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Abstract. We study an asymptotically efficient estimator for drift parameters of a one-dimensional small diffusion process with a linear drift. A martingale estimating function can be constructed for this model, and an estimator obtained from the estimating function has an explicit form. Under the situation where the sample path is observed at n regularly spaced time points $t_k = k/n$ on the interval $[0,1]$, we consider asymptotic properties of the estimator as a small dispersion parameter $\varepsilon \to 0$ and $n \to \infty$ simultaneously.

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Abbreviated Title: Estimation for a linear drift.

1 Introduction

Consider a one-dimensional diffusion process defined by the stochastic differential equation (SDE) as follows:

$$\begin{aligned} dX_t &= b(X_t,\theta)dt + \varepsilon\sigma(X_t)dw_t,\ t \in [0,1],\ \varepsilon \in (0,1], \qquad (1) \\ X_0 &= x_0, \end{aligned}$$

where x_0 and ε are known constants, $\theta \in \Theta$ with Θ being a compact convex subset of $\mathbf{R}^p$, b is an $\mathbf{R}$-valued function defined on $\mathbf{R} \times \Theta$, σ is an $\mathbf{R}$-valued function defined on $\mathbf{R}$, and w is a one-dimensional standard Wiener process. We assume that the drift b is known apart from the parameter θ. The type of data is discrete observations of the process X_t at n regularly spaced time points $t_k = k/n$ on the fixed interval $[0,1]$, that is, $\mathbf{X}_n = (X_{t_k})_{0\le k\le n}$. The asymptotics is when $\varepsilon \to 0$ and $n \to \infty$ simultaneously.

The above process (1), which is called a small diffusion process, is often applied to a model of mathematical finance, see Yoshida (1992b), Kunitomo and Takahashi (2001), Takahashi and Yoshida (2004) and references therein. Because of it, our interest is in statistical inference for small diffusion processes. For a continuously observed sample path $\mathbf{X}_1 = \{X_t; t \in [0,1]\}$, the log-likelihood function is

$$l_\varepsilon(\theta) = \frac{1}{\varepsilon^2}\int_0^1 b(X_t,\theta)\sigma^{-2}(X_t)dX_t - \frac{1}{2\varepsilon^2}\int_0^1 b(X_t,\theta)\sigma^{-2}(X_t)b(X_t,\theta)dt. \qquad (2)$$

The maximum likelihood estimator (MLE) is given by $l_\varepsilon(\hat{\theta}_\varepsilon^{(ML)}) = \sup_{\theta\in\Theta} l_\varepsilon(\theta)$. The asymptotic properties of the MLE have been studied carefully, see Kutoyants (1984, 1994) and Yoshida (1992, 2003). These continuous paths, however, are hardly observed. Therefore, in practice, statistical inference for a discretely observed small diffusion is regarded as important.

As for asymptotically efficient estimation of a drift parameter based on discrete observations, we can refer the following two papers. Genon-Catalot (1990) proposed two contrast functions. One is based on a discretization of the likelihood (2) as follows.

$$U_{\varepsilon,n}(\theta) = \frac{1}{\varepsilon^2}\sum_{k=1}^n b(X_{t_{k-1}},\theta)\sigma^{-2}(X_{t_{k-1}})\left(X_{t_k} - X_{t_{k-1}} - \frac{1}{2n}b(X_{t_{k-1}},\theta)\right). \tag{3}$$

The maximum contrast estimator (MCE) is defined by $U_{\varepsilon,n}(\hat{\theta}_{\varepsilon,n}^{(S)}) = \sup_{\theta\in\Theta} U_{\varepsilon,n}(\theta)$. She showed that the MCE $\hat{\theta}_{\varepsilon,n}^{(S)}$ has asymptotic efficiency under $(\varepsilon n)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$. The other is based on a Gaussian approximation of the sample path as follows.

$$\Lambda_{\varepsilon,n}(\theta) = -\frac{1}{2}\sum_{k=1}^n \frac{\left\{X_{t_k} - X_{t_k}^0(\theta) - \frac{H_{t_k}(\theta)}{H_{t_{k-1}}(\theta)}(X_{t_{k-1}} - X_{t_{k-1}}^0(\theta))\right\}^2}{H_{t_k}^2(\theta)\int_{t_{k-1}}^{t_k} H_s^{-2}(\theta)\sigma^2(X_s^0(\theta))ds}, \tag{4}$$

where $H_t(\theta) = \exp\left\{\int_0^t \frac{\partial b}{\partial u}(X_s^0(\theta),\theta)ds\right\}$ and $X_t^0(\theta)$ is the solution of the ordinary differential equation (ODE): $dX_t^0(\theta) = b(X_s^0(\theta),\theta)dt, X_0^0 = x_0$. The MCE $\hat{\theta}_{\varepsilon,n}^{(G)}$ defined by $\Lambda_{\varepsilon,n}(\hat{\theta}_{\varepsilon,n}^{(G)}) = \sup_{\theta\in\Theta}\Lambda_{\varepsilon,n}(\theta)$ has asymptotic efficiency under the weak condition that $\varepsilon\sqrt{n} = O(1)$ as $\varepsilon \to 0$ and $n \to \infty$. Laredo (1990) presented the following estimation procedure. First, solve the function $V(u)$ from the ODE: $\frac{\partial}{\partial u}V(u,\theta) = b(u,\theta)\sigma^{-2}(u), V(x_0,\theta) = 0$. Next, create a process $\{Y_t\}_{0\le t\le 1}$ such that $Y_t = X_{t_{k-1}} + \frac{t-t_{k-1}}{t_k - t_{k-1}}(X_{t_k} - X_{t_{k-1}})$ for $t_{k-1} \le t \le t_k$. Then, the contrast function of Laredo (1990) is as follows.

