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RELATIVE POSITION OF FOUR SUBSPACES IN A HILBERT SPACE

MASATOSHI ENOMOTO AND YASUO WATATANI

Abstract. The relative position of one subfactor of a factor has been proved quite rich since the work of Jones. We shall show that the theory of relative position of *several subspaces* of a separable infinite-dimensional Hilbert space is also rich. In finite-dimensonal case, Gelfand and Ponomarev gave a complete classification of indecomposable systems of four subspaces. We construct exotic examples of indecomposable systems of four subspaces in infinite-dimensional Hilbert spaces. We extend their Coxeter functors and defect using Fredholm index. There exist close connections with strongly irreducible operators and transitive lattices.

KEYWORDS: subspace, Hilbert space, indecomposable system, defect, Coxeter functor, strongly irreducible operator, transitive lattice

AMS SUBJECT CLASSIFICATION: 46C07, 47A15, 15A21, 16G20, 16G60.

1. Introduction

The relative position of one subfactor of a factor has been proved quite rich since the work of Jones [J] . On the other hand, the relative position of one subspace of a Hilbert space is extremely simple and determined by the dimension and the co-dimension of the subspace. But we shall show that the theory of relative position of *several subspaces* of a Hilbert space is rich as subfactor theory.

It is a well known fact that the relative position of two subspaces E and F in a Hilbert space H can be described completely up to unitary equivalence as in Araki [Ar] Dixmier [D] and Halmos [Ha1]. The Hilbert space is the direct sum of five subspaces:

$$H = (E \cap F) \oplus (\text{the rest}) \oplus (E \cap F^{\perp}) \oplus (E^{\perp} \cap F) \oplus (E^{\perp} \cap F^{\perp}).$$

In the rest part, E and F are in generic position and the relative position is described only by "the angles" between them.

We disregard "the angles" and study the still-remaining fundamental feature of the relative position of n subspaces. As it is important to study *irreducible* subfactors in subfactor theory, we should study an *indecomposable* system of n subspaces in the sense that the system can not be isomorphic to a direct sum of two non-zero systems.

On the other hand, many problems of linear algebra can be reduced to the classification of the systems of subpaces in a finite-dimensional vector space. In a finite-dimensional space, the classification of indecomposable systems of n subspaces for $n = 1, 2$ and 3 was simple. Jordan blocks give indecomposable systems of 4 subspaces. But there exist many other kinds of indecomposable systems of 4 subspaces. Therefore it was surprising that Gelfand and Ponomarev [GP] gave a complete classification of indecomposable systems of four subspaces in a finite-dimensional space over an algebraically closed field.

In this note we study relative position of n subspaces in a separable infinite-dimensional Hilbert space. The fact that the sum of closed subspaces is not necessary closed causes some troubles in several arguments in Gelfand-Ponomarev [GP]. Let H be a Hilbert space and $E_1, \dots E_n$ be n subspaces in H. Then we say that $\mathcal{S} = (H; E_1, \dots, E_n)$ is a system of n subspaces in H or a n-subspace system in H. A system $\mathcal{S}$ is called indecomposable if $\mathcal{S}$ can not be decomposed into a nontrivial direct sum. For any bounded linear operator A on a Hilbert space K, we can associate a system $\mathcal{S}_A$ of four subspaces in $H = K \oplus K$ by

$$\mathcal{S}_A = (H; K \oplus 0, 0 \oplus K, \operatorname{graph} A, \{(x, x); x \in K\}).$$

Two such systems $\mathcal{S}_A$ and $\mathcal{S}_B$ are isomorphic if and only if the two operators A and B are similar. The direct sum of such systems corresponds to the direct sum of the operators. In this sense the theory of operators is included into the theory of relative positions of four subspaces. In particular on a finite dimesional space, Jordan blocks correspond to indecomposable systems. Moreover on an infinite dimensional Hilbert space, the above system $\mathcal{S}_A$ is indecomposable if and only if A is strongly irreducible, which is an infinite-dimensional analog of a Jordan block, see, for example, a monograph by Jiang and Wang [JW]. Therefore there exist uncountably many indecomposable systems of four subspaces. But it is rather difficult to know whether there exists another kind of indecomposable system of four subspaces. One of the main result of the paper is to give uncountably many, exotic, indecomposable systems of four subspaces on an infinite-dimensional separable Hilbert space. The ℓ^2-boundedness is crucially used.

Gelfand and Ponomarev introduced an integer valued invariant $\rho(\mathcal{S})$, called *defect*, for a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces by

$$\rho(\mathcal{S}) = \sum_{i=1}^{4} \dim E_i - 2 \dim H.$$

We extend the defect to a certain class of systems of four subspaces on an infinite dimesional Hilbert space using Fredholm index. We believe that there exists an analogy between a classification of systems of subspaces and a classification of subfactors, and the defect by Gelfand and Ponomarev seems to correspond to the index by Jones

[J]. Therefore the determination of possible value of defect is also important. If a pair $N \subset M$ of factor-subfactor is finite-dimensional, then Jones index $[M : N]$ is an integer. But if $N \subset M$ is infinite-dimensional, then Jones index $[M : N]$ is a non-integer in general. One of the amazing fact was that the possible value of Jones index is in $\{4\cos^2 \frac{\pi}{n} \mid n = 3, 4, ...\} \cup [4, \infty]$. We show that a similar situation occurs for the possible value of defect. If a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces is finite-dimensional, then the defect $\rho(\mathcal{S})$ is an integer. Gelfand and Ponomarev showed that the possible value of defect $\rho(\mathcal{S})$ is exactly in $\{-2, -1, 0, 1, 2\}$. We show that the set of values of defect for indecomposable systems of four subspaces in an infinite-dimesional Hilbert spaces is exactly $\{\frac{n}{3}; n \in \mathbb{Z}\}$.

We extend Coxeter functors after Gelfand-Ponomarev and show that the Coxeter functors preserve the defect and indecomposability under certain conditions.

Halmos initiated the study of transitive lattices and gave an example of transitive lattice consisting of seven subspaces in [Ha2]. Harison-Radjavi-Rosenthal [HRR] constructed a transitive lattice consisting of six subspaces using the graph of an unbounded closed operator. Hadwin-Longstaff-Rosenthal found a transitive lattice of five non-closed linear subspaces in [HLR]. Any finite transitive lattice which consists of n subspaces of a Hilbert space H gives an indecomposable system of $n-2$ subspaces by withdrawing 0 and H, but the converse is not true. It is still unknown whether or not there exists a transitive lattice consisting of five subspaces. Therefore it is also an interesting problem to know whether there exists an indecomposable system of three subspaces in an infinite-dimensional Hilbert space.

Throughout the paper a projection means an operator e with $e^2 = e = e^*$ and an idempotent means an operator p with $p^2 = p$.

Sunder also considered n subspaces in [S]. But his interest is extremely opposite to ours. In fact he studied the decomposable case such that the Hilbert space H is an algebraic sum of the n subspaces. He solved the statistical problem of computing the canonical partial correlation coefficients between three sets of random variables.

When we announced some part of our result in US-Japan seminar at Fukuoka in 1999, we had not yet known the notion and interesting works on strong irreducible operators which are summarized in a monograph by Jiang and Wang [JW].

There seems to be interesting relations with the study of representations of $*$-algebras generated by idempotents by S. Kruglyak and Y. Samoilenko [KS] and the study on sums of projections by S. Kruglyak, V. Rabanovich and Y. Samoilenko [KRS]. But we do not know the exact implication, because their objects are different with ours.

In finite dimensional case, the classification of four subspaces is described as the classification of the representations of the extended

Dynkin diagram $D_4^{(1)}$. Recall that Gabriel [G] listed Dynkin diagrams A_n, D_n, E_6, E_7, E_8 in his theory on finiteness of indecomposable representations of quivers. We will discuss on indecomposable representations of quivers on *infinite-dimensinal Hilbert spaces* somewhere else [EW] as a continuation of this paper.

In purely algebraic setting, it is known that if a finite-dimensional algebra R is not of representation-finite type, then there exist indecomposable R-modules of infinite length as in M. Auslander [Au]. Since we consider representations on Hilbert spaces, the result in [Au] cannot be applied directly. We need several techniques in functional analysis. See a book [KR] for infinite length modules.

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2. Systems of n subspaces

We study the relative position of n subspaces in a separable Hilbert space. Let H be a Hilbert space and $E_1, \dots E_n$ be n subspaces in H. Then we say that $\mathcal{S} = (H; E_1, \dots, E_n)$ is a system of n-subspaces in H or a n-subspace system in H. Let $\mathcal{T} = (K; F_1, \dots, F_n)$ be another system of n-subspaces in a Hilbert space K. Then $\varphi : \mathcal{S} \to \mathcal{T}$ is called a homomorphism if $\varphi : H \to K$ is a bounded linear operator satisfying that $\varphi(E_i) \subset F_i$ for $i = 1, \dots, n$. And $\varphi : \mathcal{S} \to \mathcal{T}$ is called an isomorphism if $\varphi : H \to K$ is an invertible (i.e., bounded bijective) linear operator satisfying that $\varphi(E_i) = F_i$ for $i = 1, \dots, n$. We say that systems $\mathcal{S}$ and $\mathcal{T}$ are *isomorphic* if there is an isomorphism $\varphi : \mathcal{S} \to \mathcal{T}$. This means that the relative positions of n subspaces $(E_1, \dots, E_n)$ in H and $(F_1, \dots, F_n)$ in K are same under disregarding angles. We say that systems $\mathcal{S}$ and $\mathcal{T}$ are *unitarily equivalent* if the above isomorphism $\varphi : H \to K$ can be chosen to be a unitary. This means that the relative positions of n subspaces $(E_1, \dots, E_n)$ in H and $(F_1, \dots, F_n)$ in K are same with preserving the angles between the subspaces. We are interested in the relative position of subspaces up to isomorphims to study the still-remaining fundamental feature of the relative position after disregarding "the angles" .

We denote by $Hom(\mathcal{S}, \mathcal{T})$ the set of homomorphims of $\mathcal{S}$ to $\mathcal{T}$ and $End(\mathcal{S}) := Hom(\mathcal{S}, \mathcal{S})$ the set of endomorphisms on $\mathcal{S}$.

Let $G_2 = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a_1, a_2 \rangle$ be the free product of the cyclic groups of order two with generators a_1 and a_2. For two subspaces E_1 and E_2 of a Hilbert space H, let e_1 and e_2 be the projections onto E_1 and E_2. Then $u_1 = 2e_1 - I$ and $u_2 = 2e_2 - I$ are self-adjoint unitaries. Thus there is a bijective correspondence between the set $\mathcal{S}ys^2(H)$ of systems $\mathcal{S} = (H; E_1, E_2)$ of two subspaces in a Hilbert space H and the set $Rep(G_2, H)$ of unitary representations π of G_2 on H such that $\pi(a_1) = u_1$ and $\pi(a_2) = u_2$. Similarly let $G_n = \mathbb{Z}/2\mathbb{Z} * ... * \mathbb{Z}/2\mathbb{Z}$ be the n-times free product of the cyclic groups of order two. Then there is a bijective correspondence between the set $\mathcal{S}ys^n(H)$ of systems of n subspaces in a Hilbert space H and the set $Rep(G_n, H)$ of unitary representations on H of G_n on H. It is well known that if $n \geq 3$, then the group G_n is non-amenable. We should be careful that even if two systems of n subspaces are isomorphic, the corresponding unitary representations are *not* necessary to be similar, although the converse is always true.

Example 1 Let $H = \mathbb{C}^2$. Fix an angle θ with $0 < \theta < \pi/2$. Put $E_1 = \mathbb{C}(1,0)$ and $E_2 = \mathbb{C}(cos\theta, sin\theta)$. Then $\mathcal{S}_1 = (H; E_1, E_2)$ is isomorphic to $\mathcal{S}_2 = (\mathbb{C}^2; \mathbb{C}\oplus 0, 0 \oplus \mathbb{C})$. But the corresponding two unitary representations π_1 and π_2 are not similar, because $\frac{1}{2}(\pi_1(a_1)+1)\frac{1}{2}(\pi_1(a_2)+1) \neq 0$ and $\frac{1}{2}(\pi_2(a_1)+1)\frac{1}{2}(\pi_2(a_2)+1) = 0$.

We start with a known fact to recall some notation.

Lemma 2.1. *Let H be a Hilbert space and H_1 and H_2 be two subspaces of H. Then the following are equivalent:*

(1) $H = H_1 + H_2$ *and* $H_1 \cap H_2 = 0$.
(2) *There exists a closed subspace $M \subset H$ such that $(H; H_1, H_2)$ is isomorphic to $(H; M, M^\perp)$*
(3) *There exists an idempotnet $P \in B(H)$ such that $H_1 = \operatorname{Im} P$ and $H_2 = \operatorname{Im}(1 - P)$.*

Proof. The equivalence between (1) and (3) is trivial and it is immediate that (2)⇒(1) . We show that (1)⇒(2). Assume (1) and put $M = H_1$. Let e_1 be the (orthogonal) projection onto H_1. Let P be the idempotent onto H_1 along H_2, so that $P\xi = \xi_1$ for $\xi = \xi_1 + \xi_2$, $(\xi_1 \in H_1, \xi_2 \in H_2)$. Define an operator $T : H \to H$ by $T\xi = P\xi + (I - e_1)(I - P)\xi$ for $\xi \in H$. The operator P, T and T^{-1} are also writen as operator matrices

$$P = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix}, T = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \text{ and } T^{-1} = \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}$$

under the decomposion $H = H_1 \oplus H_1^\perp$. Thus T is an invertible bounded linear operator satisfying $TH_1 = H_1$ and $TH_2 = H_1^\perp$. Hence T gives an isomorphism. □

Lemma 2.2. *Let H and K be Hilbert spaces and $E \subset H$ and $F \subset K$ be closed subspaces of H and K. Let $e \in B(H)$ and $f \in B(K)$ be the projections onto E and F. Then the following are equivalent:*

(1) *There exists an invertible operator $T : H \to K$ such that $T(E) = F$.*
(2) *There exists an invertible operator $T : H \to K$ such that $e = (T^{-1}fT)e$ and $f = (TeT^{-1})f$.*

Proof. (1)⇒(2):Assume there exists an invertible operator $T : H \to K$ such that $T(E) = F$. Then for any $\xi \in H$, $Te(\xi) \in T(E) = F$. Hence $f(Te(\xi)) = Te(\xi)$. Thus $T^{-1}fTe = e$. Similarly we have $f = TeT^{-1}f$.

(2) $\Rightarrow$ (1):Assume (2). For $\xi \in E$, $T(\xi) = Te(\xi) = fTe(\xi) \in F$. Thus $T(E) \subset F$. Similarly $T^{-1}(F) \subset E$. Hence $F \subset T(E)$. Therefore $T(E) = F$. $\square$

Using the above lemma, we can describe an isomorphism between two systems of n suspaces in terms of operators only as follows:

Corollary 2.3. *Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ and $\mathcal{S}' = (H'; E'_1, \cdots, E'_n)$ be two systems of n-subspaces. Let e_i (resp. e'_i) be the projection onto E_i (resp. E'_i) . Then two systems $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic if and only if there exists an invertible operator $T : H \to H'$ such that $e_i = (T^{-1}e'_iT)e_i$ and $e'_i = (Te_iT^{-1})e'_i$ for $i = 1, \ldots, n$.*

Remark. If there exists an invertible operator $T : H \to H'$ such that $e'_i = Te_iT^{-1}$ for $i = 1, \ldots, n$, then two systems $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic. But the converse is not true as in example 1.

