorem 1.1. We begin with the following lemma.

Lemma 3.1. *Let $k(\xi)$ $(\xi > 0)$ be a non-negative, locally integrable function satisfying*

$$\int_{0+} \xi^2 k(\xi)\, d\xi < \infty \quad \text{and} \quad \int^{\infty} \xi k(\xi)\, d\xi < \infty.$$

Then it holds that, for all $y > 0$,

$$E_y^{(3)}[\int_0^{\infty} k(R_s)\, ds] < \infty.$$

Proof. The assertion is immediate from Fubini's theorem and the fact that

$$\int_0^{\infty} ds\, P_y^{(3)}(R_s \in d\xi) = \frac{2\xi}{y}(\xi \wedge y)\, d\xi.$$

□

Now we take $k(\xi) = \frac{f}{f_0}(g_0^{-1}(\xi))|(g_0^{-1})'(\xi)|^2$. Then the assumption of Lemma 3.1 is fulfilled; indeed, by making the change of variables with $\xi = g_0(z)$,

$$\int_0^{\infty} \xi^2 k(\xi)\, d\xi = \int_{\mathbb{R}} f(z) f_0(z)\, dz, \tag{3.1}$$

which is finite by (A). Applying Lemma 3.1 to this k, we see in particular that, for each $y > 0$,

$$E_y^{(3)}[\int_{a_t(R)}^{\infty} k(R_s)\, ds] < \infty, \quad t \geq 0.$$

Note that, since $a_t(R) \to \infty$ as $t \to \infty$ $P_y^{(3)}$-a.s., the LHS converges to 0 as $t \to \infty$.

Proposition 3.1. *Under the same assumption as in Theorem 1.1, it holds that, as $t \to \infty$,*

$$\sqrt{t} E_y^{(3)}[\int_{a_t(R)}^{\infty} k(R_s)\, ds] \to \frac{4}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z) f_0(z)\, dz.$$

A key step to showing Proposition 3.1 is:

Lemma 3.2. *We have the following decomposition:*

$$E_y^{(3)}[\int_{a_t(R)}^{\infty} k(R_s)\, ds] = I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_t^{\infty} ds\, E_y^{(3)}[k(R_s)], \qquad I_2(t) = \int_0^t ds\, E_y^{(3)}[\mathbf{1}_{\{A_s(R) > t\}} k(R_s)].$$

Proof. By the definition of $a_t(R)$ and by Fubini's theorem,

$$E_y^{(3)}[\int_{a_t(R)}^{\infty} k(R_s)\,ds] = E_y^{(3)}[\int_{\{s;A_s(R)>t\}} k(R_s)\,ds]$$
$$= \int_0^{\infty} ds\, E_y^{(3)}[\mathbf{1}_{\{A_s(R)>t\}} k(R_s)].$$

Now the assertion follows from the fact that $A_s(R) \geq s$ for all $s \geq 0$ (recall (2.4)). □

We have the following two propositions concerning this decomposition:

Proposition 3.2. *Under the assumption* (A)*,*

$$\sqrt{t} I_1(t) \to \frac{4}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z) f_0(z)\,dz \quad \text{as } t \to \infty.$$

Proposition 3.3. *Under the same assumption as in Theorem 1.1,*

$$\sqrt{t} I_2(t) \to 0 \quad \text{as } t \to \infty.$$

Proofs are given in Subsections 3.b and 3.c, respectively. We now easily see Proposition 3.1 follows from these:

Proof of Proposition 3.1. The assertion is an immediate consequence of Lemma 3.2, Propositions 3.2 and 3.3. □

Using Proposition 3.1, we prove Theorem 1.1:

Proof of Theorem 1.1. By the relation (2.5), we have, for each $x \in \mathbb{R}$,

$$\int_t^{\infty} \mathbb{E}_x[\frac{f}{f_0}(X_s)]\,ds = E_y^{(3)}[\int_{a_t(R)}^{\infty} k(R_s)\,ds], \quad t \geq 0.$$

Here, as before, $y = g_0(x)$. Then, by Proposition 3.1, we have

$$\int_t^{\infty} \mathbb{E}_x[\frac{f}{f_0}(X_s)]\,ds \sim t^{-1/2} \times \frac{4}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z) f_0(z)\,dz \quad \text{as } t \to \infty.$$

Here and below, for positive functions $\alpha(t), \beta(t)$ $(t > 0)$, we use the notation $\alpha(t) \sim \beta(t)$ as $t \to \infty$ to mean $\lim_{t\to\infty} \alpha(t)/\beta(t) = 1$. Since the convergence of the LHS to 0 is monotone, we may differentiate both sides with respect to t to get

$$\mathbb{E}_x[\frac{f}{f_0}(X_t)] \sim t^{-3/2} \times \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z) f_0(z)\,dz \quad \text{as } t \to \infty.$$

Now the theorem follows from this and the relation (2.2). □

The rest of the section is devoted to proving Propositions 3.2 and 3.3. In the following, every argument is done for an arbitrarily fixed $y > 0$, which means it is not necessary to relate y to the starting point of the Brownian motion B in such a way as $y = g_0(x)$. So we use below x to denote a variable, not the starting point.

