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AIC for ergodic diffusion processes from discrete observations

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Abstract. Akaike's information criterion (AIC) is proposed for evaluating statistical models constructed by the maximum likelihood estimators under the situation where the parametric models contain the true model. In order to obtain AIC, it suffices to get a log likelihood function and the maximum likelihood estimator. However, we can not generally derive AIC for discretely observed diffusion processes since the transition densities of diffusion processes do not commonly have explicit forms. This paper presents AIC type of information criterion for discretely observed ergodic diffusion processes. The information criterion is constructed by using an approximate log likelihood function and an asymptotically efficient estimator. The approximate log likelihood function is based on a result of Dacunha-Castelle and Florens-Zmirou (1986). The asymptotically efficient estimator is derived from a contrast function based on a locally Gaussian approximation.

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Abbreviated Title: AIC for diffusion processes.

1 Introduction

In this paper we consider a family of one-dimensional diffusion processes defined by the stochastic differential equations

$$\begin{aligned}dX_t &= b(X_t, \alpha)dt + \sigma(X_t, \beta)dw_t, \quad t \in [0, T], \\X_0 &= x_0,\end{aligned}\tag{1}$$

where $\theta = (\alpha, \beta) \in \Theta_\alpha \times \Theta_\beta = \Theta$ with Θ_α and Θ_β being compact convex subsets of \mathbf{R}^p and \mathbf{R}^q , respectively. Furthermore, b is an \mathbf{R} -valued function defined on $\mathbf{R} \times \Theta_\alpha$, σ is an \mathbf{R} -valued function defined on $\mathbf{R} \times \Theta_\beta$, and w is a one-dimensional standard Wiener process. We assume that the drift b and the diffusion coefficient σ are known apart from the parameters α and β . Moreover, it is assumed that the process X is ergodic for every θ with invariant probability measure μ_θ . For details of ergodic diffusion processes and the invariant probability measures, see Kutoyants (2004). The data we treat are discrete observations $\mathbf{X}_n = (X_{t_k^n})_{0 \leq k \leq n}$ with $t_k^n = kh_n$, where h_n is the discretization step. The type of asymptotics we consider is when $h_n \rightarrow 0$, $nh_n \rightarrow \infty$ and $nh_n^2 \rightarrow 0$ as $n \rightarrow \infty$.

Using an information criterion, we discuss the problem of choosing a statistical model among specified parametric models which include the true model. In order to explain the concepts

of information criteria, we treat the following simple situation for the moment. Based on the information contained in the observations $\mathbf{X}_n = x_n$, we choose a parametric model which consists of a family of probability densities $\{f(x_n, \theta); \theta \in \Theta\}$. We assume that this specified family of probability densities contain the true density $g(x_n) := f(x_n, \theta_0)$. The adopted parametric model is estimated by replacing the unknown parameter vector θ with an estimator $\hat{\theta}(\mathbf{X}_n)$, for example the maximum likelihood estimator. Then a future observation $\mathbf{Z}_n = z_n$ derived from the true density $g(z_n)$ is predicted by using the statistical model $f(z_n, \hat{\theta}(\mathbf{X}_n))$. We can also recognize that $g(z_n)$ is predicted by the statistical model $f(z_n, \hat{\theta}(\mathbf{X}_n))$. After fitting a parametric model to the observations \mathbf{X}_n , we would like to assess the closeness of the statistical model $f(z_n, \hat{\theta}(\mathbf{X}_n))$ to the true density $g(z_n)$. The estimated Kullback-Leibler information

$$I\{g(z_n); f(z_n, \hat{\theta}(\mathbf{X}_n))\} = E_{\mathbf{Z}_n} \left[\log \frac{g(\mathbf{Z}_n)}{f(\mathbf{Z}_n, \hat{\theta}(\mathbf{X}_n))} \right]$$

is used as an overall measure of the divergence of the statistical model $f(z_n, \hat{\theta}(\mathbf{X}_n))$ from the true density $g(z_n)$, conditional on the observations \mathbf{X}_n . It can be expressed as

$$I\{g(z_n); f(z_n, \hat{\theta}(\mathbf{X}_n))\} = \int g(z_n) \log g(z_n) dz_n - \int g(z_n) \log f(z_n, \hat{\theta}(\mathbf{X}_n)) dz_n. \quad (2)$$

In the same way as in Akaike (1973,1974), we use here the concept of the model selection based on minimizing the estimated Kullback-Leibler information (2). We see that the first term in the right hand side of (2) does not depend on the statistical model $f(z_n, \hat{\theta}(\mathbf{X}_n))$ while the second term $\eta(\hat{\theta}(\mathbf{X}_n)) := \int g(z_n) \log f(z_n, \hat{\theta}(\mathbf{X}_n)) dz_n$ depends on it, where $\eta(\hat{\theta}(\mathbf{X}_n))$ is called the expected log likelihood. Taking account of it, our selection rule is to choose a model which is maximizing the expected log likelihood $\eta(\hat{\theta}(\mathbf{X}_n))$ among parametric models. However, since the expected log likelihood $\eta(\hat{\theta}(\mathbf{X}_n))$ depends on the true density $g(z_n)$, we need to estimate it. The simple estimator of the expected log likelihood $\eta(\hat{\theta}(\mathbf{X}_n))$ is the estimated log likelihood $l_n(\mathbf{X}_n, \hat{\theta}(\mathbf{X}_n)) := \log f(\mathbf{X}_n, \hat{\theta}(\mathbf{X}_n))$. Usually the estimated log likelihood $l_n(\mathbf{X}_n, \hat{\theta}(\mathbf{X}_n))$ provides an optimistic assessment (overestimation) of the expected log likelihood $\eta(\hat{\theta}(\mathbf{X}_n))$ because the same observations \mathbf{X}_n are used both to estimate the unknown parameter vector θ and to evaluate the expected log likelihood $\eta(\hat{\theta}(\mathbf{X}_n))$. The bias of the estimated log likelihood appearing in estimating the expected log likelihood is given by

$$\text{bias} = E_{\mathbf{X}_n} \left[\log f(\mathbf{X}_n, \hat{\theta}(\mathbf{X}_n)) - \int g(z_n) \log f(z_n, \hat{\theta}(\mathbf{X}_n)) dz_n \right].$$

If the maximum likelihood estimator $\hat{\theta}_n^{(ML)}$ can be used, one has that under some regularity conditions, $\text{bias} = \dim(\Theta) + o(1)$ as $n \rightarrow \infty$, where $\dim(\Theta)$ denotes the dimension of a parameter space Θ . The bias corrected log likelihood is given by $\log f(\mathbf{X}_n, \hat{\theta}_n^{(ML)}) - \dim(\Theta)$. Thus, Akaike (1973, 1974) proposed Akaike's information criterion (AIC) as follows:

$$\text{AIC}(\mathbf{X}_n, \hat{\theta}_n^{(ML)}) = -2 \log f(\mathbf{X}_n, \hat{\theta}_n^{(ML)}) + 2 \dim(\Theta). \quad (3)$$

Consequently, we choose a statistical model which minimizes the value of AIC among a set of competing models. Note that AIC should be used under the situation where statistical model f contains the true model g , that is, f is a specified parametric model. For information criteria for misspecified models, we can refer Takeuchi (1976) and Konishi and Kitagawa (1996, 2003). For applications of model selection by information criteria, see, for example, Shibata (1976), Hall (1990), Burman and Nolan (1995), Burnham and Anderson (1998), Hurvich et al. (1998), Shimodaira (1998) and references therein.

