One-step estimators for diffusion processes with small dispersion parameters from discrete observations

Matsuzaki, Ryo
Graduate School of Mathematics, Kyushu University

Uchida, Masayuki
Faculty of Mathematics, Kyushu University

出版情報：MHF Preprint Series. MHF2005-10, 2005-03-07. 九州大学大学院数理学研究院
バージョン：
権利関係：

MHF Preprint Series

Kyushu University
21st Century COE Program
Development of Dynamic Mathematics with
High Functionality

One-step estimators for diffusion processes with small dispersion parameters from discrete observations

R. Matsuzaki & M. Uchida

MHF 2005-10

(Received March 7, 2005)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

One-step estimators for diffusion processes with small dispersion parameters from discrete observations

[1]Ryo Matsuzaki and [2]Masayuki Uchida
[1]Graduate School of Mathematics, Kyushu University
Ropponmatsu, Fukuoka 810-8560, Japan
[2]Faculty of Mathematics, Kyushu University
Ropponmatsu, Fukuoka 810-8560, Japan

Abstract. For a multi-dimensional diffusion process with small dispersion parameter ε, an asymptotically efficient estimator of the drift parameter is studied. When the sample path is observed at n regularly spaced time points $t_k = k/n$, $k = 0, 1, \ldots, n$, we investigate asymptotic properties of a one-step estimator derived from an approximate estimating function under the situation when $\varepsilon \to 0$ and $n \to \infty$ simultaneously.

AMS 2000 subject classifications: Primary 62M05; Secondary 62F12.

Key words and phrases: Approximate martingale estimating function, diffusion process with small noise, discrete time observation, parametric inference.

Abbreviated Title: One-step estimator for small diffusions.

1 Introduction

We consider a family of d-dimensional diffusion processes defined by the following stochastic differential equations:

$$\begin{aligned} dX_t &= b(X_t, \theta)dt + \varepsilon\sigma(X_t)dw_t, \ t \in [0, 1], \ \varepsilon \in (0, 1], \\ X_0 &= x_0, \end{aligned} \tag{1}$$

where $\theta \in \bar{\Theta}$, Θ is an open bounded convex subset of $\mathbf{R}^p$ and $\bar{\Theta}$ is the closure of Θ. Further, x_0 and ε are known constants, b is an $\mathbf{R}^d$-valued function defined on $\mathbf{R}^d \times \bar{\Theta}$, σ is an $\mathbf{R}^d \otimes \mathbf{R}^r$-valued function defined on $\mathbf{R}^d$, and w is an r-dimensional standard Wiener process. We assume that the drift b is known apart from the parameter θ. The type of data considered in this paper is discrete observations of X at n regularly spaced time points $t_k = k/n$ on the fixed interval $[0, 1]$, that is, $(X_{t_k})_{0 \le k \le n}$. The asymptotics we consider is when $\varepsilon \to 0$ and $n \to \infty$ simultaneously.

Parametric inference based on small diffusion asymptotics for diffusion processes is well-developed together with applications to mathematical finance. For continuously observed small diffusions, see Kutoyants (1984, 1994) and Yoshida (1992, 2003). Moreover, there are a number of papers on estimation of the parameter for a discretely observed diffusion process with a small dispersion parameter, see Genon-Catalot (1990), Laredo (1990), Sørensen (2000), Sørensen and Uchida (2003) and Uchida (2004a, 2004b, 2004c). As for the estimation of the drift parameter, it seems that the following three papers are fundamental. Genon-Catalot (1990) presented two kinds of contrast functions, $U(\theta)$ and $\Lambda(\theta)$ (see p. 102 and p. 103 in Genon-Catalot (1990)). She proved that an estimator obtained from the first contrast function $U(\theta)$ is asymptotically efficient under $(\varepsilon n)^{-1} \to 0$ and an estimator derived from the second contrast function $\Lambda(\theta)$ has asymptotic efficiency under $\varepsilon\sqrt{n} = O(1)$. Laredo (1990) studied an estimator which has asymptotic efficiency under $\left(\varepsilon n^2\right)^{-1} \to 0$. Recently, Uchida (2004a) proposed an approximate martingale estimating function based on a martingale estimating function proposed in Bibby and Sørensen (1995). The estimator obtained from the estimating function has asymptotic efficiency under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ for a natural number ℓ. We note that the first contrast function $U(\theta)$ in Genon-Catalot (1990) and the estimating function in Uchida (2004a) are explicitly obtained, while the contrast function in Laredo (1990) and the second contrast function $\Lambda(\theta)$ in Genon-Catalot (1990) do not generally have explicit forms. Furthermore, even if all contrast functions have explicit expressions, the estimators are not always derived from the contrast functions because of their complicated forms.

In order to overcome this difficulty, we study a one-step estimator which handles well even in such ill cases. The Newton-Raphson method is widely known as a practical method to obtain an approximate solution when we can not solve a non-random equation. The one-step estimator is constructed in the same way as this method. For details of one-step estimators, see, for example, Ferguson (1996), Van der Vaart (1998) and Sakamoto and Yoshida (1999).

This paper is organized as follows. In section 2, we propose a one-step estimator and describe the conditions that the one-step estimator has consistency, asymptotic normality and asymptotic efficiency under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$, $n \to \infty$ for a natural number ℓ. Section 3 gives two examples and simulation studies. For the first example, the contrast function in Laredo (1990) can not be explicitly obtained. The contrast function $\Lambda(\theta)$ in Genon-Catalot (1990) does not have an explicit form in both examples. The proof of the result is given in section 4.