3.b. Proof of Proposition 3.2. Here we prove Proposition 3.2.

Proof. By changing the variables with $s = tu$ in the definition of $I_1(t)$,

$$\begin{aligned}\sqrt{t}I_1(t) &= \sqrt{t} \times t \int_1^\infty du\, E_y^{(3)}[k(R_{tu})] \\ &= t^{3/2} \int_1^\infty du \int_0^\infty d\xi\, p^{(3)}(tu; y, \xi)k(\xi),\end{aligned}$$

where $p^{(3)}$ denotes the transition density of 3-dimensional Bessel process:

$$p^{(3)}(s; x, z) = \frac{1}{\sqrt{2\pi s}} \frac{z}{x} \exp\{-\frac{(z-x)^2}{2s}\}\{1 - \exp(-\frac{2xz}{s})\}, \quad s > 0,\ x, z > 0.$$

Noting the function $(1 - e^{-x})/x$ $(x > 0)$ is dominated by 1 and converges to 1 as $x \downarrow 0$, we easily see that, for each fixed u and ξ,

$$t^{3/2}p^{(3)}(tu; y, \xi) \le \frac{2\xi^2}{\sqrt{2\pi u^3}} \quad \text{for all } t > 0, \quad t^{3/2}p^{(3)}(tu; y, \xi) \to \frac{2\xi^2}{\sqrt{2\pi u^3}} \quad \text{as } t \to \infty. \tag{3.2}$$

Moreover,

$$\begin{aligned}\int_1^\infty du \int_0^\infty d\xi\, \frac{2\xi^2}{\sqrt{2\pi u^3}} k(\xi) &= \frac{2}{\sqrt{2\pi}} \int_1^\infty \frac{du}{\sqrt{u^3}} \int_0^\infty d\xi\, \xi^2 k(\xi) \\ &= \frac{4}{\sqrt{2\pi}} \int_{\mathbb{R}} dz\, f(z)f_0(z) < \infty\end{aligned}$$

by (A). The second equality follows from the relation (3.1). Now the assertion is immediate from the dominated convergence theorem. □

3.c. Proof of Proposition 3.3. Similarly to the proof of Proposition 3.2, we rewrite $\sqrt{t}I_2(t)$ as:

$$\begin{aligned}\sqrt{t}I_2(t) &= t^{3/2} \int_0^1 du \int_0^\infty P_y^{(3)}(R_{tu} \in d\xi)\, k(\xi) P_{y,tu,\xi}^{(3)}(A_{tu}(r) > t) \\ &= \int_0^1 du \int_0^\infty d\xi\, k(\xi)\psi_y(u, \xi, t),\end{aligned} \tag{3.3}$$

where we set

$$\psi_y(u, \xi, t) = t^{3/2}p^{(3)}(tu; y, \xi)P_{y,tu,\xi}^{(3)}(A_{tu}(r) > t) \tag{3.4}$$

and, for $s > 0$ and $x, z > 0$, we denote by the pair $(r = \{r_u, 0 \le u \le s\}, P^{(3)}_{x,s,z})$ a pinned 3-dimensional Bessel process over $[0, s]$ such that $P^{(3)}_{x,s,z}(r_0 = x, r_s = z) = 1$. We prove Proposition 3.3 in four steps.

Step 1. We start with the following proposition:

Proposition 3.4. *For each fixed* $0 < u < 1$ *and* $\xi > 0$*,*

$$\psi_y(u, \xi, t) \to 0 \quad \text{as } t \to \infty.$$

As was already seen in (3.2), $t^{3/2}p^{(3)}(tu; y, \xi)$ is dominated by a quantity independent of t. Therefore, rewriting the set $\{A_{tu}(r) > t\} = \{\frac{1}{tu}A_{tu}(r) > \frac{1}{u}\}$, we see the proof of Proposition 3.4 is reduced to showing the following proposition:

Proposition 3.4′ . *For each* $\varepsilon > 0$ *and* $\xi > 0$*,*

$$P^{(3)}_{y,T,\xi}(\frac{1}{T}A_T(r) > 1 + \varepsilon) \to 0 \quad \text{as } T \to \infty.$$

The proof given here relies on the fact that the FKG inequality is applicable to the laws of pinned 3-dimensional Bessel processes (see the appendix).

Lemma 3.3. *For each* $\varepsilon > 0$ *and* $x, z > 0$*,*

$$P^{(3)}_{x,T,\sqrt{T}z}(\frac{1}{T}A_T(r) \le 1 + \varepsilon) \to 1 \quad \text{as } T \to \infty.$$

In the following proof, we say that a function F defined on the path space $C([0, T]; \mathbb{R})$ is non-decreasing (resp. non-increasing) if $F(w_1) \le F(w_2)$ (resp. $F(w_1) \ge F(w_2)$) for all $w_1, w_2 \in C([0, T]; \mathbb{R})$ satisfying $w_1(t) \le w_2(t)$ for all $0 \le t \le T$.

