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A methodology for numerical simulations to a singular limit

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Abstract

A numerical methodology is proposed for a singular limit of certain reaction-diffusion systems. The limit problem arises in competition-diffusion systems and chemical reaction equations. Convergence of the semi-discrete-in-time solutions obtained by the methodology is proved. In a particular case, a convergence rate is also shown.
Keywords: reaction-diffusion systems, singular limit, operator splitting method

1 Introduction

In ecological and chemical problems, we encounter nonlinear equations of the form

$$w_t = \nabla \cdot (d(w)\nabla w) + h(w) \qquad x \in \Omega, \quad t > 0, \tag{1.1}$$

$$\frac{\partial w}{\partial \nu} = 0 \qquad x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

$$w(0, x) = w_0(x) \in L^\infty(\Omega) \qquad x \in \Omega, \tag{1.3}$$

where $d(s)$ is a step function:

$$d(s) = \begin{cases} d_1 & (s \geq 0), \\ d_2 & (s < 0). \end{cases} \tag{1.4}$$

The function $w = w(t, x)$ is real-valued, d_1 and d_2 are positive constants, Ω is a bounded region in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and ν is the unit outer normal to $\partial\Omega$. The aim of this paper is to propose a numerical methodology for (1.1)–(1.3).

One origin of (1.1)–(1.3) is in theoretical ecology. A reaction-diffusion system, specifically called a competition-diffusion system, has been studied as a model of spatially distributed competing species (see Cantrell (1996), Cosner & Lazer (1984), Dancer, Hilhorst, Mimura & Peletier (1999), Ei, Ikota & Mimura (1999), Iida, Muramatsu, Ninomiya & Yanagida (1998), Mimura & Fife (1986) for examples). Let $u_i(t, x)$ be the population density of an ith competing species U_i $(i = 1, 2, \ldots, n)$ at the time $t > 0$ and the position $x \in \Omega$. Then the competition-diffusion system is written as

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + (r_i - \sum_{j=1}^{n} a_{ij} u_j) u_i \quad (i = 1, 2, \ldots, n), \quad x \in \Omega, \quad t > 0, \tag{1.5}$$

where d_i is the diffusion rate, r_i the intrinsic growth rate, a_{ii} the intraspecific competition rate, and a_{ij} $(i \neq j)$ the interspecific competition rate between U_i and U_j. These parameters are all positive constants. We impose the homogeneous Neumann boundary condition and suppose the initial functions are non-negative.

One of our interests in (1.5) is spatially segregating patterns of the solutions, which appear when the interspecific competition rates a_{ij} $(i \neq j)$ are large. Let k be a positive parameter and put $a_{ij} = k b_{ij}$ $(i \neq j)$. Then from (1.5) we have

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + (r_i - a_{ii} u_i) u_i - k \sum_{j \neq i} b_{ij} u_i u_j \qquad (i = 1, 2, \ldots, n) \quad x \in \Omega, \quad t > 0. \tag{1.6}$$

What we have to observe is behavior of the solutions as k is large.

On the asymptotic behavior, an analytical result is known in the case $n = 2$. By putting $u = u_1$ and $v = b_{12} u_2$ the equations (1.6) lead to the following form:

$$u_t = d_1 \Delta u + f(u) u - kuv \qquad x \in \Omega, \quad t > 0, \tag{1.7}$$

$$v_t = d_2 \Delta v + g(v) v - \alpha kuv \qquad x \in \Omega, \quad t > 0, \tag{1.8}$$

where $\alpha = b_{21}$. The boundary and initial conditions are

$$\frac{\partial u}{\partial \nu} = 0, \quad \frac{\partial v}{\partial \nu} = 0 \qquad x \in \partial\Omega, \quad t > 0, \tag{1.9}$$

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x) \qquad x \in \Omega. \tag{1.10}$$

The functions f and g are written as $f(s) = a_u(R_u - s)$ and $g(s) = a_v(R_v - s)$, where $a_u = a_{11}$, $R_u = r_1/a_{11}$, $a_v = a_{22}/b_{12}$ and $R_v = r_2 b_{12}/a_{22}$. Thus they satisfy

$$f(s) > 0 \quad (0 < s < R_u), \qquad f(s) < 0 \quad (s > R_u), \tag{1.11}$$

$$g(s) > 0 \quad (0 < s < R_v), \qquad g(s) < 0 \quad (s > R_v). \tag{1.12}$$

Let $(u^{(k)}, v^{(k)})$ be a solution to (1.7)–(1.10). Dancer et al. (1999) have shown that as $k \to \infty$ the function $w^{(k)} = u^{(k)} - v^{(k)}/\alpha$ converges to a weak solution w to (1.1)–(1.3) with a function h such that

$$h(s) = \begin{cases} f(s)s & (s \geq 0), \\ g(-\alpha s)s & (s < 0). \end{cases} \tag{1.13}$$

Furthermore they proved that $u^{(k)} \to [w]^+$ and $v^{(k)} \to \alpha[w]^-$ respectively, where $[a]^\pm$ is $\max\{\pm a, 0\}$. See Proposition 2.1 in Dancer et al. (1999) for the detail. Thus (1.1)–(1.3) describe the asymptotic behavior of (1.7)–(1.10).

Another origin of (1.1)–(1.3) is chemical reaction equations, which we obtain by dropping the nonlinear terms f and g from (1.7)–(1.10). The singular limits of the chemical reaction equations have been studied by Evans and Tonegawa earlier on. Evans (1980) gave convergence proof for restrictive initial data. Tonegawa (1998) proved regularity properties of solutions to the limiting problem.

To solve (1.1)–(1.3) numerically, for example, we can apply finite volume methos (FVM) (Eymard, Gallouët, Hilhorst & Slimane 1998). FVM is capable of dealing with wide range of nonlinear diffusion equations.

Although the target of our numerical methodology, named *Threshold Competition Dynamics* (TCD), is rather restricted, it has flexibility in implementations; TCD includes a process to solve reaction-diffusion equations, to which we can apply finite difference methods, finite element methods or others. Hence, even if the shape of Ω is complicated we can use TCD without numerical difficulties. In addition, TCD has possibility to solve the singular limits of (1.6) in the case $n \geq 3$. In fact we have applied TCD to a three-component competition-diffusion system (Ikota, Mimura & Nakaki 2001).

We should note that TCD is similar to the so-called diffusion-generated approach for mean curvature flow (see Merriman, Bence & Osher (1994) and Ruuth (1998)). Its convergence has been proved by Barles & Georgelin (1995), Evans (1993), and for more general geometric motions of hypersurfaces by Ishii, Pires & Souganidis (1999).

Throughout the rest of this paper we assume h satisfies (1.13) with (1.11) and (1.12). It should be noticed that all the statements are still valid for the case $h \equiv 0$.

This paper is organized as follows. In the next section we present TCD and state our results. The results are composed of two theorems. One refers to the convergence of TCD in a general situation. The other gives a convergence rate in a restricted situation. The former theorem is proved in section 3 and the latter in section 4. In order to demonstrate practical usefulness of TCD, we perform numerical experiments in section 5. In the appendix we see notation index.

2 Results

A scheme is shown and results on it are stated in this section.

Put

$$F(s) := f(s)s, \quad G(s) := g(s)s. \tag{2.1}$$

Then the scheme that we propose is written as follows.

Threshold Competition Dynamics (TCD)

Let M be a positive integer. The approximate solution $(u_M(t,x), v_M(t,x))$ by TCD to the limiting problem of (1.7)–(1.10) as $k \to \infty$ is defined by

$$u_M(0,x) = u_0(x), \;\; v_M(0,x) = v_0(x) \quad \text{for } x \in \Omega, \tag{2.2}$$

$$u_M(t,x) = \overline{u}^j_M(t,x), \;\; v_M(t,x) = \overline{v}^j_M(t,x), \quad \text{for } t \in (t_j, t_{j+1}],\, x \in \Omega, \tag{2.3}$$

where

$$\tau := T/M, \quad t_j := j\tau \;\; (j = 0, 1, \ldots, M). \tag{2.4}$$

The functions $\overline{u}^j_M(t,x)$ and $\overline{v}^j_M(t,x)$ are constructed by the following steps:

Step 1. Put $u^0_M(x) = u_0(x)$, $v^0_M(x) = v_0(x)$ $(x \in \Omega)$.