$$\tilde{l}_{\varepsilon,n}(\theta) = \frac{1}{\varepsilon^2}\left\{V(Y_1,\theta) - \frac{1}{2}\int_0^1 b^2(Y_s,\theta)\sigma^{-2}(Y_s)ds\right\}. \tag{5}$$

She proved that the MCE $\hat{\theta}_{\varepsilon,n}^{(L)}$ defined by $\tilde{l}_{\varepsilon,n}(\hat{\theta}_{\varepsilon,n}^{(L)}) = \sup_{\theta\in\Theta}\tilde{l}_{\varepsilon,n}(\theta)$ is asymptotically efficient under $(\varepsilon n^2)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$. Among these three estimators, $\hat{\theta}_{\varepsilon,n}^{(G)}$ is the best because of the weak condition of asymptotics.

In the same way as in Bibby and Sørensen (1995), it is possible to discuss a martingale estimating function $M_{\varepsilon,n}(\theta) = (M_{\varepsilon,n}^{(i)}(\theta))_{i=1,\dots,p}$, where for $i = 1,\dots,p$,

$$M_{\varepsilon,n}^{(i)}(\theta) = \frac{1}{\varepsilon^2}\sum_{k=1}^n\left(\frac{\partial b}{\partial\theta_i}\right)(X_{t_{k-1}},\theta)\sigma^{-2}(X_{t_{k-1}})(X_{t_{k-1}} - E_\theta[X_{t_k}|X_{t_{k-1}}]). \tag{6}$$

Uchida (2004) showed that under some regularity conditions, an M-estimator $\hat{\theta}_{\varepsilon,n}^{(M)}$ obtained from the estimating equation $M_{\varepsilon,n}(\theta) = 0$ is asymptotically efficient under the weakest condition that $\varepsilon \to 0$ and $n \to \infty$. Although the conditional expectation $E_\theta[X_{t_k}|X_{t_{k-1}}]$ does not generally have an explicit form for diffusion processes, the estimator $\hat{\theta}_{\varepsilon,n}^{(M)}$ is the best of the above four estimators in the case that the conditional expectation is explicitly obtained. In particular, when the drift term is linear, that is, $b(x,\theta) = \theta_1 + \theta_2 x$, the conditional expectation always has an explicit expression.

In this paper, we consider the one-dimensional SDE with a linear drift $b(x,\theta) = \theta_1 + \theta_2 x$. Compared with SDE (1), it seems that the model we treat is somewhat restricted. However, there are a number of examples for financial models and it is an appealing model from the viewpoint of asymptotically statistical estimation.

This paper is organized as follows. In section 2, we present an explicit martingale estimating function. An M-estimator obtained from the martingale estimating function has asymptotic efficiency as $\varepsilon \to 0$ and $n \to \infty$. Section 3 gives three examples and simulation studies. Section 4 is devoted to the proof of the result stated in section 2.

2 Martingale estimating functions

In this paper, we consider the following one-dimensional SDE

$$\begin{aligned} dX_t &= (\theta_1 + \theta_2 X_t)dt + \varepsilon\sigma(X_t)dw_t,\ t \in [0,1],\ \varepsilon \in (0,1], \\ X_0 &= x_0, \end{aligned} \tag{7}$$

where x_0 and ε are known constants, w is a one-dimensional standard Wiener process, θ_1 and θ_2 are unknown parameter and assume that $\theta_2 \neq 0$. Let $\theta_0 = (\theta_{1,0}, \theta_{2,0})$ be a true value of $\theta = (\theta_1, \theta_2)$ and assume that $\theta_0 \in \Theta \subset \mathbf{R}^2$. Let X_t^0 be the solution of the ODE: $dX_t^0 = (\theta_{1,0} + \theta_{2,0}X_t^0)dt,\ X_0^0 = x_0$. Define that $C_\uparrow^\infty(\mathbf{R};\mathbf{R})$ is a space of functions h which satisfies the following conditions: (i) $h : \mathbf{R} \to \mathbf{R}$ is continuously infinitely differentiable with respect to x, (ii) for $n \geq 0$, there exists $C > 0$ such that $|\partial^n/\partial x^n h(x)| \leq C(1+|x|)^C$ for $\forall x \in \mathbf{R}$. Let $I(\theta_0) = \left(I^{(i,j)}(\theta_0)\right)_{i,j=1,2}$ denote the asymptotic Fisher information matrix, where

$$\begin{aligned} I^{(1,1)}(\theta) &= \int_0^1 \sigma^{-2}(X_s^0)ds,\ I^{(2,2)}(\theta) = \int_0^1 (X_s^0)^2\sigma^{-2}(X_s^0)ds, \\ I^{(1,2)}(\theta) &= I^{(2,1)}(\theta) = \int_0^1 (X_s^0)\sigma^{-2}(X_s^0)ds. \end{aligned}$$

We make the following assumptions.