As seen above, in order to construct AIC, it is enough to obtain the log likelihood function and the maximum likelihood estimator. For that reason, there seems no difficulty to derive AIC even if we consider diffusion processes. As a positive fact, Yoshida and Uchida (2001, 2004) obtained several types of information criteria including AIC for continuously observed diffusion processes. Unfortunately, as for AIC for discretely observed diffusion models, there are two serious problems. First, we can not explicitly obtain the log likelihood functions since the transition densities of diffusion processes do not generally have explicit forms. Because of the first difficulty, the maximum likelihood estimators can not be derived. Therefore, it is not a trivial problem to obtain AIC for diffusion models.

In order to obtain AIC type of information criteria for diffusion processes, we consider two kinds of functions. One is an approximate log likelihood function u_n based on a result of Dacunha-Castelle and Florens-Zmirou (1986). The other is a contrast function g_n based on a locally Gaussian approximation. The approximate log likelihood function u_n is used as an approximation of the log likelihood function and an asymptotically efficient estimator is derived from the contrast function g_n . The essential point is that in general we can not use the contrast function g_n as an approximation of the log likelihood function.

The rest of this paper is organized as follows. In section 2, using an approximate log likelihood function u_n and a contrast function g_n based on a locally Gaussian approximation, we propose AIC type of information criterion for discretely observed ergodic diffusion processes. In order to check that we can not generally use the contrast function g_n as an approximation of the log likelihood function, simulation studies for both the approximate log likelihood function u_n and the contrast function g_n are considered in section 3. In section 4, we study an example of model selection based on AIC including simulation results of the number of models selected by AIC. Section 5 gives conclusion of this paper and discussion on several possibilities of both an approximate log likelihood function and an asymptotically efficient estimator. Moreover, there are two directions of information criteria for discretely observed diffusion processes as future works. The results presented in section 2 are proved in section 6.

2 AIC type of information criterion

We introduce the notation used in this paper.

1. α_0 , β_0 and θ_0 denote the true values of α , β and θ , respectively.
2. For a function $f(x, \theta)$, define that $\delta_{\theta_i} f(x, \theta) = \frac{\partial}{\partial \theta_i} f(x, \theta)$, $f'(x, \theta) = \frac{\partial}{\partial x} f(x, \theta)$, $\delta_{\theta} f(x, \theta) = (\delta_{\theta_i} f(x, \theta))_{i=1, \dots, p}$ and $\delta_{\theta}^2 f(x, \theta) = \left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} f(x, \theta) \right)_{i, j=1, \dots, p+q}$.
3. E denotes the state space of X , $E \subseteq \mathbf{R}$.
4. When the distribution of X_t given $X_0 = x$ has a strictly positive density with respect to the Lebesgue measure on the state space E , we denote it by $y \mapsto p(t, x, y, \theta)$, $y \in E$.
5. Let \xrightarrow{p} be the convergence in probability and \xrightarrow{d} be the convergence in distribution.

Moreover, we define the following functions.

$$\begin{aligned} s(x, \beta) &= \int_0^x \frac{du}{\sigma(u, \beta)}, \\ B(x, \theta) &= \frac{b(x, \alpha)}{\sigma(x, \beta)} - \frac{1}{2} \sigma'(x, \beta), \\ \tilde{B}(x, \theta) &= B(s^{-1}(x, \beta), \theta), \\ \tilde{h}(x, \theta) &= \tilde{B}^2(x, \theta) + \tilde{B}'(x, \theta). \end{aligned}$$

We make three assumptions as follows.

Assumption 1 (i) Equation (1) has a unique strong solution on $[0, T]$.

(ii) $\inf_{x, \beta} \sigma^2(x, \beta) > 0$.

(iii) The process X is ergodic for every θ with invariant probability measure μ_θ . All polynomial moments of μ_θ are finite.

(iv) For all $m \geq 0$ and for all θ , $\sup_t E_\theta[|X_t|^m] < \infty$.

(v) For every θ , the functions $b(x, \alpha)$ and $\sigma(x, \beta)$ are twice continuously differentiable with respect to x and the derivatives are of polynomial growth in x , uniformly in θ .

(vi) The functions $b(x, \alpha)$ and $\sigma(x, \beta)$ and all their partial x -derivatives up to order 2 are three times differentiable with respect to θ for all x in the state space. All these derivatives with respect to θ are of polynomial growth in x , uniformly in θ .

Assumption 2 (i) $\tilde{h}(x, \theta) = O(|x|^2)$ as $x \rightarrow \infty$.

(ii) $\inf_x \tilde{h}(x, \theta) > -\infty$ for all θ .

(iii) $\sup_\theta \sup_x |\tilde{h}^3(x, \theta)| \leq M < \infty$.

(iv) There exists $\gamma > 0$ such that for every θ and $j = 1, 2$, $|\tilde{B}^j(x, \theta)| = O(|\tilde{B}|^\gamma(x, \theta))$ as $|x| \rightarrow \infty$.

Assumption 3

$$b(x, \alpha) = b(x, \alpha_0) \quad \text{for } \mu_{\theta_0} \text{ a.s. all } x \Rightarrow \alpha = \alpha_0,$$

$$\sigma(x, \beta) = \sigma(x, \beta_0) \quad \text{for } \mu_{\theta_0} \text{ a.s. all } x \Rightarrow \beta = \beta_0.$$

Remark 1 Assumptions 1 and 3 are made in order to estimate an unknown parameter θ . For more details, see the conditions in Kessler (1997). It follows from assumptions 1 and 2 that we can obtain an approximate log likelihood function based on lemma 2 in Dacunha-Castelle and Florens-Zmirou (1986).

The log likelihood function of \mathbf{X}_n is

$$l_n(\mathbf{X}_n, \theta) = \sum_{k=1}^n l(h_n, X_{t_{k-1}^n}, X_{t_k^n}, \theta),$$

where $l(t, x, y, \theta) = \log p(t, x, y, \theta)$. Define the maximum likelihood estimator

$$\hat{\theta}_n^{(ML)} = \arg \sup_{\theta} l_n(\mathbf{X}_n, \theta).$$

Then, Akaike's information criterion is as follows:

$$AIC = -2l_n(\mathbf{X}_n, \hat{\theta}_n^{(ML)}) + 2\dim(\Theta).$$

However, since the transition density p of the diffusion process X does not generally have an explicit form, we can not directly obtain the log likelihood function l_n and the maximum likelihood estimator $\hat{\theta}_n^{(ML)}$. That is why we need to obtain both an approximation of the log-likelihood function l_n and an asymptotically efficient estimator $\hat{\theta}_n$ in order to construct AIC type of information criteria for diffusion processes.