Step 2. For given $u^j_M(x)$ and $v^j_M(x)$,

(i) Find $\overline{u}_M^j(t,x)$ and $\overline{v}_M^j(t,x)$ such that

$$\begin{cases} \dfrac{\partial \overline{u}_M^j}{\partial t} = d_1 \Delta \overline{u}_M^j + F(\overline{u}_M^j) & x \in \Omega, \quad t_j < t < t_{j+1}, \\ \dfrac{\partial \overline{u}_M^j}{\partial \nu} = 0 & x \in \partial\Omega, \quad t_j < t < t_{j+1}, \\ \overline{u}_M^j(t_j, x) = u_M^j(x) & x \in \Omega, \end{cases} \tag{2.5}$$

$$\begin{cases} \dfrac{\partial \overline{v}_M^j}{\partial t} = d_2 \Delta \overline{v}_M^j + G(\overline{v}_M^j) & x \in \Omega, \quad t_j < t < t_{j+1}, \\ \dfrac{\partial \overline{v}_M^j}{\partial \nu} = 0 & x \in \partial\Omega, \quad t_j < t < t_{j+1}, \\ \overline{v}_M^j(t_j, x) = v_M^j(x) & x \in \Omega. \end{cases} \tag{2.6}$$

(ii) Define $u_M^{j+1}(x)$ and $v_M^{j+1}(x)$ by

$$u_M^{j+1}(x) = \lim_{\theta\to\infty} \widehat{u}_M^j(\theta;x), \quad v_M^{j+1}(x) = \lim_{\theta\to\infty} \widehat{v}_M^j(\theta;x), \tag{2.7}$$

where $\widehat{u}_M^j$ and $\widehat{v}_M^j$ solve

$$\begin{cases} \dfrac{d\widehat{u}_M^j}{d\theta} = -\widehat{u}_M^j \widehat{v}_M^j & x \in \Omega, \quad 0 < \theta < k\tau, \\ \dfrac{d\widehat{v}_M^j}{d\theta} = -\alpha \widehat{u}_M^j \widehat{v}_M^j & x \in \Omega, \quad 0 < \theta < k\tau, \\ \widehat{u}_M^j(0;x) = \overline{u}_M^j(x, t_{j+1}), \quad \widehat{v}_M^j(0;x) = \overline{v}_M^j(x, t_{j+1}), & x \in \Omega. \end{cases} \tag{2.8}$$

We note that an operator-splitting method is used in Step 2, that is, (1.7) and (1.8) are splitted into

$$u_t = d_1 \Delta u + F(u), \quad v_t = d_2 \Delta v + G(v), \tag{2.9}$$

and

$$\frac{du}{dt} = -kuv, \quad \frac{dv}{dt} = -k\alpha uv. \tag{2.10}$$

The main idea of TCD is Step 2 (ii). Let $\theta = kt$; then (2.10) are rewritten to (2.8). Instead of passing to the limit $k \to \infty$ in (2.10), we use the asymptotic limit $\theta \to \infty$ in a solution to (2.8). The limit is easily obtained. In fact, by using the fact that $d(u - v/\alpha)/d\theta = 0$, it follows that

$$\lim_{\theta\to\infty} \widehat{u}_M^j(\theta;x) = \left[\overline{u}_M^j(t_{j+1}, x) - \overline{v}_M^j(t_{j+1}, x)/\alpha\right]^+, \tag{2.11}$$

$$\lim_{\theta\to\infty} \widehat{v}_M^j(\theta;x) = \alpha \left[\overline{u}_M^j(t_{j+1}, x) - \overline{v}_M^j(t_{j+1}, x)/\alpha\right]^-. \tag{2.12}$$

We now define a weak solution to (1.1)–(1.3).

Definition 2.1. *We call w a weak solution if it satisfies:*

$$w \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega)), \tag{2.13}$$

$$\int_\Omega w(T)\phi(T) - \iint_{Q_T} \{w\phi_t - d(w)\nabla w \nabla \phi + h(w)\phi\} = \int_\Omega w_0 \phi(0), \tag{2.14}$$

for all $\phi \in C^1(\overline{Q}_T)$, where $Q_T = \Omega \times (0,T)$.

Remark 2.1. *There exists a unique solution to* (2.13)–(2.14) *if* $w_0 \in L^\infty(\Omega)$ *(Dancer et al. 1999).*

We are ready to state our results.

Theorem 2.1. *Suppose* $w_0 \in L^\infty(\Omega)$*. Set* $u_0 = [w_0]^+$ *and* $v_0 = \alpha[w_0]^-$*. Let* w *be a weak solution for the initial data* w_0 *and* (u_M, v_M) *an approximate solution by Threshold Competition Dynamics for the initial data* (u_0, v_0)*. Then* u_M*,* v_M *and* $w_M = u_M - v_M/\alpha$ *converge to* $[w]^+$*,* $\alpha[w]^-$ *and* w *in* $L^2(0,T;L^2(\Omega))$ *respectively as* M *tends to* ∞*.*

Moreover, if $d_1 = d_2$ we have information about the convergence rate.

Theorem 2.2. *Functions* w_0*,* u_0*,* v_0*,* w_M *and* w *are the same as those in Theorem 2.1. Assume that*

$$u_0,\, v_0 \in H^2(\Omega) \cap L^\infty(\Omega), \tag{2.15}$$

and

$$\left.\begin{aligned} &\frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ &\int_\Omega u_0 v_0 = 0. \end{aligned}\right\} \tag{2.16}$$

In addition if $d_1 = d_2$*, then*

$$\|(u_M(T) - v_M(T)/\alpha) - w(T)\|_{L^2(\Omega)} \le C(1/M)^{1/2}, \tag{2.17}$$

$$\|(u_M(T) - v_M(T)/\alpha) - w(T)\|_{L^1(\Omega)} \le C'(1/M), \tag{2.18}$$

where C *and* C' *are positive constants independent of* M*.*

3 Proof of Theorem 2.1

We use the evolution triple $H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (H^1(\Omega))^*$ (see chapter 23 in Zeidler (1990) or chapter 3 in Temam (1984)) and prove the following four lemmas.

Lemma 3.1. *Functions* u_M *and* v_M *are uniformly bounded with respect to* M *in* $L^\infty(\Omega)$*. More precisely, for any* t $(0 \le t \le T)$ u_M *and* v_M *satisfy*

$$0 \le u_M(t,x) \le \max\{R_u, \|u_0\|_{L^\infty(\Omega)}\} \qquad \text{a.e. in } \Omega, \tag{3.1}$$

$$0 \le v_M(t,x) \le \max\{R_v, \|v_0\|_{L^\infty(\Omega)}\} \qquad \text{a.e. in } \Omega. \tag{3.2}$$

Proof. We establish only (3.1) because the same argument yields (3.2).

Recall that

$$u_M^j(x) = \left[\overline{u}_M^{j-1}(t_j, x) - \overline{v}_M^{j-1}(t_j, x)/\alpha\right]^+ \le \overline{u}_M^{j-1}(t_j, x).$$

Thus it suffices to show that

$$0 \le \overline{u}_M^j(t,x) \le \max\{R_u, \|u_M^j\|_{L^\infty(\Omega)}\} \qquad \text{a.e. in } \Omega \;\; (t_j \le t \le t_{j+1}). \tag{3.3}$$

By regularizing the function $u_M^j(x)$, we have only to show (3.3) for smooth solutions to (2.5). For smooth solutions, (3.3) is easily deducible from the comparison theorem. □

Lemma 3.2. *Functions u_M, v_M and w_M are uniformly bounded in $L^2(0,T;H^1(\Omega))$ with respect to M.*

Proof. We see in the scalar distribution sense on (t_j,t_{j+1}) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\overline{u}_M^j\right)^2 = -d_1\int_\Omega \left|\nabla \overline{u}_M^j\right|^2 + \int_\Omega F(\overline{u}_M^j)\overline{u}_M^j.$$

Thus we have

$$d_1\int_{t_j}^{t_{j+1}}\int_\Omega \left|\nabla \overline{u}_M^j\right|^2 = \frac{1}{2}\left[\lim_{t\downarrow t_j}\int_\Omega (u_M)^2 - \lim_{t\uparrow t_{j+1}}\int_\Omega (u_M)^2\right] + \int_{t_j}^{t_{j+1}}\int_\Omega F(u_M)u_M. \tag{3.4}$$

