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# One-step estimators for diffusion processes with small dispersion parameters from discrete observations

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**Abstract.** For a multi-dimensional diffusion process with small dispersion parameter $\varepsilon$, an asymptotically efficient estimator of the drift parameter is studied. When the sample path is observed at $n$ regularly spaced time points $t_k = k/n$, $k = 0, 1, \ldots, n$, we investigate asymptotic properties of a one-step estimator derived from an approximate estimating function under the situation when $\varepsilon \to 0$ and $n \to \infty$ simultaneously.

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**Key words and phrases:** Approximate martingale estimating function, diffusion process with small noise, discrete time observation, parametric inference.

**Abbreviated Title:** One-step estimator for small diffusions.

## 1 Introduction

We consider a family of $d$-dimensional diffusion processes defined by the following stochastic differential equations:

$$\begin{aligned} dX_t &= b(X_t, \theta)dt + \varepsilon\sigma(X_t)dw_t, \ t \in [0, 1], \ \varepsilon \in (0, 1], \\ X_0 &= x_0, \end{aligned} \tag{1}$$

where $\theta \in \bar{\Theta}$, $\Theta$ is an open bounded convex subset of $\mathbf{R}^p$ and $\bar{\Theta}$ is the closure of $\Theta$. Further, $x_0$ and $\varepsilon$ are known constants, $b$ is an $\mathbf{R}^d$-valued function defined on $\mathbf{R}^d \times \bar{\Theta}$, $\sigma$ is an $\mathbf{R}^d \otimes \mathbf{R}^r$-valued function defined on $\mathbf{R}^d$, and $w$ is an $r$-dimensional standard Wiener process. We assume that the drift $b$ is known apart from the parameter $\theta$. The type of data considered in this paper is discrete observations of $X$ at $n$ regularly spaced time points $t_k = k/n$ on the fixed interval $[0, 1]$, that is, $(X_{t_k})_{0 \le k \le n}$. The asymptotics we consider is when $\varepsilon \to 0$ and $n \to \infty$ simultaneously.

Parametric inference based on small diffusion asymptotics for diffusion processes is well-developed together with applications to mathematical finance. For continuously observed small diffusions, see Kutoyants (1984, 1994) and Yoshida (1992, 2003). Moreover, there are a number of papers on estimation of the parameter for a discretely observed diffusion process with a small dispersion parameter, see Genon-Catalot (1990), Laredo (1990), Sørensen (2000), Sørensen and Uchida (2003) and Uchida (2004a, 2004b, 2004c). As for the estimation of the drift parameter, it seems that the following three papers are fundamental. Genon-Catalot (1990) presented two kinds of contrast functions, $U(\theta)$ and $\Lambda(\theta)$ (see p. 102 and p. 103 in Genon-Catalot (1990)). She proved that an estimator obtained from the first contrast function $U(\theta)$ is asymptotically efficient under $(\varepsilon n)^{-1} \to 0$ and an estimator derived from the second contrast function $\Lambda(\theta)$ has asymptotic efficiency under $\varepsilon\sqrt{n} = O(1)$. Laredo (1990) studied an estimator which has asymptotic efficiency under $\left(\varepsilon n^2\right)^{-1} \to 0$. Recently, Uchida (2004a) proposed an approximate martingale estimating function based on a martingale estimating function proposed in Bibby and Sørensen (1995). The estimator obtained from the estimating function has asymptotic efficiency under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ for a natural number $\ell$. We note that the first contrast function $U(\theta)$ in Genon-Catalot (1990) and the estimating function in Uchida (2004a) are explicitly obtained, while the contrast function in Laredo (1990) and the second contrast function $\Lambda(\theta)$ in Genon-Catalot (1990) do not generally have explicit forms. Furthermore, even if all contrast functions have explicit expressions, the estimators are not always derived from the contrast functions because of their complicated forms.

In order to overcome this difficulty, we study a one-step estimator which handles well even in such ill cases. The Newton-Raphson method is widely known as a practical method to obtain an approximate solution when we can not solve a non-random equation. The one-step estimator is constructed in the same way as this method. For details of one-step estimators, see, for example, Ferguson (1996), Van der Vaart (1998) and Sakamoto and Yoshida (1999).

This paper is organized as follows. In section 2, we propose a one-step estimator and describe the conditions that the one-step estimator has consistency, asymptotic normality and asymptotic efficiency under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$, $n \to \infty$ for a natural number $\ell$. Section 3 gives two examples and simulation studies. For the first example, the contrast function in Laredo (1990) can not be explicitly obtained. The contrast function $\Lambda(\theta)$ in Genon-Catalot (1990) does not have an explicit form in both examples. The proof of the result is given in section 4.