2 One-step estimator

Let θ_0 be a true value of θ and assume that $\theta_0 \in \Theta$. Let X_t^0 be the solution of the ordinary differential equation: $dX_t^0 = b(X_t^0, \theta_0)dt$, $X_0^0 = x_0$. Let $C_\uparrow^{\infty,3}(\mathbf{R}^d \times \Theta; \mathbf{R}^d)$ be a space of functions f which satisfies the following conditions: (i) $f : \mathbf{R}^d \times \Theta \longrightarrow \mathbf{R}^d$ is infinitely differentiable with respect to x and continuously differentiable with respect to θ up to order 3, (ii) for $\mathbf{n}, \nu$ satisfying $\mid \mathbf{n} \mid \geq 0, 0 \leq \mid \nu \mid \leq 3$, there exists $C > 0$ such that $\sup_{\theta\in\Theta} \mid \delta^\nu \partial^{\mathbf{n}} f(x,\theta) \mid \leq C(1+ \mid x \mid)^C$ for $\forall x \in \mathbf{R}^d$. Here, $\mathbf{n} = (n_1, n_2, \ldots, n_d)$, $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ are multi-indices, $\mid \mathbf{n} \mid = n_1 + n_2 + \cdots + n_d$, $\mid \nu \mid = \nu_1 + \nu_2 + \cdots + \nu_p$, $\partial^{\mathbf{n}} = \partial_1^{n_1}\partial_2^{n_2}\cdots\partial_d^{n_d}$, $\delta^\nu = \delta_1^{\nu_1}\partial_2^{\nu_2}\cdots\partial_p^{\nu_p}$, $\partial_i = \partial/\partial_{x_1}$ and $\delta_j = \partial/\partial_{\theta_j}$. Let $C_\uparrow^{\infty}(\mathbf{R}^d; \mathbf{R}^d \otimes \mathbf{R}^r)$ be a space of functions h which satisfies the following conditions: (i) $h : \mathbf{R}^d \longrightarrow \mathbf{R}^d \otimes \mathbf{R}^r$ is continuously infinitely differentiable with respect to x, (ii) for $\mid \mathbf{n} \mid \geq 0$, there exists $C > 0$ such that $\mid \partial^{\mathbf{n}} h(x) \mid \leq C(1+ \mid x \mid)^C$ for $\forall x \in \mathbf{R}^d$. Moreover, let σ^T be the transposition of σ, $\xrightarrow{\mathrm{P}}$ be the convergence in probability and $\xrightarrow{\mathrm{d}}$ be the convergence in distribution henceforth.

In this paper we make the following assumptions.

(A1) Equation (1) has a unique strong solution on [0,1].

(A2) For $\forall m > 0$, $\sup_{0\leq t\leq 1} E\left[\mid X_t \mid^m\right] < \infty$.

(A3) $b(x,\theta) \in C_\uparrow^{\infty,3}(\mathbf{R}^d \times \bar{\Theta}; \mathbf{R}^d), \sigma(x) \in C_\uparrow^{\infty}(\mathbf{R}^d; \mathbf{R}^d \otimes \mathbf{R}^r)$.

(A4) $\inf_x \det\left[\sigma\sigma^T(x)\right] > 0, \left[\sigma\sigma^T(x)\right]^{-1} \in C_\uparrow^{\infty}(\mathbf{R}^d; \mathbf{R}^d \otimes \mathbf{R}^d)$.

(A5) $I(\theta_0) = \left(I^{i,j}(\theta_0)\right)_{i,j=1,2,\ldots,p}$ is positive definite, where

$$I^{i,j}(\theta) \quad = \quad \int_0^1 \left[\delta_i b(X_s^0,\theta)\right]^T \left[\sigma\sigma^T(X_s^0)\right]^{-1} \left[\delta_j b(X_s^0,\theta)\right] ds.$$

For $\ell \in \mathbf{N}$, an approximate martingale estimating function $G_{\varepsilon,n,\ell}(\theta) = \left(G^{(i)}_{\varepsilon,n,\ell}(\theta)\right)_{i=1,2,\ldots,p}$ proposed in Uchida (2004a) is as follows:

$$
\begin{aligned}
G^{(i)}_{\varepsilon,n,\ell}(\theta) &= \varepsilon^{-2}\sum_{k=1}^{n}\left[\delta_i b(X_{t_{k-1}},\theta)\right]^T\left[\sigma\sigma^T(X_{t_{k-1}})\right]^{-1}P_{k,\ell}(\theta), \qquad (2)\\
P_{k,\ell}(\theta) &= X_{t_k}-\sum_{j=0}^{\ell}\frac{1}{j!n^j}\tilde{L}^j_\theta g(X_{t_{k-1}}),\\
\tilde{L}_\theta g(x) &= \sum_{i=1}^{d} b^i(x,\theta)\partial_i g(x),
\end{aligned}
$$

where $g(x)=x$ and $b^i(x,\theta)$ is the i-th element of $b(x,\theta)$. For example, when $\ell=2$,

$$G^{(i)}_{\varepsilon,n,2}(\theta)=\varepsilon^{-2}\sum_{k=1}^{n}\left[\delta_i b(X_{t_{k-1}},\theta)\right]^T\left[\sigma\sigma^T(X_{t_{k-1}})\right]^{-1}\left(X_{t_k}-X_{t_{k-1}}-\frac{1}{n}b(X_{t_{k-1}},\theta)\right).$$

An estimator obtained from the estimating function $G_{\varepsilon,n,\ell}(\theta)$ has the following asymptotic properties. For details of the result, see Uchida (2004a).

Theorem 1 *(Uchida (2004a)) Let $\ell \in \mathbf{N}$. Assume (A1)-(A5). Then, an estimator $\hat{\theta}_{\varepsilon,n,\ell}$, which solves $G_{\varepsilon,n,\ell}(\theta)=0$, exists with a probability tending to one as $\varepsilon \to 0$ and $n \to \infty$ under P_{θ_0}, and $\hat{\theta}_{\varepsilon,n,\ell} \xrightarrow{p} \theta_0$ as $\varepsilon \to 0$ and $n \to \infty$. Moreover, if $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$, $\varepsilon^{-1}(\hat{\theta}_{\varepsilon,n,\ell}-\theta_0) \xrightarrow{\mathrm{d}} N(0, I^{-1}(\theta_0))$.*

It follows from theorem 1 that an estimator derived from the estimating function $G_{\varepsilon,n,\ell}(\theta)$ has asymptotic efficiency under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$. It seems that the estimation procedure works well. However, there is a disadvantage that we can not generally obtain an explicit estimator because the estimating function $G_{\varepsilon,n,\ell}(\theta)$ has complicated form when l is large.

In order to conquer this difficulty, we suggest a one-step estimator as follows. For an initial estimator $\hat{\theta}^{(0)}_{\varepsilon,n,\ell}$, a one-step estimator $\hat{\theta}^{(1)}_{\varepsilon,n,\ell}$ is defined by

$$\hat{\theta}^{(1)}_{\varepsilon,n,\ell}=\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\left[\partial_\theta G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right]^{-1}G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}). \qquad (3)$$

We then have the following theorem.

Theorem 2 *Let $\ell \in \mathbf{N}$. Assume (A1)-(A5). Moreover, assume that an initial estimator $\hat{\theta}^{(0)}_{\varepsilon,n,\ell}$ satisfies $\varepsilon^{-1}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)=O_P(1)$ under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$. Then, under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$,*
(i) $\varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,\ell}-\theta_0)-\left[I(\theta_0)\right]^{-1}\varepsilon G_{\varepsilon,n,\ell}(\theta_0)=o_P(1)$,
(ii) $\varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,\ell}-\theta_0) \xrightarrow{\mathrm{d}} N\left(0, \left[I(\theta_0)\right]^{-1}\right)$.