Therefore we obtain

$$\begin{aligned}
d_1\iint_{Q_T}|\nabla u_M|^2 &= \frac{1}{2}\sum_{j=0}^{M-1}\left[\lim_{t\downarrow t_j}\int_\Omega (u_M)^2 - \lim_{t\uparrow t_{j+1}}\int_\Omega (u_M)^2\right] + \iint_{Q_T}F(u_M)u_M\\
&= \frac{1}{2}\int_\Omega {u_0}^2 + \frac{1}{2}\sum_{j=0}^{M-2}\left[-\lim_{t\uparrow t_{j+1}}\int_\Omega (u_M)^2 + \lim_{t\downarrow t_{j+1}}\iint_\Omega (u_M)^2\right]\\
&\qquad\qquad -\frac{1}{2}\lim_{t\uparrow T}\int_\Omega (u_M)^2 + \iint_{Q_T}F(u_M)u_M\\
&\le \frac{1}{2}\int_\Omega {u_0}^2 + \iint_{Q_T}F(u_M)u_M.
\end{aligned}\tag{3.5}$$

Combining this and (3.1) we get the stated result for u_M. The boundedness for v_M is obtained in the same way. Since $w_M = u_M - v_M/\alpha$, it is also uniformly bounded. $\square$

Lemma 3.3.

$$\iint_{Q_T} u_M v_M \to 0 \qquad \text{as } M\to\infty. \tag{3.6}$$

Proof. From the equation (25) in section 23.6 (Zeidler 1990), in each interval $[t_j,t_{j+1}]$ ($j = 0,1,\ldots,M-1$) we have

$$\begin{aligned}
\int_\Omega \overline{u}_M^j(t,\cdot)\overline{v}_M^j(t,\cdot) &= \int_\Omega \overline{u}_M^j(t,\cdot)\overline{v}_M^j(t,\cdot) - \int_\Omega u_M^j(\cdot)v_M^j(\cdot)\\
&= -(d_1+d_2)\int_{t_j}^t ds\int_\Omega \nabla u_M(s,\cdot)\cdot\nabla v_M(s,\cdot)\\
&\qquad + \int_{t_j}^t ds\int_\Omega \{F(u_M(s,\cdot))v_M(s,\cdot) + G(v_M(s,\cdot))u_M(s,\cdot)\}\\
&\le \frac{d_1+d_2}{2}\int_{t_j}^t ds\int_\Omega (|\nabla u_M(s,\cdot)|^2 + |\nabla v_M(s,\cdot)|^2) + C_1(t-t_j),
\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}
C_1 = |\Omega|\sup\{F(p)q + G(q)p \mid\\
&0\le p\le \max\{R_u,\|u_0\|_{L^\infty(\Omega)}\},\ 0\le q\le\max\{R_v,\|v_0\|_{L^\infty(\Omega)}\}\}.
\end{aligned}$$

Here notice that $u_M^j v_M^j = 0$ in Ω. Now we observe

$$\begin{aligned}
\int_{t_j}^{t_{j+1}} dt \int_{t_j}^{t} ds \int_\Omega (|\nabla u_M(s,\cdot)|^2 + |\nabla v_M(s,\cdot)|^2) \\
&\le \int_{t_j}^{t_{j+1}} dt \int_{t_j}^{t_{j+1}} ds \int_\Omega (|\nabla u_M(s,\cdot)|^2 + |\nabla v_M(s,\cdot)|^2) \\
&\le \tau \int_{t_j}^{t_{j+1}} ds \int_\Omega (|\nabla u_M(s,\cdot)|^2 + |\nabla v_M(s,\cdot)|^2).
\end{aligned}$$

Thus we have

$$\iint_{Q_T} u_M v_M \le \frac{d_1+d_2}{2}\tau \int_0^T \int_\Omega (|\nabla u_M(s,\cdot)|^2 + |\nabla v_M(s,\cdot)|^2) + \frac{C_1}{2}\tau T \to 0$$

as $M \to \infty$. □

Lemma 3.4. *The generalized derivative $\partial_t w_M$ exists and it is written as*

$$\partial_t w_M = d_1 \Delta u_M - (d_2/\alpha)\Delta v_M + F(u_M) - G(v_M)/\alpha. \tag{3.8}$$

Proof. Although u_M and v_M are generally discontinuous as functions from $[0,T]$ to $L^2(\Omega)$, w_M belongs to $C([0,T];L^2(\Omega))$. In addition $\partial_t u_M$ and $\partial_t v_M$ exist in each interval $[t_j, t_{j+1}]$. Hence, for $z \in H^1(\Omega)$ and $\psi \in C_0^\infty(0,T)$, we have

$$\begin{aligned}
\int_0^T \int_\Omega w_M z \psi_t &= \sum_{j=0}^{M-1} \Bigg[\int_\Omega w_M(t_{j+1},\cdot) z(\cdot)\psi(t_{j+1}) - \int_\Omega w_M(t_j,\cdot) z(\cdot)\psi(t_j) \\
&\qquad - \int_{t_j}^{t_{j+1}} \int_\Omega (\partial_t u_M - \partial_t v_M/\alpha) z\psi \Bigg] \\
&= -\int_0^T \int_\Omega \{d_1 \Delta u_M + F(u_M) - (d_2 \Delta v_M + G(v_M))/\alpha\} z\psi.
\end{aligned}$$

Thus the statement follows. □

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. From Lemmas 3.2 and 3.4, we observe that $\|w_M\|_{L^2(0,T;H^1(\Omega))}$ and $\|\partial_t w_M\|_{L^2(0,T;H^1(\Omega)^*)}$ are uniformly bounded with respect to M. Hence thanks to the compactness property (Theorem 2.1, chapter 3 in Temam (1984)) we obtain a subsequence from $\{w_M\}$, which is denoted by $\{w_M\}$ again, converging in $L^2(0,T;L^2(\Omega))$. We write the limit as w_∞:

$$w_M \to w_\infty \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \quad \text{as} \quad M \to \infty. \tag{3.9}$$

Note that

$$\left|u - [u - v/\alpha]^+\right|^2 \le \frac{uv}{\alpha}, \tag{3.10}$$

$$\left|v/\alpha - [u - v/\alpha]^-\right|^2 \le \frac{uv}{\alpha}. \tag{3.11}$$

Thus $u_M - [w_M]^+$ and $v_M/\alpha - [w_M]^-$ converge to 0 in $L^2(0,T;L^2(\Omega))$ from Lemma 3.3. On the other hand we see that

$$[w_M]^\pm \to [w_\infty]^\pm \qquad \text{in} \ \ L^2(0,T;L^2(\Omega)), \tag{3.12}$$

because $|[w_M]^{\pm} - [w_\infty]^{\pm}| \le |w_M - w_\infty|$. Therefore we have

$$u_M \to [w_\infty]^+ \qquad \text{in } L^2(0,T;L^2(\Omega)), \tag{3.13}$$

$$v_M/\alpha \to [w_\infty]^- \qquad \text{in } L^2(0,T;L^2(\Omega)). \tag{3.14}$$

Extracting a subsequence again if necessary we see u_M and v_M/α converge weakly in $L^2(0,T;H^1(\Omega))$ and $\partial_t w_M$ does in $L^2(0,T;H^1(\Omega)^*)$ from Lemmas 3.2 and 3.4. The limits are identical with $\partial_t w_M$, $[w_\infty]^+$ and $[w_\infty]^-$ respectively owing to (3.13), (3.14) and Lemma 3.4:

$$\begin{aligned} \partial_t w_M \to \partial_t w_\infty & \qquad \text{weakly in } L^2(0,T;H^1(\Omega)^*), \\ u_M \to [w_\infty]^+ & \qquad \text{weakly in } L^2(0,T;H^1(\Omega)), \\ v_M/\alpha \to [w_\infty]^- & \qquad \text{weakly in } L^2(0,T;H^1(\Omega)). \end{aligned}$$

In addition, from the Lebesgue's bounded integral lemma we have

$$\begin{aligned} \int_{Q_T} F(u_M)\varphi &\to \int_{Q_T} F([w_\infty]^+)\varphi, \\ \int_{Q_T} \frac{G(v_M)}{\alpha}\varphi &\to \int_{Q_T} \frac{G(\alpha[w_\infty]^-)}{\alpha}\varphi. \end{aligned}$$