## 2 One-step estimator

Let $\theta_0$ be a true value of $\theta$ and assume that $\theta_0 \in \Theta$. Let $X_t^0$ be the solution of the ordinary differential equation: $dX_t^0 = b(X_t^0, \theta_0)dt$, $X_0^0 = x_0$. Let $C_\uparrow^{\infty,3}(\mathbf{R}^d \times \Theta; \mathbf{R}^d)$ be a space of functions $f$ which satisfies the following conditions: (i) $f : \mathbf{R}^d \times \Theta \longrightarrow \mathbf{R}^d$ is infinitely differentiable with respect to $x$ and continuously differentiable with respect to $\theta$ up to order 3, (ii) for $\mathbf{n}, \nu$ satisfying $\mid \mathbf{n} \mid \geq 0, 0 \leq \mid \nu \mid \leq 3$, there exists $C > 0$ such that $\sup_{\theta \in \Theta} \mid \delta^\nu \partial^{\mathbf{n}} f(x, \theta) \mid \leq C(1+ \mid x \mid)^C$ for $\forall x \in \mathbf{R}^d$. Here, $\mathbf{n} = (n_1, n_2, \ldots, n_d)$, $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ are multi-indices, $\mid \mathbf{n} \mid = n_1 + n_2 + \cdots + n_d$, $\mid \nu \mid = \nu_1 + \nu_2 + \cdots + \nu_p$, $\partial^{\mathbf{n}} = \partial_1^{n_1}\partial_2^{n_2}\cdots\partial_d^{n_d}$, $\delta^\nu = \delta_1^{\nu_1}\partial_2^{\nu_2}\cdots\partial_p^{\nu_p}$, $\partial_i = \partial/\partial_{x_1}$ and $\delta_j = \partial/\partial_{\theta_j}$. Let $C_\uparrow^\infty(\mathbf{R}^d; \mathbf{R}^d \otimes \mathbf{R}^r)$ be a space of functions $h$ which satisfies the following conditions: (i) $h : \mathbf{R}^d \longrightarrow \mathbf{R}^d \otimes \mathbf{R}^r$ is continuously infinitely differentiable with respect to $x$, (ii) for $\mid \mathbf{n} \mid \geq 0$, there exists $C > 0$ such that $\mid \partial^{\mathbf{n}} h(x) \mid \leq C(1+ \mid x \mid)^C$ for $\forall x \in \mathbf{R}^d$. Moreover, let $\sigma^T$ be the transposition of $\sigma$, $\xrightarrow{\mathrm{P}}$ be the convergence in probability and $\xrightarrow{\mathrm{d}}$ be the convergence in distribution henceforth.

In this paper we make the following assumptions.

(A1) Equation (1) has a unique strong solution on [0,1].

(A2) For $\forall m > 0$, $\sup_{0 \leq t \leq 1} E\left[\mid X_t \mid^m\right] < \infty$.

(A3) $b(x, \theta) \in C_\uparrow^{\infty,3}(\mathbf{R}^d \times \bar{\Theta}; \mathbf{R}^d), \sigma(x) \in C_\uparrow^\infty(\mathbf{R}^d; \mathbf{R}^d \otimes \mathbf{R}^r)$.

(A4) $\inf_x \det\left[\sigma\sigma^T(x)\right] > 0, \left[\sigma\sigma^T(x)\right]^{-1} \in C_\uparrow^\infty(\mathbf{R}^d; \mathbf{R}^d \otimes \mathbf{R}^d)$.

(A5) $I(\theta_0) = \left(I^{i.j}(\theta_0)\right)_{i,j=1,2,\ldots,p}$ is positive definite, where

$$I^{i.j}(\theta) \quad = \quad \int_0^1 \left[\delta_i b(X_s^0, \theta)\right]^T \left[\sigma\sigma^T(X_s^0)\right]^{-1} \left[\delta_j b(X_s^0, \theta)\right] ds.$$

For $\ell \in \mathbf{N}$, an approximate martingale estimating function $G_{\varepsilon,n,\ell}(\theta) = \big(G^{(i)}_{\varepsilon,n,\ell}(\theta)\big)_{i=1,2,\dots,p}$ proposed in Uchida (2004a) is as follows:

$$
\begin{aligned}
G^{(i)}_{\varepsilon,n,\ell}(\theta) &= \varepsilon^{-2}\sum_{k=1}^{n}\left[\delta_i b(X_{t_{k-1}},\theta)\right]^T\left[\sigma\sigma^T(X_{t_{k-1}})\right]^{-1}P_{k,\ell}(\theta), \qquad (2)\\
P_{k,\ell}(\theta) &= X_{t_k}-\sum_{j=0}^{\ell}\frac{1}{j!n^j}\tilde{L}^j_\theta g(X_{t_{k-1}}),\\
\tilde{L}_\theta g(x) &= \sum_{i=1}^{d} b^i(x,\theta)\partial_i g(x),
\end{aligned}
$$

where $g(x)=x$ and $b^i(x,\theta)$ is the $i$-th element of $b(x,\theta)$. For example, when $\ell = 2$,

$$
G^{(i)}_{\varepsilon,n,2}(\theta) = \varepsilon^{-2}\sum_{k=1}^{n}\left[\delta_i b(X_{t_{k-1}},\theta)\right]^T\left[\sigma\sigma^T(X_{t_{k-1}})\right]^{-1}\left(X_{t_k}-X_{t_{k-1}}-\frac{1}{n}b(X_{t_{k-1}},\theta)\right).
$$

An estimator obtained from the estimating function $G_{\varepsilon,n,\ell}(\theta)$ has the following asymptotic properties. For details of the result, see Uchida (2004a).