We often want to disregard the order of the subspaces.

Definition Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ and $\mathcal{S}' = (H'; E'_1, \cdots, E'_n)$ be two systems of n-subspaces. Then we say that $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic up to a permutation of subspaces if there exists a permutation σ on $\{1, 2, \ldots, n\}$ such that $\sigma(\mathcal{S}) := (H; E_{\sigma(1)}, \cdots, E_{\sigma(n)})$ and $\mathcal{S}' = (H'; E'_1, \cdots, E'_n)$ are isomorphic, i.e., there exists a bounded invertible operator $\varphi : H \to H'$ satisfying that $\varphi(E_{\sigma(i)}) = E'_i$ for $i = 1, \ldots, n$.

3. INDECOMPOSABLE SYSTEMS

In this section we shall introduce a notion of indecomposable system, that is, a system which cannot be decomposed into a direct sum of smaller systems anymore.

Definition (direct sum) Let $\mathcal{S} = (H; E_1, \ldots, E_n)$ and $\mathcal{S}' = (H'; E'_1, \cdots, E'_n)$ be systems of n subspaces in Hilbert spaces H and H'. Then their direct sum $\mathcal{S} \oplus \mathcal{S}'$ is defined by

$$\mathcal{S} \oplus \mathcal{S}' := (H \oplus H'; E_1 \oplus E'_1, \ldots, E_n \oplus E'_n).$$

Definition(indecomposable system) A system $\mathcal{S} = (H; E_1, \ldots, E_n)$ of n subspaces is called *decomposable* if the system $\mathcal{S}$ is isomorphic to a direct sum of two non-zero systems. A system $\mathcal{S} = (H; E_1, \cdots, E_n)$ is said to be *indecomposable* if it is not decomposable.

Example 2. Let $H = \mathbb{C}^2$. Fix an angle θ with $0 < \theta < \pi/2$. Put $E_1 = \mathbb{C}(1, 0)$ and $E_2 = \mathbb{C}(cos\theta, sin\theta)$. Then $(H; E_1, E_2)$ is isomorphic to

$$(\mathbb{C}^2; \mathbb{C} \oplus 0, 0 \oplus \mathbb{C}) \cong (\mathbb{C}; \mathbb{C}, 0) \oplus (\mathbb{C}; 0, \mathbb{C}).$$

Hence $(H; E_1, E_2)$ is decomposable.

Remark. Let e_1 and e_2 be the projections onto E_1 and E_2 in the example 2 above. Then the C^*-algebra $C^*(\{e_1, e_2\})$ generated by e_1 and e_2 is exactly $B(H) \cong M_2(\mathbb{C})$. Therefore the irreducibility of $C^*(\{e_1, e_2\})$ does *not* imply the indecomposability of $(H; E_1, E_2)$. Thus seeking an indecomposable system of subspaces is much more difficult and fundamental task than showing irreducibility of the C^*-algebra generated by the corresponding projectios for the subspaces.

We can characterize decomposability of systems inside the ambient Hilbert space.

Lemma 3.1. *Let H be a Hilbert space and $\mathcal{S} = (H; E_1, \dots, E_n)$ a system of n subspaces. Then the following condition are equivalent:*

(1) *$\mathcal{S}$ is decomposable.*
(2) *there exist non-zero closed subspaces H_1 and H_2 of H such that $H_1 + H_2 = H$, $H_1 \cap H_2 = 0$ and $E_i = E_i \cap H_1 + E_i \cap H_2$ for $i = 1, \dots, n$.*

Proof. (1)$\Rightarrow$ (2): It is trivial. (2)$\Rightarrow$ (1): Assume (2). By 2.1, there exist a closed subspace $M \subset H$ (in fact we can choose $M = H_1$) and an invertible operator $T \in B(H)$ such that $T(H_1) = M$ and $T(H_2) = M^{\perp}$. Then $\mathcal{S}$ is isomorphic to a direct sum

$$(M; T(E_1 \cap H_1), \dots, T(E_n \cap H_1)) \oplus (M^{\perp}; T(E_1 \cap H_2), \dots, T(E_n \cap H_2)).$$

□

We give a condition of decomposability in terms of endomorphism algebras for the systems.

Lemma 3.2. *Let H be a Hilbert space and $\mathcal{S} = (H; E_1, \dots, E_n)$ a system of n subspaces in H. Let e_i be the projection onto E_i. Then the following are equivalent:*

(1) *There exist non-zero closed subspaces $H_1, H_2 \subset H$ such that $H = H_1 + H_2$, $H_1 \cap H_2 = (0)$ and $E_i = E_i \cap H_1 + E_i \cap H_2$, $(i = 1, \dots, n)$.*
(2) *There exists a non-trivial idempotent $R \in B(H)$ such that $R(E_i) \subset E_i$, $(i = 1, \dots, n)$.*
(3) *There exists a non-trivial idempotent $R \in B(H)$ such that $e_i R e_i = R e_i$, $(i = 1, \dots, n)$.*

Proof. (1) $\Rightarrow$ (2): Assume (1). Let R be the idempotent onto H_1 along H_2. For any $\xi \in E_i$, there exist $\xi_1 \in E_i \cap H_1$ and $\xi_2 \in E_i \cap H_2$ such that $\xi = \xi_1 + \xi_2$. Then $R(\xi) = \xi_1 \in E_i$. Thus $R(E_i) \subset E_i$.
(2) $\Rightarrow$(1): Assume (2). We put $H_1 = \operatorname{Im} R$ and $H_2 = \operatorname{Im}(I - R)$. For $\xi \in E_i$, we have $\xi = R(\xi) + (I - R)(\xi)$. Since $R(E_i) \subset E_i$, $R(\xi) \in E_i$. Then $(I - R)(\xi) = \xi - R(\xi) \in E_i$. Thus $E_i \subset E_i \cap H_1 + E_i \cap H_2$. The other inclusion "$\supset$" is trivial. (2) $\Leftrightarrow$ (3) : It is trivial. □

We put $Idem(\mathcal{S}) := \{T \in End(\mathcal{S}); T = T^2\}$.

Corollary 3.3. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Then $\mathcal{S}$ is indecomposable if and only if $Idem(\mathcal{S}) = \{0, I\}$.*

Corollary 3.4. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Let e_i be the projection of H onto E_i for $i = 1, \dots, n$. If $\mathcal{S} = (H; E_1, \dots, E_n)$ is indecomposable, then the $C^*(\{e_1, \dots, e_n\})$ generated by $e_1, \dots, e_n$ is irreducible. But the converse is not true.*

Corollary 3.5. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Let e_i be the projection of H onto E_i for $i = 1, \dots, n$. Let P be a closed subspace of H and p the projection of H onto P. If p commutes with any e_i, then*

$$E_i = E_i \cap P + E_i \cap P^\perp$$

Proof. The projection R of H onto P satisfies the condition (3) in 3.2. □

Definition. Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Let e_i be the projection of H onto E_i for $i = 1, \dots, n$. We say that $\mathcal{S}$ is a *commutative* system if the $C^*(\{e_1, \dots, e_n\})$ generated by $e_1, \dots, e_n$ is commutative. Be carefull that commutativity is *not* an isomorphic invariant as shown in Example 1. But it is meaningful that a system is isomorphic to a commutative system.

Proposition 3.6. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Assume that $\mathcal{S}$ is a commutative system. Then $\mathcal{S}$ is indecomposable if and only if $\dim H = 1$. Moreover each subset $\Lambda \subset \{1, \dots, n\}$ corresponds to a commutative system satisfying $\dim E_i = 1$ for $i \in \Lambda$ and $\dim E_i = 0$ for $i \notin \Lambda$.*

Proof. Let e_i be the projection of H onto E_i for $i = 1, \dots, n$. If $\mathcal{S}$ is a commutative, indecomposable system, then the $C^*(\{e_1, \dots, e_n\}) \subset B(H)$ is commutative and irreducible. Thus $\dim H = 1$. The converse and the rest is clear. □

Example 3. Let $H = \mathbb{C}^2$. Put $E_1 = \mathbb{C}(1, 0)$, $E_2 = \mathbb{C}(0, 1)$ and $E_3 = \mathbb{C}(1, 1)$. Then $\mathcal{S} = (H; E_1, E_2, E_3)$ is indecomposable. The system $\mathcal{S}$ is the lowest dimensional one among non-commutative indecomposable systems.

Example 4. Let $H = \mathbb{C}^3$ and $\{a_1, a_2, a_3\}$ be a linearly independent subset of H. Put $E_1 = \mathbb{C}a_1$, $E_2 = \mathbb{C}a_2$ and $E_3 = \mathbb{C}a_3$. Then $\mathcal{S} = (H; E_1, E_2, E_3)$ is decomposable. In fact, let $H_1 = E_1 \vee E_2 \neq 0$ and $H_2 = E_3 \neq 0$. Then $H_1 + H_2 = H$, $H_1 \cap H_2 = 0$ and $E_i = E_i \cap H_1 + E_i \cap H_2$, for $i = 1, 2, 3$.

Example 5. Let $H = \mathbb{C}^3$ and $\{b_1, b_2, b_3, b_4\}$ be a subset of H. Put $E_i = \mathbb{C}b_i$ for $i = 1, \ldots, 4$. Consider a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces. Then the following are equivalent:

(1) $\mathcal{S}$ is indecomposable.
(2) Any three vectors of $\{b_1, b_2, b_3, b_4\}$ is linearly independent.
(3) The set $\{b_1, b_2, b_3\}$ is linearly independent and $b_4 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$ for some scalars $\lambda_i \neq 0$ $(i = 1, 2, 3)$.

Assume that $\{u_1, u_2, u_3, u_4\} \subset H$ and $\{v_1, v_2, v_3, v_4\} \subset H$ satisfy the above condition (2). Then $\mathcal{S} = (H; \mathbb{C}u_1, \mathbb{C}u_2, \mathbb{C}u_3, \mathbb{C}u_4)$ and $\mathcal{T} = (H; \mathbb{C}v_1, \mathbb{C}v_2, \mathbb{C}v_3, \mathbb{C}v_4)$ are isomorphic.

Example 6. Let $H = \mathbb{C}^3$. Put $E_1 = \mathbb{C} \oplus \mathbb{C} \oplus 0$, $E_2 = \mathbb{C}(1, 1, 1)$ and $E_3 = \mathbb{C}(1, 2, 3)$. Then a system $\mathcal{S} = (H; E_1, E_2, E_3)$ is decomposable. In fact, let $E_1' = (E_2 \vee E_3) \cap E_1$ and $H_1 = E_1 \cap (E_1')^{\perp} \neq 0$. Let $H_2 = E_2 \vee E_3 \neq 0$. Then $H_1 + H_2 = H$, $H_1 \cap H_2 = 0$ and $E_i = E_i \cap H_1 + E_i \cap H_2$ for $i = 1, 2, 3$.

Example 7. Let $H = \mathbb{C}^3$. Put $E_1 = \mathbb{C} \oplus \mathbb{C} \oplus 0$, $E_2 = \mathbb{C}(0, 0, 1)$, $E_3 = \mathbb{C}(0, 1, 1)$ and $E_4 = \mathbb{C}(1, 0, 1)$. Then a system $\mathcal{S}_7 = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Example 8. Let $H = \mathbb{C}^3$. Put $E_1 = \mathbb{C} \oplus \mathbb{C} \oplus 0$, $E_2 = \mathbb{C}(0, 0, 1)$, $E_3 = \mathbb{C}(1, 0, 0) + \mathbb{C}(0, 1, 1)$ and $E_4 = \mathbb{C}(1, 0, 1)$. Then a system $\mathcal{S}_8 = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Example 9. Let $H = \mathbb{C}^3$. Put $E_1 = \mathbb{C} \oplus \mathbb{C} \oplus 0$, $E_2 = \mathbb{C}(0, 0, 1)$, $E_3 = \mathbb{C}(1, 0, 0) + \mathbb{C}(0, 1, 1)$ and $E_4 = \mathbb{C}(1, 0, 1) + \mathbb{C}(0, 1, 0)$. Then a system $\mathcal{S}_9 = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Example 10. Let $H = \mathbb{C}^3$. Put $E_1 = \mathbb{C}(1, 0, 0) + \mathbb{C}(0, 1, 0)$, $E_2 = \mathbb{C}(0, 1, 0) + \mathbb{C}(0, 0, 1)$ $E_3 = \mathbb{C}(1, 0, 0) + \mathbb{C}(0, 1, 1)$ and $E_4 = \mathbb{C}(0, 0, 1) + \mathbb{C}(1, 1, 0)$. Then a system $\mathcal{S}_{10} = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Remark Any two of the above indecomposable systems $\mathcal{S}_7, \ldots, \mathcal{S}_{10}$ of four subspaces are not isomorphic each other.

Example 11. Let $K = \ell^2(\mathbb{N})$ and $H = K \oplus K$. Consider a unilateral shift $S : K \to K$. Let $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$, $E_3 = \{(x, Sx) \in H; x \in K\}$ and $E_4 = \{(x, x) \in H; x \in K\}$. Then a system $\mathcal{S}_{11} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in H is indecomposable. In fact, let R be an idempotent which commutes with S. Then R is a lower triangular Toeplitz matrix. Since R is an idempotent, $R = 0$ or $R = I$.

Recall that Halmos initiated the study of transitive lattices. A complete lattice of closed subspaces of a Hilbert space H containing 0 and H is called *transitive* if every bounded operator on H leaving each subspace invariant is a scalar multiple of the identity. Halmos gave an example of transitive lattice consisting of seven subspaces in [Ha2]. Harison-Radjavi-Rosenthal [HRR] constructed a transitive lattice consisting of six subspaces using the graph of an unbounded operator. Any finite transitive lattice which consists of n subspaces gives an indecomposable system of n-2 subspaces but the converse is not true. Following the study of transitive lattices, we shall introduce the notion of transitive system.

Definition. Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Then we say that $\mathcal{S}$ is *transitive* if $End(\mathcal{S}) = \mathbb{C}I_H$. Recall that $\mathcal{S}$ is indecomposable if and only if $Idem(\mathcal{S}) = \{0, I\}$. Hence if $\mathcal{S}$ is transitive, then $\mathcal{S}$ is indecomposable. But the converse is not true. In fact the system $\mathcal{S}_{11}$ as above is indecomposable but is not transitve, because $End(\mathcal{S})$ contains $S \oplus S$.

Example 12.(Harrison-Radjavi-Rosenthal [HRR]) Let $K = \ell^2(\mathbb{Z})$ and $H = K \oplus K$. Consider a sequence $(\alpha_n)_n$ given by $\alpha_n = 1$ for $n \leq 0$ and $\alpha_n = exp((-1)^n n!)$ for $n > 1$. Consider a bilateral weighted shift $S : \mathcal{D}_T \to K$ such that $T(x_n)_n = (\alpha_{n-1}x_{n-1})_n$ with the domain $\mathcal{D}_T = \{(x_n)_n \in \ell^2(\mathbb{Z}); \sum_n |\alpha_n x_n|^2 < \infty\}$. Let $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$, $E_3 = \{(x, Tx) \in H; x \in \mathcal{D}_T\}$ and $E_4 = \{(x, x) \in H; x \in K\}$. Harrison, Radjavi and Rosental showed that $\{0, H, E_1, E_2, E_3, E_4\}$ is a transitive lattice. Hence the system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in H is transitive and in particular indecomposable.

Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a finite-dimensional vector space H. Gelfand and Ponomarev [GP] introduced the conjugate system $\mathcal{S}^* = (H^*; E'_1, \dots, E'_n)$, where $E'_i = \{f \in H^*; f(x) = 0 \ for \ all \ x \in E_i\}$. In our setting of Hilbert spaces, their conjugate system $\mathcal{S}^*$ could be replaced by the following orthogonal complement.

Definition. Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Then the orthogonal complement of $\mathcal{S}$, denoted by $\mathcal{S}^\perp$, is defined by $\mathcal{S}^\perp = (H; E_1^\perp, \dots, E_n^\perp)$. Let $\mathcal{T} = (K; F_1, \dots, F_n)$ be another system of n subspaces in a Hilbert space K and $\varphi : \mathcal{S} \to \mathcal{T}$ be a homomorphism. We define a homomorphism $\varphi^* : \mathcal{T}^\perp \to \mathcal{S}^\perp$ by $\varphi^* : K \to H$. In fact, $\varphi^*(F_i^\perp) \subset E_i^\perp$, because $\varphi(E_i) \subset F_i$.

We denote by $\mathcal{S}ys^n$ the category of the systems of n subspaces in Hilbert spaces and homomorphisms. Then we can introduce a contravariant functor $\Phi^\perp : \mathcal{S}ys^n \to \mathcal{S}ys^n$ by

$$\Phi^\perp(\mathcal{S}) = \mathcal{S}^\perp \text{ and } \Phi^\perp(\varphi) = \varphi^*.$$

Proposition 3.7. *Let H be a Hilbert space and $\mathcal{S} = (H; E_1, \dots, E_n)$ a system of n subspaces in H. Then $\mathcal{S}$ is indecomposable if and only if $\mathcal{S}^\perp$ is indecomposable.*

Proof. If $\mathcal{S}$ is decomposable, then there exists an idempotent $R \in End(\mathcal{S})$ with $R \neq 0$ and $R \neq I_H$. Since $R(E_i) \subset E_i$, we have $R^*(E_i^{\perp}) \subset E_i^{\perp}$. Thus $R^* \in End(\mathcal{S}^{\perp})$ is an idempotent with $R^* \neq 0$ and $R^* \neq I_H$, that is, $\mathcal{S}^{\perp}$ is decomposable. This implies the desired conclusion. □

Similarly we have a same fact for transitive systems.

Proposition 3.8. *Let H be a Hilbert space and $\mathcal{S} = (H; E_1, \ldots, E_n)$ a system of n subspaces in H. Then $\mathcal{S}$ is transitive if and only if $\mathcal{S}^{\perp}$ is transitive .*

4. Indecomposable systems of one subspace

It is easy to see the case of indecomposable systems of one subspace even in an infinite-dimensional Hilbert space.

Proposition 4.1. *Let H be a Hilbert space and $\mathcal{S} = (H; E)$ a system of one subspace. Then $\mathcal{S} = (H; E)$ is indecomposable if and only if $\mathcal{S} \cong (\mathbb{C}; 0)$ or $\mathcal{S} \cong (\mathbb{C}; \mathbb{C})$.*

Proof. If $E \neq 0$ and $E \neq H$, then $\mathcal{S} = (E; E) \oplus (E^{\perp}; 0)$ gives a non-trivial decomposition. Assume that $\mathcal{S}$ is indecomposable. Then $E = 0$ or $E = H$. Suppose we had $dim H \geq 2$, then there exist non-zero closed subspaces H_1 and H_2 such that $H = H_1 + H_2$ and $H_1 \cap H_2 = 0$. This gives a non-trivial decompositon of $\mathcal{S}$. The contradiction shows that $dim H = 1$. Hence $\mathcal{S} \cong (\mathbb{C}; 0)$ or $\mathcal{S} \cong (\mathbb{C}; \mathbb{C})$. The converse is trivial. □

Let $\mathcal{S} = (H; E)$ and $\mathcal{S}' = (H'; E')$ be two systems of one subspace. Then $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic if and only if $\dim E = \dim E'$ and $\operatorname{codim} E = \operatorname{codim} E'$.

5. Indecomposable systems of two subspaces

It is a well known fact that the relative position of two subspaces E_1 and E_2 in a Hilbert space H can be described completely up to unitary equivalence as in Araki [Ar], Dixmier [D] and Halmos [Ha1]. The Hilbert space H is the direct sum of five subspaces:

$$H = (E_1 \cap E_2) \oplus (\text{the rest}) \oplus (E_1 \cap E_2^{\perp}) \oplus (E_1^{\perp} \cap E_2) \oplus (E_1^{\perp} \cap E_2^{\perp}).$$

In the rest part, E_1 and E_2 are in generic position and the relative position is described only by "the angles" between them. In fact the rest part is written as $K \oplus K$ for some subspace K and there exist two positive operators $c, s \in B(K)$ with null kernels with $c^2 + s^2 = 1$ such that

$$E_1 = (E_1 \cap E_2) \oplus \operatorname{Im} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus (E_1 \cap E_2^{\perp}) \oplus 0 \oplus 0,$$

and

$$E_2 = (E_1 \cap E_2) \oplus \mathrm{Im}\begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \oplus 0 \oplus (E_1^\perp \cap E_2) \oplus 0.$$

By the functional calculus, there exists a unique positive operator θ, called the angle operator, such that $c = \cos\theta$ and $s = \sin\theta$ with $0 \leq \theta \leq \frac{\pi}{2}$.

Proposition 5.1. *Let $\mathcal{S} = (H; E_1, E_2)$ be a system of two subspaces in a Hilbert space H. Then $\mathcal{S}$ is indecomposable if and only if $\mathcal{S}$ is isomorphic to one of the following four commutative systems:*

$$\mathcal{S}_1 = (\mathbb{C}; \mathbb{C}, 0), \quad \mathcal{S}_2 = (\mathbb{C}; 0, \mathbb{C}), \mathcal{S}_3 = (\mathbb{C}; \mathbb{C}, \mathbb{C}), \quad \mathcal{S}_4 = (\mathbb{C}; 0, 0).$$

Proof. Let $e_i \in B(H)$ be the projection of H onto E_i, $i = 1, 2$ with the canonical decomposition as above. Suppose that $\dim K \geq 2$. Then there exists a projection $p \in B(K)$ with $0 \neq p \neq I_K$ satisfying p commutes with c and s. Let $H_1 := \mathrm{Im}(p \oplus p) \subset K \oplus K$ and $H_2 := H_1^\perp \cap H$. Let $p_1 \in B(H)$ be the projection of H onto H_1. Since non-trivial projection p_1 commute with e_1 and e_2, $\mathcal{S}$ is decomposable by Lemma 3.2. Therefore if $\mathcal{S}$ is indecomposable, then $\dim K \leq 1$ and only one of the five direct summands is non-zero. If the rest component were non-zero, then it is isomorphic to a decomposable one as in Example 2. Thus the rest component does not appear. One of the other part is commutative. Since $\mathcal{S}$ is indecomposable, $\mathcal{S}$ is one of $\mathcal{S}_1, \dots, \mathcal{S}_4$ by Proposition 3.6. The converse is clear. □

6. SOME PROPERTIES OF INDECOMPOSABLE SYSTEMS OF n-SUBSPACES

Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space. We denote by $\vee_{i=1}^n E_i$ the closed subspace spanned by $E_1, \dots, E_n$. If $\mathcal{S}$ is indecomposable and $\dim H \geq 2$, then it is easy to see that

$$\bigcap_{i=1}^n E_i = 0 \text{ and } \bigvee_{i=1}^n E_i = H.$$

In fact, on the contrary suppose that $M := \cap_{i=1}^n E_i \neq 0$. We choose a one-dimensional subspace $F \subset M$. Since $\dim H \geq 2$, the orthogonal decomposition $H = F \oplus F^\perp$ of the Hilbert space H gives a non-trivial decomposition of the system $\mathcal{S}$. This contradicts to that $\mathcal{S}$ is indecomposable. Hence we have $\cap_{i=1}^n E_i = 0$. Since the orthogonal complement $\mathcal{S}^\perp$ is also indecomposable, we have $\vee_{i=1}^n E_i = (\cap_{i=1}^n E_i^\perp)^\perp = H$. But we can say more as follows:

Proposition 6.1. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space. If $\mathcal{S}$ is indecomposable and* $\dim H \geq 2$*, then for any* *distinct n-1 subspaces $E_{i_1}, \dots, E_{i_{n-1}}$, we have that*

$$\bigcap_{k=1}^{n-1} E_{i_k} = 0 \text{ and } \bigvee_{k=1}^{n-1} E_{i_k} = H.$$

Proof. We may and do assume that $E_{i_1} = E_1, E_{i_2} = E_2, \dots, E_{i_{n-1}} = E_{n-1}$. On the contrary suppose that $M := \cap_{i=1}^{n-1} E_i \neq 0$. Since $\dim H \geq 2$, we can choose a one-dimensional subspace $F \subset M$. Consider two subspaces F and E_n in H. We have the following canonical decomposition into five parts:

$$F = (F \cap E_n) \oplus \operatorname{Im} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus (F \cap E_n^{\perp}) \oplus 0 \oplus 0,$$

$$E_n = (F \cap E_n) \oplus \operatorname{Im} \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \oplus 0 \oplus (F^{\perp} \cap E_n) \oplus 0.$$

We denote by $K \oplus K$ the underlying subspace of the part in generic position.
(i)(the case that $K = 0$): Since $F \cap E_n = \cap_{i=1}^{n} E_i = 0$, we have $F = F \cap E_n^{\perp}$, so that $F \subset E_n^{\perp}$. Let e_i and f be the projections of H onto E_i and F respectively. Then f commutes with each e_i. Therefore the orthogonal decomposition $H = F \oplus F^{\perp}$ of H gives a non-trivial decomposition of the system $\mathcal{S}$. This contradicts to that $\mathcal{S}$ is indecomposable. Hence $M = \cap_{i=1}^{n-1} E_i = 0$.
(ii)(the case that $K \neq 0$): Since F is one-dimensional,

$$K \oplus 0 + \operatorname{Im} \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} = K \oplus K$$

and

$$(K \oplus 0) \cap \operatorname{Im} \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} = 0.$$

Then there exists an invertible operator $T \in B(K \oplus K)$ such that $T(K \oplus 0) = K \oplus 0$, and $T(\operatorname{Im} \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}) = 0 \oplus K$.

We define an invertible operator $\varphi := I \oplus T \oplus I \oplus I \oplus I \in B(H)$. Let $E_i' := \varphi(E_i)$ for $i = 1, \dots, n$. Since $\mathcal{S}$ is indecomposabe, a new system $\mathcal{S}' := (H; E_1', \dots, E_n')$ is indecomposable. Since $F = \varphi(F)$, $F \subset \cap_{i=1}^{n-1} E_i'$ and F is orthogonal to E_n'. Let e_i' and f be the projections of H onto E_i' and F. Then f commutes with each e_i'. Therefore the orthogonal decomposition $H = F \oplus F^{\perp}$ of H gives a non-trivial decomposition of the system $\mathcal{S}'$. This contradicts to that $\mathcal{S}'$ is indecomposable. Hence $M = \cap_{i=1}^{n-1} E_i = 0$

Since the orthogonal complement $\mathcal{S}^{\perp}$ is also indecomposable, we also have $\vee_{k=1}^{n-1} E_{i_k} = H$. $\square$

Corollary 6.2. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ a system of n subspaces in a Hilbert space. If $\mathcal{S}$ is indecomposable and H is infinite-dimensional, then $^{\#}\{i; E_i$ is finite dimensional $\} \le n-2$.*

Proof. On the contrary, suppose that there were distinct n-1 finite-dimensional subspaces $E_{i_1}, \dots, E_{i_{n-1}}$. Then $H = \bigvee_{k=1}^{n-1} E_{i_k}$ is also finite-dimensional. This is a contradiction. $\square$

7. INDECOMPOSABLE SYSTEMS OF THREE SUBSPACES

Gelfand and Ponomarev ([GP]) claimed that there exist only nine, finite-dimensional, indecomposable systems of three subspaces. We shall include a direct proof of it. We do not know whether there exists an infinite-dimensional transitive systems of three subspaces. In fact it is still an unsolved problem whether there exists a transitive lattice consisting of five elements in an infinite-dimensional Hilbert space. Therefore it is worth while investigating the existence of infinite-dimensional indecomposable systems of three subspaces.

Proposition 7.1. *Let $\mathcal{S} = (H; E_1, E_2, E_3)$ be an indecomposable system of three subspaces. If H is infinite dimensional, then $E_i \neq 0$ and $E_i \neq H$ for $i = 1, 2, 3$.*

Proof. On the contrary suppose that $E_1 = 0$. Then $\mathcal{S}' = (H; E_2, E_3)$ is an indecomposable system of two subspaces. Hence by Proposition 5.1, H is finite dimensional. This is a contradiction. Hence $E_1 \neq 0$. Similary $E_i \neq 0$ and $E_i \neq H$ for $i = 1, 2, 3$. $\square$

Theorem 7.2. *Let $\mathcal{S} = (H; E_1, E_2, E_3)$ be an indecomposable system of three subspaces in a Hilbert space H. Then the following hold.*
(1)If H is infinite-dimensional, then for any $i \neq j$, $E_i \cap E_j = 0$ and $E_i + E_j$ is a non-closed dense subspace of H. In particular each E_i is infinite-dimensional.
(2)[GP] *If H is finite-dimensional, then $\mathcal{S}$ is isomorphic to one of the following eight commutaitve systems $\mathcal{S}_1, \dots, \mathcal{S}_8$ and one non-commutative system $\mathcal{S}_9$:*

$$\mathcal{S}_1 = (\mathbb{C}; 0, 0, 0), \quad \mathcal{S}_2 = (\mathbb{C}; \mathbb{C}, 0, 0), \quad \mathcal{S}_3 = (\mathbb{C}; 0, \mathbb{C}, 0),$$
$$\mathcal{S}_4 = (\mathbb{C}; 0, 0, \mathbb{C}), \quad \mathcal{S}_5 = (\mathbb{C}; \mathbb{C}, \mathbb{C}, 0), \quad \mathcal{S}_6 = (\mathbb{C}; \mathbb{C}, 0, \mathbb{C}),$$
$$\mathcal{S}_7 = (\mathbb{C}; 0, \mathbb{C}, \mathbb{C}), \ \ \mathcal{S}_8 = (\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C}), \ \ \mathcal{S}_9 = (\mathbb{C}^2; \mathbb{C}(1,0), \mathbb{C}(0,1), \mathbb{C}(1,1)).$$