Proof of Lemma 3.3. Since $(g_0^{-1})'$ is non-increasing, $A_T(r)$ is non-increasing in r, hence the indicator function of the set $\{\frac{1}{T}A_T(r) \le 1 + \varepsilon\}$ is non-decreasing in r. So, by the FKG inequality, we see $P^{(3)}_{x,T,\eta}(\frac{1}{T}A_T(r) \le 1 + \varepsilon)$ is non-decreasing in η. By using this, we have

$$\begin{aligned} P^{(3)}_x(\frac{1}{T}A_T(R) \le 1 + \varepsilon, R_T \le \sqrt{T}z) &= \int_0^{\sqrt{T}z} P^{(3)}_x(R_t \in d\eta)P^{(3)}_{x,T,\eta}(\frac{1}{T}A_T(r) \le 1 + \varepsilon) \\ &\le P^{(3)}_{x,T,\sqrt{T}z}(\frac{1}{T}A_T(r) \le 1 + \varepsilon)P^{(3)}_x(R_T \le \sqrt{T}z). \end{aligned}$$

Dividing both sides by $P^{(3)}_x(R_T \le \sqrt{T}z)$, we obtain:

$$\frac{P^{(3)}_x(\frac{1}{T}A_T(R) \le 1 + \varepsilon, R_T \le \sqrt{T}z)}{P^{(3)}_x(R_T \le \sqrt{T}z)} \le P^{(3)}_{x,T,\sqrt{T}z}(\frac{1}{T}A_T(r) \le 1 + \varepsilon). \tag{3.5}$$

Since, as $T \to \infty$, $A_T(R)/T \to 1$ $P_x^{(3)}$-a.s. (recall (2.4)), the convergence in probability is implied:

$$\lim_{T\to\infty} P_x^{(3)}(\frac{1}{T}A_T(R) \le 1+\varepsilon) = 1.$$

We also note that, by the scaling property,

$$\lim_{T\to\infty} P_x^{(3)}(R_T \le \sqrt{T}z) = P_0^{(3)}(R_1 \le z) > 0.$$

Combining these, we see that the LHS of (3.5) converges to 1 as $T \to \infty$, and so does the RHS. This shows the lemma. □

By using this lemma, we prove Proposition 3.4′:

Proof of Proposition 3.4′. Conditionally on $r_{T/2} = \eta$, the process $\{r_t, 0 \le t \le T\}$ is identical in law with the process $r^1 \bullet r^2$ defined by:

$$(r^1 \bullet r^2)(t) = \begin{cases} r^1(t), & 0 \le t \le \frac{T}{2}, \\ r^2(T-t), & \frac{T}{2} \le t \le T, \end{cases}$$

where r^1 (resp. r^2) is a pinned 3-dimensional Bessel process over $[0, T/2]$ with $r^1(0) = y, r^1(T/2) = \eta$ (resp. with $r^2(0) = \xi, r^2(T/2) = \eta$), and r^1 and r^2 are taken to be independent. It then holds that

$$\begin{aligned} &P_{y,T,\xi}^{(3)}(\frac{1}{T}A_T(r) > 1+\varepsilon) \\ &= \int_0^\infty P_{y,T,\xi}^{(3)}(r_{\frac{T}{2}} \in d\eta)\, P_{y,\frac{T}{2},\eta}^{(3)} \otimes P_{\xi,\frac{T}{2},\eta}^{(3)}(\frac{1}{T}A_T(r^1 \bullet r^2) > 1+\varepsilon). \end{aligned} \tag{3.6}$$

Note that the integrand on the RHS is non-increasing in η by the FKG inequality (recall the argument in the proof of Lemma 3.3). Therefore, using the FKG inequality again, we see that (3.6) is dominated by

$$\int_0^\infty P_{0,T,0}^{(3)}(r_{\frac{T}{2}} \in d\eta)\, P_{y,\frac{T}{2},\eta}^{(3)} \otimes P_{\xi,\frac{T}{2},\eta}^{(3)}(\frac{1}{T}A_T(r^1 \bullet r^2) > 1+\varepsilon). \tag{3.7}$$

Changing the variables with $\eta = \sqrt{T}z$, and noting

$$\{\frac{1}{T}A_T(r^1 \bullet r^2) > 1+\varepsilon\} \subset \{\frac{2}{T}A_{\frac{T}{2}}(r^1) > 1+\varepsilon\} \cup \{\frac{2}{T}A_{\frac{T}{2}}(r^2) > 1+\varepsilon\},$$

we see further that (3.7) is dominated by

$$\int_0^\infty P_{0,1,0}^{(3)}(r_{\frac{1}{2}} \in dz) \left\{1 - P_{y,\frac{T}{2},\sqrt{T}z}^{(3)}(\frac{2}{T}A_{\frac{T}{2}}(r^1) \le 1+\varepsilon) P_{\xi,\frac{T}{2},\sqrt{T}z}^{(3)}(\frac{2}{T}A_{\frac{T}{2}}(r^2) \le 1+\varepsilon)\right\},$$

which converges to 0 as $T \to \infty$ by Lemma 3.3. So the proposition is proved. □