By theorem 2, in order to get an asymptotically efficient estimator it is essential to obtain an initial estimator with the property that $\varepsilon^{-1}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)=O_P(1)$ under $\left(\varepsilon n^\ell\right)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$. One of sufficient conditions of an initial estimator is that $\varepsilon^{-1}(\hat{\theta}^{(0)}_{\varepsilon,n}-\theta_0)=O_P(1)$ as $\varepsilon \to 0$ and $n \to \infty$. Note that the estimator $\hat{\theta}^{(0)}_{\varepsilon,n}$ is independent of ℓ. For estimators satisfying that $\varepsilon^{-1}(\hat{\theta}^{(0)}_{\varepsilon,n}-\theta_0)=O_P(1)$ as $\varepsilon \to 0$ and $n \to \infty$, see Uchida (2004b, 2004c).

3 Examples

In this section, we give two examples and examine the performance of the one-step estimator through simulation studies. One thousand sample paths are generated by the Euler-Maruyama scheme, see Kloeden and Platen (1992). We set that $\varepsilon = 0.01, 0.05, 0.1$ and $n = 10, 50$. In order to obtain a one-step estimator $\hat{\theta}^{(1)}_{\varepsilon,n,3}$, the approximate martingale estimating function defined by (2) is treated with $\ell = 3$. For an initial estimator, we use the estimator obtained from an approximate martingale estimating function based on an eigenfunction, see Uchida (2004c). In order to evaluate the estimator $\hat{\theta}^{(1)}_{\varepsilon,n,3}$, we also calculate the estimator obtained from the first contrast function $U(\theta)$ in Genon-Catalot (1990) and the estimator in Laredo (1990), which are denoted by $\hat{\theta}^{(S)}_{\varepsilon,n}$ and $\hat{\theta}^{(L)}_{\varepsilon,n}$, respectively. In both examples, the second contrast function $\Lambda(\theta)$ in Genon-Catalot (1990) does not have an explicit form. As for the contrast function $l_\varepsilon(\theta)$ presented by Laredo (1990), it is not explicitly obtained in the first example. For each of the estimators, the means and standard deviations are computed.

3.1 Non-linear model 1

We consider a non-linear model defined by the stochastic diffusion equation,

$$dX_t \;=\; \theta \cos X_t dt + \varepsilon X_t dw_t,\; t \in [0,1],\; \varepsilon \in (0,1],\; X_0 = x_0,$$

where x_0 and ε are known constants and θ is an unknown parameter. Let X^0_t be the solution of the differential equation, $dX^0_t = \theta \cos X^0_t dt, X^0_0 = x_0$. Since X^0_t does not have an explicit form, we can not explicitly get the contrast function $\Lambda(\theta)$. The contrast function $l_\varepsilon(\theta)$ is not also explicitly derived for this diffusion model. Note that in order to obtain $l_\varepsilon(\theta)$, we need to have the function $V(x,\theta)$ satisfying that

$$V(x,\theta) - V(x_0,\theta) = \int_{x_0}^{x} \frac{\theta \cos u}{u^2} du.$$

Unfortunately, the function $V(x,\theta)$ can not have an explicit expression. For this reason, we treat the estimating function defined by (2).

For $\ell = 3$, the estimating function $G_{\varepsilon,n,3}(\theta)$ in (2) is given by

$$\begin{aligned} G_{\varepsilon,n,3}(\theta) \;=\; & \frac{1}{\varepsilon^2}\Bigg\{ \sum_{k=1}^{n} \frac{\cos X_{t_{k-1}}}{X^2_{t_{k-1}}}(X_{t_k} - X_{t_{k-1}}) - \frac{\theta}{n}\sum_{k=1}^{n} \frac{\cos^2 X_{t_{k-1}}}{X^2_{t_{k-1}}} \\ & + \frac{\theta^2}{2n^2}\sum_{k=1}^{n} \frac{\cos^2 X_{t_{k-1}} \sin X_{t_{k-1}}}{X^2_{t_{k-1}}} - \frac{\theta^3}{6n^3}\sum_{k=1}^{n} \frac{\cos^2 X_{t_{k-1}}(\sin^2 X_{t_{k-1}} - \cos^2 X_{t_{k-1}})}{X_{t_{k-1}}} \Bigg\}. \end{aligned}$$

Since the estimating equation $G_{\varepsilon,n,3}(\theta) = 0$ is not solvable, we consider a one-step estimator.

In the same way as in Uchida (2004c), an initial estimator is obtained from the estimating function

$$H_{\varepsilon,n}(\theta) = \sum_{k=1}^{n} \frac{\cos X_{t_{k-1}}}{X^2_{t_{k-1}}} \left\{ \varphi(X_{t_k}) - e^{-\lambda(\theta)/n}\varphi(X_{t_{k-1}}) \right\},$$

where $\lambda(\theta) = \theta$ and

$$\varphi(x) = \frac{\cos(x/2) + \sin(x/2)}{\cos(x/2) - \sin(x/2)}.$$

It then follows that the initial estimator $\hat{\theta}_{\varepsilon,n}^{(0)}$ satisfying that $H_{\varepsilon,n}(\theta)=0$ is given by

$$\hat{\theta}_{\varepsilon,n}^{(0)} = n\left\{\log\sum_{k=1}^{n}\frac{\cos X_{t_{k-1}}\varphi(X_{t_k})}{X_{t_{k-1}}^2} - \log\sum_{k=1}^{n}\frac{\cos X_{t_{k-1}}\varphi(X_{t_{k-1}})}{X_{t_{k-1}}^2}\right\}.$$

By the result of Uchida (2004c), the estimator $\hat{\theta}_{\varepsilon,n}^{(0)}$ has asymptotic normality as $\varepsilon\to 0$ and $n\to\infty$. Therefore, theorem 2 implies that the one-step estimator $\hat{\theta}_{\varepsilon,n,3}^{(1)}$ defined by (3) is asymptotically efficient, namely, under $(\varepsilon n^3)^{-1}\to 0$ as $\varepsilon\to 0$ and $n\to\infty$,

$$\varepsilon^{-1}(\hat{\theta}_{\varepsilon,n,3}^{(1)}-\theta_0) \xrightarrow{\mathrm{d}} N(0, I(\theta_0)^{-1}),$$

where

$$I(\theta_0) = \int_0^1 \frac{\cos^2 X_t^0}{(X_t^0)^2}dt.$$

Table 1 shows the simulation results for the two estimators, $\hat{\theta}_{\varepsilon,n}^{(S)}$ and $\hat{\theta}_{\varepsilon,n,3}^{(1)}$, in the situation where $\theta=1$ and $x_0=0.5$. For the case that $n=10$, $\hat{\theta}_{\varepsilon,n}^{(S)}$ has a considerable bias while $\hat{\theta}_{\varepsilon,n,3}^{(1)}$ is still unbiased. When $n=50$, it seems that there is no big difference between the two estimators. We see that $\hat{\theta}_{\varepsilon,n,3}^{(1)}$ performs quite well in all cases.