Proof. If $\dim H = 1$, then $\mathcal{S}$ is commutative. Hence if $\mathcal{S}$ is isomorphic to one of $\mathcal{S}_1, \dots, \mathcal{S}_8$. Therefore we may assume that $\mathcal{S}$ is indecomposable and $\dim H \ge 2$. Then, by Proposition 6.1, for any $i \neq j$, $E_i \cap E_j = 0$ and $E_i + E_j$ is a dense subspace of H. We claim that if $E_1 + E_2 = H$, then H is finite-dimensional and $\mathcal{S}$ is isomorphic to $\mathcal{S}_9$. It is enough to show the claim to prove the theorem. In fact, assume that the claim holds. (1)If H is infinite-dimensional, then $E_1 + E_2$ is not closed. Similarly for any $i \neq j$, $E_i + E_j$ is not closed. (2)If H is finite-dimensional, then $E_1 + E_2 = H$. Thus $\mathcal{S}$ is isomorphic to $\mathcal{S}_9$. We shall show the claim. Since $E_1 \cap E_2 = 0$ and $E_1 + E_2 = H$, there exists $T \in B(H)^{-1}$ such that $T(E_1) = E_1$ and $T(E_2) = E_1^{\perp}$. Therefore we may assume that $E_2 = E_1^{\perp}$ to show the claim. Considering the canonical decomposition for two subspaces E_1 and E_3, we have the following descripton of three subspaces:

$$E_1 = (E_1 \cap E_3) \oplus \operatorname{Im}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus (E_1 \cap E_3^{\perp}) \oplus 0 \oplus 0,$$

$$E_3 = (E_1 \cap E_3) \oplus \operatorname{Im}\begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \oplus 0 \oplus (E_1^{\perp} \cap E_3) \oplus 0,$$

$$E_2 = E_1^{\perp} = 0 \oplus \operatorname{Im}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0 \oplus (E_1^{\perp} \cap E_3) \oplus (E_1^{\perp} \cap E_3^{\perp}),$$

where the underlying Hilbert space H is decomposed into five parts

$$H = (E_1 \cap E_3) \oplus (K \oplus K) \oplus (E_1 \cap E_3^{\perp}) \oplus (E_1^{\perp} \cap E_3) \oplus (E_1^{\perp} \cap E_3^{\perp}).$$

If two parts of the above five parts were non-zero, then $\mathcal{S}$ can be decomposed non-trivially. This contradicts to that $\mathcal{S}$ is indecomposable. Hence only one of the above five parts is non-zero. If the part $K \oplus K = 0$, then $\mathcal{S}$ is commutative. Since $\mathcal{S}$ is indecomposable, $\dim H = 1$. This contradicts to that $\dim H \geq 2$. Hence the only the part $K \oplus K \neq 0$. If $\dim K = 1$, then it is clear that $\mathcal{S}$ is isomorphic to $\mathcal{S}_9$. If $\dim K \geq 2$, then there exists a projection $p \in B(K)$ with $0 \neq p \neq I_K$ satisfying p commute with c and s. Let $H_1 := \operatorname{Im}(p \oplus p) \subset K \oplus K = H$ and $H_2 := H_1^{\perp} \cap H$. Let $p_1, e_1, e_2, e_3 \in B(H)$ be the projections of H onto H_1, E_1, E_2, E_3 respectively. Since non-trivial projection p_1 commute with e_1, e_2 and e_3, $\mathcal{S}$ is decomposable by Lemma 3.2. This is a contradiciton. Hence the case that $\dim K \geq 2$ does not occur. We have shown the claim. □

8. Operator systems

We can associate a system of four subspaces for any operator.

Definition. (bounded operator system) We say that a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces is a *bounded operator system* if there exist a Hilbert space K_1, K_2 and bounded operators $T : K_1 \to K_2$, $S : K_2 \to K_1$ such that $H = K_1 \oplus K_2$ and

$$E_1 = K_1 \oplus 0, \ E_2 = 0 \oplus K_2,$$

$$E_3 = \{(x, Tx); x \in K_1\}, \ E_4 = \{(Sy, y); y \in K_2\}.$$

We denote by $\mathcal{S}_{T,S}$ the above operator system $\mathcal{S}$. We often identify E_1 with K_1 and E_2 with K_2. In particular we associate an operator system $\mathcal{S}_T := \mathcal{S}_{T,I} = (H; E_1, E_2, E_3, E_4)$ for any single operator $T \in B(K)$ such that $H = K \oplus K$ and

$$E_1 = K \oplus 0, E_2 = 0 \oplus K, E_3 = \{(x, Tx); x \in K\}, E_4 = \{(y, y); y \in K\}.$$

We shall study a relation between the system $\mathcal{S}_T$ of four subspaces and a single operator T.

Proposition 8.1. *Let $\mathcal{S}_{T,S} = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system associated with $T : K_1 \to K_2$ and $S : K_2 \to K_1$. Then*

$$\begin{aligned} End(\mathcal{S}_{T,S}) = \{ &A_1 \oplus A_2 \in B(H); A_1 \in B(K_1), A_2 \in B(K_2), \\ &A_1 S = S A_2,\ A_2 T = T A_1 \}, \textit{ and} \end{aligned}$$

$$\begin{aligned} Idem(\mathcal{S}_{T,S}) = \{ &A_1 \oplus A_2 \in B(H); A_1 \in B(K_1), A_2 \in B(K_2), \\ &A_1 S = S A_2,\ A_2 T = T A_1, A_1^2 = A_1,\ A_2^2 = A_2 \} \end{aligned}$$

Proof. Let $A \in End(\mathcal{S})$. Since $A(E_1) \subset E_1$ and $A(E_2) \subset E_2$, we have $A = A_1 \oplus A_2$ for some $A_1 \in B(K_1), A_2 \in B(K_2)$. Since $A(E_3) \subset E_3$, for any $x \in K_1$, $(A_1 \oplus A_2)(x, Tx) \in E_3$. Thus $(A_1 x, A_2 T x) = (y, Ty)$ for some $y \in K_2$. Therefore $A_2 T x = T A_1 x$. Thus $A_2 T = T A_1$. Similarly $A(E_3) \subset E_3$ implies $A_1 S = S A_2$. The converse is clear. We get the equality for $Idem(\mathcal{S})$ immediately. $\square$

Corollary 8.2. *Let $\mathcal{S}_T = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system associated with a single operator $T \in B(K)$. Then*

$$End(\mathcal{S}_T) = \{ B \oplus B \in B(H); B \in B(K),\ BT = TB \}, \textit{ and}$$

$$Idem(\mathcal{S}_T) = \{ B \oplus B \in B(H); B \in B(K),\ BT = TB, B^2 = B \}.$$

Definition. Recall that a bounded operator T on a Hilbert space K is called *strongly irreducible* if there do not exist two non-trivial subspaces $M \subset K$ and $N \subset K$ such that $T(M) \subset M$, $T(N) \subset N$, $M \cap N = 0$ and $M + N = K$. We also see that T is strongly irreducible if and only if there does not exist any non-trivial idempotent P such that $PT = TP$. See a monograph [JW] by Jiang and Wang.

Corollary 8.3. *Let $\mathcal{S}_T = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system associated with a single operator $T \in B(K)$. Then $\mathcal{S}_T$ is indecomposable if and only if T is strongly irreducible.*

Example. Let $K = \ell^2(\mathbb{N})$ and $S \in B(K)$ be the unilateral shift. Let $P \in B(K)$ be an idempotent which commutes with S. Then P is a lower triangular Toeplitz matrix. Since P is an idempotent, we have $P = 0$ or $P = I$ as in Lemma 10.1. Thus S is strongly irreducible, as already known, for example, in [JW], and $\mathcal{S}_S$ is indecomposable.

Corollary 8.4. *Let $\mathcal{S}_T = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system associated with a single operator $T \in B(K)$. If $\mathcal{S}_T$ is decomposable, then T has a non-trivial invariant subspace.*

Proof. Let $\mathcal{S}_T$ be decomposable. Then there exists a non-trivial idempotent P such that $PT = TP$. Then $\operatorname{Im} P$ is a non-trivial invariant subspace. $\square$

Proposition 8.5. *Let $\mathcal{S}_T = (H; E_1, E_2, E_3, E_4)$ and $\mathcal{S}_{T'} = (H'; E'_1, E'_2, E'_3, E'_4)$ be bounded operator systems associated with operators $T \in B(K)$ and $T' \in B(K')$. Then $\mathcal{S}_T$ and $\mathcal{S}_{T'}$ are isomorphic if and only if T and T' are similar.*

Proof. Assume that $\mathcal{S}_T$ and $\mathcal{S}_{T'}$ are isomorphic. Then there exists a bounded invertible operator $A : H \to H'$ with $A(E_i) = E'_i$ for $i = 1, 2, 3, 4$. Since $A(E_i) = E'_i$ for $i = 1, 2, 4$, we have $A = B \oplus B$ for some invertible operator $B : K \to K'$. And $A(E_3) \subset E_3$ implies that $BT = T'B$, that is, T and T' are similar. The converse is clear. $\square$

Remark. The above proposition shows that the classification of systems of four subspaces contains the classification of operators up to similarity in a certain sense.

Example.(an uncountable family of indecomposable systems of four subspaces) Let $K = \ell^2(\mathbb{N})$ and $H = K \oplus K$. Consider a unilateral shift $S : K \to K$. For a parameter $\alpha \in \mathbb{C}$, let $E_1 = K \oplus 0, E_2 = 0 \oplus K, E_3 = \{(x, (S+\alpha I)x) | x \in K\}$ and $E_4 = \{(x, x) | x \in K\}$. Then the system $\mathcal{S}_\alpha = (H; E_1, E_2, E_3, E_4)$ of four subspaces are indecomposable. If $\alpha \neq \beta$, then $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ are not isomorphic, because the spectra $\sigma(S + \alpha) \neq \sigma(S + \beta)$ and $S + \alpha I$ and $S + \beta I$ are not similar. Thus we can easily construct an uncountable family $(\mathcal{S}_\alpha)_{\alpha \in \mathbb{C}}$ of indecomposable systems of four subspaces.

As the single operator case, we also obtain the following:

Proposition 8.6. *Let $\mathcal{S}_{T,S} = (H; E_1, E_2, E_3, E_4)$ and $\mathcal{S}_{T',S'} = (H'; E'_1, E'_2, E'_3, E'_4)$ be bounded operator systems associated with operators $S \in B(K_2, K_1), T \in B(K_1, K_2)$, $S' \in B(K'_2, K'_1), T' \in B(K'_1, K'_2)$. Then $\mathcal{S}_{T,S}$ and $\mathcal{S}_{T',S'}$ are isomorphic if and only if there exist bounded invertible operators $A_1 : K_1 \to K'_1$ and $A_2 : K_2 \to K'_2$ such that $A_1 S = S' A_2$ and $A_2 T = T' A_1$.*

Proposition 8.7. *Let $\mathcal{S}_{T,S} = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system associated with operators $S \in B(K_2, K_1), T \in B(K_1, K_2)$.* ! ! *Then the orthogonal complement of the system $\mathcal{S}_{T,S}$ is isomorphic to another bounded operator system up to a permutation of subspaces and given by*

$$\mathcal{S}_{T,S}^\perp \cong \sigma_{1,2}\sigma_{3,4}\mathcal{S}_{-S^*,-T^*},$$

where $\sigma_{i,j}$ is a transposition of i and j.

Proof. It is evident from the fact $\{(x, Tx) \in K_1 \oplus K_2; x \in K_1\}^\perp = \{(-T^* y, y) \in K_1 \oplus K_2; y \in K_2\}$ and etc. $\square$

Proposition 8.8. *Let $\mathcal{S}_{T,S} = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system associated with operators $S \in B(K_2, K_1), T \in B(K_1, K_2)$. If T*

is invertible, then $\mathcal{S}_{T,S}$ *is isomorphic to* $\mathcal{S}_{I,TS}$*. If* S *is invertible, then* $\mathcal{S}_{T,S}$ *is isomorphic to* $\mathcal{S}_{ST,I}$.

Proof. Let T be invertible. Define an invertible operator $\varphi : K_1 \oplus K_2 \to K_2 \oplus K_2$ by $\varphi(x, y) = (Tx, y)$. Then $\varphi(E_1) = \varphi(K_1 \oplus 0) = K_2 \oplus 0$. $\varphi(E_2) = \varphi(0 \oplus K_2) = 0 \oplus K_2$. Since $\varphi(x, Tx) = (Tx, Tx)$, $\varphi(E_3) = \varphi(\mathrm{graph}\, T) = \{(y, y); y \in K_2\}$. Because $\varphi(Sy, y) = (TSy, y)$, $\varphi(E_4) = \varphi(\mathrm{cograph}\, S) = \{(TSy, y); y \in K_2\} = \mathrm{cograph}\, TS$. Hence $\mathcal{S}_{T,S}$ is isomorphic to $\mathcal{S}_{I,TS}$. If S is invertible, use an invertible operator $\psi : K_1 \oplus K_2 \to K_1 \oplus K_1$ defined by $\psi(x, y) = (x, Sy)$. □

Bounded operator systems can be extended to (unbounded) closed operator systems.

Definition.(closed operator systems) We say that a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces is a *closed operator system* if there exist Hilbert spaces K_1, K_2 and closed operators $T : K_1 \supset D(T) \to K_2$, $S : K_2 \supset D(S) \to K_1$ such that $H = K_1 \oplus K_2$ and $E_1 = K_1 \oplus 0$,

$$E_2 = 0 \oplus K_2,\ E_3 = \{(x, Tx); x \in D(T)\}, E_4 = \{(Sy, y); y \in D(S)\}.$$

We also denote by $\mathcal{S}_{T,S}$ the above operator system $\mathcal{S}$.

We shall give a characterization of (densely defined) closed operator systems.

Proposition 8.9. *Let* $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ *be a system of four subspaces in a Hilbert space* H*. Then the following are equivalent:*

(1) $\mathcal{S}$ *is isomorphic to a closed operator system* $\mathcal{S}_{T,S}$ *for some closed operators* $T : E_1 \supset D(T) \to E_2$ *and* $S : E_2 \supset D(S) \to E_1$.

(2) $E_1 + E_2 = H$ *and* $E_i \cap E_j = 0$ *for* $(i, j) = (1, 2), (2, 3)$ *and* $(4, 1)$.

Moreover if these conditions are satisfied, then $D(T) := E_1 \cap (E_3 + E_2)$ *and* $D(S) := E_2 \cap (E_4 + E_1)$.

Proof. (1)⇒(2): It is trivial. (2)⇒(1): By Lemma 2.1, we may assume that $E_2 = E_1^{\perp}$. Put $K_1 = E_1$ and $K_2 = E_2$. Then $H = E_1 \oplus E_2$. Since $E_3 \cap E_2 = 0$, for any $x_1 \in E_1 \cap (E_3 + E_2)$, there exist unique $x_3 \in E_3$ and $x_2 \in E_2$ such that $x_1 = x_3 - x_2$. Define a linear operator $T : E_1 \supset D(T) \to E_2$ by $Tx_1 = x_2$ with a domain $D(T) := E_1 \cap (E_3 + E_2)$. Since $E_1 + E_2 = H$, for any $x_3 \in E_3$ there exist $x_1 \in E_1$ and $x_2 \in E_2$ with $x_3 = x_1 + x_2$. This implies that $\mathrm{graph}\, T = E_3$. Hence T is a closed operator. Similarly there exists a closed operator $S : E_2 \supset D(S) \to E_1$ with a domain $D(S) := E_2 \cap (E_4 + E_1)$. □

Corollary 8.10. *Let* $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ *be a system of four subspaces in a Hilbert space* H*. Then the following are equivalent:*

(1) $\mathcal{S}$ *is isomorphic to a closed operator system* $\mathcal{S}_{T,S}$ *for some densely defined closed operators* $T : E_1 \supset D(T) \to E_2$ *and* $S : E_2 \supset D(S) \to E_1$.

(2) $E_1 + E_2 = H$ *and* $E_i \cap E_j = 0$ *for* $(i, j) = (1, 2), (2, 3), (4, 1)$, $E_1 \cap (E_3 + E_2)$ *is dense in* E_1 , $E_2 \cap (E_4 + E_1)$ *is dense in* E_2

We immediately have a characterization of bounded operator systems.