Step 2. First we introduce the cut-off of $|(g_0^{-1})'|^2$:

$$\theta_y(x) = |(g_0^{-1})'(x \wedge y)|^2 - |(g_0^{-1})'(y)|^2, \quad x > 0.$$

Here $\wedge$ means the minimum. We fix $u_0 \in (0,1)$ in such a way that $u_0 < 1/|(g_0^{-1})'(y)|^2$. We divide the strip $\{(u,\xi); 0 < u < 1, \xi > 0\}$ into three regions:

$$D_1 = (0,u_0) \times (0,y), \qquad D_2 = (0,u_0) \times [y,\infty), \qquad D_3 = [u_0,1) \times (0,\infty).$$

In this step, we prove:

Proposition 3.5. *For each fixed* $0 < u < 1,\ \xi > 0$,

$$\psi_y(u,\xi,t) \le \Psi_y(u,\xi) \quad \textit{for all } t > 0,$$

where

$$\Psi_y(u,\xi) = \begin{cases} C_1 \dfrac{\xi}{\sqrt{u}} \left(\displaystyle\int_0^\xi z^2\theta_y(z)\,dz + \xi \int_\xi^y z\theta_y(z)\,dz \right) & \textit{on } D_1, \\ C_2 \dfrac{\xi}{\sqrt{u}} & \textit{on } D_2, \\ \dfrac{2\xi^2}{\sqrt{2\pi u^3}} & \textit{on } D_3, \end{cases}$$

with constants C_1, C_2 *independent of* u *and* ξ*:*

$$C_1 = 8/\{\sqrt{2\pi}y^2(1 - u_0|(g_0^{-1})'(y)|^2)\}, \qquad C_2 = C_1 \int_0^y z^2\theta_y(z)\,dz.$$

Remark 3.1. The constant C_2 above is finite; to see this, we only have to check, by the definition of θ, $\int_{0+} z^2|(g_0^{-1})'(z)|^2\,dz < \infty$, which is immediate from (v) of Lemma 2.1.

The bound on D_3 is obvious (recall (3.2)). So we keep $u < u_0$ for a while and will not indicate this unless it is necessary. Since y is fixed, we often suppress it from the notation; e.g., we write θ for θ_y. Put $tu = T$.

Lemma 3.4. *It holds that*

$$P^{(3)}_{y,T,\xi}\left(\frac{1}{T}A_T(r) > \frac{1}{u}\right) \le C_3 u E^{(3)}_{y,T,\xi}\left[\frac{1}{T}\int_0^T \theta(r_s)\,ds\right].$$

Here $C_3 = 1/(1 - u_0|(g_0^{-1})'(y)|^2)$.

Proof. Note that the following inclusions hold:

$$\begin{aligned} \{\frac{1}{T}A_T(r) > \frac{1}{u}\} &\subset \{\frac{1}{T}\int_0^T |(g_0^{-1})'(r_s \wedge y)|^2\,ds > \frac{1}{u}\} \\ &= \{\frac{1}{T}\int_0^T \theta(r_s)\,ds > \frac{1 - u|(g_0^{-1})'(y)|^2}{u}\} \\ &\subset \{\frac{1}{T}\int_0^T \theta(r_s)\,ds > \frac{1 - u_0|(g_0^{-1})'(y)|^2}{u}\}, \end{aligned}$$

Here, for the first line, we used the fact that $(g_0^{-1})'$ is non-increasing (Lemma 2.1 (iii)), and the definition of θ for the second. Now the assertion follows from Chebyshev's inequality. □

By using this lemma, we shall prove:

Lemma 3.5. *$\psi(u,\xi,t)$ is dominated by*

$$C_3 \frac{\xi}{y\sqrt{2\pi u}} \int_0^y dz\, \theta(z) \int_{|z-\xi|}^{z+\xi} da \left(\exp\{-\frac{(a+y-z)^2}{2T}\} - \exp\{-\frac{(a+y+z)^2}{2T}\} \right).$$

Proof. By Lemma 3.4, and by the definition (3.4) of $\psi(u,\xi,t)$,

$$\psi(u,\xi,t) \le C_3 t^{1/2} p^{(3)}(T;y,\xi) E^{(3)}_{y,T,\xi}[\int_0^T \theta(r_s)\, ds].$$

Using the law of r at time s, we see:

$$E^{(3)}_{y,T,\xi}[\int_0^T \theta(r_s)\, ds] = \int_0^T ds \int_0^y dz\, \theta(z) \frac{p^{(3)}(s;y,z)p^{(3)}(T-s;z,\xi)}{p^{(3)}(T;y,\xi)}.$$