Now we observe for $\varphi \in C^1(\overline{Q}_T)$ that

$$\begin{aligned} \int_0^T \int_\Omega (\partial_t w_M)\varphi = -\int_0^T \int_\Omega d_1 \nabla u_M \nabla\varphi + \int_0^T \int_\Omega d_2 \frac{\nabla v_M}{\alpha}\nabla\varphi \\ + \int_0^T \int_\Omega F(u_M)\varphi - \int_0^T \int_\Omega \frac{G(v_M)}{\alpha}\varphi. \end{aligned}$$

Thus we get

$$\int_0^T \int_\Omega (\partial_t w_\infty)\varphi = -\int_0^T \int_\Omega d(w_\infty)\nabla w_\infty \nabla\varphi + h(w_\infty)\varphi. \tag{3.15}$$

On the other hand, for $\widetilde{\varphi} \in C^1(\overline{Q}_T)$ satisfying $\widetilde{\varphi}(T,\cdot) = 0$ we see

$$\int_0^T \int_\Omega (\partial_t w_M)\widetilde{\varphi} = -\int_\Omega w_0(\cdot)\widetilde{\varphi}(0,\cdot) - \iint_{Q_T} w_M \widetilde{\varphi}_t. \tag{3.16}$$

Passing to the limit along a subsequence, we have

$$\int_0^T \int_\Omega (\partial_t w_\infty)\widetilde{\varphi} = -\int_\Omega w_0(\cdot)\widetilde{\varphi}(0,\cdot) - \iint_{Q_T} w_\infty \widetilde{\varphi}_t. \tag{3.17}$$

This implies $w_\infty(0,\cdot) = w_0(\cdot)$. Consequently w_∞ is a weak solution to (1.1)–(1.3). Since the limit is unique, the original whole sequence $\{w_M\}$ converges to w_∞. □

4 Proof of Theorem 2.2

Throughout this section we assume that the conditions for Theorem 2.2 are satisfied. Set

$$e_j^{(p)} := \|w_M(t_j,\cdot) - w(t_j,\cdot)\|_{L^p(\Omega)} \qquad (p = 1,2). \tag{4.1}$$

Our strategy is to deduce a recursive inequality for $e_j^{(p)}$.

In this section we choose a positive constant R_1 so that

$$R_1 > \max\{\|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)}, R_u, R_v\}. \tag{4.2}$$

In addition we set

$$R_2 = \max\left\{\sup_{0\le p\le R_1}|F(p)|, \sup_{0\le p\le R_1}|G(p)|\right\}, \tag{4.3}$$

$$R_3 = \max\left\{\sup_{0\le p\le R_1}|F'(p)|, \sup_{0\le p\le R_1}|G'(p)|\right\}. \tag{4.4}$$

Here we state the convergence theorem obtained by Dancer et al. (1999).

Proposition 4.1 (Dancer et al. (1999)). *Suppose*

$$u_0^{(k)},\ v_0^{(k)} \in C(\overline{\Omega}), \tag{4.5}$$

$$0 \le u_0^{(k)} \le R_1, \quad 0 \le v_0^{(k)} \le R_1, \tag{4.6}$$

$$u_0^{(k)} \to u_0, \ \ v_0^{(k)} \to v_0, \quad \text{weakly in } L^2(\Omega) \text{ as } k \to \infty. \tag{4.7}$$

Denote solutions to (1.7)–(1.9) *with initial conditions* $u(0,\cdot) = u_0^{(k)}$ *and* $v(0,\cdot) = v_0^{(k)}$ *by* $u^{(k)}$ *and* $v^{(k)}$. *Set* $w^{(k)} = u^{(k)} - v^{(k)}/\alpha$. *Then there exists a weak solution* w *to* (2.13)–(2.14) *such that it satisfies*

$$u^{(k)} \to [w]^+, \ \ v^{(k)} \to \alpha[w]^- \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{4.8}$$

$$u^{(k)} \to [w]^+, \ \ v^{(k)} \to \alpha[w]^- \quad \text{in } L^1(Q_T), \tag{4.9}$$

$$u^{(k)}v^{(k)} \to 0 \quad \text{in } L^1(Q_T), \tag{4.10}$$

$$w^{(k)} \to w \quad \text{in } L^2(Q_T). \tag{4.11}$$

In order to prove Theorem 2.2 we work within the framework of the following spaces:

$$\hat{C}^2(\overline{\Omega}) \stackrel{\text{def}}{=} \left\{ v \in C^2(\overline{\Omega})\,; \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \right\}, \tag{4.12}$$

$$\hat{C}^3(\overline{\Omega}) \stackrel{\text{def}}{=} \left\{ u \in C^3(\overline{\Omega})\,; \quad \frac{\partial u}{\partial \nu} = 0, \quad \frac{\partial \Delta u}{\partial \nu} = 0 \ \text{ on } \partial\Omega \right\}. \tag{4.13}$$

The next proposition is obtained by putting $N = 2$, $N' = 0$, $m = 1$, $q = 1$, $r = 0$, $k = 2$ and $l = 0$ in Theorem 4.1 of Mora (1983).

Proposition 4.2. *The equations* (1.7)–(1.9) *determines a semiflow of class* C^0 *on the space* $\hat{C}^3(\overline{\Omega})$.

Here we prove several lemmas.

Lemma 4.1. *Suppose* $u_0^{(k)} \in \hat{C}^3(\overline{\Omega})$ *and* $v_0^{(k)} \in \hat{C}^3(\overline{\Omega})$. *Then solutions* $u^{(k)}$ *and* $v^{(k)}$ *to* (1.7)–(1.9), *with initial conditions* $u^{(k)}(0,\cdot) = u_0^{(k)}$ *and* $v^{(k)}(0,\cdot) = v_0^{(k)}$, *satisfy*

$$\begin{aligned} &\sup_{0\le t\le T}\left(\int_\Omega |u_t^{(k)}(t,\cdot)| + \int_\Omega |v_t^{(k)}(t,\cdot)/\alpha|\right) \\ &\qquad \le e^{R_3 T}\int_\Omega \left(|d_1\Delta u_0^{(k)}| + |d_2 \Delta v_0^{(k)}/\alpha| + |F(u_0^{(k)})| + |G(v_0^{(k)})/\alpha| + 2k u_0^{(k)} v_0^{(k)}\right). \end{aligned} \tag{4.14}$$

Proof. An argument similar to that in Evans (1980) provides the following inequality:

$$\int_\Omega (|u_t(T,\cdot)| + |v_t(T,\cdot)/\alpha|) \\ \leq \int_\Omega (|u_t(0,\cdot)| + |v_t(0,\cdot)/\alpha|) + R_3 \iint_{Q_T} (|u_t| + |v_t/\alpha|).$$

Whence we get (4.14) from the Gronwall's inequality. □

Lemma 4.2. *If the nonnegative functions u_0 and v_0 satisfy* (2.15) *and* (2.16), *then there exist* $\{k_i\}_{i=1}^\infty \subset \mathbb{R}_+$, $\{u_0^{(i)}\}_{i=1}^\infty \subset \hat{C}^3(\overline{\Omega})$ *and* $\{v_0^{(i)}\}_{i=1}^\infty \subset \hat{C}^3(\overline{\Omega})$ *such that*

$$k_i \to \infty \qquad \text{as } i \to \infty, \tag{4.15}$$

$$0 \leq u_0^{(i)} \leq R_1, \quad 0 \leq v_0^{(i)} \leq R_1 \quad (i = 1, 2, \dots), \tag{4.16}$$

$$u_0^{(i)} \to u_0, \quad v_0^{(i)} \to v_0 \quad \text{in } H^1(\Omega) \qquad \text{as } i \to \infty, \tag{4.17}$$

$$\int_\Omega |d_1 \Delta u_0^{(i)}| + \int_\Omega |d_2 \Delta v_0^{(i)}/\alpha| \leq C_2 \qquad (i = 1, 2, \dots), \tag{4.18}$$

$$k_i \int_\Omega u_0^{(i)} v_0^{(i)} \leq C_3 \qquad (i = 1, 2, \dots), \tag{4.19}$$

where C_2, C_3 are independent of i.