**Theorem 1** *(Uchida (2004a)) Let $\ell \in \mathbf{N}$. Assume (A1)-(A5). Then, an estimator $\hat{\theta}_{\varepsilon,n,\ell}$, which solves $G_{\varepsilon,n,\ell}(\theta)=0$, exists with a probability tending to one as $\varepsilon \to 0$ and $n \to \infty$ under $P_{\theta_0}$, and $\hat{\theta}_{\varepsilon,n,\ell} \xrightarrow{p} \theta_0$ as $\varepsilon \to 0$ and $n \to \infty$. Moreover, if $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$, $\varepsilon^{-1}(\hat{\theta}_{\varepsilon,n,\ell}-\theta_0) \xrightarrow{\mathrm{d}} N(0, I^{-1}(\theta_0))$.*

It follows from theorem 1 that an estimator derived from the estimating function $G_{\varepsilon,n,\ell}(\theta)$ has asymptotic efficiency under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$. It seems that the estimation procedure works well. However, there is a disadvantage that we can not generally obtain an explicit estimator because the estimating function $G_{\varepsilon,n,\ell}(\theta)$ has complicated form when $l$ is large.

In order to conquer this difficulty, we suggest a one-step estimator as follows. For an initial estimator $\hat{\theta}^{(0)}_{\varepsilon,n,\ell}$, a one-step estimator $\hat{\theta}^{(1)}_{\varepsilon,n,\ell}$ is defined by

$$
\hat{\theta}^{(1)}_{\varepsilon,n,\ell} = \hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \left[\partial_\theta G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right]^{-1} G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}). \qquad (3)
$$

We then have the following theorem.

**Theorem 2** *Let $\ell \in \mathbf{N}$. Assume (A1)-(A5). Moreover, assume that an initial estimator $\hat{\theta}^{(0)}_{\varepsilon,n,\ell}$ satisfies $\varepsilon^{-1}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0) = O_P(1)$ under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$. Then, under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$,*
*(i) $\varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,\ell}-\theta_0) - \left[I(\theta_0)\right]^{-1}\varepsilon G_{\varepsilon,n,\ell}(\theta_0) = o_P(1)$,*
*(ii) $\varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,\ell}-\theta_0) \xrightarrow{\mathrm{d}} N\Big(0, \left[I(\theta_0)\right]^{-1}\Big)$.*

By theorem 2, in order to get an asymptotically efficient estimator it is essential to obtain an initial estimator with the property that $\varepsilon^{-1}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0) = O_P(1)$ under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$. One of sufficient conditions of an initial estimator is that $\varepsilon^{-1}(\hat{\theta}^{(0)}_{\varepsilon,n}-\theta_0) = O_P(1)$ as $\varepsilon \to 0$ and $n \to \infty$. Note that the estimator $\hat{\theta}^{(0)}_{\varepsilon,n}$ is independent of $\ell$. For estimators satisfying that $\varepsilon^{-1}(\hat{\theta}^{(0)}_{\varepsilon,n}-\theta_0) = O_P(1)$ as $\varepsilon \to 0$ and $n \to \infty$, see Uchida (2004b, 2004c).

## 3 Examples

In this section, we give two examples and examine the performance of the one-step estimator through simulation studies. One thousand sample paths are generated by the Euler-Maruyama scheme, see Kloeden and Platen (1992). We set that $\varepsilon = 0.01, 0.05, 0.1$ and $n = 10, 50$. In order to obtain a one-step estimator $\hat{\theta}^{(1)}_{\varepsilon,n,3}$, the approximate martingale estimating function defined by (2) is treated with $\ell = 3$. For an initial estimator, we use the estimator obtained from an approximate martingale estimating function based on an eigenfunction, see Uchida (2004c). In order to evaluate the estimator $\hat{\theta}^{(1)}_{\varepsilon,n,3}$, we also calculate the estimator obtained from the first contrast function $U(\theta)$ in Genon-Catalot (1990) and the estimator in Laredo (1990), which are denoted by $\hat{\theta}^{(S)}_{\varepsilon,n}$ and $\hat{\theta}^{(L)}_{\varepsilon,n}$, respectively. In both examples, the second contrast function $\Lambda(\theta)$ in Genon-Catalot (1990) does not have an explicit form. As for the contrast function $l_\varepsilon(\theta)$ presented by Laredo (1990), it is not explicitly obtained in the first example. For each of the estimators, the means and standard deviations are computed.

### 3.1 Non-linear model 1

We consider a non-linear model defined by the stochastic diffusion equation,

$$dX_t \;=\; \theta \cos X_t dt + \varepsilon X_t dw_t,\ t \in [0,1],\ \varepsilon \in (0,1],\ X_0 = x_0,$$

where $x_0$ and $\varepsilon$ are known constants and $\theta$ is an unknown parameter. Let $X^0_t$ be the solution of the differential equation, $dX^0_t = \theta \cos X^0_t dt, X^0_0 = x_0$. Since $X^0_t$ does not have an explicit form, we can not explicitly get the contrast function $\Lambda(\theta)$. The contrast function $l_\varepsilon(\theta)$ is not also explicitly derived for this diffusion model. Note that in order to obtain $l_\varepsilon(\theta)$, we need to have the function $V(x,\theta)$ satisfying that

$$V(x,\theta) - V(x_0,\theta) = \int_{x_0}^{x} \frac{\theta \cos u}{u^2} du.$$

Unfortunately, the function $V(x,\theta)$ can not have an explicit expression. For this reason, we treat the estimating function defined by (2).