As an approximation of log likelihood function l_n , we use the following approximate log likelihood function based on lemma 2 in Dacunha-Castelle and Florens-Zmirou (1986).

$$u_n(\mathbf{X}_n, \theta) = \sum_{k=1}^n u(h_n, X_{t_{k-1}^n}, X_{t_k^n}, \theta),$$

where

$$u(t, x, y, \theta) = -\frac{1}{2} \log(2\pi t) - \log \sigma(y, \beta) - \frac{[S(x, y, \beta)]^2}{2t} + H(x, y, \theta) + t\tilde{g}(x, y, \theta). \quad (4)$$

Here,

$$\begin{aligned} S(x, y, \beta) &= \int_x^y \frac{du}{\sigma(u, \beta)}, \\ H(x, y, \theta) &= \int_x^y \left\{ \frac{b(u, \alpha)}{\sigma^2(u, \beta)} - \frac{1}{2} \frac{\sigma'(u, \beta)}{\sigma(u, \beta)} \right\} du, \\ \tilde{g}(x, y, \theta) &= -\frac{1}{2} \left\{ C(x, \theta) + C(y, \theta) + \frac{1}{3} B(x, \theta) B(y, \theta) \right\}, \\ C(x, \theta) &= \frac{1}{3} [B(x, \theta)]^2 + \frac{1}{2} [B(x, \theta)]' \sigma(x, \beta). \end{aligned}$$

Next, in order to derive an asymptotically efficient estimator, we use the contrast function based on locally Gaussian approximation as follows:

$$g_n(\mathbf{X}_n, \theta) = \sum_{k=1}^n g(h_n, X_{t_{k-1}^n}, X_{t_k^n}, \theta),$$

where

$$g(t, x, y, \theta) = -\frac{1}{2} \log(2\pi t) - \log \sigma(x, \beta) - \frac{[y - x - tb(x, \alpha)]^2}{2t\sigma^2(x, \beta)}. \quad (5)$$

We then define the maximum contrast estimator as

$$\hat{\theta}_n^{(C)} = \arg \sup_{\theta} g_n(\mathbf{X}_n, \theta).$$

For a process \mathbf{Z}_n which is independent of (but has the same distribution as) the observed process \mathbf{X}_n ,

$$\begin{aligned} &u_n(\mathbf{X}_n, \hat{\theta}_n^{(C)}(\mathbf{X}_n)) - E_{\mathbf{Z}_n}[l_n(\mathbf{Z}_n, \hat{\theta}_n^{(C)}(\mathbf{X}_n))] \\ &= u_n(\mathbf{X}_n, \hat{\theta}_n^{(C)}(\mathbf{X}_n)) - u_n(\mathbf{X}_n, \theta_0) \end{aligned} \quad (6)$$

$$+ u_n(\mathbf{X}_n, \theta_0) - E_{\mathbf{Z}_n}[l_n(\mathbf{Z}_n, \theta_0)] \quad (7)$$

$$+ E_{\mathbf{Z}_n}[l_n(\mathbf{Z}_n, \theta_0)] - E_{\mathbf{Z}_n}[l_n(\mathbf{Z}_n, \hat{\theta}_n^{(C)}(\mathbf{X}_n))]. \quad (8)$$

Under the regularity conditions, one has

$$\begin{aligned} (6) &= [D^{1/2} \delta_{\theta} u_n(\mathbf{X}_n, \theta_0)]^T D^{-1/2} (\hat{\theta}_n^{(C)}(\mathbf{X}_n) - \theta_0) \\ &\quad + \frac{1}{2} [D^{-1/2} (\hat{\theta}_n^{(C)}(\mathbf{X}_n) - \theta_0)]^T [D^{1/2} \delta_{\theta}^2 u_n(\mathbf{X}_n, \theta_0) D^{1/2}] \\ &\quad \times D^{-1/2} (\hat{\theta}_n^{(C)}(\mathbf{X}_n) - \theta_0) + o_p(1), \\ (8) &= - \left[D^{1/2} E_{\mathbf{Z}_n} [\delta_{\theta} l_n(\mathbf{Z}_n, \theta_0)] \right]^T D^{-1/2} (\hat{\theta}_n^{(C)}(\mathbf{X}_n) - \theta_0) \\ &\quad - \frac{1}{2} [D^{-1/2} (\hat{\theta}_n^{(C)}(\mathbf{X}_n) - \theta_0)]^T D^{1/2} E_{\mathbf{Z}_n} [\delta_{\theta}^2 l_n(\mathbf{Z}_n, \theta_0)] D^{1/2} \\ &\quad \times D^{-1/2} (\hat{\theta}_n^{(C)}(\mathbf{X}_n) - \theta_0) + o_p(1), \end{aligned}$$

where A^T is the transpose of A for a vector A , D is the following $(p+q) \times (p+q)$ matrix

$$D = \begin{pmatrix} \frac{1}{nh_n} I_p & 0 \\ 0 & \frac{1}{n} I_q \end{pmatrix},$$

and I_p is the $p \times p$ identity matrix.

Let $I(\theta_0)$ denote the Fisher information matrix as follows:

$$I(\theta_0) = \begin{pmatrix} (I_b^{ij}(\theta_0))_{i,j=1,\dots,p} & 0 \\ 0 & (I_\sigma^{ij}(\theta_0))_{i,j=1,\dots,q} \end{pmatrix},$$

where

$$I_b^{ij}(\theta_0) = \int_{\mathbf{R}} \frac{\delta_{\alpha_i} b(x, \alpha_0) \delta_{\alpha_j} b(x, \alpha_0)}{\sigma^2(x, \beta_0)} \mu_{\theta_0}(dx),$$

$$I_\sigma^{ij}(\theta_0) = 2 \int_{\mathbf{R}} \frac{\delta_{\beta_i} \sigma(x, \beta_0) \delta_{\beta_j} \sigma(x, \beta_0)}{\sigma^2(x, \beta_0)} \mu_{\theta_0}(dx).$$

In order to obtain our main result, we need the following four lemmas.