The second integral is taken only over $(0,y)$ because, by definition, $\theta(z)=0$ for $z \ge y$. We also note that

$$\begin{aligned}\int_0^T ds\, p^{(3)}(s;y,z)p^{(3)}(T-s;z,\xi) &= \frac{\xi}{y} \int_{|z-y|}^{z+y} db \int_{|z-\xi|}^{z+\xi} da\, \frac{a+b}{\sqrt{2\pi T^3}} \exp\{-\frac{(a+b)^2}{2T}\} \\ &= \frac{\xi}{y} \int_{|z-\xi|}^{z+\xi} \frac{da}{\sqrt{2\pi T}} \left(\exp\{-\frac{(a+y-z)^2}{2T}\} - \exp\{-\frac{(a+y+z)^2}{2T}\} \right)\end{aligned}$$

for $z<y$. Combining these yields the lemma. □

Now we are prepared to prove Proposition 3.5.

Proof of Proposition 3.5. The bound for the case $(u,\xi) \in D_3$ follows from the former of (3.2). For the other two cases, we use the following fact: for $0<\alpha<\beta$, the function $e^{-\alpha x} - e^{-\beta x}$ $(x \ge 0)$ is bounded from above by $1-(\alpha/\beta)$. Using this, we easily see that, for each $a>0$ and $z<y$,

$$\exp\{-\frac{(a+y-z)^2}{2T}\} - \exp\{-\frac{(a+y+z)^2}{2T}\} \le \frac{4z}{a+y+z} \quad \text{for all } T>0.$$

Combining this with Lemma 3.5, we have, for all $t>0$,

$$\psi(u,\xi,t) \le 4C_3 \frac{\xi}{y\sqrt{2\pi u}} \int_0^y dz\, z\theta(z) \int_{|z-\xi|}^{z+\xi} \frac{da}{a+y+z}.$$

Note that the integral with respect to da above is dominated by $2(z \wedge \xi)/y$; indeed,

$$\begin{aligned}\int_{|z-\xi|}^{z+\xi} \frac{da}{a+y+z} &= \log(1 + \frac{z+\xi-|z-\xi|}{|z-\xi|+y+z}) \\ &\leq \frac{z+\xi-|z-\xi|}{|z-\xi|+y+z} \\ &\leq \frac{z+\xi-|z-\xi|}{y}.\end{aligned}$$

Now the bounds for the cases D_1 and D_2 follow from these. □

Step 3. The purpose of this step is to show the following:

Proposition 3.6. *Under the same assumption as in Theorem 1.1,*

$$\int_0^1 du \int_0^\infty d\xi\, k(\xi)\Psi(u,\xi) < \infty.$$

Once this proposition is shown, then, combining this with Propositions 3.4 and 3.5, we see Proposition 3.3 follows immediately from the dominated convergence theorem.

The integrability of $k(\xi)\Psi(u,\xi)$ on D_2 and D_3 is obvious; indeed, by definition,

$$\int_{D_i} dud\xi\, k(\xi)\Psi(u,\xi) = \begin{cases} C_2 \displaystyle\int_0^{u_0} \frac{du}{\sqrt{u}} \int_y^\infty d\xi\, \xi k(\xi), & i = 2, \\ \displaystyle\frac{2}{\sqrt{2\pi}} \int_{u_0}^1 \frac{du}{\sqrt{u^3}} \int_0^\infty d\xi\, \xi^2 k(\xi), & i = 3, \end{cases}$$

both of which are finite by the relation (3.1) and the assumption (A). So we need only to prove the integrability on D_1. For this purpose, we prove the following proposition first.

Proposition 3.7. *Under the same assumption as in Theorem 1.1, it holds that*

$$\int_{0+} d\xi\, \xi^2 k(\xi) \int_\xi^y dz\, z|(g_0^{-1})'(z)|^2 < \infty. \tag{3.8}$$

To see this proposition holds, first note that, by changing the variables, the LHS of (3.8) is rewritten as:

$$\int_{-\infty} d\eta\, f(\eta) f_0(\eta) \int_\eta^{x^*} dz\, \frac{f_0(z)^2}{g_0(z)}. \tag{3.9}$$

Here we write $x^* = g_0^{-1}(y)$. Recall $\gamma_0 = \sup\{\gamma \geq 0; \liminf_{x\to-\infty} |x|^{-2\gamma}\varphi(x) > 0\}$.