Assumption 1 *(i) Equation (1) has a unique strong solution on [0,1]. (ii) For* $\forall m > 0$, $\sup_{0\leq t\leq 1} E\left[\mid X_t \mid^m\right] < \infty$. *(iii)* $\sigma(x) \in C_\uparrow^\infty(\mathbf{R};\mathbf{R})$. *(iv)* $\inf_x \sigma^2(x) > 0$. *(v)* $I(\theta_0) = \left(I^{(i,j)}(\theta_0)\right)_{i,j=1,2}$ *is positive definite.*

Since Ito's formula yields that

$$X_t = e^{\theta_2 t}\left[X_0 - \frac{\theta_1}{\theta_2}(e^{-\theta_2 t} - 1) + \varepsilon\int_0^t e^{-\theta_2 s}\sigma(X_s)dw_s\right],$$

the conditional expectation is as follows:

$$E\left[X_{\frac{1}{n}}|X_0 = x\right] = e^{\frac{\theta_2}{n}}x + \frac{\theta_1}{\theta_2}(e^{\frac{\theta_2}{n}} - 1).$$

We then have the martingale estimating function $M_{\varepsilon,n}(\theta)$ of the model (7),

$$M_{\varepsilon,n}(\theta) = \begin{pmatrix} \varepsilon^{-2}\sum_{k=1}^n \sigma^{-2}(X_{t_{k-1}})\left\{X_{t_k} - e^{\frac{\theta_2}{n}}X_{t_{k-1}} - \frac{\theta_1}{\theta_2}(e^{\frac{\theta_2}{n}} - 1)\right\} \\ \varepsilon^{-2}\sum_{k=1}^n X_{t_{k-1}}\sigma^{-2}(X_{t_{k-1}})\left\{X_{t_k} - e^{\frac{\theta_2}{n}}X_{t_{k-1}} - \frac{\theta_1}{\theta_2}(e^{\frac{\theta_2}{n}} - 1)\right\} \end{pmatrix}. \tag{8}$$

Let $\hat{\theta}_{\varepsilon,n}^{(M)} = (\hat{\theta}_{\varepsilon,n}^{(1)}, \hat{\theta}_{\varepsilon,n}^{(2)})$ be the solution of the estimating equation $M_{\varepsilon,n}(\hat{\theta}_{\varepsilon,n}^{(M)}) = 0$. It then follows that

$$\begin{aligned}
\hat{\theta}_{\varepsilon,n}^{(1)} &= \left[n \log \left\{ \frac{g_2 g_3 - g_1 g_4}{(g_2)^2 - g_1 g_5} \right\} \right] \frac{1}{g_1} \left\{ \frac{g_2 g_3 - g_1 g_4}{(g_2)^2 - g_1 g_5} - 1 \right\}^{-1} \left[g_3 - g_2 \left\{ \frac{g_2 g_3 - g_1 g_4}{(g_2)^2 - g_1 g_5} \right\} \right], \\
\hat{\theta}_{\varepsilon,n}^{(2)} &= n \log \left\{ \frac{g_2 g_3 - g_1 g_4}{(g_2)^2 - g_1 g_5} \right\},
\end{aligned}$$

where

$$\begin{aligned}
g_1 &= \sum_{k=1}^{n} \frac{1}{\sigma^2(X_{t_{k-1}})}, \quad g_2 = \sum_{k=1}^{n} \frac{X_{t_{k-1}}}{\sigma^2(X_{t_{k-1}})}, \quad g_3 = \sum_{k=1}^{n} \frac{X_{t_k}}{\sigma^2(X_{t_{k-1}})}, \\
g_4 &= \sum_{k=1}^{n} \frac{X_{t_k} X_{t_{k-1}}}{\sigma^2(X_{t_{k-1}})}, \quad g_5 = \sum_{k=1}^{n} \frac{X_{t_{k-1}}^2}{\sigma^2(X_{t_{k-1}})}.
\end{aligned}$$

The main result is as follows.

Theorem 1 *Suppose that assumption 1 holds true. Then, as $\varepsilon \to 0$ and $n \to \infty$, $\hat{\theta}_{\varepsilon,n}^{(M)} \to \theta_0$ in probability and $\varepsilon^{-1}(\hat{\theta}_{\varepsilon,n}^{(M)} - \theta_0) \to N\Big(0, I^{-1}(\theta_0)\Big)$ in distribution.*

It follows from theorem 1 that the estimator $\hat{\theta}_{\varepsilon,n}^{(M)}$ is asymptotically efficient under the general condition that $\varepsilon \to 0$ and $n \to \infty$. This means that in the sense of the asymptotics with respect to ε and n, the estimator $\hat{\theta}_{\varepsilon,n}^{(M)}$ is better than the three estimators, $\hat{\theta}_{\varepsilon,n}^{(S)}$, $\hat{\theta}_{\varepsilon,n}^{(G)}$ and $\hat{\theta}_{\varepsilon,n}^{(L)}$.

3 Examples

In this section, we study the asymptotic behaviour of our estimators for three examples through simulations. In all examples, for each $\varepsilon = 0.1, 0.05, 0.01$ and $n = 5, 10, 50$, we simulated 1000 independent sample paths with $\theta = \theta_0$ (true parameter value) and the initial value x_0. The simulations were done by using the Euler-Maruyama scheme, see Kloeden and Platen (1992). For each sample path, the estimator $\hat{\theta}_{\varepsilon,n}^{(M)}$ in theorem 1 was calculated. In order to evaluate the estimator $\hat{\theta}_{\varepsilon,n}^{(M)}$, we also calculate the three estimators, $\hat{\theta}_{\varepsilon,n}^{(S)}$, $\hat{\theta}_{\varepsilon,n}^{(G)}$ and $\hat{\theta}_{\varepsilon,n}^{(L)}$. For the resulting 1000 values of the estimators, the means and the standard deviations of the estimators were computed.