Lemma 1 *Suppose that assumptions 1 and 2 hold true. Then, as $nh_n^2 \rightarrow 0$,*

$$E[u_n(\mathbf{X}_n, \theta_0) - l_n(\mathbf{X}_n, \theta_0)] = o(1).$$

Lemma 2 *Suppose that assumptions 1 and 2 hold true. Then, as $nh_n^2 \rightarrow 0$,*

$$D^{1/2}[\delta_\theta u_n(\mathbf{X}_n, \theta_0) - \delta_\theta g_n(\mathbf{X}_n, \theta_0)] = o_p(1).$$

Lemma 3 (Kessler (1997)) *Suppose that assumptions 1 and 3 hold true. Then, as $nh_n^2 \rightarrow 0$,*

$$(i) D^{-1/2}(\hat{\theta}_n^{(C)} - \theta_0) = I^{-1}(\theta_0) D^{1/2}(\delta_\theta g_n)(\mathbf{X}_n, \theta_0) + o_p(1),$$

$$(ii) D^{1/2}(\delta_\theta g_n)(\mathbf{X}_n, \theta_0) \xrightarrow{d} N(0, I(\theta_0)).$$

Lemma 4 *Suppose that assumptions 1 and 2 hold true. Then, as $nh_n^2 \rightarrow 0$,*

$$D^{1/2}(\delta_\theta^2 u_n)(\mathbf{X}_n, \theta_0) D^{1/2} \xrightarrow{p} -I(\theta_0).$$

The main result is as follows.

Theorem 1 *Suppose that assumptions 1, 2 and 3 hold true. Then, as $nh_n^2 \rightarrow 0$,*

$$E_{\mathbf{X}_n} [u_n(\mathbf{X}_n, \hat{\theta}_n^{(C)}) - E_{\mathbf{Z}_n} [l_n(\mathbf{Z}_n, \hat{\theta}_n^{(C)}(\mathbf{X}_n))]] = \dim(\Theta) + o(1).$$

Remark 2 (i) *By theorem 1, AIC type of information criterion for diffusion processes is*

$$AIC = -2u_n(\mathbf{X}_n, \hat{\theta}_n^{(C)}) + 2\dim(\Theta).$$

(ii) *Instead of the maximum contrast estimator $\hat{\theta}_n^{(C)}$, we can also use the approximate maximum likelihood estimator $\hat{\theta}_n^{(AML)}$ derived from the approximate log likelihood function u_n . Under assumptions 1-3, $\hat{\theta}_n^{(AML)}$ has the same properties as lemma 3, that is, $\hat{\theta}_n^{(AML)}$ is asymptotically efficient. Therefore, even if $\hat{\theta}_n^{(AML)}$ is used, we can make the same assertion as theorem 1 and*

$$AIC = -2u_n(\mathbf{X}_n, \hat{\theta}_n^{(AML)}) + 2\dim(\Theta).$$

However, as seen in several examples put later, u_n has a complicated expression while g_n is a simple form. Thus, as for an asymptotically efficient estimator, it is better to use g_n than u_n .

3 Simulation studies on approximate log likelihoods

After getting theorem 1, we immediately have the following question. "Is it possible to use the contrast function g_n based on a locally Gaussian approximation as an approximation of the log likelihood function l_n ?" The answer is negative. In order to understand this fact, we examine the asymptotic behaviours of $E_{\theta_0}[g_n(\mathbf{X}_n, \theta_0) - l_n(\mathbf{X}_n, \theta_0)]$ and $E_{\theta_0}[u_n(\mathbf{X}_n, \theta_0) - l_n(\mathbf{X}_n, \theta_0)]$ through simulations, which are done for each $T = nh_n = 10, 30, 50$ and $h_n = 1/100, 1/1000$. Three models we simulate are the Ornstein-Uhlenbeck process, the Radial Ornstein-Uhlenbeck process and the Cox-Ingersoll-Ross process, which have explicit transition densities. For details of the transition densities of the three models, see Karlin and Taylor (1981). For a true parameter value θ_0 and an initial value x_0 , 5000 independent sample paths are generated by the Milstein scheme. For the Milstein scheme, see Kloeden and Platen (1992).

3.1 The Ornstein-Uhlenbeck process

Consider the Ornstein-Uhlenbeck process defined by the following stochastic differential equation

$$dX_t = -\alpha X_t dt + \beta dw_t, \quad X_0 = x_0,$$

where $\alpha > 0$ and $\beta > 0$ are unknown parameters.

It follows from (4) and (5) that

$$\begin{aligned} g(t, x, y, \theta) &= -\frac{1}{2} \log(2\pi t) - \frac{1}{2} \log(\beta^2) - \frac{(y - x + t\alpha x)^2}{2t\beta^2}, \\ u(t, x, y, \theta) &= -\frac{1}{2} \log(2\pi t) - \frac{1}{2} \log(\beta^2) - \frac{(y - x)^2}{2t\beta^2} - \frac{\alpha}{2\beta^2}(y^2 - x^2) \\ &\quad - \frac{t}{2} \left\{ \frac{\alpha^2}{3\beta^2}(x^2 + y^2 + xy) - \alpha \right\}. \end{aligned}$$

The log likelihood function l_n is obtained from

$$\begin{aligned} l(t, x, y, \theta) &= \log p(t, x, y, \theta), \\ p(t, x, y, \theta) &= \frac{1}{\sqrt{\pi\beta^2(1 - \exp\{-2\alpha t\})/\alpha}} \exp \left[\frac{-(y - \exp\{-\alpha t\}x)^2}{\beta^2(1 - \exp\{-2\alpha t\})/\alpha} \right]. \end{aligned}$$

In tables 1 and 2, both u_n and g_n have good approximations of l_n for all cases. It is worth mentioning that u_n is better than g_n . For the the Ornstein-Uhlenbeck model, it seems that g_n can be substituted as an approximation of l_n instead of u_n .

Table 1: *The Ornstein-Uhlenbeck process. Means of $g_n - l_n$ and $u_n - l_n$ for 5000 independent simulated sample paths with $\alpha_0 = 1$, $\beta_0 = 2$ and $x_0 = 10$.*

T	h_n	$E[g_n - l_n]$	$E[u_n - l_n]$
10	1/100	-0.02436	0.00826
	1/1000	-0.00227	0.00083
30	1/100	-0.06979	0.02488
	1/1000	-0.00790	0.00249
50	1/100	-0.11241	0.04151
	1/1000	-0.01186	0.00416

Table 2: *The Ornstein-Uhlenbeck process. Means of $g_n - l_n$ and $u_n - l_n$ for 5000 independent simulated sample paths with $\alpha_0 = 2$, $\beta_0 = 5$ and $x_0 = 10$.*

T	h_n	$E[g_n - l_n]$	$E[u_n - l_n]$
10	1/100	-0.10091	0.03310
	1/1000	-0.00922	0.00333
30	1/100	-0.28569	0.09944
	1/1000	-0.02453	0.00999
50	1/100	-0.49459	0.16578
	1/1000	-0.04084	0.01665

3.2 The Radial Ornstein-Uhlenbeck process

We consider the Radial Ornstein-Uhlenbeck process defined by

$$dX_t = (\theta X_t^{-1} - X_t)dt + dw_t, \quad X_0 = x_0,$$

where $\theta > 0$ is an unknown parameter.