Proof. We use the heat equation with the homogeneous Neumann boundary condition as a mollifier. From the standard regularity argument (see Brezis (1983) for example), we observe that

$$e^{t\Delta} u_0,\ e^{t\Delta} v_0 \in \hat{C}^3(\overline{\Omega}) \qquad \text{for } t > 0,$$
$$e^{t\Delta} u_0 \to u_0,\ e^{t\Delta} v_0 \to v_0 \qquad \text{in } H^2(\Omega) \text{ as } t \downarrow 0.$$

Moreover we have $0 \leq e^{t\Delta} u_0 \leq \|u_0\|_{L^\infty(\Omega)}$, $0 \leq e^{t\Delta} v_0 \leq \|v_0\|_{L^\infty(\Omega)}$. Then we set

$$u_0^{(i)} = e^{t\Delta} u_0 \Big|_{t=1/i}, \qquad v_0^{(i)} = e^{t\Delta} v_0 \Big|_{t=1/i}.$$

Since

$$\int_\Omega u_0^{(i)} v_0^{(i)} \to \int_\Omega u_0 v_0 = 0,$$

we have

$$k_i \stackrel{\text{def}}{=} \frac{1}{1/i + \int_\Omega u_0^{(i)} v_0^{(i)}} \to \infty \quad \text{as } i \to \infty.$$

Hence we prove the statement. □

Lemma 4.3. *If the conditions* (4.15)–(4.19) *hold, then solutions $u^{(k_i)}$ and $v^{(k_i)}$ to* (1.7)–(1.9) *for $k = k_i$, $u^{(k_i)}(0,\cdot) = u_0^{(i)}$ and $v^{(k_i)}(0,\cdot) = v_0^{(i)}$ satisfy*

$$\sup_{0 \leq t \leq T} \int_\Omega \left(|u_t^{(k_i)}| + |v_t^{(k_i)}/\alpha| \right) \leq C_4, \tag{4.20}$$

$$\sup_{0 \leq t \leq T} k_i \int_\Omega u^{(k_i)} v^{(k_i)} \leq C_5, \tag{4.21}$$

$$\sup_{0 \leq t \leq T} \left(d_1 \int_\Omega |\nabla u^{(k_i)}|^2 + d_2 \int_\Omega |\nabla v^{(k_i)}/\alpha|^2 \right) \leq C_6, \tag{4.22}$$

$$\sup_{0 \leq t \leq T} \left(d_1 \int_\Omega |\Delta u^{(k_i)}| + d_2 \int_\Omega |\Delta v^{(k_i)}/\alpha| \right) \leq C_7, \tag{4.23}$$

where C_5, C_6 and C_7 are positive constants independent of k_i.

Proof. From Lemmas 4.1 and 4.2 we obtain (4.20) by putting $C_4 = e^{R_3 T}(C_2+(1+\alpha^{-1})R_2+2C_3)$. Thus we have

$$k_i \int_\Omega u^{(k_i)} v^{(k_i)} = -\int_\Omega u_t^{(k_i)} + \int_\Omega F(u^{(k_i)}) \le C_5,$$

where $C_5 = C_4 + R_2|\Omega|$.

Next, multiplying the both hand sides of (1.7) by $u^{(k_i)}$ and integrating them over Ω, we observe

$$\begin{aligned} d_1 \int_\Omega |\nabla u^{(k_i)}|^2 &= -\int_\Omega u^{(k_i)} u_t^{(k_i)} + \int_\Omega F(u^{(k_i)}) u^{(k_i)} - k_i \int_\Omega (u^{(k_i)})^2 v^{(k_i)} \\ &\le R_1 C_4 + R_1 R_2 |\Omega|. \end{aligned}$$

Note that $\|u^{(k_i)}\|_{L^\infty(\Omega)} \le R_1$ from the comparison theorem. A similar estimate holds for $d_2 \int_\Omega |\nabla v^{(k_i)}|^2$. Setting $C_6 = (1+\alpha^{-1})R_1 C_4 + (1+\alpha^{-2})R_1 R_2|\Omega|$ provides (4.22).

Finally we see

$$\begin{aligned} d_1 \int_\Omega |\Delta u^{(k_i)}| &= \int_\Omega |u_t^{(k_i)} - F(u^{(k_i)}) + k_i u^{(k_i)} v^{(k_i)}| \\ &\le C_4 + R_2|\Omega| + C_5. \end{aligned}$$

We obtain the estimate for $d_2 \int_\Omega |\Delta v^{(k_i)}|$ likewise. Putting $C_7 = 2C_4 + (1+\alpha^{-1})R_2|\Omega| + 2C_5$ we prove (4.23). □

Proposition 4.3. *Suppose the same conditions as those in Lemma 4.3 hold. Let w be the weak solution to* (1.1)–(1.3)*. Then w satisfies the followings:*

$$u^{(k_i)} \to [w]^+ \qquad \text{in } C([0,T]; L^2(\Omega)), \tag{4.24}$$

$$v^{(k_i)}/\alpha \to [w]^- \qquad \text{in } C([0,T]; L^2(\Omega)), \tag{4.25}$$

$$w^{(k_i)} = u^{(k_i)} - v^{(k_i)}/\alpha \to w \qquad \text{in } C([0,T]; L^2(\Omega)). \tag{4.26}$$

Proof. We see for $t \ge s$

$$\begin{aligned} &\int_\Omega \left| u^{(k_i)}(t,\cdot) - u^{(k_i)}(s,\cdot) \right|^2 \\ &\qquad = \int_\Omega \left\{ (u^{(k_i)}(t,x) - u^{(k_i)}(s,x)) \int_s^t u_t^{(k_i)}(\sigma,x)\, d\sigma \right\} dx \\ &\qquad \le 2R_1 \int_s^t \left(\int_\Omega \left| u_t^{(k_i)}(\sigma,x) \right| dx \right) d\sigma \\ &\qquad \le 2R_1 C_4 (t-s). \end{aligned} \tag{4.27}$$

Hence $u^{(k_i)}$ is uniformly bounded in $C^{1/2}([0,T]; L^2(\Omega))$ and so is $v^{(k_i)}$. Therefore $u^{(k_i)}$ and $v^{(k_i)}$ are equicontinuous in $C([0,T]; L^2(\Omega))$. Moreover from (4.22) $u^{(k_i)}(t,\cdot)$ and $v^{(k_i)}(t,\cdot)$ are contained in a compact subset of $L^2(\Omega)$. By Ascoli-Arzela's theorem we obtain a subsequence converging in $C([0,T]; L^2(\Omega))$. In view of Proposition 4.1 the limit of $w^{(k_i)}$ along the subsequence is the weak solution. Uniqueness of the limit assures the convergence of the original sequence. □

From this proposition we immediately have the next lemma.

Lemma 4.4. *For any positive ε, there exist $k = k(\varepsilon) > 0$ and $u_0^{(\varepsilon)}$, $v_0^{(\varepsilon)} \in \hat{C}^3(\bar{\Omega})$ such that solutions $u^{(k(\varepsilon))}$, $v^{(k(\varepsilon))}$ to* (1.7)–(1.9) *with the initial conditions $u^{(k(\varepsilon))}(0,\cdot) = u_0^{(\varepsilon)}$ and $v^{(k(\varepsilon))} = v_0^{(\varepsilon)}$ satisfy the following:*

$$\|w^{(k(\varepsilon))} - w\|_{C([0,T];L^p(\Omega))} \le \varepsilon, \tag{4.28}$$

$$\|u^{(k(\varepsilon))} - [w]^+\|_{C([0,T];L^p(\Omega))} \le \varepsilon, \tag{4.29}$$

$$\|v^{(k(\varepsilon))}/\alpha - [w]^-\|_{C([0,T];L^p(\Omega))} \le \varepsilon, \tag{4.30}$$

where $w^{(k(\varepsilon))} = u^{(k(\varepsilon))} - v^{(k(\varepsilon))}/\alpha$ and $p = 1, 2$. Moreover $k(\varepsilon)$, $u^{(k(\varepsilon))}$ and $v^{(k(\varepsilon))}$ satisfy the inequalities (4.20)–(4.23).