3.1 The Ornstein-Uhlenbeck process

Consider the one-dimensional diffusion process defined by the SDE

$$dX_t = -\theta X_t dt + \varepsilon dw_t, \; t \in [0,1], \; \varepsilon \in (0,1], \; X_0 = x_0, \tag{9}$$

where x_0 and ε are known constants and $\theta > 0$ is an unknown parameter. This diffusion process is a version of the Ornstein-Uhlenbeck process. By setting that $\varepsilon = 0$, the dynamical system is $X_t^0(\theta) = x_0 e^{-\theta t}$. Furthermore, the asymptotic Fisher information is $I(\theta_0) = x_0^2(1 - e^{-2\theta_0})/(2\theta_0)$.

Since the first contrast function in Genon-Catalot (1990) is

$$U_{\varepsilon,n}(\theta) = -\frac{1}{\varepsilon^2} \sum_{k=1}^{n} \theta X_{t_{k-1}} \left(X_{t_k} - X_{t_{k-1}} + \frac{1}{2n} \theta X_{t_{k-1}} \right),$$

the MCE is given by

$$\hat{\theta}^{(S)}_{\varepsilon,n} = -\frac{\sum_{k=1}^n X_{t_{k-1}}(X_{t_k} - X_{t_{k-1}})}{\frac{1}{n}\sum_{k=1}^n X^2_{t_{k-1}}}.$$

The second contrast function in Genon-Catalot (1990) is

$$\Lambda_{\varepsilon,n}(\theta) = -\frac{\theta}{1-e^{-2\theta/n}}\sum_{k=1}^n (X_{t_k} - e^{-\theta/n}X_{t_{k-1}})^2.$$

Since the MCE does not have an explicit form, we will obtain an approximate solution by using numerical analysis. The contrast function in Laredo (1990) is given by

$$\tilde{l}_\varepsilon(\theta) = \frac{1}{2\varepsilon^2}\left\{-\theta(X_1^2 - x_0^2) - \theta^2\left\{\frac{1}{n}\sum_{k=1}^n X_{t_{k-1}}X_{t_k} + \frac{1}{3n}\sum_{k=1}^n (X_{t_k} - X_{t_{k-1}})^2\right\}\right\}.$$

Solving $\partial_\theta \tilde{l}_\varepsilon(\theta) = 0$, we obtain the MCE

$$\hat{\theta}^{(L)}_{\varepsilon,n} = -\frac{\frac{1}{2}(X_1^2 - x_0^2)}{\frac{1}{n}\sum_{k=1}^n X_{t_{k-1}}X_{t_k} + \frac{1}{3n}\sum_{k=1}^n (X_{t_k} - X_{t_{k-1}})^2}.$$

The estimator obtained from the martingale estimating function is

$$\hat{\theta}^{(M)}_{\varepsilon,n} = n\left\{\log\left(\sum X^2_{t_{k-1}}\right) - \log\left(\sum X_{t_k}X_{t_{k-1}}\right)\right\}.$$

Table 1 shows the means and the standard deviations of the four estimators for $\theta = 10$ and $x_0 = 20$. For the case that $n \leq 10$, though $\hat{\theta}^{(S)}_{\varepsilon,n}$ and $\hat{\theta}^{(L)}_{\varepsilon,n}$ have considerable biases, $\hat{\theta}^{(G)}_{\varepsilon,n}$ and $\hat{\theta}^{(M)}_{\varepsilon,n}$ perform quite well. For the case that n is large, there is no difference between the three estimators, $\hat{\theta}^{(L)}_{\varepsilon,n}$, $\hat{\theta}^{(G)}_{\varepsilon,n}$ and $\hat{\theta}^{(M)}_{\varepsilon,n}$. In this model, both $\hat{\theta}^{(G)}_{\varepsilon,n}$ and $\hat{\theta}^{(M)}_{\varepsilon,n}$ are acceptable. It is worth mentioning that $\hat{\theta}^{(M)}_{\varepsilon,n}$ has an simple expression while $\hat{\theta}^{(G)}_{\varepsilon,n}$ does not have an explicit form.

Table 1: (The Ornstein-Uhlenbeck process) The mean and standard deviation of the estimators, which are determined from 1000 independent simulated sample paths for $\theta = 10$ and $x_0 = 20$.

		$\hat{\theta}^{(S)}_{\varepsilon,n}$		$\hat{\theta}^{(L)}_{\varepsilon,n}$		$\hat{\theta}^{(G)}_{\varepsilon,n}$		$\hat{\theta}^{(M)}_{\varepsilon,n}$	
ε	n	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.
0.10	5	4.33002	0.00553	6.39373	0.00971	10.04956	0.04121	10.04987	0.04121
0.10	10	6.33991	0.00960	8.66041	0.01662	10.05079	0.02622	10.05101	0.02622
0.10	50	9.10495	0.01846	9.98337	0.02213	10.05047	0.02258	10.05070	0.02257
0.05	5	4.33006	0.00276	6.39385	0.00486	10.04998	0.02062	10.05006	0.02062
0.05	10	6.33979	0.00480	8.66030	0.00831	10.05061	0.01311	10.05066	0.01311
0.05	50	9.10479	0.00923	9.98336	0.01106	10.05045	0.01129	10.05051	0.01129
0.01	5	4.33009	0.00055	6.39392	0.00097	10.05027	0.00413	10.05027	0.00413
0.01	10	6.33970	0.00096	8.66017	0.00166	10.05040	0.00262	10.05040	0.00262
0.01	50	9.10468	0.00185	9.98327	0.00221	10.05037	0.00226	10.05037	0.00226

3.2 The geometric Brownian motion

We treat the one-dimensional diffusion process defined by

$$dX_t \quad = \quad \theta X_t dt + \varepsilon X_t dw_t,\ t \in [0,1],\ \varepsilon \in (0,1],\ X_0 = x_0, \tag{10}$$

where $\theta > 0$ is an unknown parameter, x_0 and ε are known constants. This model is called the geometric Brownian motion. Note that the dynamical system with $\varepsilon = 0$ is given by $X_t^0(\theta) = x_0 e^{\theta t}$ and the asymptotic Fisher information is $I(\theta_0) = 1$.