The contrast function g_n , the approximate log likelihood function u_n and the log likelihood function l_n are constructed by

$$\begin{aligned} g(t, x, y, \theta) &= -\frac{1}{2} \log(2\pi t) - \frac{(y - x - t(\theta x^{-1} - x))^2}{2t}, \\ u(t, x, y, \theta) &= -\frac{1}{2} \log(2\pi t) - \frac{(y - x)^2}{2t} + \theta \log\left(\frac{y}{x}\right) - \frac{1}{2}(y^2 - x^2) \\ &\quad - \frac{t}{2} \left\{ \frac{1}{3} \left(\frac{\theta}{x} - x\right)^2 - \frac{1}{2} \left(\frac{\theta}{x^2} + 1\right) + \frac{1}{3} \left(\frac{\theta}{y} - y\right)^2 - \frac{1}{2} \left(\frac{\theta}{y^2} + 1\right) \right. \\ &\quad \left. + \frac{1}{3} \left(\frac{\theta}{x} - x\right) \left(\frac{\theta}{y} - y\right) \right\}, \\ l(t, x, y, \theta) &= \log p(t, x, y), \end{aligned}$$

respectively, where

$$p(t, x, y, \theta) = \frac{(y/x)^\theta \sqrt{xy} \exp\{-y^2 + (\theta + \frac{1}{2})t\}}{\sinh(t)} \exp\left[\frac{-(x^2 + y^2)}{\exp\{2t\} - 1}\right] I_{\theta - \frac{1}{2}}\left(\frac{xy}{\sinh(t)}\right)$$

and I_ν is a modified Bessel function with index ν .

In tables 3 and 4, g_n has a small bias when $T = 30, 50$ and $h = 1/100$, while u_n has a good approximation for all cases. It follows from these tables that g_n is not suitable for an approximation of l_n when $T = 30, 50$ and $h = 1/100$.

Table 3: *The Radial Ornstein-Uhlenbeck process. Means of $g_n - l_n$ and $u_n - l_n$ for 5000 independent simulated sample paths with $\theta_0 = 2$ and $x_0 = 10$.*

T	h_n	$E[g_n - l_n]$	$E[u_n - l_n]$
10	1/100	-0.24315	0.05307
	1/1000	-0.10851	0.00544
30	1/100	-0.86191	0.18446
	1/1000	-0.37434	0.01892
50	1/100	-1.45814	0.31463
	1/1000	-0.64074	0.03222

Table 4: *The Radial Ornstein-Uhlenbeck process. Means of $g_n - l_n$ and $u_n - l_n$ for 5000 independent simulated sample paths with $\theta_0 = 100$ and $x_0 = 10$.*

T	h_n	$E[g_n - l_n]$	$E[u_n - l_n]$
10	1/100	-0.09708	0.03347
	1/1000	-0.01881	0.00334
30	1/100	-0.29819	0.10038
	1/1000	-0.05279	0.01002
50	1/100	-0.52674	0.16733
	1/1000	-0.09156	0.01671

3.3 The Cox-Ingersoll-Ross process

Consider the Cox-Ingersoll-Ross process defined by the following stochastic differential equation

$$dX_t = -\alpha_1(X_t - \alpha_2)dt + \beta\sqrt{X_t}dw_t, \quad X_0 = x_0,$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ and $\beta > 0$ are unknown parameters.

The contrast function g_n and the approximate log likelihood function u_n are obtained from

$$g(t, x, y, \theta) = -\frac{1}{2}\log(2\pi t) - \frac{1}{2}\log(\beta^2 x) - \frac{(y - x + t\alpha_1(x - \alpha_2))^2}{2t\beta^2 x}, \quad (9)$$

$$\begin{aligned} u(t, x, y, \theta) = & -\frac{1}{2}\log(2\pi t) - \frac{1}{2}\log(\beta^2 y) - \frac{2(\sqrt{y} - \sqrt{x})^2}{t\beta^2} - \frac{\alpha_1(y - x)}{\beta^2} + \left(\frac{\alpha_1\alpha_2}{\beta^2} - \frac{1}{4}\right)\log\left(\frac{y}{x}\right) \\ & - \frac{t}{2}\left[\frac{1}{3}\left\{-\frac{\alpha_1}{\beta}\sqrt{x} + \left(\frac{\alpha_1\alpha_2}{\beta} - \frac{\beta}{4}\right)\frac{1}{\sqrt{x}}\right\}^2 + \frac{1}{2}\left\{-\frac{\alpha_1}{2} - \frac{1}{2}\left(\alpha_1\alpha_2 - \frac{\beta^2}{4}\right)\frac{1}{x}\right\}\right. \\ & + \frac{1}{3}\left\{-\frac{\alpha_1}{\beta}\sqrt{y} + \left(\frac{\alpha_1\alpha_2}{\beta} - \frac{\beta}{4}\right)\frac{1}{\sqrt{y}}\right\}^2 + \frac{1}{2}\left\{-\frac{\alpha_1}{2} - \frac{1}{2}\left(\alpha_1\alpha_2 - \frac{\beta^2}{4}\right)\frac{1}{y}\right\} \\ & \left. + \frac{1}{3}\left\{-\frac{\alpha_1}{\beta}\sqrt{x} + \left(\frac{\alpha_1\alpha_2}{\beta} - \frac{\beta}{4}\right)\frac{1}{\sqrt{x}}\right\}\left\{-\frac{\alpha_1}{\beta}\sqrt{y} + \left(\frac{\alpha_1\alpha_2}{\beta} - \frac{\beta}{4}\right)\frac{1}{\sqrt{y}}\right\}\right], \quad (10) \end{aligned}$$

respectively. The log likelihood function l_n is constructed by

$$\begin{aligned} l(t, x, y, \theta) &= \log p(t, x, y, \theta), \\ p(t, x, y, \theta) &= \frac{\gamma(y/x)^{\frac{1}{2}\nu} \exp\{\frac{1}{2}\alpha_1\nu t - \gamma y\}}{1 - \exp\{-\alpha_1 t\}} \exp\left[\frac{-\gamma(x+y)}{\exp\{\alpha_1 t\} - 1}\right] I_\nu\left(\frac{\gamma\sqrt{xy}}{\sinh(\frac{1}{2}\alpha_1 t)}\right), \end{aligned}$$

where $\gamma = 2\alpha_1\beta^{-2}$, $\nu = \gamma\alpha_2 - 1$ and I_ν is a modified Bessel function with index ν .

In tables 5 and 6, u_n is a good approximation for all cases, while g_n is a considerable bias for all cases. These results show that we can not use g_n instead of u_n .