For $\ell = 3$, the estimating function $G_{\varepsilon,n,3}(\theta)$ in (2) is given by

$$\begin{aligned} G_{\varepsilon,n,3}(\theta) &= \frac{1}{\varepsilon^2}\left\{ \sum_{k=1}^{n} \frac{\cos X_{t_{k-1}}}{X^2_{t_{k-1}}}(X_{t_k} - X_{t_{k-1}}) - \frac{\theta}{n}\sum_{k=1}^{n}\frac{\cos^2 X_{t_{k-1}}}{X^2_{t_{k-1}}} \right. \\ &\quad \left. + \frac{\theta^2}{2n^2}\sum_{k=1}^{n}\frac{\cos^2 X_{t_{k-1}} \sin X_{t_{k-1}}}{X^2_{t_{k-1}}} - \frac{\theta^3}{6n^3}\sum_{k=1}^{n}\frac{\cos^2 X_{t_{k-1}}(\sin^2 X_{t_{k-1}} - \cos^2 X_{t_{k-1}})}{X_{t_{k-1}}} \right\}. \end{aligned}$$

Since the estimating equation $G_{\varepsilon,n,3}(\theta) = 0$ is not solvable, we consider a one-step estimator.

In the same way as in Uchida (2004c), an initial estimator is obtained from the estimating function

$$H_{\varepsilon,n}(\theta) = \sum_{k=1}^{n} \frac{\cos X_{t_{k-1}}}{X^2_{t_{k-1}}}\left\{\varphi(X_{t_k}) - e^{-\lambda(\theta)/n}\varphi(X_{t_{k-1}})\right\},$$

where $\lambda(\theta) = \theta$ and

$$\varphi(x) = \frac{\cos(x/2) + \sin(x/2)}{\cos(x/2) - \sin(x/2)}.$$

It then follows that the initial estimator $\hat{\theta}^{(0)}_{\varepsilon,n}$ satisfying that $H_{\varepsilon,n}(\theta) = 0$ is given by

$$\hat{\theta}^{(0)}_{\varepsilon,n} = n\left\{\log\sum_{k=1}^{n}\frac{\cos X_{t_{k-1}}\varphi(X_{t_k})}{X^2_{t_{k-1}}} - \log\sum_{k=1}^{n}\frac{\cos X_{t_{k-1}}\varphi(X_{t_{k-1}})}{X^2_{t_{k-1}}}\right\}.$$

By the result of Uchida (2004c), the estimator $\hat{\theta}^{(0)}_{\varepsilon,n}$ has asymptotic normality as $\varepsilon \to 0$ and $n \to \infty$. Therefore, theorem 2 implies that the one-step estimator $\hat{\theta}^{(1)}_{\varepsilon,n,3}$ defined by (3) is asymptotically efficient, namely, under $(\varepsilon n^3)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$,

$$\varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,3} - \theta_0) \xrightarrow{\mathrm{d}} N(0, I(\theta_0)^{-1}),$$

where

$$I(\theta_0) = \int_0^1 \frac{\cos^2 X^0_t}{(X^0_t)^2}dt.$$

Table 1 shows the simulation results for the two estimators, $\hat{\theta}^{(S)}_{\varepsilon,n}$ and $\hat{\theta}^{(1)}_{\varepsilon,n,3}$, in the situation where $\theta = 1$ and $x_0 = 0.5$. For the case that $n = 10$, $\hat{\theta}^{(S)}_{\varepsilon,n}$ has a considerable bias while $\hat{\theta}^{(1)}_{\varepsilon,n,3}$ is still unbiased. When $n = 50$, it seems that there is no big difference between the two estimators. We see that $\hat{\theta}^{(1)}_{\varepsilon,n,3}$ performs quite well in all cases.

Table 1: (Non-linear model) The mean and standard deviation of the estimators, which are determined from 1000 independent simulated sample paths for $\theta = 1$ and $x_0 = 0.5$.

| | | $\hat{\theta}^{(S)}_{\varepsilon,n}$ | | $\hat{\theta}^{(1)}_{\varepsilon,n,3}$ | |
|---|---|---|---|---|---|
| $n$ | $\varepsilon$ | mean | s.d. | mean | s.d. |
| 0.10 | 10 | 0.975953 | 0.105133 | 1.007391 | 0.112336 |
| 0.10 | 50 | 1.002238 | 0.110424 | 1.008725 | 0.111927 |
| 0.05 | 10 | 0.970827 | 0.052081 | 1.001763 | 0.055602 |
| 0.05 | 50 | 0.995715 | 0.054673 | 1.002085 | 0.055405 |
| 0.01 | 10 | 0.969524 | 0.010350 | 1.000317 | 0.011044 |
| 0.01 | 50 | 0.993944 | 0.010865 | 1.000280 | 0.011009 |