Table 1: (Non-linear model) The mean and standard deviation of the estimators, which are determined from 1000 independent simulated sample paths for $\theta=1$ and $x_0=0.5$.

		$\hat{\theta}_{\varepsilon,n}^{(S)}$		$\hat{\theta}_{\varepsilon,n,3}^{(1)}$	
n	ε	mean	s.d.	mean	s.d.
0.10	10	0.975953	0.105133	1.007391	0.112336
0.10	50	1.002238	0.110424	1.008725	0.111927
0.05	10	0.970827	0.052081	1.001763	0.055602
0.05	50	0.995715	0.054673	1.002085	0.055405
0.01	10	0.969524	0.010350	1.000317	0.011044
0.01	50	0.993944	0.010865	1.000280	0.011009

3.2 Non-linear model 2

We consider another non-linear model defined by the stochastic differential equation

$$dX_t = \left(\frac{\alpha}{X_t}-\beta X_t\right)dt+\varepsilon dw_t,\ X_0=x_0,$$

where x_0 and ε are known constants and α and β are unknown parameters. Here we assume that the state space is the positive real line. Furthermore, we set that $\theta=(\alpha,\beta)$. Let X_t^0 denote the solution of the differential equation, $dX_t^0=\left(\alpha_0/X_t^0-\beta_0X_t^0\right)dt, X_0^0=x_0$. As in the previous subsection, the second contrast function $\Lambda(\theta)$ can not be explicitly obtained because X_t^0 does not have an explicit form. We can explicitly derive the contrast function $l_\varepsilon(\theta)$, but we note that the asymptotics which the estimator $\hat{\theta}_{\varepsilon,n}^{(L)}$ works is under $(\varepsilon n^2)^{-1}\to 0$ as $\varepsilon\to 0$ and $n\to\infty$. In this subsection, we consider the estimating function which works under $(\varepsilon n^3)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$.

By (2) with $\ell = 3$, the two-dimensional estimating function $G_{\varepsilon,n,3}(\theta) = (G^{(1)}_{\varepsilon,n,3}(\theta), G^{(2)}_{\varepsilon,n,3}(\theta))$ is as follows.

$$\begin{aligned} G^{(1)}_{\varepsilon,n,3}(\theta) &= \frac{1}{\varepsilon^2}\sum_{k=1}^{n} \frac{1}{X_{t_{k-1}}} \left[X_{t_k} - X_{t_{k-1}} - \frac{1}{n}\left(\frac{\alpha}{X_{t_{k-1}}} - \beta \right) - \frac{1}{2n^2}\left(-\frac{\alpha^2}{X^3_{t_{k-1}}} + \beta^2 X_{t_{k-1}} \right) \right. \\ & \left. -\frac{1}{6n^3}\left(\frac{3\alpha^3}{X^5_{t_{k-1}}} + \frac{\alpha\beta^2}{X_{t_{k-1}}} - \frac{3\alpha^2\beta}{X^3_{t_{k-1}}} - \beta^3 X_{t_{k-1}} \right) \right], \\ G^{(2)}_{\varepsilon,n,3}(\theta) &= -\frac{1}{\varepsilon^2}\sum_{k=1}^{n} X_{t_{k-1}} \left[X_{t_k} - X_{t_{k-1}} - \frac{1}{n}\left(\frac{\alpha}{X_{t_{k-1}}} - \beta \right) - \frac{1}{2n^2}\left(-\frac{\alpha^2}{X^3_{t_{k-1}}} + \beta^2 X_{t_{k-1}} \right) \right. \\ & \left. -\frac{1}{6n^3}\left(\frac{3\alpha^3}{X^5_{t_{k-1}}} + \frac{\alpha\beta^2}{X_{t_{k-1}}} - \frac{3\alpha^2\beta}{X^3_{t_{k-1}}} - \beta^3 X_{t_{k-1}} \right) \right]. \end{aligned}$$

As seen in the previous subsection, it is too difficult to derive an explicit estimator from the estimating function. Therefore, we need to take a one-step estimator.

In order to obtain an initial estimator, we consider the estimating functions of Uchida (2004c) as follows.

$$\begin{aligned} H^{(1)}_{\varepsilon,n}(\theta) &= \sum_{k=1}^{n} \frac{1}{X_{t_{k-1}}} \left\{ \varphi(X_{t_k}, \theta) - e^{-\lambda(\theta)/n} \varphi(X_{t_{k-1}}, \theta) \right\}, \\ H^{(2)}_{\varepsilon,n}(\theta) &= \sum_{k=1}^{n} X_{t_{k-1}} \left\{ \varphi(X_{t_k}, \theta) - e^{-\lambda(\theta)/n} \varphi(X_{t_{k-1}}, \theta) \right\}, \end{aligned}$$

where $\varphi(x, \theta) = \alpha - \beta x^2$ and $\lambda(\theta) = 2\beta$. By solving the estimating equation that $H_{\varepsilon,n}(\theta) = 0$, the estimators for α and β are

$$\begin{aligned} \hat{\alpha}^{(0)}_{\varepsilon,n} &= \hat{\beta}^{(0)}_{\varepsilon,n} \frac{A - e^{-2\hat{\beta}^{(0)}_{\varepsilon,n}/n} B}{(1 - e^{-2\hat{\beta}^{(0)}_{\varepsilon,n}/n}) C}, \\ \hat{\beta}^{(0)}_{\varepsilon,n} &= -\frac{n}{2} \log \frac{AD - CE}{BD - CF}, \end{aligned}$$

respectively, where

$$\begin{aligned} A &= \sum_{k=1}^{n} \frac{X^2_{t_k}}{X_{t_{k-1}}}, \ B = \sum_{k=1}^{n} X_{t_{k-1}}, \ C = \sum_{k=1}^{n} \frac{1}{X_{t_{k-1}}}, \\ D &= \sum_{k=1}^{n} X_{t_k}, \ E = \sum_{k=1}^{n} X_{t_{k-1}} X^2_{t_k}, \ F = \sum_{k=1}^{n} X^3_{t_{k-1}}. \end{aligned}$$