The first contrast function in Genon-Catalot (1990) is

$$U_{\varepsilon,n}(\theta) = \frac{1}{\varepsilon^2} \sum_{k=1}^{n} \frac{\theta}{X_{t_{k-1}}} (X_{t_k} - X_{t_{k-1}} - \frac{\theta}{2n} X_{t_{k-1}}).$$

Solving the estimating equation $\partial_\theta U_{\varepsilon,n}(\theta) = 0$, we have the MCE

$$\hat{\theta}_{\varepsilon,n}^{(S)} = \sum_{k=1}^{n} \frac{X_{t_k} - X_{t_{k-1}}}{X_{t_{k-1}}}.$$

Although the second contrast function in Genon-Catalot (1990) is given by

$$\Lambda_{\varepsilon,n}(\theta) = -\frac{n}{2x_0^2} \sum_{k=1}^{n} (e^{-\theta t_k} X_{t_k} - e^{-\theta t_{k-1}} X_{t_{k-1}})^2,$$

the MCE can not be explicitly derived. As in the previous subsection, we need to compute an approximate estimator. The contrast function in Laredo (1990) is described as

$$\tilde{l}_\varepsilon(\theta) = \frac{1}{\varepsilon^2} \left\{ \theta \log |X_1| - \theta \log |x_0| - \frac{\theta^2}{2} \right\}$$

and the MCE is

$$\hat{\theta}_{\varepsilon,n}^{(L)} = \log |X_1| - \log |x_0|.$$

The estimator based on the martingale estimating function is given by

$$\hat{\theta}_{\varepsilon,n}^{(M)} = n \left\{ \log \left(\sum \frac{X_{t_k}}{X_{t_{k-1}}} \right) - \log n \right\}.$$

Table 2 gives the means and the standard deviations of the four estimators in the situation where $\theta = 3$ and $x_0 = 2$. Since $\hat{\theta}_{\varepsilon,n}^{(S)}$ has a considerable bias in all cases, it should not be used in this setting. For the case that $\varepsilon = 0.1$, both $\hat{\theta}_{\varepsilon,n}^{(G)}$ and $\hat{\theta}_{\varepsilon,n}^{(L)}$ have small biases while $\hat{\theta}_{\varepsilon,n}^{(M)}$ performs quite well. Here we note that because $\hat{\theta}_{\varepsilon,n}^{(L)} = \log |X_1| - \log |x_0|$, $\hat{\theta}_{\varepsilon,n}^{(L)}$ is independent of n. This means that $\hat{\theta}_{\varepsilon,n}^{(L)}$ has asymptotic efficiency as $\varepsilon \to 0$. For this reason, $\hat{\theta}_{\varepsilon,n}^{(L)}$ is not a good estimator in the situation that ε is not so small. If $n = 50$ and $\varepsilon \geq 0.05$, we see that $\hat{\theta}_{\varepsilon,n}^{(G)}$ has a considerable bias. For the case that $n = 50$, there seems no big difference between the three estimators $\hat{\theta}_{\varepsilon,n}^{(L)}$, $\hat{\theta}_{\varepsilon,n}^{(G)}$ and $\hat{\theta}_{\varepsilon,n}^{(M)}$. Therefore we conclude that $\hat{\theta}_{\varepsilon,n}^{(M)}$ is better than the others in all cases.

Table 2: (The geometric Brownian motion) The mean and standard deviation of the estimators, which are determined from 1000 independent simulated sample paths for $\theta = 3$ and $x_0 = 2$.

		$\hat{\theta}^{(S)}_{\varepsilon,n}$		$\hat{\theta}^{(L)}_{\varepsilon,n}$		$\hat{\theta}^{(G)}_{\varepsilon,n}$		$\hat{\theta}^{(M)}_{\varepsilon,n}$	
ε	n	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.
0.10	5	4.10311	0.18635	2.99098	0.10200	3.01052	0.10506	2.99484	0.10244
0.10	10	3.49317	0.13819	2.99098	0.10200	3.03649	0.10952	2.99546	0.10246
0.10	50	3.08758	0.10858	2.99098	0.10200	3.24911	0.13646	2.99589	0.10228
0.05	5	4.10279	0.09299	2.99449	0.05100	2.99928	0.05159	2.99545	0.05110
0.05	10	3.49284	0.06894	2.99449	0.05100	3.00566	0.05243	2.99561	0.05111
0.05	50	3.08731	0.05422	2.99449	0.05100	3.05678	0.05681	2.99572	0.05107
0.01	5	4.10250	0.01857	2.99550	0.01020	2.99569	0.01022	2.99554	0.01020
0.01	10	3.49259	0.01377	2.99550	0.01020	2.99595	0.01024	2.99555	0.01020
0.01	50	3.08711	0.01083	2.99550	0.01020	2.99797	0.01036	2.99555	0.01020