### 3.2 Non-linear model 2

We consider another non-linear model defined by the stochastic differential equation

$$dX_t = \left(\frac{\alpha}{X_t} - \beta X_t\right)dt + \varepsilon dw_t,\ X_0 = x_0,$$

where $x_0$ and $\varepsilon$ are known constants and $\alpha$ and $\beta$ are unknown parameters. Here we assume that the state space is the positive real line. Furthermore, we set that $\theta = (\alpha, \beta)$. Let $X^0_t$ denote the solution of the differential equation, $dX^0_t = \left(\alpha_0/X^0_t - \beta_0 X^0_t\right)dt, X^0_0 = x_0$. As in the previous subsection, the second contrast function $\Lambda(\theta)$ can not be explicitly obtained because $X^0_t$ does not have an explicit form. We can explicitly derive the contrast function $l_\varepsilon(\theta)$, but we note that the asymptotics which the estimator $\hat{\theta}^{(L)}_{\varepsilon,n}$ works is under $(\varepsilon n^2)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$. In this subsection, we consider the estimating function which works under $(\varepsilon n^3)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$.

By (2) with $\ell = 3$, the two-dimensional estimating function $G_{\varepsilon,n,3}(\theta) = (G^{(1)}_{\varepsilon,n,3}(\theta), G^{(2)}_{\varepsilon,n,3}(\theta))$ is as follows.

$$
\begin{aligned}
G^{(1)}_{\varepsilon,n,3}(\theta) &= \frac{1}{\varepsilon^2}\sum_{k=1}^n \frac{1}{X_{t_{k-1}}}\left[X_{t_k} - X_{t_{k-1}} - \frac{1}{n}\left(\frac{\alpha}{X_{t_{k-1}}} - \beta\right) - \frac{1}{2n^2}\left(-\frac{\alpha^2}{X^3_{t_{k-1}}} + \beta^2 X_{t_{k-1}}\right)\right.\\
&\quad \left. -\frac{1}{6n^3}\left(\frac{3\alpha^3}{X^5_{t_{k-1}}} + \frac{\alpha\beta^2}{X_{t_{k-1}}} - \frac{3\alpha^2\beta}{X^3_{t_{k-1}}} - \beta^3 X_{t_{k-1}}\right)\right],\\
G^{(2)}_{\varepsilon,n,3}(\theta) &= -\frac{1}{\varepsilon^2}\sum_{k=1}^n X_{t_{k-1}}\left[X_{t_k} - X_{t_{k-1}} - \frac{1}{n}\left(\frac{\alpha}{X_{t_{k-1}}} - \beta\right) - \frac{1}{2n^2}\left(-\frac{\alpha^2}{X^3_{t_{k-1}}} + \beta^2 X_{t_{k-1}}\right)\right.\\
&\quad \left. -\frac{1}{6n^3}\left(\frac{3\alpha^3}{X^5_{t_{k-1}}} + \frac{\alpha\beta^2}{X_{t_{k-1}}} - \frac{3\alpha^2\beta}{X^3_{t_{k-1}}} - \beta^3 X_{t_{k-1}}\right)\right].
\end{aligned}
$$

As seen in the previous subsection, it is too difficult to derive an explicit estimator from the estimating function. Therefore, we need to take a one-step estimator.

In order to obtain an initial estimator, we consider the estimating functions of Uchida (2004c) as follows.

$$
\begin{aligned}
H^{(1)}_{\varepsilon,n}(\theta) &= \sum_{k=1}^n \frac{1}{X_{t_{k-1}}}\left\{\varphi(X_{t_k},\theta) - e^{-\lambda(\theta)/n}\varphi(X_{t_{k-1}},\theta)\right\},\\
H^{(2)}_{\varepsilon,n}(\theta) &= \sum_{k=1}^n X_{t_{k-1}}\left\{\varphi(X_{t_k},\theta) - e^{-\lambda(\theta)/n}\varphi(X_{t_{k-1}},\theta)\right\},
\end{aligned}
$$

where $\varphi(x,\theta) = \alpha - \beta x^2$ and $\lambda(\theta) = 2\beta$. By solving the estimating equation that $H_{\varepsilon,n}(\theta) = 0$, the estimators for $\alpha$ and $\beta$ are

$$
\begin{aligned}
\hat{\alpha}^{(0)}_{\varepsilon,n} &= \hat{\beta}^{(0)}_{\varepsilon,n}\frac{A - e^{-2\hat{\beta}^{(0)}_{\varepsilon,n}/n}B}{(1 - e^{-2\hat{\beta}^{(0)}_{\varepsilon,n}/n})C},\\
\hat{\beta}^{(0)}_{\varepsilon,n} &= -\frac{n}{2}\log\frac{AD - CE}{BD - CF},
\end{aligned}
$$

respectively, where

$$
\begin{aligned}
A &= \sum_{k=1}^n \frac{X^2_{t_k}}{X_{t_{k-1}}},\ B = \sum_{k=1}^n X_{t_{k-1}},\ C = \sum_{k=1}^n \frac{1}{X_{t_{k-1}}},\\
D &= \sum_{k=1}^n X_{t_k},\ E = \sum_{k=1}^n X_{t_{k-1}}X^2_{t_k},\ F = \sum_{k=1}^n X^3_{t_{k-1}}.
\end{aligned}
$$