Lemma 3.6. (i) *If $\gamma_0 \le 1$, then there exists a constant $c > 0$ such that*

$$\int_{\eta}^{x^*} dz\, \frac{f_0(z)^2}{g_0(z)} \le c(1+|\eta|) \quad \textit{for all } \eta \le x^*.$$

(ii) *If $\gamma_0 > 1$, then*

$$\int_{-\infty}^{x^*} dz\, \frac{f_0(z)^2}{g_0(z)} < \infty.$$

Once this lemma is shown, then Proposition 3.7 follows immediately:

Proof of Proposition 3.7. Using (i) of Lemma 3.6, we see that, in the case $\gamma_0 \le 1$, (3.9) is dominated by

$$c \int_{-\infty} d\eta\, f(\eta) f_0(\eta)(1+|\eta|).$$

Note that this is finite by the assumption (B). For the case $\gamma_0 > 1$, we may bound (3.9) from above by

$$\int_{-\infty} d\eta\, f(\eta) f_0(\eta) \times \int_{-\infty}^{x^*} dz\, \frac{f_0(z)^2}{g_0(z)}.$$

Note that this is also finite by the assumption (A) and (ii) of Lemma 3.6. So the proposition is proved. □

With the help of Proposition 3.7, we give a proof of Proposition 3.6, the main objective of this step:

Proof of Proposition 3.6. We have already seen above that $k(\xi)\Psi(u,\xi)$ is integrable on $D_2 \cup D_3$. For the integrability on $D_1 = (0,u_0) \times (0,y)$, it suffices to prove, by the definition of Ψ,

$$\int_0^y d\xi\, \xi k(\xi) \int_0^{\xi} dz\, z^2 \theta(z) < \infty, \tag{3.10}$$

$$\int_0^y d\xi\, \xi^2 k(\xi) \int_{\xi}^{y} dz\, z\theta(z) < \infty. \tag{3.11}$$

Note that, by (v) of Lemma 2.1, we may find a constant $c > 0$ such that

$$\int_0^{\xi} z^2 |(g_0^{-1})'(z)|^2\, dz \le c\xi$$

for every sufficiently small ξ. Therefore

$$\int_{0+} d\xi\, \xi k(\xi) \int_0^{\xi} dz\, z^2 |(g_0^{-1})'(z)|^2 \le c \int_{0+} d\xi\, \xi^2 k(\xi),$$

which is finite by the relation (3.1) and the assumption (A). From this and the definition of θ, (3.10) follows. (3.11) is a consequence of Proposition 3.7 and the definition of θ. □

It now remains to prove Lemma 3.6. To this end, we prepare the following lemma:

Lemma 3.7. *Suppose that there exists a $\gamma \geq 0$ such that*

$$\liminf_{x\to-\infty} |x|^{-2\gamma}\varphi(x) > 0.$$

Then there exist constants $a < 0$ and $c > 0$ such that

$$\frac{f_0(x)^2}{g_0(x)} \leq c|x|^{-\gamma} \quad \textit{for all } x < a.$$

Proof. By the assumption, there exist $a < 0, c > 0$ such that $\varphi(z) \geq c|z|^{2\gamma}$ for all $z < a$. Combining this with $f_0'' = 2\varphi f_0$, we see that, for all $z < a$, $f_0''(z) \geq 2c|z|^{2\gamma} f_0(z)$. Multiplying both sides by $f_0' > 0$, we have

$$f_0''(z)f_0'(z) \geq 2c|z|^{2\gamma} f_0(z)f_0'(z) \quad \text{for all } z < a.$$

Integrating both sides over $(-\infty, x)$ for $x < a$, we see:

$$\begin{aligned}
\frac{1}{2}f_0'(x)^2 &\geq 2c\int_{-\infty}^{x} |z|^{2\gamma} f_0(z)f_0'(z)\,dz \\
&= c|x|^{2\gamma} f_0(x)^2 + 2c\gamma \int_{-\infty}^{x} |z|^{2\gamma-1} f_0(z)^2\,dz \\
&\geq c|x|^{2\gamma} f_0(x)^2.
\end{aligned}$$

Here we used integration by parts formula for the equality. (As was seen in Remark 2.1, f_0 decays exponentially or faster at $-\infty$, provided that $\liminf_{x\to-\infty}\varphi(x) > 0$. So the assumption here also ensures $\lim_{x\to-\infty}|x|^{2\gamma} f_0(x)^2 = 0$.) We thus obtain $f_0(x)/f_0'(x) \leq \sqrt{2c}|x|^{-\gamma}$ for all $x < a$. Note that, by (iv) of Lemma 2.1, $f_0^2/g_0 \leq f_0/f_0'$. Combining these ends the proof. □

Using this lemma, we prove Lemma 3.6:

Proof of Lemma 3.6. For the case (i), we may apply Lemma 3.7 with $\gamma = 0$ and get

$$\int_{\eta}^{a} dz\,\frac{f_0(z)^2}{g_0(z)} \leq c(a + |\eta|) \quad \text{for all } \eta < a,$$

for some $a < 0$ and $c > 0$. This implies (i). For the case (ii), we may take $1 < \gamma < \gamma_0$ so that $\liminf_{x\to-\infty}|x|^{-2\gamma}\varphi(x) > 0$. Applying Lemma 3.7 to this γ yields, in particular,

$$\int_{-\infty} dz\,\frac{f_0(z)^2}{g_0(z)} < \infty;$$

indeed, by Lemma 3.7, for some $a < 0$ and $c > 0$,

$$\int_{-\infty}^{a} dz\,\frac{f_0(z)^2}{g_0(z)} \leq c\int_{-\infty}^{a}\frac{dz}{|z|^{\gamma}} < \infty.$$