3.3 The Cox-Ingersoll-Ross process

The Cox-Ingersoll-Ross model is defined by the following one-dimensional SDE

$$dX_t \;=\; (\alpha + \beta X_t)dt + \varepsilon\sqrt{X_t}dw_t,\ t \in [0,1],\ \varepsilon \in (0,1],\ X_0 = x_0, \tag{11}$$

where x_0 and ε are known constants, α and β are unknown parameters and we assume that $\alpha > 0$ and $\beta < 0$. Let $\theta = (\alpha, \beta)$. The dynamical system is given by $X_t^0(\theta) = \left[(\alpha + \beta x_0)e^{\beta t} - \alpha\right]/\beta$. The components of the asymptotic Fisher information matrix are

$$\begin{aligned} I^{(1,1)}(\theta_0) &= -\frac{1}{\alpha_0}\left\{\beta_0 + \log(\beta_0 C - \alpha_0) - \log(\beta_0 C e^{\beta_0} - \alpha_0)\right\}, \\ I^{(2,2)}(\theta_0) &= \frac{1}{\beta_0}(Ce^{\beta_0} - \alpha_0 - C),\ I^{(1,2)}(\theta_0) = I^{(2,1)}(\theta_0) = 1, \end{aligned}$$

where $C = x_0 + \alpha_0/\beta_0$.

The first contrast function in Genon-Catalot (1990) is expressed as follows:

$$U_{\varepsilon,n}(\theta) = \frac{1}{\varepsilon^2}\sum_{k=1}^n \frac{\alpha + \beta X_{t_{k-1}}}{X^2_{t_{k-1}}}\left\{X_{t_k} - X_{t_{k-1}} - \frac{1}{2n}(\alpha + \beta X_{t_{k-1}})\right\}.$$

The solutions which satisfy $\partial_\alpha U_{\varepsilon,n}(\theta) = \partial_\beta U_{\varepsilon,n}(\theta) = 0$ are

$$\hat{\alpha}^{(S)}_{\varepsilon,n} = \frac{\sum_{k=1}^n (X_{t_k} - X_{t_{k-1}}) - \frac{1}{n}\sum_{k=1}^n X_{t_{k-1}}\sum_{k=1}^n \frac{X_{t_k} - X_{t_{k-1}}}{X_{t_{k-1}}}}{1 - \frac{1}{n^2}\sum_{k=1}^n X_{t_{k-1}}\sum_{k=1}^n X^{-1}_{t_{k-1}}},$$

$$\hat{\beta}^{(S)}_{\varepsilon,n} = \frac{\sum_{k=1}^n \frac{X_{t_k} - X_{t_{k-1}}}{X_{t_{k-1}}} - \frac{1}{n}\sum_{k=1}^n (X_{t_k} - X_{t_{k-1}})\sum_{k=1}^n X^{-1}_{t_{k-1}}}{1 - \frac{1}{n^2}\sum_{k=1}^n X_{t_{k-1}}\sum_{k=1}^n X^{-1}_{t_{k-1}}}.$$

The second contrast function in Genon-Catalot (1990) is

$$\Lambda_{\varepsilon,n}(\theta) = -\frac{1}{2}\sum_{k=1}^n \frac{\left\{X_{t_k} - X^0_{t_k} - e^{\beta/n}(X_{t_{k-1}} - X^0_{t_{k-1}})\right\}^2}{e^{2\beta t_k}\int_{t_{k-1}}^{t_k} e^{-2\beta s}\frac{1}{\beta}\left[(\alpha + \beta x_0)e^{\beta s} - \alpha\right] ds}$$

and the MCE is obtained by a numerical method. The contrast function in Laredo (1990) is

$$\tilde{l}_\varepsilon(\theta) = \frac{1}{\varepsilon^2}\left\{\alpha \log|X_1| + \beta X_1 - \alpha\log|x_0| - \beta x_0 - \frac{1}{2}(\alpha^2 I + 2\alpha\beta + \beta^2 J)\right\},$$

where

$$I = \frac{1}{n}\sum_{k=1}^{n}\frac{\log|X_{t_k}| - \log|X_{t_{k-1}}|}{X_{t_k} - X_{t_{k-1}}}, \quad J = \frac{1}{2n}\sum_{k=1}^{n}(X_{t_k} + X_{t_{k-1}})$$

and the MCEs can be expressed as

$$\begin{aligned}
\hat{\alpha}^{(L)}_{\varepsilon,n} &= \frac{1}{IJ-1}\left[J(\log|X_1| - \log|x_0|) - X_1 + x_0\right],\\
\hat{\beta}^{(L)}_{\varepsilon,n} &= \frac{1}{IJ-1}\left[I(X_1 - x_0) - \log|X_1| + \log|x_0|\right].
\end{aligned}$$

By using the martingale estimating functions, the estimators are given by

$$\begin{aligned}
\hat{\alpha}^{(M)}_{\varepsilon,n} &= \frac{n\sum_{k=1}^{n} X_{t_k} - \sum \frac{X_{t_k}}{X_{t_{k-1}}}\sum_{k=1}^{n} X_{t_{k-1}}}{n\sum_{k=1}^{n}\frac{X_{t_k}}{X_{t_{k-1}}} - \sum_{k=1}^{n}\frac{1}{X_{t_{k-1}}}\sum_{k=1}^{n} X_{t_{k-1}} - n^2 + \sum_{k=1}^{n}\frac{1}{X_{t_{k-1}}}\sum_{k=1}^{n} X_{t_{k-1}}} n\hat{\beta}^{(M)}_{\varepsilon,n},\\
\hat{\beta}^{(M)}_{\varepsilon,n} &= n\left\{\log\left(n\sum_{k=1}^{n}\frac{X_{t_k}}{X_{t_{k-1}}} - \sum_{k=1}^{n}\frac{1}{X_{t_{k-1}}}\sum_{k=1}^{n} X_{t_k}\right) - \log\left(n^2 - \sum_{k=1}^{n}\frac{1}{X_{t_{k-1}}}\sum_{k=1}^{n} X_{t_{k-1}}\right)\right\}.
\end{aligned}$$