Corollary 8.11. *Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces in a Hilbert space H. Then the following are equivalent:*

(1) *$\mathcal{S}$ is isomorphic to a bounded operator system.*
(2) *$E_i + E_j = H$ and $E_i \cap E_j = 0$ for $(i, j) = (1, 2), (2, 3)$ and $(4, 1)$*

Proof. (1)⇒(2): It is trivial. (2)⇒(1): Since $E_3 + E_2 = H$, we have $D(T) = E_1 \cap (E_3 + E_2) = E_1$. Because graph $T = E_3$ is closed , T is bounded by the closed graph theorem. Similarly $E_4 + E_1 = H$ implies that $D(S) = E_2$ and S is bounded. □

Corollary 8.12. *Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces in a Hilbert space H. Then the following are equivalent:*

(1) *$\mathcal{S}$ is isomorphic to a bounded operator system associated with a single operator.*
(2) *$E_i + E_j = H$ and $E_i \cap E_j = 0$ for $(i, j) = (1, 2), (2, 3), (4, 1)$ and $(2, 4)$.*

Proof. (1)⇒(2): It is trivial. (2)⇒(1): By the preceding Corollary, $\mathcal{S}$ is isomorphic to a bounded operator system $\mathcal{S}_{T,S}$. Since $E_2 \cap E_4 = 0$, S is one to one. Since $E_2 + E_4 = H$, S is onto. Therefore $\mathcal{S}_{T,S}$ is isomorphic to a bounded operator system $\mathcal{S}_{ST,I} = \mathcal{S}_{ST}$ associated with a single operator ST by Proposition 8.8. □

Proposition 8.13. *Let $\mathcal{S}_T = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system associated with a single operator $T \in B(K)$. Then $\mathcal{S}_T$ is transitive if and only if $\dim K = 1$. If it is so, then $\mathcal{S}_T$ is isomorphic to*

$$(\mathbb{C}^2; \mathbb{C} \oplus 0, 0 \oplus \mathbb{C}, \{(x, \lambda x); x \in \mathbb{C}\}, \{(x, x); x \in \mathbb{C}\})$$

for some $\lambda \in \mathbb{C}$.

Proof. Recall that $\mathcal{S}_T$ is transitive if

$$End(\mathcal{S}_T) = \{B \oplus B \in B(H); B \in B(K),\ BT = TB\} = \mathbb{C}I.$$

Hence $\mathcal{S}_T$ is transitive if and only if $\{T\}' := \{B \in B(K);\ BT = TB\} = \mathbb{C}I$ if and only if $\dim K = 1$.

□

But certain unbounded operators on an infnite dimensional Hilbert space give transitive systems of four subspaces.

Example(Harrison-Radjavi-Rosenthal [HRR]) Let $K = \ell^2(\mathbb{Z})$ and $H = K \oplus K$. Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence given by $a_n = 1$ for $n \leq 0$ and $a_n = exp((-1)^n n!)$ for $n \geq 1$. Define a bilateral weighted shift $T : K \supset D(T) \to K$ by $(Tx)_n = a_{n-1}x_{n-1}$ with the domain $D(T) = \{(x_n)_n \in \ell^2(\mathbb{Z}); \sum_n |a_n x_n|^2 < \infty\}$. Let $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$, $E_3 = \{(x, Tx) \in K \oplus K; x \in D(T)\}$, and $E_4 = \{(x, x) \in K \oplus K; x \in K\}$. Harrison, Radjavi and Rosenthal showed that $\{H, E_1, E_2, E_3, E_4, 0\}$ is a transitive lattice in [HRR]. Hence $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ is a transitive system of four subspaces.

We can extend their example to construct uncountably many transitive systems.

Lemma 8.14. *Let $\mathcal{S}_T = (H; E_1, E_2, E_3, E_4)$ and $\mathcal{S}_{T'} = (H'; E_1', E_2', E_3', E_4')$ be closed operator systems associated with operators $T : D(T) \to K$, $T' : D(T') \to K'$. Then $\mathcal{S}_T$ and $\mathcal{S}_{T'}$ are isomorphic if and only if T and T' are similar.*

Proof. The proof is as same as bounded operators if we see the domains of the closed operators carefully. $\square$

Example. Let $K = \ell^2(\mathbb{Z})$ and $H = K \oplus K$. For a fixed number $\alpha > 1$, let $(w_n)_{n\in\mathbb{Z}} = (w_n(\alpha))_{n\in\mathbb{Z}}$ be a sequence given by $w_n = 1$ for $n \leq 0$ and $w_n = exp((-\alpha)^n)$for $(n \geq 1)$. Define a bilateral weighted shift $T_\alpha : K \supset D_\alpha \to K$ by $(T_\alpha x)_n = w_{n-1}x_{n-1}$ with the domain $D_\alpha = \{(x_n)_n \in \ell^2(\mathbb{Z}); \sum_n |w_n x_n|^2 < \infty\}$. Let $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$, $E_3^\alpha = \{(x, T_\alpha x) \in K \oplus K; x \in D_\alpha$, and $E_4 = \{(x, x) \in K \oplus K; x \in K\}$.

Proposition 8.15. *If $\alpha > 1$, then the above system $\mathcal{S}_\alpha = (H; E_1, E_2, E_3^\alpha, E_4)$ is a transitive system. Furthermore if $\alpha \neq \beta$, then $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ are not isomorphic.*

Proof. Let $V \in Hom(\mathcal{S}_\alpha, \mathcal{S}_\beta)$. Since $V(E_i) \subset E_i$ for $i = 1, 2, 4$, $V = A \oplus A$ for some $A = (a_{ij})_{ij} \in B(K)$. Since $V(E_3^\alpha) \subset E_3^\beta$ and $e_n \in D_\alpha$,

$$(A \oplus A)(e_n, T_\alpha e_n) = (Ae_n, AT_\alpha e_n) \in E_3^\beta.$$

Hence $AT_\alpha e_n = T_\beta A e_n$. Comparing $(m+1)$-th component, we have $w_n(\alpha)a_{m+1,n+1} = w_m(\beta)a_{m,n}$, that is,

$$a_{m+1,n+1} = \frac{w_m(\beta)}{w_n(\alpha)} a_{m,n}.$$

Therefore for any $k \in \mathbb{N}$,

$$a_{m+k,n+k} = \frac{w_m(\beta) \dots w_{m+k-1}(\beta)}{w_n(\alpha) \dots w_{n+k-1}(\alpha)} a_{m,n} = exp(c_k(m, n))a_{m,n},$$

where

$$\begin{aligned} c_k(m,n) &= ((-\beta)^m + \cdots + (-\beta)^{m+k-1}) - ((-\alpha)^n + \cdots + (-\alpha)^{n+k-1}) \\ &= \frac{(-\beta)^m(1 - (-\beta)^k)}{1 + \beta} - \frac{(-\alpha)^n(1 - (-\alpha)^k)}{1 + \alpha} \ . \end{aligned}$$

(i)(the case when $\alpha = \beta$): Putting $n = m$, we have $c_k(m, m) = 0$. Hence the diagonal of A is constant. If A were not a multiple of the identitiy, then there exist distinct m and n with $a_{m,n} \neq 0$. According to $m < n$ or $m > n$, for a sufficient large k,

$$c_k(m,n) = ((-\alpha)^m + \cdots + (-\alpha)^n) - ((-\alpha)^{m+k-1} + \cdots + (-\alpha)^{n+k-1})$$

or

$$c_k(m,n) = -((-\alpha)^n + \cdots + (-\alpha)^m) + ((-\alpha)^{n+k-1} + \cdots + (-\alpha)^{m+k-1}).$$

In either case we have $\limsup_k c_k(m,n) = \infty$. Hence $a_{m+k,n+k}$ is not bounded as $k \to \infty$. This contradicts to that A is bounded. Therefore A is a scalar. We have shown that $\mathcal{S}_\alpha$ is a transitive system.
(ii)the case when $\alpha \neq \beta$: We may and do assume that $1 < \alpha < \beta$. If A were not equal to 0, then there exist m and n with $a_{m,n} \neq 0$. Since

$$c_k(m,n) = \frac{(-\beta)^m(1-(-\beta)^k)}{1+\beta}\{1 - \frac{(-\alpha)^n(1+\beta)(1-(-\alpha)^k)}{(-\beta)^m(1+\alpha)(1-(-\beta)^k)}\} ,$$

we have $\limsup_k c_k(m,n) = \infty$. This contradicts to that A is bounded. Therefore $A = 0$. We have shown that $Hom(\mathcal{S}_\alpha, \mathcal{S}_\beta) = 0$. Therefore $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ are not isomorphic. □

Proposition 8.16. *Let* $\mathcal{S}_{T,S} = (H; E_1, E_2, E_3, E_4)$ *be a bounded operator system associated with operators* $S \in B(K_2, K_1), T \in B(K_1, K_2)$. *Then* $\mathcal{S}$ *is transitive if and only if* $\mathcal{S}$ *is isomorphic to* $(\mathbb{C}; \mathbb{C}, 0, \mathbb{C}, 0)$, $(\mathbb{C}; 0, \mathbb{C}, 0, \mathbb{C})$, $(\mathbb{C}^2; \mathbb{C}\oplus 0, 0\oplus\mathbb{C}, \{(x,x); x \in \mathbb{C}\}, 0\oplus\mathbb{C})$ *or* $(\mathbb{C}^2; \mathbb{C}\oplus 0, 0\oplus\mathbb{C}, \{(x,\lambda x); x \in \mathbb{C}\}, \{(x,x); x \in \mathbb{C}\})$ *for some* $\lambda \in \mathbb{C}$.

Proof. Suppose that $\mathcal{S} = \mathcal{S}_{T,S}$ is transitive. If $\dim H = 1$, then $\mathcal{S}$ is isomorphic to $(\mathbb{C}; \mathbb{C}, 0, \mathbb{C}, 0)$ or $(\mathbb{C}; 0, \mathbb{C}, 0, \mathbb{C})$. We assume that $\dim H \geq 2$. Since $ST \oplus TS \in End(\mathcal{S}_{T,S})$ and $\mathcal{S}$ is transitive, there exists $\lambda \in \mathbb{C}$ such that $ST = \lambda I_{K_1}$ and $TS = \lambda I_{K_2}$.

In the case that $\lambda \neq 0$, T and S are invertible and $S = \lambda T^{-1}$. By Proposition 8.8, $\mathcal{S}_{T,S}$ is isomorphic to $\mathcal{S}_{\lambda I_{K_1}, I_{K_1}}$. Applying Proposition 8.13, $\mathcal{S}$ is isomorphic to

$$(\mathbb{C}^2; \mathbb{C}\oplus 0, 0\oplus\mathbb{C}, \{(x,\lambda x); x \in \mathbb{C}\}, \{(x,x); x \in \mathbb{C}\})$$

for some $\lambda \in \mathbb{C}$.

In the case that $\lambda = 0$, we have $ST = 0$ and $TS = 0$. Since $SS^* \oplus S^*S, T^*T \oplus TT^* \in End(\mathcal{S}_{T,S})$ and $\mathcal{S}$ is transitive, we have $SS^* = \alpha I_{K_1}, S^*S = \alpha I_{K_2}, T^*T = \beta I_{K_1}$ and $TT^* = \beta I_{K_2}$. Because $ST = 0$, $\alpha\beta = 0$. Hence $\alpha = 0$ or $\beta = 0$, so that $S = 0$ or $T = 0$. If $T = 0$, then a subsystem $(H; K_1 \oplus 0, 0 \oplus K_2, \{(Sy, y); y \in K_2\})$ of three subspaces is transitive. Since $\dim H \geq 2$, the subsystem is isomorphic to $(\mathbb{C}^2; \mathbb{C}\oplus 0, 0\oplus\mathbb{C}, \{(x,x); x \in \mathbb{C}\})$. Hence $\mathcal{S}$ is isomorphic to $(\mathbb{C}^2; \mathbb{C}\oplus 0, 0\oplus\mathbb{C}, \mathbb{C}\oplus 0, \{(x,x); x \in \mathbb{C}\})$. Similarly if $S = 0$, then $\mathcal{S}$ is isomorphic to $(\mathbb{C}^2; \mathbb{C}\oplus 0, 0\oplus\mathbb{C}, \{(x,x); x \in \mathbb{C}\}, 0\oplus\mathbb{C})$. The converse is clear. □

9. classification theorem by Gelfand-Ponomarev

One of the main problem to attack is a classification of indecomposable systems $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in a Hilbert space H. In the case when H is finite-dimensional, Gelfand and Ponomarev completely classified indecomposable systems and gave a complete list of them in [GP]. The important numerical invariants are *dim* H and the defect defined by

$$\rho(\mathcal{S}) := \sum_{i=1}^{4} dim\ E_i - 2dim\ H.$$

Theorem 9.1 (Gelfand-Ponomarev [GP]). *The set of possible values of the defect $\rho(\mathcal{S})$ for indecomposable systems $\mathcal{S}$ of four subspaces in a finite-dimensional space is exactly the set* $\{-2, -1, 0, 1, 2\}$.

The defect characterizes an essential feature of the system. If $\rho(\mathcal{S}) = 0$, then $\mathcal{S}$ is isomorphic to a bounded operator system up to permutation of subspaces , that is, there exists a permutation σ on $\{1, 2, 3, 4\}$ and a pair of linear operators $A : E \to F$ and $B : F \to E$ such that $H = E \oplus F$, $E_{\sigma(1)} = E \oplus 0$, $E_{\sigma(2)} = 0 \oplus F$, $E_{\sigma(3)} = \{(x, Ax) \in H; x \in E\}$ and $E_4 = \{(By, y) \in H; y \in F\}$. If $\rho(\mathcal{S}) = \pm 1$, $\mathcal{S}$ is represented up to permutation by $H = E \oplus F$, $E_1 = E \oplus 0$, $E_2 = 0 \oplus F$, E_3 and E_4 are subspaces of H that do not reduced to the graphs of the operators as in the case that $\rho(\mathcal{S}) = 0$. A system with $\rho(\mathcal{S}) = \pm 2$ cannot be described in the above forms.

Following [GP], we recall the canonical forms of indecomposable systems $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in a finite-dimensional space H up to permutation in the following: (A) the case when *dim* $H = 2k$ for some positive integer k.