It follows from the result of Uchida (2004c) that the estimator $\hat{\theta}^{(0)}_{\varepsilon,n} = (\hat{\alpha}^{(0)}_{\varepsilon,n}, \hat{\beta}^{(0)}_{\varepsilon,n})$ is asymptotically normal as $\varepsilon \to 0$ and $n \to \infty$. Therefore, theorem 2 yields that the one-step estimator $\hat{\theta}^{(1)}_{\varepsilon,n,3}$ defined by (3) has asymptotic efficiency, that is,

$$\varepsilon^{-1}(\hat{\theta}^{(1)}_{\varepsilon,n,3} - \theta_0) \xrightarrow{\mathrm{d}} N(0, I(\theta_0)^{-1})$$

under $(\varepsilon n^3)^{-1} \to 0$ as $\varepsilon \to 0$ and $n \to \infty$, where

$$I^{(1,1)}(\theta_0) = \int_0^1 \frac{1}{(X_t^0)^2} dt,\ I^{(1,2)}(\theta_0) = I^{(2,1)}(\theta_0) = -1,\ I^{(2,2)}(\theta_0) = \int_0^1 (X_t^0)^2 dt.$$

In the setting that $\alpha = 10$, $\beta = 1$ and $x_0 = 1$, the simulation results of the three estimators for α and β are given in tables 2 and 3, respectively. For the case that $n = 10$, both $\hat{\alpha}_{\varepsilon,n}^{(S)}$ and $\hat{\alpha}_{\varepsilon,n}^{(L)}$ have considerable biases while $\hat{\alpha}_{\varepsilon,n,3}^{(1)}$ works well. In the situation where n is not so large, it is reasonable to use $\hat{\alpha}_{\varepsilon,n,3}^{(1)}$ for this model. For the situation where $n = 50$, $\hat{\alpha}_{\varepsilon,n}^{(S)}$ still has a small bias while both $\hat{\alpha}_{\varepsilon,n}^{(L)}$ and $\hat{\alpha}_{\varepsilon,n,3}^{(1)}$ are unbiased with small variances. We can say that $\hat{\alpha}_{\varepsilon,n,3}^{(1)}$ is better than the others in all cases. For the simulation results of β, we observe the same phenomenon as α. In the case that $n = 10$, both $\hat{\beta}_{\varepsilon,n}^{(S)}$ and $\hat{\beta}_{\varepsilon,n}^{(L)}$ have such serious significant biases that we can not use them in this setting. For the case that $n = 50$, $\hat{\beta}_{\varepsilon,n}^{(S)}$ has a small bias and there seems no difference between $\hat{\beta}_{\varepsilon,n}^{(L)}$ and $\hat{\beta}_{\varepsilon,n,3}^{(1)}$. It is worth mentioning that $\hat{\beta}_{\varepsilon,n,3}^{(1)}$ performs quite well even if n is not so large and ε is not so small.

Table 2: (Non-linear model 2) The mean and standard deviation (s.d.) of the three estimators determined from 1000 independent simulated sample paths for $\alpha = 10$, $\beta = 1$ and $x_0 = 1$.

		$\hat{\alpha}_{\varepsilon,n}^{(S)}$		$\hat{\alpha}_{\varepsilon,n}^{(L)}$		$\hat{\alpha}_{\varepsilon,n,3}^{(1)}$	
ε	n	mean	s.d.	mean	s.d.	mean	s.d.
0.10	10	6.98508	0.03127	8.98324	0.04847	10.44851	0.08243
0.10	50	9.29033	0.04892	9.97759	0.05622	10.01142	0.05666
0.05	10	6.98469	0.01563	8.98286	0.02424	10.44687	0.04121
0.05	50	9.28963	0.02445	9.97724	0.02810	10.01059	0.02832
0.01	10	6.98442	0.00313	8.98254	0.00485	10.44591	0.00824
0.01	50	9.28920	0.00489	9.97690	0.00562	10.01008	0.00566

Table 3: (Non-linear model 2) The mean and standard deviation (s.d.) of the three estimators determined from 1000 independent simulated sample paths for $\alpha = 10$, $\beta = 1$ and $x_0 = 1$.

		$\hat{\beta}_{\varepsilon,n}^{(S)}$		$\hat{\beta}_{\varepsilon,n}^{(L)}$		$\hat{\beta}_{\varepsilon,n,3}^{(1)}$	
ε	n	mean	s.d.	mean	s.d.	mean	s.d.
0.10	10	0.59291	0.00811	0.82724	0.01050	1.05620	0.01610
0.10	50	0.90386	0.01084	0.99485	0.01194	1.00092	0.01202
0.05	10	0.59284	0.00406	0.82717	0.00525	1.05593	0.00805
0.05	50	0.90371	0.00542	0.99477	0.00597	1.00075	0.00601
0.01	10	0.59278	0.00081	0.82711	0.00105	1.05576	0.00161
0.01	50	0.90362	0.00108	0.99470	0.00119	1.00065	0.00120

4 Proof

In order to prove the result, we introduce some notation and two lemmas. For lemmas 1 and 2 put later on, we set that $K_{\varepsilon,n,\ell}(\theta) = \left(K^{i,j}_{\varepsilon,n,\ell}(\theta)\right)_{i,j=1,2,\ldots,p}$ and $K(\theta) = \left(K^{i,j}(\theta)\right)_{i,j=1,2,\ldots,p}$, where $K^{i,j}_{\varepsilon,n,\ell}(\theta) = \delta_j G^{(i)}_{\varepsilon,n,\ell}(\theta)$,

$$K^{i,j}(\theta) \quad = \quad \int_0^1 \left[\delta_j \delta_i b(X^0_s, \theta)\right]^T \left[\sigma\sigma^T(X^0_s)\right]^{-1} B(X^0_s, \theta_0, \theta) ds - I^{i,j}(\theta)$$

and $B(x, \theta_0, \theta) = b(x, \theta_0) - b(x, \theta)$.