It follows from the result of Uchida (2004c) that the estimator $\hat{\theta}^{(0)}_{\varepsilon,n} = (\hat{\alpha}^{(0)}_{\varepsilon,n}, \hat{\beta}^{(0)}_{\varepsilon,n})$ is asymptotically normal as $\varepsilon \to 0$ and $n \to \infty$. Therefore, theorem 2 yields that the one-step estimator $\hat{\theta}^{(1)}_{\varepsilon,n,3}$ defined by (3) has asymptotic efficiency, that is,

$$
\varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,3} - \theta_0) \xrightarrow{\mathrm{d}} N(0, I(\theta_0)^{-1})
$$

under $(\varepsilon n^3)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$, where

$$I^{(1,1)}(\theta_0) = \int_0^1 \frac{1}{(X_t^0)^2} dt,\ I^{(1,2)}(\theta_0) = I^{(2,1)}(\theta_0) = -1,\ I^{(2,2)}(\theta_0) = \int_0^1 (X_t^0)^2 dt.$$

In the setting that $\alpha = 10$, $\beta = 1$ and $x_0 = 1$, the simulation results of the three estimators for $\alpha$ and $\beta$ are given in tables 2 and 3, respectively. For the case that $n = 10$, both $\hat{\alpha}_{\varepsilon,n}^{(S)}$ and $\hat{\alpha}_{\varepsilon,n}^{(L)}$ have considerable biases while $\hat{\alpha}_{\varepsilon,n,3}^{(1)}$ works well. In the situation where $n$ is not so large, it is reasonable to use $\hat{\alpha}_{\varepsilon,n,3}^{(1)}$ for this model. For the situation where $n = 50$, $\hat{\alpha}_{\varepsilon,n}^{(S)}$ still has a small bias while both $\hat{\alpha}_{\varepsilon,n}^{(L)}$ and $\hat{\alpha}_{\varepsilon,n,3}^{(1)}$ are unbiased with small variances. We can say that $\hat{\alpha}_{\varepsilon,n,3}^{(1)}$ is better than the others in all cases. For the simulation results of $\beta$, we observe the same phenomenon as $\alpha$. In the case that $n = 10$, both $\hat{\beta}_{\varepsilon,n}^{(S)}$ and $\hat{\beta}_{\varepsilon,n}^{(L)}$ have such serious significant biases that we can not use them in this setting. For the case that $n = 50$, $\hat{\beta}_{\varepsilon,n}^{(S)}$ has a small bias and there seems no difference between $\hat{\beta}_{\varepsilon,n}^{(L)}$ and $\hat{\beta}_{\varepsilon,n,3}^{(1)}$. It is worth mentioning that $\hat{\beta}_{\varepsilon,n,3}^{(1)}$ performs quite well even if $n$ is not so large and $\varepsilon$ is not so small.

Table 2: (Non-linear model 2) The mean and standard deviation (s.d.) of the three estimators determined from 1000 independent simulated sample paths for $\alpha = 10$, $\beta = 1$ and $x_0 = 1$.

| | | $\hat{\alpha}_{\varepsilon,n}^{(S)}$ | | $\hat{\alpha}_{\varepsilon,n}^{(L)}$ | | $\hat{\alpha}_{\varepsilon,n,3}^{(1)}$ | |
|---|---|---|---|---|---|---|---|
| $\varepsilon$ | $n$ | mean | s.d. | mean | s.d. | mean | s.d. |
| 0.10 | 10 | 6.98508 | 0.03127 | 8.98324 | 0.04847 | 10.44851 | 0.08243 |
| 0.10 | 50 | 9.29033 | 0.04892 | 9.97759 | 0.05622 | 10.01142 | 0.05666 |
| 0.05 | 10 | 6.98469 | 0.01563 | 8.98286 | 0.02424 | 10.44687 | 0.04121 |
| 0.05 | 50 | 9.28963 | 0.02445 | 9.97724 | 0.02810 | 10.01059 | 0.02832 |
| 0.01 | 10 | 6.98442 | 0.00313 | 8.98254 | 0.00485 | 10.44591 | 0.00824 |
| 0.01 | 50 | 9.28920 | 0.00489 | 9.97690 | 0.00562 | 10.01008 | 0.00566 |

Table 3: (Non-linear model 2) The mean and standard deviation (s.d.) of the three estimators determined from 1000 independent simulated sample paths for $\alpha = 10$, $\beta = 1$ and $x_0 = 1$.