So the assertion (ii) is also proved. □

Step 4. We are now in a position to prove Proposition 3.3:

Proof of Proposition 3.3. Recall the expression (3.3) of $\sqrt{t}I_2(t)$. We then see that the proposition is a consequence of Propositions 3.4, 3.5 and 3.6, and the dominated convergence theorem. □

4 A remark on Corollary 1.1

We shall consider taking φ as f in Corollary 1.1. Then we see every assumption in Theorem 1.1 is fulfilled; indeed, by the equation $(1/2)f_0'' = \varphi f_0$,

$$\begin{aligned}\int_{\mathbb{R}} \varphi(z) f_0(z)\, dz &= \frac{1}{2}\int_{\mathbb{R}} f_0''(z)\, dz \\ &= \frac{1}{2}\{f_0'(+\infty) - f_0'(-\infty)\} = \frac{1}{2} < \infty,\end{aligned}$$

and, from integration by parts, it is also seen that, for all $a < 0$,

$$\begin{aligned}\int_{-\infty}^{a} |z|\varphi(z) f_0(z)\, dz &= \frac{1}{2}\int_{-\infty}^{a} |z| f_0''(z)\, dz \\ &= \frac{1}{2}(|a| f_0'(a) + f_0(a)) < \infty.\end{aligned}$$

As a consequence, (1.4) holds with $f = \varphi$:

$$\lim_{t\to\infty} \sqrt{t} \int_t^\infty ds\, E_x[\varphi(B_s) \exp\{-\int_0^s \varphi(B_u)\, du\}] = \sqrt{\frac{2}{\pi}} f_0(x).$$

Note that

$$E_x[\varphi(B_s) \exp\{-\int_0^s \varphi(B_u)\, du\}] = -\frac{d}{ds} E_x[\exp\{-\int_0^s \varphi(B_u)\, du\}].$$

Moreover, since φ can be bounded from below by $c\mathbf{1}_{(-\infty,a)}$ for some $a < 0$ and $c > 0$ by the condition (P2), it can be easily checked that

$$\limsup_{s\to\infty} E_x[\exp\{-\int_0^s \varphi(B_u)\, du\}] \le \limsup_{s\to\infty} E_x[\exp\{-c\int_0^s \mathbf{1}_{(-\infty,a)}(B_u)\, du\}] = 0,$$

with the help of the scaling property of Brownian motion. Combining these, we have

$$\lim_{t\to\infty} \sqrt{t} E_x[\exp\{-\int_0^t \varphi(B_s)\, ds\}] = \sqrt{\frac{2}{\pi}} f_0(x),$$

which partly recovers the result of [6, Section 3].

Appendix

In this appendix, we prove the FKG inequality is applicable to the laws of pinned 3-dimensional Bessel processes (or, more precisely, to the discretized measures of them). For the formulation of the FKG inequality, we refer to [5, 7].

For $t > 0$ and $x, y > 0$, let $q(t; x, y)$ denote the transition density function of absorbing Brownian motion:

$$q(t; x, y) = \frac{2}{\sqrt{2\pi t}} \exp(-\frac{x^2 + y^2}{2t}) \sinh(\frac{xy}{t}).$$

Note that

$$p^{(3)}(t; x, y) = \frac{y}{x} q(t; x, y). \tag{A.1}$$

Lemma A.1. *For each fixed $t > 0$, it holds that*

$$q(t; x_1 \vee y_1, x_2 \vee y_2) q(t; x_1 \wedge y_1, x_2 \wedge y_2) \geq q(t; x_1, x_2) q(t; y_1, y_2) \tag{A.2}$$

for all $(x_1, x_2), (y_1, y_2) \in (0, \infty) \times (0, \infty)$. Here $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$.

Proof. We divide the case into four cases: (i) $x_1 \geq y_1$, $x_2 \geq y_2$; (ii) $x_1 \leq y_1$, $x_2 \leq y_2$; (iii) $x_1 \geq y_1$, $x_2 \leq y_2$; (iv) $x_1 \leq y_1$, $x_2 \geq y_2$. In both cases (i) and (ii), (A.2) holds as an equality. So, by symmetry, we only need to consider either (iii) or (iv). Here we give a proof in the case (iii). By the definition of $q(t; x, y)$, the proof is reduced to showing the following: for $x_1 \geq y_1$ and $x_2 \leq y_2$,

$$\sinh(\frac{x_1 y_2}{t}) \sinh(\frac{x_2 y_1}{t}) \geq \sinh(\frac{x_1 x_2}{t}) \sinh(\frac{y_1 y_2}{t}). \tag{A.3}$$