There exist no indecomposable systems $\mathcal{S}$ with $\rho(\mathcal{S}) = \pm 2$. Let H be a space with a basis $\{e_1, \dots, e_k, f_1, \dots, f_k\}$.
(1)$\mathcal{S}_3(2k, -1) = (H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = -1$

$$\begin{aligned} H &= [e_1, \dots, e_k, f_1, \dots, f_k], \\ E_1 &= [e_1, \dots, e_k],\ E_2 = [f_1, \dots, f_k], \\ E_3 &= [(e_2 + f_1), \dots, (e_k + f_{k-1})], \\ E_4 &= [(e_1 + f_1), \dots, (e_k + f_k)]. \end{aligned}$$

(2)$S_3(2k, 1) = (H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = 1$

$$\begin{aligned} H &= [e_1, \dots, e_k, f_1, \dots, f_k], \\ E_1 &= [e_1, \dots, e_k],\ E_2 = [f_1, \dots, f_k], \\ E_3 &= [e_1, (e_2 + f_1), \dots, (e_k + f_{k-1}), f_k], \\ E_4 &= [(e_1 + f_1), \dots, (e_k + f_k)]. \end{aligned}$$

(3)$\mathcal{S}_{1,3}(2k,0) = (H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = 0$

$$
\begin{aligned}
&H = [e_1, \dots, e_k, f_1, \dots, f_k],\\
&E_1 = [e_1, \dots, e_k],\ E_2 = [f_1, \dots, f_k],\\
&E_3 = [e_1, (e_2 + f_1), \dots, (e_k + f_{k-1})],\\
&E_4 = [(e_1 + f_1), \dots, (e_k + f_k)].
\end{aligned}
$$

(4)$\mathcal{S}(2k, 0; \lambda) = (H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = 0$

$$
\begin{aligned}
&H = [e_1, \dots, e_k, f_1, \dots, f_k],\\
&E_1 = [e_1, \dots, e_k],\ E_2 = [f_1, \dots, f_k],\\
&E_3 = [(e_1 + \lambda f_1), (e_2 + f_1 + \lambda f_2), \dots, (e_k + f_{k-1} + \lambda f_k)],\\
&E_4 = [(e_1 + f_1), \dots, (e_k + f_k)].
\end{aligned}
$$

Every other system $\mathcal{S}_i(2k, \rho)$, $\mathcal{S}_{i,j}(2k, 0)$ can be obtained from the systems $\mathcal{S}_3(2k, \rho)$, $\mathcal{S}_{i,3}(2k, 0)$ by a suitable permutation of the subspaces. Let $\sigma_{i,j}$ be the transposition (i, j). We put $\mathcal{S}_i(2k, \rho) = \sigma_{3,i}\mathcal{S}_3(2k, \rho)$ for $\rho = -1, 1$. We also define $\mathcal{S}_{i,j}(2k, 0) = \sigma_{1,i}\sigma_{3,j}\mathcal{S}_{1,3}(2k, 0)$ for $i, j \in \{1, 2, 3, 4\}$.

(B)the case $dim\ H = 2k + 1$ is odd for some integer $k \geq 0$. Let H be a space with a basis $\{e_1, \dots, e_k, e_{k+1}, f_1, \dots, f_k\}$.

(5)$\mathcal{S}_1(2k + 1, -1) = (H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = -1$

$$
\begin{aligned}
&H = [e_1, \dots, e_k, e_{k+1}, f_1, \dots, f_k],\\
&E_1 = [e_1, \dots, e_k, e_{k+1}],\ E_2 = [f_1, \dots, f_k],\\
&E_3 = [(e_2 + f_1), \dots, (e_{k+1} + f_k)],\\
&E_4 = [(e_1 + f_1), \dots, (e_k + f_k)].
\end{aligned}
$$

(6)$\mathcal{S}_2(2k + 1, 1) = (H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = 1$

$$
\begin{aligned}
&H = [e_1, \dots, e_k, e_{k+1}, f_1, \dots, f_k],\\
&E_1 = [e_1, \dots, e_k, e_{k+1}],\ E_2 = [f_1, \dots, f_k],\\
&E_3 = [e_1, (e_2 + f_1), \dots, (e_{k+1} + f_k)],\\
&E_4 = [(e_1 + f_1), \dots, (e_k + f_k), e_{k+1}].
\end{aligned}
$$

(7)$\mathcal{S}_{1,3}(2k + 1, 0) = (H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = 0$

$$
\begin{aligned}
&H = [e_1, \dots, e_k, e_{k+1}, f_1, \dots, f_k],\\
&E_1 = [e_1, \dots, e_k, e_{k+1}],\ E_2 = [f_1, \dots, f_k],\\
&E_3 = [e_1, (e_2 + f_1), \dots, (e_{k+1} + f_k)],\\
&E_4 = [(e_1 + f_1), \dots, (e_k + f_k)].
\end{aligned}
$$

(8)$\mathcal{S}(2k + 1, -2) = (H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = -2$

$$
\begin{aligned}
&H = [e_1, \dots, e_k, e_{k+1}, f_1, \dots, f_k],\\
&E_1 = [e_1, \dots, e_k],\ E_2 = [f_1, \dots, f_k],\\
&E_3 = [(e_2 + f_1), \dots, (e_{k+1} + f_k)],\\
&E_4 = [(e_1 + f_2), \dots, (e_{k-1} + f_k), (e_k + e_{k+1})].
\end{aligned}
$$

(9)$\mathcal{S}(2k+1,2) = (H; E_1, E_2, E_3, E_4)$ with $\rho(\mathcal{S}) = 2$

$$\begin{aligned} &H = [e_1, \dots, e_k, e_{k+1}, f_1, \dots, f_k], \\ &E_1 = [e_1, \dots, e_k, e_{k+1}], \quad E_2 = [f_1, \dots, f_k, e_{k+1}], \\ &E_3 = [e_1, (e_2 + f_1), \dots, (e_{k+1} + f_k)], \\ &E_4 = [f_1, (e_1 + f_2), \dots, (e_{k-1} + f_k), (e_k + e_{k+1})]. \end{aligned}$$

We put $\mathcal{S}_i(2k+1,-1) = \sigma_{1,i}\mathcal{S}_1(2k+1,-1)$, $\mathcal{S}_i(2k+1,+1) = \sigma_{2,i}\mathcal{S}_2(2k+1,1)$, $\mathcal{S}_{i,j}(2k+1,0) = \sigma_{1,i}\sigma_{3,j}\mathcal{S}_{1,3}(2k+1,0)$ for $i, j \in \{1,2,3,4\}$.

Theorem 9.2 (Gelfand-Ponomarev [GP])**.** *If a system $\mathcal{S}$ of four subspaces in a finite-dimensional H is indecomposable, then $\mathcal{S}$ is isomorphic to one of the following systems:*
$\mathcal{S}_{i,j}(m,0)$, $(i < j, i, j \in \{1,2,3,4\}, m = 1,2,...)$; $\mathcal{S}(2k,0;\lambda)$, $(\lambda \in \mathbb{C}, \lambda \neq 0, \lambda \neq 1, k = 1,2,...)$, $\mathcal{S}_i(m,-1)$, $\mathcal{S}_i(m,1)$, $(i \in \{1,2,3,4\}, m = 1,2,...)$; $\mathcal{S}(2k+1,-2)$, $\mathcal{S}(2k+1,+2)$, $(k = 0,1,...)$.

Remark. It is known that if $\mathcal{S}$ is an indecomposable system of four subspaces in the above Theorem satisfying $\rho(\mathcal{S}) \neq 0$, then $\mathcal{S}$ is transitive, for example, see [B].

10. Exotic indecomposable systems of four subspaces

In this section we shall construct uncountably many, exotic, indecomposable systems of four subspaces, that is, indecomposable systems which are not isomorphic to any closed operator system under any permutaion of subspaces.

Exotic examples. Let $L = \ell^2(\mathbb{N})$ with a standard basis $\{e_1, e_2, \dots\}$. Put $K = L \oplus L$ and $H = K \oplus K = L \oplus L \oplus L \oplus L$. Consider a unilateral shift $S : L \to L$ by $Se_n = e_{n+1}$ for $n = 1, 2, \dots$. For a fixed paramater $\gamma \in \mathbb{C}$ with $|\gamma| \geq 1$, we consider an operator

$$T_\gamma = \begin{pmatrix} \gamma S^* & I \\ 0 & S \end{pmatrix} \in B(K) = B(L \oplus L).$$

Let $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$,
$E_3 = \{(x, T_\gamma x) \in K \oplus K; x \in K\} + \mathbb{C}(0,0,0,e_1) = \operatorname{graph} T_\gamma + \mathbb{C}(0,0,0,e_1)$, and $E_4 = \{(x,x) \in K \oplus K; x \in K\}$. Consider a system $\mathcal{S}_\gamma = (H; E_1, E_2, E_3, E_4)$. We shall show that $\mathcal{S}_\gamma$ is indecomposable. If $|\gamma| > 1$, then $\mathcal{S}_\gamma$ is not isomorphic to any closed operator systems under any permutation. We could regard the system $\mathcal{S}_\gamma$ is a one-dimensional "deformation" of an operator system. First we start with an easy fact.

Lemma 10.1. *Assume that a bounded operator $A \in B(\ell^2(\mathbb{N}))$ is represented as an upper triangular matrix $A = (a_{ij})_{ij}$ by a standard basis $\{e_1, e_2, \dots\}$. If the diagonal is constant λ, i.e., $a_{ii} = \lambda$ for $i = 1, \dots,$ and A is an idempotent, then $A = 0$ or $A = I$.*

Proof. Put $N = A - \lambda I$. Then N is an upper triangular matrix with zero diagonal. Comparing the diagonals for

$$\lambda I + N = A = A^2 = \lambda^2 I + 2\lambda N + N^2,$$

we have $\lambda^2 = \lambda$. Hence $\lambda = 0$ or 1. If $\lambda = 0$, then $N^2 = N$. Since N is an idempotent and an upper triangular matrix with zero diagonal, $N = 0$, that is, $A = 0$. If $\lambda = 1$, then $(I - A)$ is an idempotent and an upper triangular matrix with zero diagonal, $I - A = 0$, that is, $A = I$. □

Theorem 10.2. *If $|\gamma| \geq 1$, then the above system $\mathcal{S}_\gamma = (H; E_1, E_2, E_3, E_4)$ is indecomposable.*

Proof. We shall show that $\{V \in End(\mathcal{S}_\gamma); V^2 = V\} = \{0, I\}$. Let $V \in End(\mathcal{S}_\gamma)$ satisfy $V^2 = V$. Since $V(E_i) \subset E_i$ for $i = 1, 2, 4$, we have

$$V = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \in B(H) \quad \text{for some } \; U \in B(K)$$

We write

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in B(K),$$

for some $A = (a_{ij})_{ij},\ B = (b_{ij})_{ij},\ C = (c_{ij})_{ij},\ D = (d_{ij})_{ij} \in B(K)$. We shall investigate the condition that $V(E_3) \subset E_3$. Since $E_3 = \operatorname{graph} T_\gamma + \mathbb{C}(0, 0, 0, e_1)$, E_3 is spanned by

$$\{ \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} e_m \\ 0 \\ \gamma e_{m-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_n \\ e_n \\ e_{n+1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_1 \end{pmatrix}; m = 2, 3, \ldots, n = 1, 2, \ldots \}.$$

We may write

$$E_3 = \{ \begin{pmatrix} (\lambda_n)_n \\ (\mu_n)_n \\ (\gamma\lambda_{n+1} + \mu_n)_n \\ (\alpha, (\mu_n)_n) \end{pmatrix}; \lambda_n, \mu_n, \alpha \in \mathbb{C}, \sum_n |\lambda_n|^2 < \infty, \sum_n |\mu_n|^2 < \infty \}$$

Since $(e_1, 0, 0, 0) \in E_3$, we have

$$\begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & D \end{pmatrix} \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} Ae_1 \\ Ce_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (\lambda_m)_m \\ (\mu_m)_m \\ (\gamma\lambda_{m+1} + \mu_m)_m \\ (\alpha, (\mu_m)_m) \end{pmatrix} \in E_3.$$

Then, for any $m = 1, 2, \ldots$, we have $c_{m1} = \mu_m = 0$. Moreover $0 = \gamma\lambda_{m+1} + \mu_m = \gamma\lambda_{m+1}$. Hence $\lambda_{m+1} = 0$ because $\gamma \neq 0$. Therefore $a_{m+1,1} = \lambda_{m+1} = 0$. Thus the first column of C is zero and the first column of A is zero except a_{11}. We shall show that $C = 0$ and A is an upper triangular Toeplitz matrix with by the induction of n-th columns. !!The case when $n = 1$ is already shown. Assume that the assertion hold for n-th columns. Since $(e_{n+1}, 0, \gamma e_n, 0) \in E_3$, we have

$$\begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & D \end{pmatrix} \begin{pmatrix} e_{n+1} \\ 0 \\ \gamma e_n \\ 0 \end{pmatrix} = \begin{pmatrix} Ae_{n+1} \\ Ce_{n+1} \\ \gamma Ae_n \\ \gamma Ce_n \end{pmatrix} = \begin{pmatrix} (\lambda_m)_m \\ (\mu_m)_m \\ (\gamma\lambda_{m+1} + \mu_m)_m \\ (\alpha, (\mu_m)_m) \end{pmatrix} \in E_3.$$

Then $c_{m,n+1} = \mu_m = \gamma c_{m+1,n} = 0$. And $\gamma a_{m,n} = \gamma\lambda_{m+1} + \mu_m = \gamma\lambda_{m+1}$. Since $\gamma \neq 0$, $a_{m,n} = \lambda_{m+1} = a_{m+1,n+1}$. Thus we have shown that $C = 0$ and A is an upper triangular Toeplitz matrix. Since V is an idempotent, so is

$$U = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Hence A is also an idempotent. By Lemma 10.1, we have two cases $A = 0$ or $A = I$.
(i)the case $A = 0$: we shall show that $B = D = 0$. This immediately implies $U = 0$, so that $V = 0$.
(ii)the case $A = I$: Since $I - V \in End(\mathcal{S}_\gamma)$ is is also an idempotent and it can be reduced to the case (i) and we have $V = I$.

Hence we may assume that $A = 0$. Since U is an idempotent, D is also an idempotent. Since $(0, 0, 0, e_1) \in E_3$, we have

$$\begin{pmatrix} 0 & B & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & 0 & D \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ Be_1 \\ De_1 \end{pmatrix} = \begin{pmatrix} (\lambda_m)_m \\ (\mu_m)_m \\ (\gamma\lambda_{m+1} + \mu_m)_m \\ (\alpha, (\mu_m)_m) \end{pmatrix} \in E_3.$$

Then, for any $m = 1, 2, \ldots$, we have $\mu_m = \lambda_m = 0$. Hence $b_{m1} = \gamma\lambda_{m+1} + \mu_m = 0$ and $d_{m+1,1} = \mu_m = 0$. Thus the first column of B is zero and the first column of D is zero except d_{11}. We shall show that D is an upper triangular Toeplitz matrix by the induction of $n-$th columns. !!The case when $n = 1$ is already shown. Assume that the assertion hold for $n-$th columns. Since $(0, e_n, e_n, e_{n+1}) \in E_3$,

$$\begin{pmatrix} 0 & B & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & 0 & D \end{pmatrix} \begin{pmatrix} 0 \\ e_n \\ e_n \\ e_{n+1} \end{pmatrix} = \begin{pmatrix} Be_n \\ De_n \\ Be_{n+1} \\ De_{n+1} \end{pmatrix} = \begin{pmatrix} (\lambda_m)_m \\ (\mu_m)_m \\ (\gamma\lambda_{m+1} + \mu_m)_m \\ (\alpha, (\mu_m)_m) \end{pmatrix} \in E_3.$$

We have $d_{m+1,n+1} = \mu_m = d_{mn}$. Hence D is an upper triangular Toeplitz matrix. Since D is also an idempotent, $D = O$ or $D = I$ by Lemma 10.1.