Table 5: *The Cox-Ingersoll-Ross process. Means of $g_n - l_n$ and $u_n - l_n$ for 5000 independent simulated sample paths with $\alpha_{1,0} = 1$, $\alpha_{2,0} = 5$, $\beta_0 = 2$ and $x_0 = 10$.*

T	h_n	$E[g_n - l_n]$	$E[u_n - l_n]$
10	1/100	-9.23405	0.01440
	1/1000	-31.57794	0.00145
30	1/100	-28.95825	0.04840
	1/1000	-98.57695	0.00489
50	1/100	-48.62306	0.08193
	1/1000	-165.32443	0.00830

Table 6: *The Cox-Ingersoll-Ross process. Means of $g_n - l_n$ and $u_n - l_n$ for 5000 independent simulated sample paths with $\alpha_{1,0} = 1$, $\alpha_{2,0} = 10$, $\beta_0 = 2$ and $x_0 = 10$.*

T	h_n	$E[g_n - l_n]$	$E[u_n - l_n]$
10	1/100	-5.63288	0.01009
	1/1000	-20.92659	0.00100
30	1/100	-17.06086	0.03066
	1/1000	-63.01867	0.00305
50	1/100	-28.42232	0.05122
	1/1000	-105.04559	0.00510

4 Example of model selection based on AIC

As an example of model selection based on AIC, we treat the following setting. The true model is

$$dX_t = -(X_t - 10)dt + 2\sqrt{X_t}dw_t,$$

where $X_0 = 10$ and $t \in [0, T]$. We consider the following three statistical models:

$$dX_t = -\alpha_1(X_t - \alpha_2)dt + \beta\sqrt{X_t}dw_t, \quad (11)$$

$$dX_t = -\alpha_1(X_t - \alpha_2)dt + \sqrt{\beta_1 + \beta_2 X_t}dw_t, \quad (12)$$

$$dX_t = -\alpha_1(X_t - \alpha_2)dt + (\beta_1 + \beta_2 X_t)^{\beta_3}dw_t, \quad (13)$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$, $\beta_1 \geq 0$, $\beta_2 > 0$ and $\beta_3 \geq 0$.

As seen in section 3, the contrast function $g^{(1)}(t, x, y, \theta)$ of the model (11) can be used (9). It follows from (5) that the contrast functions for the models (12) and (13) are

$$g^{(2)}(t, x, y, \theta) = -\frac{1}{2}\log(2\pi t) - \frac{1}{2}\log(\beta_1 + \beta_2 x) - \frac{(y - x + t\alpha_1(x - \alpha_2))^2}{2t(\beta_1 x + \beta_2)}, \quad (14)$$

$$g^{(3)}(t, x, y, \theta) = -\frac{1}{2}\log(2\pi t) - \beta_3 \log(\beta_1 + \beta_2 x) - \frac{(y - x + t\alpha_1(x - \alpha_2))^2}{2t(\beta_1 x + \beta_2)^{2\beta_3}}, \quad (15)$$

respectively. For the approximate log-likelihood function $u^{(1)}(t, x, y, \theta)$ of the model (11), we have already obtained (10) in section 3. Moreover, by (4), the approximate log-likelihood functions of the models (12) and (13) are

$$\begin{aligned} u^{(2)}(t, x, y, \theta) &= -\frac{1}{2}\log(2\pi t) - \frac{1}{2}\log(\beta_1 + \beta_2 y) - \frac{2(\sqrt{\beta_1 + \beta_2 y} - \sqrt{\beta_1 + \beta_2 x})^2}{t\beta_2^2} \\ &\quad - \frac{\alpha_1(y - x)}{\beta_2} + \left(\frac{\alpha_1\alpha_2}{\beta_2} + \frac{\alpha_1\beta_1}{\beta_2^2} - \frac{1}{4} \right) \log \left(\frac{\beta_1 + \beta_2 y}{\beta_1 + \beta_2 x} \right) \\ &\quad - \frac{t}{2} \left[C(x, \theta) + C(y, \theta) + \frac{1}{3}B(x, \theta)B(y, \theta) \right], \end{aligned} \quad (16)$$

where

$$\begin{aligned} B(x, \theta) &= \frac{-\alpha_1 x + \alpha_1 \alpha_2 - \beta_2/4}{\sqrt{\beta_1 + \beta_2 x}}, \\ C(x, \theta) &= \frac{1}{3} \left\{ \frac{-\alpha_1 x + \alpha_1 \alpha_2 - \beta_2/4}{\sqrt{\beta_1 + \beta_2 x}} \right\}^2 + \frac{1}{2} \left\{ \frac{-\alpha_1 \beta_2 x/2 - \alpha_1 \beta_1 - \alpha_1 \alpha_2 \beta_2/2 + \beta_2^2/8}{\beta_1 + \beta_2 x} \right\}, \end{aligned}$$

and

$$\begin{aligned}
u^{(3)}(t, x, y, \theta) &= -\frac{1}{2} \log(2\pi t) - \beta_3 \log(\beta_1 + \beta_2 y) - \frac{\left\{ (\beta_1 + \beta_2 y)^{1-\beta_3} - (\beta_1 + \beta_2 x)^{1-\beta_3} \right\}^2}{2t(1-\beta_3)^2 \beta_2^2} \\
&\quad - \alpha_1 \left[\frac{(-\beta_1 + \beta_2 y(1-2\beta_3) - 2\alpha_2 \beta_2(1-\beta_3))}{2\beta_2^2(1-\beta_3)(1-2\beta_3)(\beta_1 + \beta_2 y)^{2\beta_3-1}} \right. \\
&\quad \left. - \frac{(-\beta_1 + \beta_2 x(1-2\beta_3) - 2\alpha_2 \beta_2(1-\beta_3))}{2\beta_2^2(1-\beta_3)(1-2\beta_3)(\beta_1 + \beta_2 x)^{2\beta_3-1}} \right] - \frac{\beta_3}{2} \log \left(\frac{\beta_1 + \beta_2 y}{\beta_1 + \beta_2 x} \right) \\
&\quad - \frac{t}{2} \left[C(x, \theta) + C(y, \theta) + \frac{1}{3} B(x, \theta) B(y, \theta) \right], \tag{17}
\end{aligned}$$

where

$$\begin{aligned}
B(x, \theta) &= \frac{-\alpha_1(x - \alpha_2)}{(\beta_1 + \beta_2 x)^{\beta_3}} - \frac{\beta_2 \beta_3}{2(\beta_1 + \beta_2 x)^{1-\beta_3}}, \\
C(x, \theta) &= \frac{1}{3} \left\{ \frac{-\alpha_1(x - \alpha_2)}{(\beta_1 + \beta_2 x)^{\beta_3}} - \frac{\beta_2 \beta_3}{2(\beta_1 + \beta_2 x)^{1-\beta_3}} \right\}^2 \\
&\quad + \frac{1}{2} \left\{ -\alpha_1 + \frac{\alpha_1(x - \alpha_2) \beta_2 \beta_3}{\beta_1 + \beta_2 x} - \frac{\beta_2^2 \beta_3 (\beta_3 - 1)}{2(\beta_1 + \beta_2 x)^{2-2\beta_3}} \right\},
\end{aligned}$$

respectively. Note that $u^{(3)}(t, x, y, \theta)$ is obtained under the assumption $\beta_3 \neq 1/2$. When $\beta_3 = 1/2$, it suffices to consider the model (12).