Lemma 4.5. *Consider the following equations in each interval $[t_j, t_{j+1}]$:*

$$\frac{\partial \bar{\bar{u}}_M^{j,\varepsilon}}{\partial t} = d_1 \Delta \bar{\bar{u}}_M^{j,\varepsilon} + F(\bar{\bar{u}}_M^{j,\varepsilon}), \qquad t_j < t \le t_{j+1}, \quad x \in \Omega, \tag{4.31}$$

$$\frac{\partial \bar{\bar{v}}_M^{j,\varepsilon}}{\partial t} = d_2 \Delta \bar{\bar{v}}_M^{j,\varepsilon} + G(\bar{\bar{v}}_M^{j,\varepsilon}), \qquad t_j < t \le t_{j+1}, \quad x \in \Omega, \tag{4.32}$$

$$\frac{\partial \bar{\bar{u}}_M^{j,\varepsilon}}{\partial \nu} = \frac{\partial \bar{\bar{v}}_M^{j,\varepsilon}}{\partial \nu} = 0, \qquad t_j < t \le t_{j+1}, \quad x \in \partial\Omega, \tag{4.33}$$

$$\bar{\bar{u}}_M^{j,\varepsilon}(t_j, x) = u^{(k(\varepsilon))}(t_j, x), \tag{4.34}$$

$$\bar{\bar{v}}_M^{j,\varepsilon}(t_j, x) = v^{(k(\varepsilon))}(t_j, x). \tag{4.35}$$

Set

$$\bar{\bar{w}}_M^{j,\varepsilon} = \bar{\bar{u}}_M^{j,\varepsilon} - \bar{\bar{v}}_M^{j,\varepsilon}/\alpha. \tag{4.36}$$

If $d_1 = d_2$, the following inequality holds:

$$\|w_M(t_{j+1},\cdot) - \bar{\bar{w}}_M^{j,\varepsilon}(t_{j+1},\cdot)\|_{L^p(\Omega)} \le (e_j^{(p)} + \varepsilon)(1 + E\tau), \quad (p = 1, 2), \tag{4.37}$$

where E is independent of M, j and ε.

Proof. Throughout this proof we assume $p = 1, 2$. Using the Duhamel formula we have

$$\begin{aligned}
&\|w_M(t_{j+1},\cdot) - \bar{\bar{w}}_M^{j,\varepsilon}(t_{j+1},\cdot)\|_{L^p(\Omega)} \\
&\le \|e^{(t_{j+1}-t_j)d_1\Delta}(w_M(t_j,\cdot) - \bar{\bar{w}}_M^{j,\varepsilon}(t_j,\cdot)\|_{L^p(\Omega)} \\
&\qquad + \left\| \int_{t_j}^{t_{j+1}} e^{(t_{j+1}-s)d_1\Delta}(F(\overline{u}_M^j(s,\cdot)) - F(\bar{\bar{u}}_M^{j,\varepsilon}(s,\cdot))ds \right\|_{L^p(\Omega)} \\
&\qquad + \frac{1}{\alpha} \left\| \int_{t_j}^{t_{j+1}} e^{(t_{j+1}-s)d_1\Delta}(G(\overline{v}_M^j(s,\cdot)) - G(\bar{\bar{v}}_M^{j,\varepsilon}(s,\cdot))ds \right\|_{L^p(\Omega)} \\
&= I + II + III.
\end{aligned}$$

Let w be the weak solution to (1.1)–(1.3). Recall that $\|e^{td_1\Delta}z\|_{L^p(\Omega)} \le \|z\|_{L^p(\Omega)}$ for $z \in L^p(\Omega)$ ($p = 1, 2$). Then we observe

$$\begin{aligned}
I &\le \|w_M(t_j,\cdot) - \bar{\bar{w}}_M^{j,\varepsilon}(t_j,\cdot)\|_{L^p(\Omega)} \\
&\le \|w_M(t_j,\cdot) - w(t_j,\cdot)\|_{L^p(\Omega)} + \|w(t_j,\cdot) - \bar{\bar{w}}_M^{j,\varepsilon}(t_j,\cdot)\|_{L^p(\Omega)} \\
&\le e_j^{(p)} + \varepsilon.
\end{aligned}$$

Next we estimate II. For $t_j < t \le t_{j+1}$ by means of the Duhamel formula we obtain

$$
\begin{aligned}
\|\overline{u}^j_M(t,\cdot)-&\bar{\bar{u}}^{j,\varepsilon}_M(t,\cdot)\|_{L^p(\Omega)} \\
&\le \|\overline{u}^j_M(t_j,\cdot) - \bar{\bar{u}}^{j,\varepsilon}_M(t_j,\cdot)\|_{L^p(\Omega)} + \int_{t_j}^t \|F(\overline{u}^j_M(s,\cdot)) - F(\bar{\bar{u}}^{j,\varepsilon}_M(s,\cdot))\|_{L^p(\Omega)} ds \\
&\le \|u^j_M(\cdot) - [w(t_j,\cdot)]^+\|_{L^p(\Omega)} + \|[w(t_j,\cdot)]^+ - \bar{\bar{u}}^{j,\varepsilon}_M(t_j,\cdot)\|_{L^p(\Omega)} \\
&\qquad\qquad + \int_{t_j}^t \|F(\overline{u}^j_M(s,\cdot)) - F(\bar{\bar{u}}^{j,\varepsilon}_M(s,\cdot))\|_{L^p(\Omega)} ds \\
&\le e^{(p)}_j + \varepsilon + R_3 \int_{t_j}^t \|\overline{u}^j_M(s,\cdot) - \bar{\bar{u}}^{j,\varepsilon}_M(s,\cdot)\|_{L^p(\Omega)} ds.
\end{aligned}
$$

Recall that $\overline{u}^j_M$ and $\overline{v}^j_M$ are defined by (2.5)–(2.6). Hence we obtain by Gronwall's inequality

$$
\|\overline{u}^j_M(t,\cdot) - \bar{\bar{u}}^{j,\varepsilon}_M(t,\cdot)\|_{L^p(\Omega)} \le (e^{(p)}_j + \varepsilon) e^{R_3(t-t_j)}.
$$

Therefore

$$
II \le R_3 \int_{t_j}^{t_{j+1}} \|\overline{u}^j_M(s,\cdot) - \bar{\bar{u}}^{j,\varepsilon}_M(s,\cdot)\|_{L^p(\Omega)} \le R_3 (e^{(p)}_j + \varepsilon) e^{R_3 T} \tau.
$$

In a similar fashion we have

$$
III \le (R_3/\alpha)(e^{(p)}_j + \varepsilon) e^{R_3 T} \tau.
$$

Setting $E = (1 + 1/\alpha) R_3 e^{R_3 T}$ we complete the proof. □

Lemma 4.6. *Suppose $\bar{\bar{w}}^{j,\varepsilon}_M$ is given by (4.36). If $d_1 = d_2$, $\bar{\bar{w}}^{j,\varepsilon}_M$ satisfies*

$$
\|\bar{\bar{w}}^{j,\varepsilon}_M(t_{j+1},\cdot) - w^{(k(\varepsilon))}(t_{j+1},\cdot)\|_{L^2(\Omega)} \le C_8 \tau^{3/2}, \tag{4.38}
$$

$$
\|\bar{\bar{w}}^{j,\varepsilon}_M(t_{j+1},\cdot) - w^{(k(\varepsilon))}(t_{j+1},\cdot)\|_{L^1(\Omega)} \le C_9 \tau^2, \tag{4.39}
$$

where C_8 and C_9 are independent of M, j and ε.