If $D = 0$, then $U = U^2 = 0$. Thus $B = 0$, and the assertion is verified. We shall show that the case when $D = I$ will not occur. On the contrary, suppose that $D = I$. We have

$$V\begin{pmatrix}0\\0\\0\\e_1\end{pmatrix} = \begin{pmatrix}0 & B & 0 & 0\\0 & I & 0 & 0\\0 & 0 & 0 & B\\0 & 0 & 0 & I\end{pmatrix}\begin{pmatrix}0\\0\\0\\e_1\end{pmatrix} = \begin{pmatrix}0\\0\\Be_1\\e_1\end{pmatrix} = \begin{pmatrix}(\lambda_m)_m\\(\mu_m)_m\\(\gamma\lambda_{m+1}+\mu_m)_m\\(\alpha,(\mu_m)_m)\end{pmatrix} \in E_3.$$

Then, for any $m = 1, 2, \ldots$, we have $\mu_m = \lambda_m = 0$. Hence $b_{m1} = \gamma\lambda_{m+1} + \mu_m = 0$ Thus the first column of B is zero. We shall show that B should be the following form by the induction of $n-$th columns:

$$B = \begin{pmatrix}
0 & 1 & 0 & \gamma & 0 & \gamma^2 & 0 & \gamma^3 & 0 & \cdots\\
0 & 0 & 1 & 0 & \gamma & 0 & \gamma^2 & 0 & \gamma^3 & \cdots\\
0 & 0 & 0 & 1 & 0 & \gamma & 0 & \gamma^2 & 0 & \ddots\\
0 & 0 & 0 & 0 & 1 & 0 & \gamma & 0 & \gamma^2 & \ddots\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \gamma & 0 & \ddots\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \gamma & \ddots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ddots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ddots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix},$$

that is, $b_{ij} = \gamma^{k-1}$ if $j > i$ and $j - i = 2k - 1$, and $b_{ij} = 0$ if otherwise.

The case when $n = 1$ is already shown. Assume that the assertion hold for n-th columns. Since

$$\begin{pmatrix}0 & B & 0 & 0\\0 & I & 0 & 0\\0 & 0 & 0 & B\\0 & 0 & 0 & I\end{pmatrix}\begin{pmatrix}0\\e_n\\e_n\\e_{n+1}\end{pmatrix} = \begin{pmatrix}Be_n\\e_n\\Be_{n+1}\\e_{n+1}\end{pmatrix} = \begin{pmatrix}(\lambda_m)_m\\(\mu_m)_m\\(\gamma\lambda_{m+1}+\mu_m)_m\\(\alpha,(\mu_m)_m)\end{pmatrix} \in E_3,$$

for any $m = 1, 2, \ldots$, we have $\mu_m = \delta_{m,n}$. And

$$b_{m,n+1} = \gamma\lambda_{m+1} + \mu_m = \gamma\lambda_{m+1} + \delta_{m,n},$$

that is,

$$((n+1)\text{-th column of } B) = \gamma S^*(n\text{ -th column of } B) + e_n.$$

By the induction we have shown that B is the above form. But then

$$\|B^*e_1\|^2 = \|(\text{the first row of } B)\|^2 = \sum_{k=1}^{\infty} |\gamma|^{2(k-1)} = \infty,$$

because $|\gamma| \geq 1$. This contradicts to that B is bounded. Therefore $D \neq I$. This finishes the proof. $\square$

Theorem 10.3. *If $|\beta| \geq 1$, $|\gamma| \geq 1$ and $|\beta| \neq |\gamma|$, then the above systems $\mathcal{S}_\beta = (H; E_1, E_2, E_3^\beta, E_4)$ and $\mathcal{S}_\gamma = (H; E_1, E_2, E_3^\gamma, E_4)$ are not isomorphic.*

Proof. On the contrary, suppose that there were an isomorphism $V : \mathcal{S}_\beta \to \mathcal{S}_\gamma$. We shall show a contradiction. We may and do assume that $|\beta| > |\gamma|$. Since $V(E_i) = E_i$ for $i = 1, 2, 4$, we have

$$V = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \in B(H) \quad \text{for some invertible } \ U \in B(K)$$

We write

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in B(K),$$

for some $A = (a_{ij})_{ij},\ B = (b_{ij})_{ij},\ C = (c_{ij})_{ij},\ D = (d_{ij})_{ij} \in B(K)$. We shall investigate the condition that $V(E_3^\beta) = E_3^\gamma$. Since $E_3^\beta = \operatorname{graph} T_\beta + \mathbb{C}(0,0,0,e_1)$, E_3^β is spanned by

$$\{ \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} e_m \\ 0 \\ \beta e_{m-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_n \\ e_n \\ e_{n+1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_1 \end{pmatrix} ; m = 2, 3, \dots, n = 1, 2, \dots \}.$$

We also write

$$E_3^\gamma = \{ \begin{pmatrix} (\lambda_n)_n \\ (\mu_n)_n \\ (\gamma\lambda_{n+1} + \mu_n)_n \\ (\alpha, (\mu_n)_n) \end{pmatrix} ; \lambda_n, \mu_n, \alpha \in \mathbb{C}, \sum_n |\lambda_n|^2 < \infty, \sum_n |\mu_n|^2 < \infty \}.$$

Since $(e_1, 0, 0, 0) \in E_3^\beta$, we have

$$0 \neq \begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & D \end{pmatrix} \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} Ae_1 \\ Ce_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (\lambda_m)_m \\ (\mu_m)_m \\ (\gamma\lambda_{m+1} + \mu_m)_m \\ (\alpha, (\mu_m)_m) \end{pmatrix} \in E_3^\gamma.$$

Then, for any $m = 1, 2, \dots$, we have $c_{m1} = \mu_m = 0$. Moreover $0 = \gamma\lambda_{m+1} + \mu_m = \gamma\lambda_{m+1}$. Hence $\lambda_{m+1} = 0$ because $\gamma \neq 0$. Therefore $a_{m+1,1} = \lambda_{m+1} = 0$. Thus the first column of C is zero and the first column of A is zero except a_{11}. Since $Ae_1 \neq 0$, $a_{11} \neq 0$. We shall show that $C = 0$ and A is an upper triangular matrix satisfying

$$a_{i+1,j+1} = \frac{\beta}{\gamma} a_{ij} \quad \text{if } i \leq j$$

and $a_{ij} = 0$ if $i > j$, by the induction of n-th columns. !! The case when $n = 1$ is already shown. Assume that the assertion hold for n-th columns. Since $(e_{n+1}, 0, \beta e_n, 0) \in E_3^\beta$, we have

$$\begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & D \end{pmatrix} \begin{pmatrix} e_{n+1} \\ 0 \\ \beta e_n \\ 0 \end{pmatrix} = \begin{pmatrix} Ae_{n+1} \\ Ce_{n+1} \\ \beta Ae_n \\ \beta Ce_n \end{pmatrix} = \begin{pmatrix} (\lambda_m)_m \\ (\mu_m)_m \\ (\gamma\lambda_{m+1} + \mu_m)_m \\ (\alpha, (\mu_m)_m) \end{pmatrix} \in E_3^\gamma.$$

Then we have $c_{m,n+1} = \mu_m = \beta c_{m+1,n} = 0$. Moreover

$$\beta a_{m,n} = \gamma\lambda_{m+1} + \mu_m = \gamma\lambda_{m+1} = \gamma a_{m+1,n+1}.$$

Since $\gamma \neq 0$, $a_{m+1,n+1} = \frac{\beta}{\gamma} a_{m,n}$. This completes the induction. Then we have

$$|a_{nn}| = |\frac{\beta}{\gamma}|^{n-1}|a_{11}| \to \infty,$$

because $a_{11} \neq 0$ and $|\frac{\beta}{\gamma}| > 1$. But This contradicts to that the operator A is bounded. Therefore $\mathcal{S}_\beta$ and $\mathcal{S}_\gamma$ are not isomorphic. $\square$

Next we shall show that if $\gamma > 1$, then $\mathcal{S}_\gamma$ is not isomorphic to any closed operator system. We introduce a necessary criterion for the purpose.

Definition(intersection diagram) Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of fours subspaces. The *intersection diagram* for a system $\mathcal{S}$ is an undirected graph $\Gamma_\mathcal{S} = (\Gamma^0_\mathcal{S}, \Gamma^1_\mathcal{S})$ with the set of vertices $\Gamma^0_\mathcal{S}$ and the set of edges $\Gamma^1_\mathcal{S}$ defined by $\Gamma^0_\mathcal{S} = \{1, 2, 3, 4\}$ and for $i \neq j \in \{1, 2, 3, 4\}$

$$\circ_i \text{ --- } \circ_j \quad \text{if and only if } E_i \cap E_j = 0.$$

Lemma 10.4. *Let $\mathcal{S} = \mathcal{S}_{T,S} = (H; E_1, E_2, E_3, E_4)$ be a closed operator system. Then the intersection diagram $\Gamma_\mathcal{S}$ for the system $\mathcal{S}$ contains*

$$\circ_4 \text{ --- } \circ_1 \text{ --- } \circ_2 \text{ --- } \circ_3 \ ,$$

that is, $E_4 \cap E_1 = 0$, $E_1 \cap E_2 = 0$ and $E_2 \cap E_3 = 0$. In particular, then the intersection diagram $\Gamma_\mathcal{S}$ is a connected graph.

Proof. It follows form Proposition 8.9. $\square$

Proposition 10.5. *If $\gamma > 1$, then the system $\mathcal{S}_\gamma$ is not isomorphic to any closed operator system under any permutation of subspaces.*

Proof. It is clear that $E_4 \cap E_1 = 0$, $E_1 \cap E_2 = 0$ and $E_2 \cap E_4 = 0$. Since $(e_1, 0, 0, 0) \in E_1 \cap E_3$, we have $E_1 \cap E_3 \neq 0$. Because $(0, 0, 0, e_4) \in E_2 \cap E_3$, we have $E_2 \cap E_3 \neq 0$. Since $|\gamma| > 1$, $a := (1, \gamma^{-1}, \gamma^{-2}, \gamma^{-3}, ...,) \in \ell^2(\mathbb{N})$. Then $(a, 0, a, 0) \in E_3 \cap E_4$, so that $E_3 \cap E_4 \neq 0$. Therefore the vertex 3 is not connected to any other vertices $1, 2, 4$. Thus the intersection diagram $\Gamma_\mathcal{S}$ is not a connected graph. This implies that $\mathcal{S}_\gamma$ is not isomorphic to any closed operator system under any permutation of subspaces. $\square$

Combining the preceeding two propositions , we have the existence of uncountably many, exotic, indecomposable systems of four subspaces.

Theorem 10.6. *There exists uncountably many, indecomposable systems of four subspaces which are not isomorphic to any closed operator system under any permutation of subspaces.*

Proof. A family $\{\mathcal{S}_\gamma; \gamma > 1, \gamma \in \mathbb{R}\}$ of indecomposable systems above is a desired one. $\square$

11. Defects for systems of four subspaces.

Gelfand and Ponomarev introduced an integer valued invariant $\rho(\mathcal{S})$, called *defect*, for a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces by

$$\rho(\mathcal{S}) = \sum_{i=1}^{4} \dim E_i - 2\dim H.$$

They showed that if a system of four subspaces is indecomposable, then the possible value of the defect $\rho(\mathcal{S})$ is one of five values $\{-2, -1, 0, 1, 2\}$ We shall extend their notion of defect for a certain class of systems relating with Fredholm index.

Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. We first introduce elementary numerical invariants

$$m_{ij} = \dim(E_i \cap E_j) \text{ and } m_{ijk} = \dim(E_i \cap E_j \cap E_k).$$

Similarly put

$$n_{ij} = \dim((E_i + E_j)^\perp) \text{ and } n_{ijk} = \dim((E_i + E_j + E_k)^\perp).$$

If $\mathcal{S}$ is indecomposable and $\dim H \geq 2$, then $m_{ijk} = 0$ and $n_{ijk} = 0$ by Proposition 6.1.

If H is finite dimensional, then

$$\begin{aligned}
&\dim E_i + \dim E_j - \dim H \\
&= \dim(E_i + E_j) + \dim(E_i \cap E_j) - (\dim(E_i + E_j) + \dim((E_i + E_j)^\perp)) \\
&= \dim(E_i \cap E_j) - \dim((E_i + E_j)^\perp)
\end{aligned}$$

In order to make the numerical invariant unchanged under any permutation of subspaces, counting ${}_4C_2 = 6$ pairs of subspaces

$$(E_1, E_2), (E_1, E_3), (E_1, E_4), (E_2, E_3), (E_2, E_4), (E_3, E_4),$$

we have the following expression of the defect:

$$\begin{aligned}
\rho(\mathcal{S}) &= \sum_{i=1}^{4} \dim E_i - 2\dim H \\
&= \frac{1}{3} \sum_{1 \leq i < j \leq 4} (\dim E_i + \dim E_j - \dim H) \\
&= \frac{1}{3} \sum_{1 \leq i < j \leq 4} (\dim(E_i \cap E_j) - \dim((E_i + E_j)^\perp)).
\end{aligned}$$

Definition Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. For any distinct $i, j = 1, 2, 3, 4$, define an adding operator

$$A_{ij} : E_i \oplus E_j \ni (x, y) \to x + y \in H.$$

Then

$$\operatorname{Ker} A_{ij} = \{(x, -x) \in E_i \oplus E_j; x \in E_i \cap E_j\}$$

and

$$\operatorname{Im} A_{ij} = E_i + E_j.$$

We say $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ is a *Fredholm* system if A_{ij} is a Fredholm operator for any $i, j = 1, 2, 3, 4$ with $i \neq j$. Then $\operatorname{Im} A_{ij} = E_i + E_j$ is closed and

$$\operatorname{Index} A_{ij} = \dim \operatorname{Ker} A_{ij} - \dim \operatorname{Ker} A_{ij}^* = \dim(E_i \cap E_j) - \dim((E_i + E_j)^\perp).$$

T. Kato called the number $dim(E_i \cap E_j) - \dim((E_i + E_j)^\perp)$ the index of the pair E_i, E_j in ([K];IV section 4).