Therefore, AIC for each model (11), (12), (13) is as follows.

$$\begin{aligned}
AIC_1(\mathbf{X}_n, \hat{\theta}_n^{(1)}) &= -2u^{(1)}(\mathbf{X}_n, \hat{\theta}_n^{(1)}) + 2 \times 3, \\
AIC_2(\mathbf{X}_n, \hat{\theta}_n^{(2)}) &= -2u^{(2)}(\mathbf{X}_n, \hat{\theta}_n^{(2)}) + 2 \times 4, \\
AIC_3(\mathbf{X}_n, \hat{\theta}_n^{(3)}) &= -2u^{(3)}(\mathbf{X}_n, \hat{\theta}_n^{(3)}) + 2 \times 5,
\end{aligned}$$

where $\hat{\theta}_n^{(i)}$ is obtained from the contrast function $g_n^{(i)}$ for $i = 1, 2, 3$.

We examine the number of models selected by AIC among competing models (11), (12), (13) for 1000 independent sample paths generated by the Milstein scheme through simulations. The simulations are done for each $T = 10, 30, 50$ and $h_n = 1/100, 1/500$.

By table 7, we see that model 1 is selected with high frequency as the best model for all cases. However, we must note that model 2 is selected in a significant probability. This fact implies that AIC is not a tool for estimating the true model. Note that AIC is a tool to choose the best model among competing models from the aspect of both model-fitting and prediction.

Table 7: *The number of models selected by AIC for 1000 independent simulated sample paths.*

T	h_n	model 1	model 2	model 3
10	1/100	761	185	54
	1/500	744	194	62
30	1/100	803	185	12
	1/500	673	238	89
50	1/100	769	186	45
	1/500	610	275	115

5 Conclusion and discussion

In order to get AIC type of information criterion, we first use the contrast function g_n based on a locally Gaussian approximation to obtain an efficient estimator. Next, for an approximation of log likelihood l_n , it is better to use the approximate log likelihood u_n based on Dacunha-Castelle and Florens-Zmirou (1986) than the locally Gaussian approximation g_n . As for the CIR model, we should not use g_n as an approximation of l_n because g_n has a considerable bias.

In this paper, we proposed AIC type of information criterion under assumptions 1-3. In order to obtain the information criterion, however, the most important point is to show the lemmas 1-4 presented in section 2. For this reason, it is possible to consider AIC type of information criterion by replacing assumptions 1-3 with the following assumption.

Assumption 4 *There exist functions u_n and g_n satisfying lemmas 1-4.*

Under assumption 4 and appropriate regularity conditions, it is possible to make the same assertion as theorem 1. Therefore, there is a possibility of having both other approximate log likelihood functions and other asymptotically efficient estimators. For approximate log likelihood functions, the essential point is to satisfy lemma 1. Nicolau (2002) considered an approximate likelihood function by means of a simulation-based technique based on a results of Dacunha-Castelle and Florens-Zmirou (1986). Aït-Sahalia (2002) presented an explicit sequence of closed-form transition densities by using Hermite expansion. It seems that under appropriate regularity conditions, their approximate log likelihood functions satisfy lemma 1. Moreover, although the contrast function g_n generally has a considerable bias for an approximation of l_n , the bias corrected contrast function \tilde{g}_n may be available to an approximation of l_n . For asymptotically efficient estimators satisfying lemma 3, there are a number of works for various diffusion models, see Florens-Zmirou (1989), Yoshida (1992), Bibby and Sørensen (1995), Aït-Sahalia and Mykland (2003, 2004), Bibby et al. (2004) and reference therein.

With regard to future projects, there are three directions. One is to obtain AIC by replacing the assumption that $nh_n^2 \rightarrow 0$ with a weaker asymptotics, for example $nh_n^3 \rightarrow 0$. Another is to extend the results of this paper to a multi-dimensional diffusion model. The third is to consider information criteria for a misspecified diffusion model, that is, Takeuchi's information criterion (TIC) presented in Takeuchi (1976) and the generalized information criterion (GIC) considered in Konishi and Kitagawa (1996). For the first objective, it seems that there is no difficulty under the situation when $nh_n^3 \rightarrow 0$. Using the contrast function $l_{3,n}$ presented in Kessler (1997) and the third order approximate log likelihood function $u_{3,n}$ based on a results of Dacunha-Castelle and Florens-Zmirou (1986), we will be able to show that $l_{3,n}$ and $u_{3,n}$ satisfy assumption 4 under $nh_n^3 \rightarrow 0$. However, there is no doubt that $l_{3,n}$ and $u_{3,n}$ have very complicated expressions. For a multi-dimensional diffusion model, an asymptotically efficient estimator can be obtained from a multi-dimensional version of the contrast function based on a locally Gaussian approximation in the same way as the case of a one-dimensional diffusion model. Therefore, it is important to consider an approximate log likelihood function satisfying lemma 1 for a multi-dimensional diffusion model. One possibility is to use the approximate log likelihood function based on a result of Aït-Sahalia (2003). As for information criteria for misspecified diffusion models, we will need to compose a statistically asymptotic theory of parametric estimation for misspecified diffusion models from discrete observations. Using the misspecified version of estimators together with non-parametric estimators of both drift and diffusion coefficient functions for discretely observed diffusion models, we will be able to obtain TIC and GIC for misspecified diffusion models. However, it seems that there are a lot of difficulties in order to prove the desired result.

6 Proof

Let R denote a function $(0, 1] \times \mathbf{R}$ for which there exists a $C > 0$ such that $|R(h_n, x)| \leq h_n(1 + |x|)^C$ for all n, x .

Proof of Lemma 1.

$$E[u_n(\mathbf{X}_n, \theta_0) - l_n(\mathbf{X}_n, \theta_0)] = \sum_{k=1}^n E \left[E[u(X_{t_{k-1}^n}, X_{t_k^n}, \theta_0) - l(X_{t_{k-1}^n}, X_{t_k^n}, \theta_0) | X_{t_{k-1}^n}] \right].$$

It follows from (3.7) and lemma 2 in Dacunha-Castelle and Florens-Zmirou (1986) that

$$E[u(X_{t_{k-1}^n}, X_{t_k^n}, \theta_0) - l(X_{t_{k-1}^n}, X_{t_k^n}, \theta_0) | X_{t_{k-1}^n}] = R(h_n^2, X_{t_{k-1}^n}).$$

Thus, as $n \rightarrow \infty$, $h_n \rightarrow 0$ and $nh_n^2 \rightarrow 0$,

$$\begin{aligned} E[u_n(\mathbf{X}_n, \theta_0) - l_n(\mathbf{X}_n, \theta_0)] &= \sum_{k=1}^n h_n^2 E[R(1, X_{t_{k-1}^n})] \\ &\leq nh_n^2 \frac{1}{n} \sum_{k=1}^n C \rightarrow 0. \end{aligned}$$

This completes the proof.