Rewriting (A.3) as

$$\frac{\sinh(\frac{y_2}{t} x_1)}{\sinh(\frac{x_2}{t} x_1)} \geq \frac{\sinh(\frac{y_2}{t} y_1)}{\sinh(\frac{x_2}{t} y_1)},$$

we see that it suffices to prove, for $\beta > \alpha > 0$,

$$\frac{\sinh(\beta x)}{\sinh(\alpha x)} \text{ is non-decreasing in } x > 0.$$

This can be easily checked as:

$$\frac{d}{dx} \left\{ \frac{\sinh(\beta x)}{\sinh(\alpha x)} \right\} = \frac{(\beta^2 - \alpha^2) x}{2\{\sinh(\alpha x)\}^2} \left(\frac{\sinh\{(\beta + \alpha) x\}}{(\beta + \alpha) x} - \frac{\sinh\{(\beta - \alpha) x\}}{(\beta - \alpha) x} \right) \geq 0,$$

where the last inequality follows from the fact that $\sinh(y)/y$ is non-decreasing in $y > 0$. So the lemma is proved. ∎

For $T > 0$, let $\Delta = \{0 < t_1 < \cdots < t_n < T\}$ be a partition of the interval $[0, T]$. For $a, b > 0$, we denote by $\Phi_\Delta(x; a, b)$ ($x = (x_i)_{1\le i\le n}$) the finite-dimensional distribution function of the pinned 3-dimensional Bessel process $P^{(3)}_{a,T,b}$ taken at the time sequence $(t_i)_{1\le i\le n}$:

$$\Phi_\Delta(x; a, b) = \frac{p^{(3)}(t_1; a, x_1)p^{(3)}(t_2 - t_1; x_1, x_2) \times \cdots \times p^{(3)}(T - t_n; x_n, b)}{p^{(3)}(T; a, b)}.$$

The next lemma shows $\Phi_\Delta(\cdot; a, b)$ fulfills the assumption of [5, Theorem 3]:

Lemma A.2. *For $a \ge a' > 0$ and $b \ge b' > 0$, it holds that*

$$\Phi_\Delta(x \vee y; a, b)\Phi_\Delta(x \wedge y; a', b') \ge \Phi_\Delta(x; a, b)\Phi_\Delta(y; a', b')$$

for all $x = (x_i)_{1\le i\le n} \in (0, \infty)^n$ and $y = (y_i)_{1\le i\le n} \in (0, \infty)^n$. Here $x \vee y = (x_i \vee y_i)_{1\le i\le n}$ and $x \wedge y = (x_i \wedge y_i)_{1\le i\le n}$.

Proof. Note that, by the relation (A.1), $\Phi_\Delta(x; a, b)$ is rewritten as

$$\Phi_\Delta(x; a, b) = \frac{q(t_1; a, x_1)q(t_2 - t_1; x_1, x_2) \times \cdots \times q(T - t_n; x_n, b)}{q(T; a, b)}.$$

Therefore the assertion follows immediately from Lemma A.1. □

Remark A.1. It is easily checked that the assertion of this lemma still holds even if either a' or b' is (or, both of them are) equal to 0; in that case, $\Phi_\Delta(x; a', b')$ should be replaced by, say, if $a' = 0$,

$$\Phi_\Delta(x; 0, b') = \frac{\widetilde{q}(t_1; x_1)q(t_2 - t_1; x_1, x_2) \times \cdots \times q(T - t_n; x_n, b')}{\widetilde{q}(T; b')},$$

where

$$\widetilde{q}(t; x) = \frac{x}{\sqrt{2\pi t^3}} \exp(-\frac{x^2}{2t}).$$

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References

[1] C. Donati-Martin, H. Matsumoto, M. Yor, The law of geometric Brownian motion and its integral, revisited; application to conditional moments, in Mathematical Finance, Bachelier Congress 2000, H. Geman, D. Madan, S.R. Pliska, T. Vorst (Eds.), 221–243, Springer, Berlin (2002)

[2] Y. Hariya, M. Yor, Limiting distributions associated with moments of exponential Brownian functionals, Studia Sci. Math. Hungar. **41**, 193–242 (2004)

[3] S. Kotani, Analytic approach to Yor's formula of exponential additive functionals of Brownian motion, in Itô's stochastic calculus and probability theory, N. Ikeda, S. Watanabe, M. Fukushima, H. Kunita (Eds.), 185–195, Springer, Tokyo (1996)

[4] N.N. Lebedev, Special Functions and their Applications, Dover, New York (1972)

[5] C.J. Preston, A generalization of the FKG inequality, Comm. Math. Phys. **36**, 233–241 (1974)

[6] B. Roynette, P. Vallois, M. Yor, Limiting laws associated with Brownian motion perturbed by normalized exponential weights, C. R. Acad. Sci. Paris, Ser. I **337**, 667–673 (2003)

[7] B. Simon. Functional Integration and Quantum Physics, Academic Press, New York (1979)

[8] M. Yor, Loi de l'indice du lacet brownien et distribution de Hartman-Watson, Z. Wahrscheinlichkeits. **53**, 71–95 (1980)

[9] M. Yor, On some exponential functionals of Brownian motion, Adv. Appl. Probab. **24**, 509–531 (1992)

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