Lemma 1 (Uchida (2004a)) Let $\ell \in \mathbf{N}$. Assume (A1)-(A4). Then, as $\varepsilon \to 0$ and $n \to \infty$,

$$\sup_{\theta \in \bar{\Theta}} \left|\varepsilon^2 K_{\varepsilon,n,\ell}(\theta) - K(\theta)\right| \xrightarrow{\mathrm{P}} 0.$$

Lemma 2 (Uchida (2004a)) Let $\ell \in \mathbf{N}$. Assume (A1)-(A4). If $\left(\varepsilon n^\ell\right)^{-1} \to 0$, then, as $\varepsilon \to 0$ and $n \to \infty$,

$$\varepsilon G_{\varepsilon,n,\ell}(\theta_0) \xrightarrow{\mathrm{d}} N\left(0, I(\theta_0)\right).$$

For proofs of lemmas 1 and 2, see Uchida (2004a).

Proof of Theorem 2. Following the proof of asymptotic efficiency for one-step estimators presented in Yoshida (2004), we will prove theorem 2. Consider the following event A_0 defined by

$$A_0 = \left\{\hat{\theta}^{(0)}_{\varepsilon,n,\ell} \in \bar{\Theta},\ \delta_\theta G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) \in A,\ \hat{\theta}^{(1)}_{\varepsilon,n,\ell} \in \bar{\Theta}\right\},$$

where A is a whole set of non-singular matrices. First of all, we will show that $P\left(A_0\right) \longrightarrow 1$. It follows from the assumption of the initial estimator $\hat{\theta}^{(0)}_{\varepsilon,n,\ell}$ that

$$P(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} \in \bar{\Theta}) \longrightarrow 1. \tag{4}$$

By using the mean value theorem,

$$\left|\varepsilon^2 \left[G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) - G_{\varepsilon,n,\ell}(\theta_0)\right]\right| = \left|\int_0^1 \varepsilon^2 \delta_\theta G_{\varepsilon,n,\ell}\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) du\right| \left|\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0\right|.$$

Note that for $0 \le u \le 1$, $P(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0) \in \bar{\Theta}) \longrightarrow 1$. Lemma 1 yields that

$$\begin{aligned}
& \left|\int_0^1 \varepsilon^2 \delta_\theta G_{\varepsilon,n,\ell}\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) du - I(\theta_0)\right| \\
\le \quad & \int_0^1 \left|\varepsilon^2 \delta_\theta G_{\varepsilon,n,\ell}\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) - K\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right)\right| du \\
& + \int_0^1 \left|K\left(\theta_0 + u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) - I(\theta_0)\right| du \\
\xrightarrow{\mathrm{P}} \quad & 0.
\end{aligned}$$

By the above estimates, $\left|\varepsilon^2\left[G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-G_{\varepsilon,n,\ell}(\theta_0)\right]\right|\xrightarrow{\mathrm{P}}0$. Lemma 2 implies that

$$\varepsilon^2 G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\xrightarrow{\mathrm{P}}0. \tag{5}$$

By (4) and lemma 1, one has $\left|\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right|\xrightarrow{\mathrm{P}}0$. Further, the continuity of $K(\theta)$ yields that $\left|K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\theta_0)\right|\xrightarrow{\mathrm{P}}0$. By using these results,

$$\left|\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\theta_0)\right| \leq \left|\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right|+\left|K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\theta_0)\right|\xrightarrow{\mathrm{P}}0.$$

Noting that $K(\theta_0)=-I(\theta_0)$, we have $\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\xrightarrow{\mathrm{P}}-I(\theta_0)$. Therefore,

$$P\left(\delta_\theta G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\in A\right)\longrightarrow 1. \tag{6}$$

Since it follows from (5), (6) and the consistency of $\hat{\theta}^{(0)}_{\varepsilon,n,\ell}$ that

$$\hat{\theta}^{(1)}_{\varepsilon,n,\ell}=\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\left[\varepsilon^2\delta_\theta G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right]^{-1}\varepsilon^2 G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\xrightarrow{\mathrm{P}}\theta_0,$$

one has

$$P\big(\hat{\theta}^{(1)}_{\varepsilon,n,\ell}\in\bar{\Theta}\big)\longrightarrow 1. \tag{7}$$

Thus, by (4), (6) and (7) that $P(A_0)\longrightarrow 1$. From this fact, it suffices to consider the estimates on the event A_0 under the asymptotics we treat.

From the Taylor expansion at $\theta=\theta_0$, we have

$$\varepsilon G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}) = \varepsilon G_{\varepsilon,n,\ell}(\theta_0)+\left[\int_0^1 K_{\varepsilon,n,\ell}\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)du\right]\varepsilon(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0). \tag{8}$$

Let $\triangle_{\varepsilon,n}=\varepsilon\left[\delta_\theta G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right](\hat{\theta}^{(1)}_{\varepsilon,n,\ell}-\theta_0)+\varepsilon G_{\varepsilon,n,\ell}(\theta_0)$. Using (3) and (8), we then have

$$\begin{aligned}\triangle_{\varepsilon,n} &= \varepsilon\left[K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right]\left\{\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\left[K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right]^{-1}G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-\theta_0\right\}\\ &\quad+\varepsilon G_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-\left[\int_0^1 K_{\varepsilon,n,\ell}\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)du\right]\varepsilon(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\\ &= \varepsilon^2\left[K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-\int_0^1 K_{\varepsilon,n,\ell}\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)du\right]\varepsilon^{-1}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0).\end{aligned}$$

Moreover,

$$\begin{aligned}&\left|\varepsilon^2\left[K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-\int_0^1 K_{\varepsilon,n,\ell}\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)du\right]\right|\\ \leq\ &\int_0^1\left|\varepsilon^2 K_{\varepsilon,n,\ell}(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})\right|du\\ &+\int_0^1\left|K(\hat{\theta}^{(0)}_{\varepsilon,n,\ell})-K\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)\right|du\\ &+\int_0^1\left|\varepsilon^2 K_{\varepsilon,n,\ell}\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)-K\left(\theta_0+u(\hat{\theta}^{(0)}_{\varepsilon,n,\ell}-\theta_0)\right)\right|du.\end{aligned}$$

It follows from lemma 1 that

$$\int_0^1 \left|\varepsilon^2 K_{\varepsilon,n,\ell}(\hat\theta^{(0)}_{\varepsilon,n,\ell}) - K(\hat\theta^{(0)}_{\varepsilon,n,\ell})\right| du \xrightarrow{\mathrm{P}} 0,$$
$$\int_0^1 \left|\varepsilon^2 K_{\varepsilon,n,\ell}\left(\theta_0 + u(\hat\theta^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) - K\left(\theta_0 + u(\hat\theta^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right)\right| du \xrightarrow{\mathrm{P}} 0.$$