Proof of Lemma 2. Set that for $i = 1, \dots, p + q$,

$$\xi_k^i := d_i [\delta_{\theta_i} u(h_n, X_{t_{k-1}^n}, X_{t_k^n}, \theta_0) - \delta_{\theta_i} g_n(h_n, X_{t_{k-1}^n}, X_{t_k^n}, \theta_0)],$$

where $d_i = 1/\sqrt{nh_n}$ for $i = 1, \dots, p$ and $d_i = 1/\sqrt{n}$ for $i = p + 1, \dots, p + q$. By lemma 9 in Genon-Catalot and Jacod (1993), it is enough to prove that for $i, j = 1, \dots, p + q$,

$$\begin{aligned} \sum_{k=1}^n E[\xi_k^i | X_{t_{k-1}^n}] &\xrightarrow{p} 0, \\ \sum_{k=1}^n E[(\xi_k^i)^2 | X_{t_{k-1}^n}] &\xrightarrow{p} 0 \end{aligned}$$

as $n \rightarrow \infty$, $h_n \rightarrow 0$ and $nh_n^2 \rightarrow 0$.

For $i = 1, \dots, p$, one has that

$$\begin{aligned} \xi_k^i &= \frac{1}{\sqrt{nh_n}} \left[\delta_{\alpha_i} H(X_{t_{k-1}^n}, X_{t_k^n}, \theta_0) + h_n \delta_{\alpha_i} \tilde{g}(X_{t_{k-1}^n}, X_{t_k^n}, \theta_0) \right. \\ &\quad \left. - (\delta_{\alpha_i} b)(X_{t_{k-1}^n}, \alpha_0) \frac{X_{t_k^n} - X_{t_{k-1}^n} - h_n b(X_{t_{k-1}^n}, \alpha_0)}{\sigma^2(X_{t_{k-1}^n}, \beta_0)} \right]. \end{aligned}$$

By the Ito-Taylor expansion based on the generator $L_{\theta_0} = b(x, \alpha_0)(\partial/\partial x) + \frac{1}{2}\sigma(x, \beta_0)(\partial/\partial x)^2$,

$$\begin{aligned} E[\xi_k^i | X_{t_{k-1}^n}] &= \frac{1}{\sqrt{nh_n}} \left[\delta_{\alpha_i} H(X_{t_{k-1}^n}, X_{t_{k-1}^n}, \theta_0) + h_n L_{\theta_0} \delta_{\alpha_i} H(X_{t_{k-1}^n}, X_{t_{k-1}^n}, \theta_0) \right. \\ &\quad + h_n \delta_{\alpha_i} \tilde{g}(X_{t_{k-1}^n}, X_{t_{k-1}^n}, \theta_0) + R(h_n^2, X_{t_{k-1}^n}) \\ &\quad \left. - \frac{(\delta_{\alpha_i} b)(X_{t_{k-1}^n}, \alpha_0)}{\sigma^2(X_{t_{k-1}^n}, \beta_0)} E[X_{t_k^n} - X_{t_{k-1}^n} - h_n b(X_{t_{k-1}^n}, \alpha_0) | X_{t_{k-1}^n}] \right]. \end{aligned}$$

For details of the Ito-Taylor expansion based on the generator L_{θ_0} , see Dacunha-Castelle and Florens-Zmirou (1986), Florens-Zmirou (1989) and Kessler (1997). Therefore,

$$\begin{aligned}\sum_{k=1}^n E[\xi_k^i | X_{t_{k-1}^n}] &= \frac{1}{\sqrt{nh_n}} \sum_{k=1}^n R(h_n^2, X_{t_{k-1}^n}) \\ &= \sqrt{nh_n^3} \frac{1}{n} \sum_{k=1}^n R(1, X_{t_{k-1}^n}) \xrightarrow{p} 0.\end{aligned}$$

Next, we obtain that for $i = 1, \dots, p$,

$$\begin{aligned}\sum_{k=1}^n E[(\xi_k^i)^2 | X_{t_{k-1}^n}] &= \frac{1}{nh_n} \sum_{k=1}^n R(h_n^2, X_{t_{k-1}^n}) \\ &= h_n \frac{1}{n} \sum_{k=1}^n R(1, X_{t_{k-1}^n}) \xrightarrow{p} 0.\end{aligned}$$

In the same way as the proof for $i = 1, \dots, p$, we can obtain the results for $i = p+1, \dots, p+q$. This completes the proof.

Proof of Lemma 3. See the proof of theorem 1 in Kessler (1997).

Proof of Lemma 4. Set that for $i, j = 1, \dots, p+q$,

$$\eta_k^{ij} := d_i d_j (\delta_{\theta_i} \delta_{\theta_j} u_n)(h_n, X_{t_{k-1}^n}, X_{t_k^n}, \theta_0),$$

where d_i is defined in the proof of lemma 2. In the same way as in lemma 2, we can show that

$$\begin{aligned}\sum_{k=1}^n E[\eta_k^{ij} | X_{t_{k-1}^n}] &\xrightarrow{p} -I^{ij}(\theta_0), \\ \sum_{k=1}^n E[(\eta_k^{ij})^2 | X_{t_{k-1}^n}] &\xrightarrow{p} 0\end{aligned}$$

as $n \rightarrow \infty$, $h_n \rightarrow 0$ and $nh_n^2 \rightarrow 0$. This completes the proof.

Proof of Theorem 1.

$$\begin{aligned}& E_{\mathbf{X}_n} \left[u_n(\mathbf{X}_n, \hat{\theta}_n^{(C)}(\mathbf{X}_n)) - E_{\mathbf{Z}_n} [l_n(\mathbf{Z}_n, \hat{\theta}_n^{(C)}(\mathbf{X}_n))] \right] \\ &= E_{\mathbf{X}_n} \left[u_n(\mathbf{X}_n, \hat{\theta}_n^{(C)}(\mathbf{X}_n)) - u_n(\mathbf{X}_n, \theta_0) \right] \tag{18} \\ &\quad + E_{\mathbf{X}_n} \left[u_n(\mathbf{X}_n, \theta_0) - E_{\mathbf{Z}_n} [l_n(\mathbf{Z}_n, \theta_0)] \right] \tag{19} \\ &\quad + E_{\mathbf{X}_n} \left[E_{\mathbf{Z}_n} [l_n(\mathbf{Z}_n, \theta_0)] - E_{\mathbf{Z}_n} [l_n(\mathbf{Z}_n, \hat{\theta}_n^{(C)}(\mathbf{X}_n))] \right]. \tag{20}\end{aligned}$$

Lemma 1 implies that (19) = $o(1)$. By (6) and lemmas 2-4, one has that

$$(18) = \frac{1}{2} \text{tr}[I(\theta_0)I^{-1}(\theta_0)] + o(1).$$

It follows from (8) and lemmas 2-4 that

$$(20) = \frac{1}{2} \text{tr}[I(\theta_0)I^{-1}(\theta_0)] + o(1).$$

This completes the proof.

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