| | | $\hat{\beta}_{\varepsilon,n}^{(S)}$ | | $\hat{\beta}_{\varepsilon,n}^{(L)}$ | | $\hat{\beta}_{\varepsilon,n,3}^{(1)}$ | |
|---|---|---|---|---|---|---|---|
| $\varepsilon$ | $n$ | mean | s.d. | mean | s.d. | mean | s.d. |
| 0.10 | 10 | 0.59291 | 0.00811 | 0.82724 | 0.01050 | 1.05620 | 0.01610 |
| 0.10 | 50 | 0.90386 | 0.01084 | 0.99485 | 0.01194 | 1.00092 | 0.01202 |
| 0.05 | 10 | 0.59284 | 0.00406 | 0.82717 | 0.00525 | 1.05593 | 0.00805 |
| 0.05 | 50 | 0.90371 | 0.00542 | 0.99477 | 0.00597 | 1.00075 | 0.00601 |
| 0.01 | 10 | 0.59278 | 0.00081 | 0.82711 | 0.00105 | 1.05576 | 0.00161 |
| 0.01 | 50 | 0.90362 | 0.00108 | 0.99470 | 0.00119 | 1.00065 | 0.00120 |

# 4 Proof

In order to prove the result, we introduce some notation and two lemmas. For lemmas 1 and 2 put later on, we set that $K_{\varepsilon,n,\ell}(\theta) = \left(K^{i,j}_{\varepsilon,n,\ell}(\theta)\right)_{i,j=1,2,\ldots,p}$ and $K(\theta) = \left(K^{i,j}(\theta)\right)_{i,j=1,2,\ldots,p}$, where $K^{i,j}_{\varepsilon,n,\ell}(\theta) = \delta_j G^{(i)}_{\varepsilon,n,\ell}(\theta)$,

$$K^{i,j}(\theta) \quad = \quad \int_0^1 \left[\delta_j \delta_i b(X^0_s, \theta)\right]^T \left[\sigma\sigma^T(X^0_s)\right]^{-1} B(X^0_s, \theta_0, \theta) ds - I^{i,j}(\theta)$$

and $B(x, \theta_0, \theta) = b(x, \theta_0) - b(x, \theta)$.

**Lemma 1** (Uchida (2004a)) Let $\ell \in \mathbf{N}$. Assume (A1)-(A4). Then, as $\varepsilon \to 0$ and $n \to \infty$,

$$\sup_{\theta \in \bar{\Theta}} \left|\varepsilon^2 K_{\varepsilon,n,\ell}(\theta) - K(\theta)\right| \xrightarrow{\mathrm{P}} 0.$$

**Lemma 2** (Uchida (2004a)) Let $\ell \in \mathbf{N}$. Assume (A1)-(A4). If $\left(\varepsilon n^\ell\right)^{-1} \to 0$, then, as $\varepsilon \to 0$ and $n \to \infty$,

$$\varepsilon G_{\varepsilon,n,\ell}(\theta_0) \xrightarrow{\mathrm{d}} N\left(0, I(\theta_0)\right).$$

For proofs of lemmas 1 and 2, see Uchida (2004a).

**Proof of Theorem 2**. Following the proof of asymptotic efficiency for one-step estimators presented in Yoshida (2004), we will prove theorem 2. Consider the following event $A_0$ defined by

$$A_0 = \left\{\hat{\theta}^{(0)}_{\varepsilon,n,\ell} \in \bar{\Theta},\ \delta_\theta G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) \in A,\ \hat{\theta}^{(1)}_{\varepsilon,n,\ell} \in \bar{\Theta}\right\},$$

where $A$ is a whole set of non-singular matrices. First of all, we will show that $P\left(A_0\right) \longrightarrow 1$. It follows from the assumption of the initial estimator $\hat{\theta}^{(0)}_{\varepsilon,n,\ell}$ that

$$P(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} \in \bar{\Theta}) \longrightarrow 1. \tag{4}$$

By using the mean value theorem,

$$\left|\varepsilon^2 \left[G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) - G_{\varepsilon,n,\ell}(\theta_0)\right]\right| = \left|\int_0^1 \varepsilon^2 \delta_\theta G_{\varepsilon,n,\ell}\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) du\right| \left|\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0\right|.$$

Note that for $0 \le u \le 1$, $P(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0) \in \bar{\Theta}) \longrightarrow 1$. Lemma 1 yields that

$$\begin{aligned}
& \left|\int_0^1 \varepsilon^2 \delta_\theta G_{\varepsilon,n,\ell}\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) du - I(\theta_0)\right| \\
\le\ & \int_0^1 \left|\varepsilon^2 \delta_\theta G_{\varepsilon,n,\ell}\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) - K\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right)\right| du \\
& + \int_0^1 \left|K\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) - I(\theta_0)\right| du \\
\xrightarrow{\mathrm{P}}\ & 0.
\end{aligned}$$

By the above estimates, $\left|\varepsilon^2\left[G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-G_{\varepsilon,n,\ell}(\theta_0)\right]\right| \xrightarrow{\mathrm{P}} 0$. Lemma 2 implies that

$$\varepsilon^2 G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) \xrightarrow{\mathrm{P}} 0. \tag{5}$$

By (4) and lemma 1, one has $\left|\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right| \xrightarrow{\mathrm{P}} 0$. Further, the continuity of $K(\theta)$ yields that $\left|K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\theta_0)\right| \xrightarrow{\mathrm{P}} 0$. By using these results,

$$\left|\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\theta_0)\right| \;\leq\; \left|\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right| + \left|K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\theta_0)\right| \xrightarrow{\mathrm{P}} 0.$$