For $\alpha = 1$, $\beta = -3$ and $x_0 = 10$, the simulation results of the four estimators for α and β are given in tables 3 and 4, respectively. For the case that $n = 5$, $\hat{\alpha}^{(S)}_{\varepsilon,n}$ and $\hat{\alpha}^{(L)}_{\varepsilon,n}$ has considerable biases while both $\hat{\alpha}^{(G)}_{\varepsilon,n}$ and $\hat{\alpha}^{(M)}_{\varepsilon,n}$ work well. We can say that $\hat{\alpha}^{(M)}_{\varepsilon,n}$ is a good estimator with a small variance in all cases. Speaking of β, we see that $\hat{\beta}^{(M)}_{\varepsilon,n}$ performs very well in all cases. Note that both $\hat{\alpha}^{(L)}_{\varepsilon,n}$ and $\hat{\beta}^{(L)}_{\varepsilon,n}$ also work very well when $n = 50$.

Table 3: (The Cox-Ingersoll-Ross process) The mean and standard deviation of the estimators for α, which are determined from 1000 independent simulated sample paths for $\alpha = 1$, $\beta = -3$ and $x_0 = 10$.

		$\hat{\alpha}^{(S)}_{\varepsilon,n}$		$\hat{\alpha}^{(L)}_{\varepsilon,n}$		$\hat{\alpha}^{(G)}_{\varepsilon,n}$		$\hat{\alpha}^{(M)}_{\varepsilon,n}$	
ε	n	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.
0.10	5	0.75954	0.16780	0.65683	0.21300	0.99537	0.71616	1.01175	0.22945
0.10	10	0.87598	0.19225	0.91668	0.22059	1.02786	0.62927	1.01485	0.22578
0.10	50	0.98675	0.21811	1.00129	0.22440	1.30557	0.55896	1.01685	0.22541
0.05	5	0.75527	0.08348	0.65610	0.10621	1.00736	0.35551	1.00502	0.11408
0.05	10	0.86876	0.09558	0.91547	0.10995	1.01521	0.31698	1.00593	0.11220
0.05	50	0.97686	0.10850	1.00001	0.11187	1.08924	0.30132	1.00653	0.11212
0.01	5	0.75315	0.01664	0.65503	0.02120	1.00406	0.07078	1.00195	0.02273
0.01	10	0.86552	0.01904	0.91402	0.02194	1.00443	0.06332	1.00203	0.02235
0.01	50	0.97258	0.02162	0.99844	0.02232	1.00748	0.06175	1.00209	0.02234

Table 4: (The Cox-Ingersoll-Ross process) The mean and standard deviation of the estimators for β, which are determined from 1000 independent simulated sample paths for $\alpha = 1$, $\beta = -3$ and $x_0 = 10$.

		$\hat{\beta}^{(S)}_{\varepsilon,n}$		$\hat{\beta}^{(L)}_{\varepsilon,n}$		$\hat{\beta}^{(G)}_{\varepsilon,n}$		$\hat{\beta}^{(M)}_{\varepsilon,n}$	
ε	n	mean	s.d.	mean	s.d.	mean	s.d.	mean	s.d.
0.10	5	-2.25774	0.04769	-2.82332	0.07713	-3.00078	0.14510	-3.00403	0.08697
0.10	10	-2.59519	0.06264	-2.95597	0.08204	-3.00369	0.13070	-3.00491	0.08460
0.10	50	-2.91691	0.07978	-3.00013	0.08458	-3.04132	0.11921	-3.00552	0.08473
0.05	5	-2.25777	0.02382	-2.82455	0.03855	-3.00376	0.07245	-3.00352	0.04343
0.05	10	-2.59456	0.03129	-2.95712	0.04099	-3.00444	0.06594	-3.00380	0.04225
0.05	50	-2.91551	0.03985	-3.00125	0.04226	-3.01484	0.06336	-3.00399	0.04231
0.01	5	-2.25824	0.00476	-2.82571	0.00771	-3.00454	0.01446	-3.00419	0.00868
0.01	10	-2.59494	0.00625	-2.95824	0.00819	-3.00457	0.01320	-3.00422	0.00844
0.01	50	-2.91577	0.00796	-3.00235	0.00845	-3.00501	0.01297	-3.00424	0.00846

4 Proof

Proof of Theorem 1. In order to prove theorem 1, it suffices to show (A1)-(A6) in Uchida (2004). It is so easy to prove (A1)-(A5) that we will do just (A6). That is,

$$\sup_{\theta\in\bar{\Theta}}\left|\left\{\sum_{k=1}^{n}\frac{\partial b}{\partial\theta_i}(X_{t_{k-1}},\theta)\sigma^{-2}(X_{t_{k-1}})\frac{\partial F}{\partial\theta_j}(X_{t_{k-1}},\theta)\right\}-I^{(i,j)}(\theta)\right|\to 0$$

in probability as $\varepsilon\to 0$ and $n\to\infty$, where $b(x,\theta)=\theta_1+\theta_2 x$ and $F(x,\theta)=E_\theta[X_{1/n}|X_0=x]$. It follows from Uchida (2004) that

$$\sup_{\theta\in\bar{\Theta}}\left|\frac{1}{n}\left\{\sum_{k=1}^{n}\frac{\partial b}{\partial\theta_i}(X_{t_{k-1}},\theta)\sigma^{-2}(X_{t_{k-1}})\frac{\partial b}{\partial\theta_i}(X_{t_{k-1}},\theta)\right\}-I^{(i,j)}(\theta)\right|\to 0$$