Proof. Recall the inequalities (4.21)–(4.23). Then from an argument similar to that in Lemma 4.1 we obtain

$$
\sup_{t_j \le t \le t_{j+1}} \left(\int_\Omega \left| \partial_t \bar{\bar{u}}^{j,\varepsilon}_M(t,\cdot) \right| + \int_\Omega \left| \partial_t \bar{\bar{v}}^{j,\varepsilon}_M(t,\cdot)/\alpha \right| \right) \le C_{10},
$$

where C_{10} is independent of ε, j, M. Noting that $u^{(k(\varepsilon))}(t_j,\cdot) = \bar{\bar{u}}^{j,\varepsilon}_M(t_j,\cdot)$ we have

$$
\begin{aligned}
\int_\Omega &|u^{(k(\varepsilon))}(s,\cdot) - \bar{\bar{u}}^{j,\varepsilon}_M(s,\cdot)|^2 \\
&= \int_\Omega \left\{ (u^{(k(\varepsilon)}(s,x) - \bar{\bar{u}}^{j,\varepsilon}_M(s,x)) \int_{t_j}^s (\partial_t u^{(k(\varepsilon))}(\sigma,x) - \partial_t \bar{\bar{u}}^{j,\varepsilon}_M(\sigma,x))\, d\sigma \right\} dx \\
&\le 2R_1 \int_{t_j}^s \int_\Omega \left(|\partial_t u^{(k(\varepsilon))}(\sigma,x)| + |\partial_t \bar{\bar{u}}^{j,\varepsilon}_M(\sigma,x)| \right) dx d\sigma \\
&\le 2R_1 (C_4 + C_{10})(s - t_j).
\end{aligned}
$$

Thus we get

$$
\|(u^{(k(\varepsilon))}(s,\cdot) - \bar{\bar{u}}^{j,\varepsilon}_M(s,\cdot))\|_{L^2(\Omega)} \le \sqrt{2R_1(C_4 + C_{10})}(s - t_j)^{1/2}.
$$

Likewise we observe

$$\|u^{(k(\varepsilon))}(s,\cdot) - \bar{\bar{u}}^{j,\varepsilon}_M(s,\cdot)\|_{L^1(\Omega)} \le (C_4 + C_{10})(s - t_j),$$
$$\|(v^{(k(\varepsilon))}(s,\cdot) - \bar{\bar{v}}^{j,\varepsilon}_M(s,\cdot))/\alpha\|_{L^2(\Omega)} \le \sqrt{2(R_1/\alpha)(C_4 + C_{10})}(s - t_j)^{1/2},$$
$$\|(v^{(k(\varepsilon))}(s,\cdot) - \bar{\bar{v}}^{j,\varepsilon}_M(s,\cdot))/\alpha\|_{L^1(\Omega)} \le (C_4 + C_{10})(s - t_j).$$

Here we use the Duhamel formula again. The condition $d_1 = d_2$ makes the term kuv vanish so that we have

$$\begin{aligned}
&\left\|w^{(k(\varepsilon))}(t_{j+1},\cdot) - \bar{\bar{w}}^{j,\varepsilon}_M(t_{j+1},\cdot)\right\|_{L^2(\Omega)} \\
&\qquad \le \int_{t_j}^{t_{j+1}} \|F(u^{(k(\varepsilon))}(s,\cdot)) - F(\bar{\bar{u}}^{j,\varepsilon}_M(s,\cdot))\|_{L^2(\Omega)}\,ds \\
&\qquad\qquad + \frac{1}{\alpha}\int_{t_j}^{t_{j+1}} \|G(v^{(k(\varepsilon))}(s,\cdot)) - G(\bar{\bar{v}}^{j,\varepsilon}_M(s,\cdot))\|_{L^2(\Omega)}\,ds \\
&\qquad \le R_3\int_{t_j}^{t_{j+1}} \|u^{(k(\varepsilon))}(s,\cdot) - \bar{\bar{u}}^{j,\varepsilon}_M(s,\cdot)\|_{L^2(\Omega)}\,ds \\
&\qquad\qquad + \frac{R_3}{\alpha}\int_{t_j}^{t_{j+1}} \|v^{(k(\varepsilon))}(s,\cdot) - \bar{\bar{v}}^{j,\varepsilon}_M(s,\cdot)\|_{L^2(\Omega)}\,ds \\
&\qquad \le \frac{2}{3}(1 + \alpha^{-1/2})R_3(\sqrt{2R_1(C_4 + C_{10})})\tau^{3/2}.
\end{aligned}$$

Analogously we get

$$\|w^{(k(\varepsilon))}(t_{j+1},\cdot) - \bar{\bar{w}}^{j,\varepsilon}_M(t_{j+1},\cdot)\|_{L^1(\Omega)} \le R_3(C_4 + C_{10})\tau^2,$$

which completes the proof. □

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. From (4.28), (4.37) and (4.38) we observe

$$\begin{aligned}
e^{(2)}_{j+1} &= \|w_M(t_{j+1},\cdot) - w(t_{j+1},\cdot)\|_{L^2(\Omega)} \\
&\le \|w_M(t_{j+1},\cdot) - \bar{\bar{w}}^{j,\varepsilon}_M(t_{j+1},\cdot)\|_{L^2(\Omega)} \\
&\qquad + \|\bar{\bar{w}}^{j,\varepsilon}_M(t_{j+1},\cdot) - w^{(k(\varepsilon))}\|_{L^2(\Omega)} \\
&\qquad\qquad + \|w^{(k(\varepsilon))}(t_{j+1},\cdot) - w(t_{j+1},\cdot)\|_{L^2(\Omega)} \\
&\le (e^{(2)}_j + \varepsilon)(1 + E\tau) + C_8\tau^{3/2} + \varepsilon.
\end{aligned}$$

Recall that we can choose arbitrary small $\varepsilon > 0$ for the above inequality. Whence we have

$$e^{(2)}_{j+1} \le (1 + E\tau)e^{(2)}_j + C_8\tau^{3/2}.$$

Consequently we are led to

$$e^{(2)}_M \le \frac{C_8\tau^{1/2}}{E}((1 + E\tau)^M - 1) \tag{4.40}$$
$$\le \frac{C_8}{E}(T/M)^{1/2}(e^{ET} - 1) \equiv C(1/M)^{1/2}. \tag{4.41}$$

In a similar way we arrive at

$$e^{(1)}_M \le C'(1/M),$$

thereby completing the proof. □

5 Numerical Experiments

In this section we perform numerical experiments to observe how TCD converges. This section is composed of two parts. First, we pose a test problem we used in this section. After describing the numerical scheme to TCD, we report numerical results.

5.1 A Test Problem

We consider the singular limit of the following system

$$\begin{cases} u_t = d_1 \Delta u + (r_1 - a_1 u)u - kb_1 uv, \\ v_t = d_2 \Delta v + (r_2 - a_2 v)v - kb_2 uv, \end{cases} \quad (x, y) \in \Omega = [0, X] \times [0, Y], 0 < t < T. \tag{5.1}$$

We impose the Neumann boundary condition on u and v. The initial functions $u|_{t=0}$ and $v|_{t=0}$ are shown in Fig. 1. In this test problem, the parameters in (5.1) are fixed as

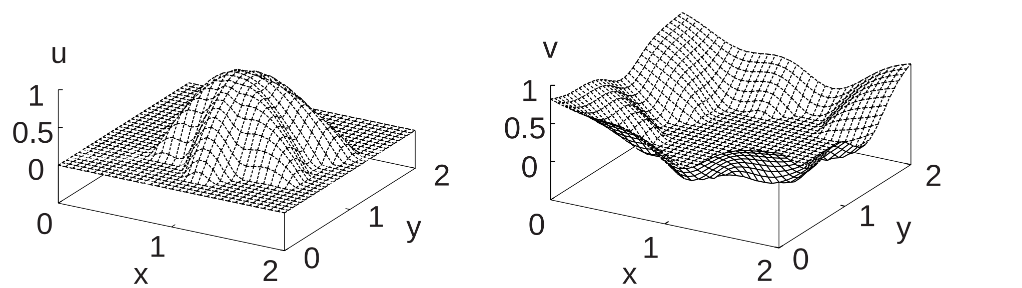

Figure 1: Initial functions $u|_{t=0}$ (*left*) and $v|_{t=0}$ (*right*) used in our computations.

$$\begin{array}{llll} d_1 = 0.7, & r_1 = 2.5, & a_1 = 1, & b_1 = 1.2 \\ d_2 = 1.5, & r_2 = 1, & a_2 = 1, & b_2 = 3.5, \\ k = 10^{10}, & X = 2, & Y = 2, & T = 0.1. \end{array}$$

Since no exact solutions of (5.1) are available, the numerical solution on the uniform 256×256 mesh and the time increment 2×10^{-7} will be referred as an "exact" solution. By our careful numerical computations we conclude that the value $k = 10^{10}$ is large enough to regard the numerical result as the singular limit solution of (5.1). The "exact" solution, which is shown in Fig. 2, is computed by the standard finite difference scheme. The zero-level sets of $w = u - v/\alpha$

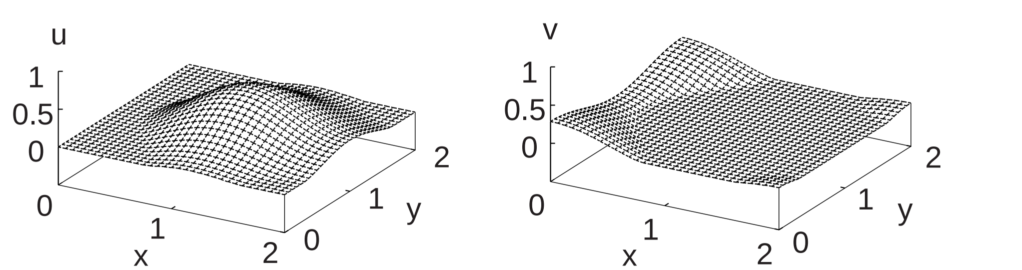

Figure 2: "Exact" solutions $u|_{t=0.1}$ (*left*) and $v|_{t=0.1}$ (*right*).

is displayed in Fig. 3.