Furthermore, by the continuity of $K(\theta)$,

$$\int_0^1 \left|K(\hat\theta^{(0)}_{\varepsilon,n,\ell}) - K\left(\theta_0 + u(\hat\theta^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right)\right| du \xrightarrow{\mathrm{P}} 0.$$

Therefore,

$$\varepsilon^2 \left[K_{\varepsilon,n,\ell}(\hat\theta^{(0)}_{\varepsilon,n,\ell}) - \int_0^1 K_{\varepsilon,n,\ell}\left(\theta_0 + u(\hat\theta^{(0)}_{\varepsilon,n,\ell} - \theta_0)\right) du\right] = o_P(1),$$

and $\triangle_{\varepsilon,n} = o_P(1) \times O_P(1) = o_P(1)$. Since it follows from an easy computation that

$$\left[\varepsilon^2 K_{\varepsilon,n,\ell}(\hat\theta^{(0)}_{\varepsilon,n,\ell})\right]^{-1} \triangle_{\varepsilon,n} \quad = \quad \varepsilon^{-1}(\hat\theta^{(1)}_{\varepsilon,n,\ell} - \theta_0) + \varepsilon^{-1}\left[K_{\varepsilon,n,\ell}(\hat\theta^{(0)}_{\varepsilon,n,\ell})\right]^{-1} G_{\varepsilon,n,\ell}(\theta_0),$$

one has

$$\begin{aligned} &\varepsilon^{-1}(\hat\theta^{(1)}_{\varepsilon,n,\ell} - \theta_0) - [I(\theta_0)]^{-1}\varepsilon G_{\varepsilon,n,\ell}(\theta_0) \\ = \quad &\left[\varepsilon^2 K_{\varepsilon,n,\ell}(\hat\theta^{(0)}_{\varepsilon,n})\right]^{-1} \triangle_{\varepsilon,n} - \left\{\left[\varepsilon^2 K_{\varepsilon,n,\ell}(\hat\theta^{(0)}_{\varepsilon,n})\right]^{-1} + [I(\theta_0)]^{-1}\right\}\varepsilon G_{\varepsilon,n,\ell}(\theta_0). \end{aligned}$$

By the results that $\triangle_{\varepsilon,n} \xrightarrow{\mathrm{P}} 0$, and $\varepsilon^2 K_{\varepsilon,n,\ell}(\hat\theta^{(0)}_{\varepsilon,n}) \xrightarrow{\mathrm{P}} -I(\theta_0)$, we obtain that

$$\varepsilon^{-1}(\hat\theta^{(1)}_{\varepsilon,n,\ell} - \theta_0) - \left[I(\theta_0)\right]^{-1}\varepsilon G_{\varepsilon,n,\ell}(\theta_0) = o_P(1).$$

This completes the proof of (i).

Finally, it follows from (i) and lemma 2 that

$$\begin{aligned} \varepsilon^{-1}(\hat\theta^{(1)}_{\varepsilon,n,\ell} - \theta_0) \quad &= \quad \varepsilon^{-1}(\hat\theta^{(1)}_{\varepsilon,n,\ell} - \theta_0) - [I(\theta_0)]^{-1}\varepsilon G_{\varepsilon,n,\ell}(\theta_0) + [I(\theta_0)]^{-1}\varepsilon G_{\varepsilon,n,\ell}(\theta_0) \\ &\xrightarrow{\mathrm{d}} \quad N\left(0, [I(\theta_0)]^{-1}\right). \end{aligned}$$

We complete the proof.

Acknowledgements

This work was supported by the Ministry of Education, Culture, Sports, Science and Technology of Japan under Kyushu University 21st Century COE Program, Development of Dynamic Mathematics with High Functionality. The second author was partially supported by the Japan Society for the Promotion of Science under Grants-in-Aid for Scientific Researches.

References

Bibby, B. M., Sørensen, M. (1995). Martingale estimating functions for discretely observed diffusion processes. *Bernoulli* **1**, 17–39.

Ferguson, T. S. (1996). *A course in large sample theory*. Chapman & Hall/CRC.

Genon-Catalot, V. (1990). Maximum contrast estimation for diffusion processes from discrete observations. *Statistics* **21**, 99–116.

Kloeden, P. E., Platen, E. (1992). *Numerical solution of stochastic differential equations.* Springer-Verlag, New York.

Kutoyants, Yu. A. (1984). Parameter estimation for stochastic processes. Prakasa Rao, B.L.S. (ed.) Heldermann, Berlin.

Kutoyants, Yu. A. (1994). *Identification of dynamical systems with small noise.* Kluwer, Dordrecht.

Laredo, C. F. (1990). A sufficient condition for asymptotic sufficiency of incomplete observations of a diffusion process. *Ann. Statist.* **18**, 1158–1171.

Sakamoto, Y., Yoshida, N. (1999). Asymptotic expansion and efficiency of one-step estimator. preprint.

Sørensen, M. (2000). Small dispersion asymptotics for diffusion martingale estimating functions. Preprint No. 2000-2, Department of Statistics and Operations Research, University of Copenhagen. http://www.stat.ku.dk/research/preprint/

Sørensen, M., Uchida, M. (2003). Small diffusion asymptotics for discretely sampled stochastic differential equations. *Bernoulli* **9**, 1051–1069.

Uchida, M. (2004a). Estimation for discretely observed small diffusions based on approximate martingale estimating functions. *Scand. J. Statist.* **31**, 553-566.

Uchida, M. (2004b). Martingale estimating functions based on eigenfunctions for discretely observed small diffusions. submitted.

Uchida, M. (2004c). Approximate martingale estimating functions under small perturbations of dynamical systems. submitted.

Van der Vaart, A. W. (1998). *Asymptotic statistics.* Cambridge university press.

Yoshida, N. (1992). Asymptotic expansion of maximum likelihood estimators for small diffusions via the theory of Malliavin-Watanabe. *Probab. Theory Relat. Fields* **92**, 275–311.

Yoshida, N. (2003). Conditional expansions and their applications. *Stochastic Process. Appl.* **107**, 53–81.

Yoshida, N. (2004). *Stochastic analysis and Statistics.* Springer. in preparation.