in probability as $\varepsilon\to 0$ and $n\to\infty$. Thus, it is enough to show that

$$\sup_{\theta\in\bar{\Theta}}\left|\sum_{k=1}^{n}\frac{\partial b}{\partial\theta_i}(X_{t_{k-1}},\theta)\sigma^{-2}(X_{t_{k-1}})\left\{\frac{\partial F}{\partial\theta_j}(X_{t_{k-1}},\theta)-\frac{1}{n}\frac{\partial b}{\partial\theta_j}(X_{t_{k-1}},\theta)\right\}\right|\to 0 \qquad (12)$$

in probability as $\varepsilon\to 0$ and $n\to\infty$. Setting that for $i,j=1,2$,

$$\alpha_{i,j}(x,\theta)=\frac{\partial b}{\partial\theta_i}(x,\theta)\sigma^{-2}(x)\left\{\frac{\partial F}{\partial\theta_j}(x,\theta)-\frac{1}{n}\frac{\partial b}{\partial\theta_j}(x,\theta)\right\},$$

one has

$$\begin{aligned}
\alpha_{1,1}(x,\theta) &= \frac{1}{\sigma^2(x)}\left[\frac{1}{\theta_2}(e^{\frac{\theta_2}{n}}-1)-\frac{1}{n}\right],\\
\alpha_{2,1}(x,\theta) &= \frac{x}{\sigma^2(x)}\left[\frac{1}{\theta_2}(e^{\frac{\theta_2}{n}}-1)-\frac{1}{n}\right],\\
\alpha_{1,2}(x,\theta) &= \frac{1}{\sigma^2(x)}\left[\frac{1}{n}xe^{\frac{\theta_2}{n}}-\frac{\theta_1}{\theta_2^2}(e^{\frac{\theta_2}{n}}-1)+\frac{\theta_1}{n\theta_2}e^{\frac{\theta_2}{n}}-\frac{1}{n}x\right],\\
\alpha_{2,2}(x,\theta) &= \frac{x}{\sigma^2(x)}\left[\frac{1}{n}xe^{\frac{\theta_2}{n}}-\frac{\theta_1}{\theta_2^2}(e^{\frac{\theta_2}{n}}-1)+\frac{\theta_1}{n\theta_2}e^{\frac{\theta_2}{n}}-\frac{1}{n}x\right].
\end{aligned}$$

It is obvious that for $i,j=1,2$,

$$\left|\sum_{k=1}^{n} \alpha_{i,j}(X_{t_{k-1}},\theta)\right| \to 0$$

in probability as $\varepsilon \to 0$ and $n \to \infty$. Let C be a constant independent of ε and n. Define that $\theta_A = (\theta_{A1}, \theta_{A2})$ and $\theta_B = (\theta_{B1}, \theta_{B2})$. In order to prove (12), it is sufficient to show the following inequalities (cf. theorem 20 in Appendix I of Ibragimov and Has'minskii (1981)): For $i,j=1,2$ and $m>1$,

$$E\left[\left\{\sum_{k=1}^{n} \alpha_{i,j}(X_{t_{k-1}},\theta)\right\}^{2m}\right] \leq C \tag{13}$$

$$E\left[\left\{\sum_{k=1}^{n} \left[\alpha_{i,j}(X_{t_{k-1}},\theta_A) - \alpha_{i,j}(X_{t_{k-1}},\theta_B)\right]\right\}^{2m}\right] \leq C\left|\theta_A - \theta_B\right|^{2m}. \tag{14}$$

We only show that $\alpha_{1,1}$ satisfies the above two inequalities. For the proof of (13),

$$\begin{aligned}
E\left[\left\{\sum_{k=1}^{n} \alpha_{1,1}(X_{t_{k-1}},\theta)\right\}^{2m}\right] &= E\left[\left\{\sum_{k=1}^{n} \frac{1}{\sigma^2(X_{t_{k-1}})}\left[\frac{1}{\theta_2}(e^{\frac{\theta_2}{n}}-1) - \frac{1}{n}\right]\right\}^{2m}\right] \\
&\leq C\frac{1}{n}\left\{\sum_{k=1}^{n} E\left[\left\{\frac{1}{\sigma^2(X_{t_{k-1}})}\right\}^{2m}\right]\right\} \\
&\leq C.
\end{aligned}$$

Moreover, for the proof of (14),

$$\begin{aligned}
& E\left[\left\{\sum_{k=1}^{n} \left[\alpha_{1,1}(X_{t_{k-1}},\theta_A) - \alpha_{1,1}(X_{t_{k-1}},\theta_B)\right]\right\}^{2m}\right] \\
\leq\ & n^{2m-1}\sum_{k=1}^{n} E\left[\left\{\frac{1}{\sigma^2(X_{t_{k-1}})}\left[\frac{1}{\theta_{A2}}(e^{\frac{\theta_{A2}}{n}}-1) - \frac{1}{\theta_{B2}}(e^{\frac{\theta_{B2}}{n}}-1)\right]\right\}^{2m}\right] \\
\leq\ & Cn^{-1}(\theta_{A2}-\theta_{B2})^{2m}\left\{\sum_{k=1}^{n} E\left[\frac{1}{\left[\sigma(X_{t_{k-1}})\right]^{4m}}\right]\right\} \\
\leq\ & C\left|\theta_A - \theta_B\right|^{2m}.
\end{aligned}$$

The rest can be shown in the same way as in the proof of $\alpha_{1,1}$. This completes the proof.

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