Noting that $K(\theta_0)=-I(\theta_0)$, we have $\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) \xrightarrow{\mathrm{P}} -I(\theta_0)$. Therefore,

$$P\left(\delta_\theta G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) \in A\right) \longrightarrow 1. \tag{6}$$

Since it follows from (5), (6) and the consistency of $\hat{\theta}^{(0)}_{\varepsilon,n,\ell}$ that

$$\hat{\theta}^{(1)}_{\varepsilon,n,\ell} = \hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \left[\varepsilon^2\delta_\theta G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right]^{-1}\varepsilon^2 G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) \xrightarrow{\mathrm{P}} \theta_0,$$

one has

$$P\big(\hat{\theta}^{(1)}_{\varepsilon,n,\ell} \in \bar{\Theta}\big) \longrightarrow 1. \tag{7}$$

Thus, by (4), (6) and (7) that $P\left(A_0\right) \longrightarrow 1$. From this fact, it suffices to consider the estimates on the event $A_0$ under the asymptotics we treat.

From the Taylor expansion at $\theta=\theta_0$, we have

$$\varepsilon G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) \;=\; \varepsilon G_{\varepsilon,n,\ell}(\theta_0) + \left[\int_0^1 K_{\varepsilon,n,\ell}\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)du\right]\varepsilon(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0). \tag{8}$$

Let $\triangle_{\varepsilon,n}=\varepsilon\left[\delta_\theta G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right](\hat{\theta}^{(1)}_{\varepsilon,n,\ell}-\theta_0)+\varepsilon G_{\varepsilon,n,\ell}(\theta_0)$. Using (3) and (8), we then have

$$\begin{aligned}
\triangle_{\varepsilon,n} \;&=\; \varepsilon\left[K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right]\left\{\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\left[K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right]^{-1}G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-\theta_0\right\}\\
&\quad +\varepsilon G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-\left[\int_0^1 K_{\varepsilon,n,\ell}\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)du\right]\varepsilon(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\\
&=\; \varepsilon^2\left[K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-\int_0^1 K_{\varepsilon,n,\ell}\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)du\right]\varepsilon^{-1}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left|\varepsilon^2\left[K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-\int_0^1 K_{\varepsilon,n,\ell}\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)du\right]\right|\\
\leq\;& \int_0^1\left|\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right|du\\
&+\int_0^1\left|K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)\right|du\\
&+\int_0^1\left|\varepsilon^2 K_{\varepsilon,n,\ell}\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)-K\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)\right|du.
\end{aligned}$$

It follows from lemma 1 that

$$\int_0^1 \left|\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) - K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right| du \xrightarrow{\mathrm{P}} 0,$$
$$\int_0^1 \left|\varepsilon^2 K_{\varepsilon,n,\ell}\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) - K\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right)\right| du \xrightarrow{\mathrm{P}} 0.$$

Furthermore, by the continuity of $K(\theta)$,

$$\int_0^1 \left|K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) - K\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right)\right| du \xrightarrow{\mathrm{P}} 0.$$

Therefore,

$$\varepsilon^2 \left[K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) - \int_0^1 K_{\varepsilon,n,\ell}\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) du\right] = o_P(1),$$

and $\triangle_{\varepsilon,n} = o_P(1) \times O_P(1) = o_P(1)$. Since it follows from an easy computation that

$$\left[\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right]^{-1} \triangle_{\varepsilon,n} = \varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,\ell} - \theta_0) + \varepsilon^{-1}\left[K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right]^{-1} G_{\varepsilon,n,\ell}(\theta_0),$$

one has

$$\begin{aligned} & \varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,\ell} - \theta_0) - [I(\theta_0)]^{-1}\varepsilon G_{\varepsilon,n,\ell}(\theta_0) \\ = & \left[\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n})\right]^{-1} \triangle_{\varepsilon,n} - \left\{\left[\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n})\right]^{-1} + [I(\theta_0)]^{-1}\right\} \varepsilon G_{\varepsilon,n,\ell}(\theta_0). \end{aligned}$$

By the results that $\triangle_{\varepsilon,n} \xrightarrow{\mathrm{P}} 0$, and $\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n}) \xrightarrow{\mathrm{P}} -I(\theta_0)$, we obtain that

$$\varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,\ell} - \theta_0) - \left[I(\theta_0)\right]^{-1}\varepsilon G_{\varepsilon,n,\ell}(\theta_0) = o_P(1).$$

This completes the proof of (i).

Finally, it follows from (i) and lemma 2 that

$$\begin{aligned} \varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,\ell} - \theta_0) &= \varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,\ell} - \theta_0) - [I(\theta_0)]^{-1}\varepsilon G_{\varepsilon,n,\ell}(\theta_0) + [I(\theta_0)]^{-1}\varepsilon G_{\varepsilon,n,\ell}(\theta_0) \\ &\xrightarrow{\mathrm{d}} N\left(0, [I(\theta_0)]^{-1}\right). \end{aligned}$$

We complete the proof.


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