List of MHF Preprint Series, Kyushu University

21st Century COE Program
Development of Dynamic Mathematics with High Functionality

MHF2003-1 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Yoshitaka WATANABE
A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems

MHF2003-2 Masahisa TABATA & Daisuke TAGAMI
Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients

MHF2003-3 Tomohiro ANDO, Sadanori KONISHI & Seiya IMOTO
Adaptive learning machines for nonlinear classification and Bayesian information criteria

MHF2003-4 Kazuhiro YOKOYAMA
On systems of algebraic equations with parametric exponents

MHF2003-5 Masao ISHIKAWA & Masato WAKAYAMA
Applications of Minor Summation Formulas III, Plücker relations, Lattice paths and Pfaffian identities

MHF2003-6 Atsushi SUZUKI & Masahisa TABATA
Finite element matrices in congruent subdomains and their effective use for large-scale computations

MHF2003-7 Setsuo TANIGUCHI
Stochastic oscillatory integrals - asymptotic and exact expressions for quadratic phase functions -

MHF2003-8 Shoki MIYAMOTO & Atsushi YOSHIKAWA
Computable sequences in the Sobolev spaces

MHF2003-9 Toru FUJII & Takashi YANAGAWA
Wavelet based estimate for non-linear and non-stationary auto-regressive model

MHF2003-10 Atsushi YOSHIKAWA
Maple and wave-front tracking — an experiment

MHF2003-11 Masanobu KANEKO
On the local factor of the zeta function of quadratic orders

MHF2003-12 Hidefumi KAWASAKI
Conjugate-set game for a nonlinear programming problem

MHF2004-1 Koji YONEMOTO & Takashi YANAGAWA
Estimating the Lyapunov exponent from chaotic time series with dynamic noise

MHF2004-2 Rui YAMAGUCHI, Eiko TSUCHIYA & Tomoyuki HIGUCHI
State space modeling approach to decompose daily sales of a restaurant into time-dependent multi-factors

MHF2004-3 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Cubic pencils and Painlevé Hamiltonians

MHF2004-4 Atsushi KAWAGUCHI, Koji YONEMOTO & Takashi YANAGAWA
Estimating the correlation dimension from a chaotic system with dynamic noise

MHF2004-5 Atsushi KAWAGUCHI, Kentarou KITAMURA, Koji YONEMOTO, Takashi YANAGAWA & Kiyofumi YUMOTO
Detection of auroral breakups using the correlation dimension

MHF2004-6 Ryo IKOTA, Masayasu MIMURA & Tatsuyuki NAKAKI
A methodology for numerical simulations to a singular limit

MHF2004-7 Ryo IKOTA & Eiji YANAGIDA
Stability of stationary interfaces of binary-tree type

MHF2004-8 Yuko ARAKI, Sadanori KONISHI & Seiya IMOTO
Functional discriminant analysis for gene expression data via radial basis expansion

MHF2004-9 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Hypergeometric solutions to the q - Painlevé equations

MHF2004-10 Raimundas VIDŪNAS
Expressions for values of the gamma function

MHF2004-11 Raimundas VIDŪNAS
Transformations of Gauss hypergeometric functions

MHF2004-12 Koji NAKAGAWA & Masakazu SUZUKI
Mathematical knowledge browser

MHF2004-13 Ken-ichi MARUNO, Wen-Xiu MA & Masayuki OIKAWA
Generalized Casorati determinant and Positon-Negaton-Type solutions of the Toda lattice equation

MHF2004-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCO
Generating function associated with the determinant formula for the solutions of the Painlevé II equation

MHF2004-15 Kouji HASHIMOTO, Ryohei ABE, Mitsuhiro T. NAKAO & Yoshitaka WATANABE
Numerical verification methods of solutions for nonlinear singularly perturbed problem

MHF2004-16 Ken-ichi MARUNO & Gino BIONDINI
Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete versions

MHF2004-17 Ryuei NISHII & Shinto EGUCHI
Supervised image classification in Markov random field models with Jeffreys divergence

MHF2004-18 Kouji HASHIMOTO, Kenta KOBAYASHI & Mitsuhiro T. NAKAO
Numerical verification methods of solutions for the free boundary problem

MHF2004-19 Hiroki MASUDA
Ergodicity and exponential β-mixing bounds for a strong solution of Lévy-driven stochastic differential equations

MHF2004-20 Setsuo TANIGUCHI
The Brownian sheet and the reflectionless potentials

MHF2004-21 Ryuei NISHII & Shinto EGUCHI
Supervised image classification based on AdaBoost with contextual weak classifiers

MHF2004-22 Hideki KOSAKI
On intersections of domains of unbounded positive operators

MHF2004-23 Masahisa TABATA & Shoichi FUJIMA
Robustness of a characteristic finite element scheme of second order in time increment

MHF2004-24 Ken-ichi MARUNO, Adrian ANKIEWICZ & Nail AKHMEDIEV
Dissipative solitons of the discrete complex cubic-quintic Ginzburg-Landau equation

MHF2004-25 Raimundas VIDŪNAS
Degenerate Gauss hypergeometric functions

MHF2004-26 Ryo IKOTA
The boundedness of propagation speeds of disturbances for reaction-diffusion systems

MHF2004-27 Ryusuke KON
Convex dominates concave: an exclusion principle in discrete-time Kolmogorov systems

MHF2004-28 Ryusuke KON
Multiple attractors in host-parasitoid interactions: coexistence and extinction

MHF2004-29 Kentaro IHARA, Masanobu KANEKO & Don ZAGIER
Derivation and double shuffle relations for multiple zeta values

MHF2004-30 Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Generalized partitioned quantum cellular automata and quantization of classical CA

MHF2005-1 Hideki KOSAKI
Matrix trace inequalities related to uncertainty principle

MHF2005-2 Masahisa TABATA
Discrepancy between theory and real computation on the stability of some finite element schemes

MHF2005-3 Yuko ARAKI & Sadanori KONISHI
Functional regression modeling via regularized basis expansions and model selection

MHF2005-4 Yuko ARAKI & Sadanori KONISHI
Functional discriminant analysis via regularized basis expansions

MHF2005-5 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Point configurations, Cremona transformations and the elliptic difference Painlevé equations

MHF2005-6 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Construction of hypergeometric solutions to the q - Painlevé equations

MHF2005-7 Hiroki MASUDA
Simple estimators for non-linear Markovian trend from sampled data:
I. ergodic cases

MHF2005-8 Hiroki MASUDA & Nakahiro YOSHIDA
Edgeworth expansion for a class of Ornstein-Uhlenbeck-based models

MHF2005-9 Masayuki UCHIDA
Approximate martingale estimating functions under small perturbations of dynamical systems

MHF2005-10 Ryo MATSUZAKI & Masayuki UCHIDA
One-step estimators for diffusion processes with small dispersion parameters from discrete observations