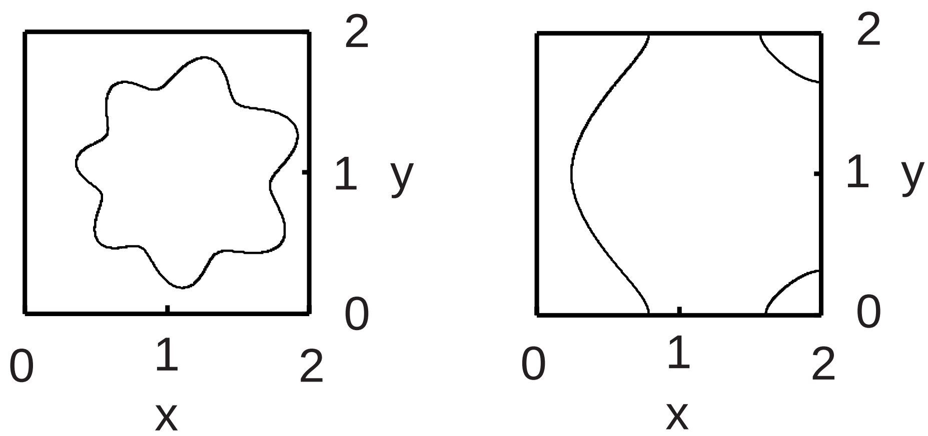

Figure 3: The zero-level sets of $w = u - v/\alpha$ at $t = 0$ (*left*) and $t = 0.1$ (*right*).

5.2 Numerical Scheme and Results

Let M_X, M_Y and M_T be positive integers. We denote the spatial mesh sizes by $\delta x = X/M_X$ and $\delta y = Y/M_Y$, the time increment by $\delta t = T/M_T$. Let $u_{i,j}^n$ ($0 \le i \le M_X$, $0 \le j \le M_Y$ and $0 \le n \le M_T$) be an approximation to $u(x,y,t)$ at the location $(x,y) = (i\delta x, j\delta y)$ and the time $t = \delta t$. We define operators L by

$$Lu_{i,j}^n = \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\delta y^2}. \tag{5.2}$$

We introduce an implementation of TCD. For Step 1 of TCD, we use the following semi-implicit scheme:

$$\begin{cases} \dfrac{u_{i,j}^{n+1/2} - u_{i,j}^n}{\delta t} = d_1 L u_{i,j}^{n+1/2} + (r_1 - a_1 u_{i,j}^{n+1/2}) u_{i,j}^n, \\ \dfrac{v_{i,j}^{n+1/2} - v_{i,j}^n}{\delta t} = d_2 L v_{i,j}^{n+1/2} + (r_2 - a_2 v_{i,j}^{n+1/2}) v_{i,j}^n \end{cases} \tag{5.3}$$

for $0 \le i \le M_X$, $0 \le i \le M_Y$. Here we define

$$\begin{aligned} &u_{-1,j}^{n+1/2} = u_{1,j}^{n+1/2}, \quad && u_{M_X+1,j}^{n+1/2} = u_{M_X-1,j}^{n+1/2}, \quad && u_{i,-1}^{n+1/2} = u_{i,1}^{n+1/2}, \\ &u_{i,M_Y+1}^{n+1/2} = u_{i,M_Y-1}^{n+1/2}, && v_{-1,j}^{n+1/2} = v_{1,j}^{n+1/2}, && v_{M_X+1,j}^{n+1/2} = v_{M_X-1,j}^{n+1/2}, \\ &v_{i,-1}^{n+1/2} = v_{i,1}^{n+1/2}, && v_{i,M_Y+1}^{n+1/2} = v_{i,M_Y-1}^{n+1/2}, \end{aligned} \tag{5.4}$$

which are led by the Neumann boundary condition.

Step 2 of TCD can be written by

$$\begin{cases} u_{i,j}^{n+1} = [u_{i,j}^{n+1/2} - v_{i,j}^{n+1/2}/\alpha]^+, \\ v_{i,j}^{n+1} = \alpha[u_{i,j}^{n+1/2} - v_{i,j}^{n+1/2}/\alpha]^-, \end{cases} \tag{5.5}$$

where $a = b_2/b_1$.

Now we report the results of numerical simulations by the above implementation of TCD. We choose $M_X = M_Y = 128$ and $M_T = 1600, 3200, 6400, \ldots, 51200$. Fig. 4 shows the relative errors E_u and E_v at $t = 0.1$ defined by

$$\begin{aligned} E_u &= \max_{(i,j)\in\Lambda} |u_{i,j}^{M_T} - u(i\delta x, j\delta y, 0.1)| \Big/ \max_{(i,j)\in\Lambda} |u(i\delta x, j\delta y, 0.1)|, \\ E_v &= \max_{(i,j)\in\Lambda} |v_{i,j}^{M_T} - v(i\delta x, j\delta y, 0.1)| \Big/ \max_{(i,j)\in\Lambda} |v(i\delta x, j\delta y, 0.1)|, \end{aligned} \tag{5.6}$$

where $(u(x,y,t),\, v(x,y,t))$ is the "exact" solution and $\Lambda = \{(i,j) \in \mathbb{Z}^2 \mid 0 \le i \le M_X,\, 0 \le j \le M_Y\}$. One can observe that E_u and E_v tends to zero as $\delta t \to 0$.

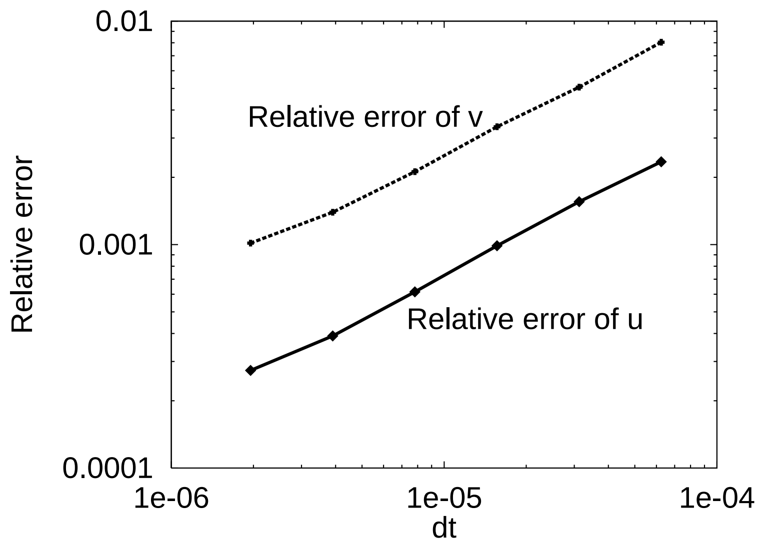

Figure 4: The relative errors versus the time increment.

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A Notation Index

T **:** a time interval.

M **:** a time step.

τ **:** a time mesh size; $\tau = T/M$.

$\overline{u}_M^j$, $\overline{v}_M^j$ **:** auxiliary functions associated with Step 1 of TCD; defined by (2.5), (2.6).

$\widehat{u}_M^j$, $\widehat{v}_M^j$ **:** auxiliary functions associated with Step 2 of TCD; see (2.8).

u_M, v_M **:** approximate functions to the singular limit via TCD; see (2.3).

w_M **:** an approximate function to (1.1)–(1.3) via TCD; $w_M = u_M - v_M/\alpha$.

u_M^j, v_M^j **:** approximate functions to the singular limit via TCD at the time $t = t_j$; see (2.7), (2.11) and (2.12).

$\bar{\bar{u}}_M^{j,\varepsilon}$, $\bar{\bar{v}}_M^{j,\varepsilon}$ **:** auxiliary functions; see (4.31)–(4.35)

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