Definition We say $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ is a *quasi-Fredholm* system if $E_i \cap E_j$ and $(E_i + E_j)^\perp$ are finite-dimensional for any $i \neq j$. In the case we define the *defect* $\rho(\mathcal{S})$ of $\mathcal{S}$ by

$$\begin{aligned}\rho(\mathcal{S}) &:= \frac{1}{3} \sum_{1 \leq i < j \leq 4} (\dim(E_i \cap E_j) - \dim(E_i + E_j)^\perp)) \\ &= \frac{1}{3} \sum_{1 \leq i < j \leq 4} (\dim(E_i \cap E_j) - \operatorname{codim} \overline{E_i + E_j})\end{aligned}$$

which coincides with the Gelfand-Ponomarev original defect if H is finite-dimensional. Moreover, if $\mathcal{S}$ is a Fredholm system, then it is a quasi-Fredholm system and

$$\rho(\mathcal{S}) = \frac{1}{3} \sum_{1 \leq i < j \leq 4} \operatorname{Index} A_{ij}.$$

Proposition 11.1. *Let $\mathcal{S}_T = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system associated with a single operator $T \in B(K)$. Then $\mathcal{S}_T$ is a Fredholm system if and only if T and $T - I$ are Fredholm operators. If the condition is satisfied, then the defect is given by*

$$\rho(\mathcal{S}_T) = \frac{1}{3}(\operatorname{Index} T + \operatorname{Index}(T - I))$$

Similarly $\mathcal{S}_T$ is a quasi-Fredholm system if and only if $\operatorname{Ker} T$*,* $\operatorname{Ker} T^*$*,* $\operatorname{Ker}(T - I)$ *and* $\operatorname{Ker}(T - I)^*$ *are finte-dimensional. If the condition is satisfied, then the defect is given by*

$$\rho(\mathcal{S}_T) = \frac{1}{3}(\dim \operatorname{Ker} T - \dim \operatorname{Ker} T^* + \dim \operatorname{Ker}(T - I) - \dim \operatorname{Ker}(T - I)^*)$$

Proof. It is clear that $E_i \cap E_j = 0$ and $E_i + E_j = H$ for $(i, j) = (1, 2), (1, 4), (2, 4), (2, 3)$. Since $\operatorname{Ker} A_{13} = E_1 \cap E_3 = \operatorname{Ker} T \oplus 0$ and $(\operatorname{Im} A_{13})^\perp = (E_1 + E_3)^\perp = (K \oplus \operatorname{Im} T)^\perp$, they are finite-dimensional if and only if $\operatorname{Ker} T$ and $(\operatorname{Im} T)^\perp = \operatorname{Ker} T^*$ are finite-dimensional. And $\operatorname{Im} A_{13}$ is closed if and only if $\operatorname{Im} T$ is closed. We transform E_3 and E_4 by an invertible operator $R = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \in B(H) = B(K \oplus K)$, then

$R(E_3) = \{(x,(T-I)x) \in\ K \oplus K; x \in K\}$ and $R(E_4) = K \oplus 0$. Hence $R(E_3 \cap E_4) = \mathrm{Ker}(T-I) \oplus 0$ and $R(E_3+E_4) = K \oplus \mathrm{Im}(T-I)$. Then

$$\begin{aligned}\dim((E_3+E_4)^{\perp}) &= \mathrm{codim}\ \overline{E_3+E_4} \\ &= \mathrm{codim}\ \overline{R(E_3)+R(E_4))} = dim((R(E_3+E_4))^{\perp})\end{aligned}$$

Thus $E_3 \cap E_4$ and $(E_3+E_4)^{\perp}$ are finite-dimensional if and only if $\mathrm{Ker}(T-I)$ and $(\mathrm{Im}(T-I))^{\perp} = \mathrm{Ker}(T-I)^*$ are finite-dimensional. And $\mathrm{Im}\, A_{13} = E_3 + E_4$ is closed if and only if $Im(T-I)$ is closed. It follows the desired conclusion. □

We shall show that the defect could have a fractional value.
Example. Let S be a unilateral shift on $K = \ell^2(\mathbb{N})$. Then the operator system $\mathcal{S}_S$ is an indecomposable. It is not a Fredholm system but a quasi-Fredholm system and $\rho(\mathcal{S}_S) = -\frac{1}{3}$. The operator system $\mathcal{S}_{S+\frac{1}{2}I}$ is a Fredholm system and $\rho(\mathcal{S}_{S+\frac{1}{2}I}) = -\frac{2}{3}$. Moreover $(\mathcal{S}_{T+\alpha I})_{\alpha\in\mathbb{C}}$ is uncountable family of indecomposable , quasi-Fredholm systems. Fredholm systems among them and their defect are given by

$$\rho(\mathcal{S}_{S+\alpha I}) = \begin{cases} -\frac{2}{3}, & (|\alpha| < 1 \text{ and } |\alpha - 1| < 1) \\ -\frac{1}{3}, & (|\alpha| < 1 \text{ and } |\alpha - 1| > 1) \text{ or } (|\alpha| > 1 \text{ and } |\alpha - 1| < 1) \\ 0, & (|\alpha| > 1 \text{ and } |\alpha - 1| > 1). \end{cases}$$

Corollary 11.2. *Let $\mathcal{S}_T = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system associated with a single operator $T \in B(K)$. If $\mathcal{S}_T$ is a Fredholm system, then $\mathcal{S}_{T^*}$ is a Fredholm system and $\rho(\mathcal{S}_{T^*}) = -\rho(\mathcal{S}_T)$. Similarly If $\mathcal{S}_T$ is a quasi-Fredholm system then $\mathcal{S}_{T^*}$ is a quasi-Fredholm system and $\rho(\mathcal{S}_{T^*}) = -\rho(\mathcal{S}_T)$.*

Proof. Use the fact that T is Fredholm if and only if T^* is a Fredholm, and then $\mathrm{Index}\, T^* = -\mathrm{Index}\, T$. □

Proposition 11.3. *Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. If $\mathcal{S}$ is a Fredholm system, then the orthogonal complement $\mathcal{S}^{\perp} = (H; E_1^{\perp}, E_2^{\perp}, E_3^{\perp}, E_4^{\perp})$ is a Fredholm system and $\rho(\mathcal{S}^{\perp}) = -\rho(\mathcal{S})$. Similarly if $\mathcal{S}$ is a quasi-Fredholm system then $\mathcal{S}^{\perp}$ is a quasi-Fredholm system and $\rho(\mathcal{S}^{\perp}) = -\rho(\mathcal{S})$.*

Proof. Recall elementary facts that $E_i^{\perp} \cap E_j^{\perp} = (E_i + E_j)^{\perp}$ and $(E_i^{\perp} + E_j^{\perp})^{\perp} = E_i \cap E_j$. The only non-trivial thing is to know that $E_i + E_j$ is closed if and only if $E_i^{\perp} + E_j^{\perp}$ is closed, see, for example, ([K];IV Theorem 4.8). □

Example. For $\gamma \in \mathbb{C}$ with $|\gamma| \geq 1$, let $\mathcal{S}_\gamma = (H; E_1, E_2, E_3, E_4)$ be an exotic system of four subspaces in Theorem 10.2. Then $\mathcal{S}_\gamma$ is a quasi-Fredholm system and

$$\rho(\mathcal{S}_\gamma) = \frac{1}{3}(\mathrm{Index}\, A_{13} + \mathrm{Index}\, A_{23} + \mathrm{Index}\, A_{34}) = \frac{1}{3}(1+1+1) = 1.$$

In fact, $E_1 \cap E_3 = \mathbb{C}(e_1, 0, 0, 0)$, $E_2 \cap E_3 = \mathbb{C}(0, 0, 0, e_1)$ and $E_4 \cap E_3 = \mathbb{C}(a, 0, a, 0)$, where $a = (\gamma^{n-1})_n \in L = \ell^2(\mathbb{N})$. All the other terms are zeros.

Definition. Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. We say that $\mathcal{S}$ is *non-degenerate* if $E_i + E_j = H$ and $E_i \cap E_j = 0$ for $i \neq j$. Then $\mathcal{S}$ is clearly a Fredholm system with the defect $\rho(\mathcal{S}) = 0$. Thus the defect measures the failure from being non-degenerate.

Proposition 11.4. *Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. Then $\mathcal{S}$ is non-degenerate if and only if $\mathcal{S}^\perp$ is non-degenerate.*

Proof. It follows from the fact that $E_i + E_j = H$ if and only if $E_i^\perp \cap E_j^\perp = 0$. □

Proposition 11.5. *Let $\mathcal{S}_{T,S}$ be a bounded operator system. Then $\mathcal{S}_{T,S}$ is a Fredholm system if and only if S, T and $ST - I$ are Fredholm operators. And if the condition is satisfied, then*

$$\rho(\mathcal{S}_{T,S}) = \frac{1}{3}(\operatorname{Index} T + \operatorname{Index} S + \operatorname{Index}(ST - I)).$$

Proof. It is clear that $E_i \cap E_j = 0$ and $E_i + E_j = H$ for $(i, j) = (1, 2), (1, 4), (2, 3)$. Since $\operatorname{Ker} A_{13} = E_1 \cap E_3 = \operatorname{Ker} T \oplus 0$ and $(\operatorname{Im} A_{13})^\perp = (E_1 + E_3)^\perp = (K_1 \oplus \operatorname{Im} T)^\perp$, they are finite-dimensional if and only if $\operatorname{Ker} T$ and $(\operatorname{Im} T)^\perp = \operatorname{Ker} T^*$ are finite-dimensional. And $\operatorname{Im} A_{13}$ is closed if and only if $\operatorname{Im} T$ is closed. Similarly $\operatorname{Ker} A_{24} = E_2 \cap E_4 = 0 \oplus \operatorname{Ker} S$ and $(\operatorname{Im} A_{24})^\perp = (E_2 + E_4)^\perp = (\operatorname{Im} S \oplus K_2)^\perp$. Hence they are finite-dimensional if and only if $\operatorname{Ker} S$ and $(\operatorname{Im} S)^\perp = \operatorname{Ker} S^*$ are finite-dimensional. And $\operatorname{Im} A_{24}$ is closed if and only if $\operatorname{Im} S$ is closed. Nextly,

$$\operatorname{Ker} A_{34} = E_3 \cap E_4 = \{(x, Tx) \in K_1 \oplus K_2; x \in \operatorname{Ker}(ST - I)\}.$$

$$\operatorname{Im} A_{34} = \{\begin{pmatrix} x + Sy \\ Tx + y \end{pmatrix}; x \in K_1, y \in K_2\} = \begin{pmatrix} I & S \\ T & I \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}; x \in K_1, y \in K_2\}.$$

Multiplying invertible operator matrices from both sides, we have

$$\begin{pmatrix} I & -S \\ 0 & I \end{pmatrix}\begin{pmatrix} I & S \\ T & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -T & I \end{pmatrix} = \begin{pmatrix} I - ST & 0 \\ 0 & I \end{pmatrix}.$$

Hence $\operatorname{Im} A_{34}$ is closed if and only if $\operatorname{Im}(ST - I)$ is closed, and $(\operatorname{Im} A_{34})^\perp$ is finite-dimensional if and only if $(\operatorname{Im}(ST - I))^\perp$ is finite-dimensional. Now it is easy to see the desired conclusons. □

Let $\mathcal{S}$ and $\mathcal{S}'$ be two quasi-Fredholm systems of four subspaces. Then it is evident that $\mathcal{S} \oplus \mathcal{S}'$ is also a quasi-Fredholm system and

$$\rho(\mathcal{S} \oplus \mathcal{S}') = \rho(\mathcal{S}) + \rho(\mathcal{S}').$$

Therefore we should investigate the possible values of the defect for indecomposable systems.

Theorem 11.6. *The set of the possible values of the defect of indecomposable systems of four subspaces is exactly* $\mathbb{Z}/3$

Proof. Let S be a unilateral shift on $L = \ell^2(\mathbb{N})$. Let $K = L \otimes \mathbb{C}^n$ and $H = K \oplus K$. For a positive integer n, put

$$V = \begin{pmatrix} S & 0 & 0 & \cdots & 0 \\ I & S & 0 & \cdots & 0 \\ 0 & I & S & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I & S \end{pmatrix} \in M_n(\mathbb{C}) \otimes B(L) = B(K).$$

Let $\mathcal{S}_V = (H; E_1, E_2, E_3, E_4)$ be the operator system associated with a single operator V. We shall show that $\mathcal{S}_V$ is indecomposable. Let $T = (T_{ij})_{ij} \in B(K)$ be an idempotent which commutes with V. It is enough to show that $T = 0$ or $T = I$.

Since $VT = TV$, we have

$$ST_{11} = T_{11}S + T_{12}, \ \ldots \ ST_{1(n-1)} = T_{1(n-1)}S + T_{1n}, \ T_{1n}S = ST_{1n}.$$

By the Kleinecke-Shirokov theorem, T_{1n} is a quasinilpotent. Since T_{1n} commutes with a unilateral shift S, T_{1n} is a Toeplitz operator. Then $\|T_{1n}\| = r(T_{1n}) = 0$. Thus $T_{1n} = 0$ by [Ha3]. Inductively we can show that $T_{12} = T_{13} = \cdots = T_{1n} = 0$. Similar argument shows that T is a lower triangular operator matrix, i.e., $T_{ij} = 0$ for $i < j$. Since $T^2 = T$, we have $T_{ii}^2 = T_{ii}$ for $i = 1, \cdots, n$. The diagonal of $VT = TV$ shows that each T_{ii} commutes with a unilatral shift S. This implies that $T_{ii} = 0$ or I as in Lemma 10.1.
(i)the case that $T_{11} = 0$: The 2-1th component of $VT = TV$ shows that $T_{22} = ST_{21} - T_{21}S$. Hence T_{22} cannot be I. Thus $T_{22} = 0$. Similarly we can show that $T_{ii} = 0$ for $i = 1, \ldots, n$. Thus the diagonal of operator matrix T is zero. Furthermore T is a lower triangular operator matrix and idempotent. Hence $T = O$.
(ii) the case that $T_{11} = I$: Considering $I - T$ instead of T, we can use the case (i) and shows that $T = I$. Therefore $\mathcal{S}_V$ is indecomposable.

The defect is given by

$$\begin{aligned}\rho(\mathcal{S}_V) &= \frac{1}{3}(\dim \operatorname{Ker} V - \dim \operatorname{Ker} V^* + \dim \operatorname{Ker}(V - I) - \dim \operatorname{Ker}(V - I)^*) \\ &= \frac{1}{3}(0 - n + 0 - 0) = \frac{-n}{3}.\end{aligned}$$

In fact,

$$\operatorname{Ker} V^* = \{(a, -S^*a, (-S^*)^2 a, \ldots, (-S^*)^{n-1}) \in (\ell^2(\mathbb{N}))^n; a \in \operatorname{Ker} S^{*n}\}$$

is n-dimensional.

Similarly $\mathcal{S}_{V^*}$ is an indecomposable system with $\rho(\mathcal{S}_{V^*}) = \frac{n}{3}$.

For $n = 0$, consider an indecomposable system $\mathcal{S}_{S+3I}$ as in Example after Proposition 11.1. Then $\rho(\mathcal{S}_{S+3I}) = 0$.

Therefore the defect for indecomposable systems of four subspaces can take any value in $\mathbb{Z}/3$. $\square$

Remark. Indecomposablity of the system $\mathcal{S}_V$ can also be derived by Theorem 3.4 in [JW], although we give our direct proof.

Corollary 11.7. *For any $n \in \mathbb{Z}$ there exist uncountable family of indecomposable systems $\mathcal{S}$ of four subspaces with the same defect $\rho(\mathcal{S}) = \frac{n}{3}$.*

Proof. For a positive integer n, consider a family $(\mathcal{S}_{V+\alpha I})_{\alpha\in(0,1)}$ and $(\mathcal{S}_{V^*+\alpha I})_{\alpha\in(0,1)}$ of bounded operator systems similarly as in the above theorem. Then any $\mathcal{S}_{V+\alpha I}$ is also indecomposable and

$$\rho(\mathcal{S}_{V+\alpha I}) = \frac{1}{3}(0 - n + 0 - 0) = \frac{-n}{3}.$$

If $\alpha \neq \beta$, then the spectrum $\sigma(V + \alpha I) \neq \sigma(V + \beta I)$. Since $V + \alpha I$ and $V + \beta I$ are not similar, $\mathcal{S}_{V+\alpha I}$ and $\mathcal{S}_{V+\beta I}$ are not isomorphic each other.

We also have $\rho(\mathcal{S}_{V^*+\alpha I}) = \frac{n}{3}$. ! ! And they are not isomorphic each other.

For $n = 0$, consider a family $(\mathcal{S}_{S+3I+\alpha I})_{\alpha\in[0,1]}$ in Example after Proposition 8.5. They are indecomposable , not isomorphic each other and $\rho(\mathcal{S}_{S+3I+\alpha I}) = 0$. $\square$

12. Coxeter functors

In [GP] Gelfand and Ponomarev introduced two functors Φ^+ and Φ^- on the category of systems $\mathcal{S}$ of n subspaces in finite-dimensional vector spaces. They used the functors Φ^+ and Φ^- to give a complete classification of indecomposable systems of four subspaces with defect $\rho(\mathcal{S}) \neq 0$ in finite-dimensional vector spaces. If the defect $\rho(\mathcal{S}) < 0$, then there exists a positive integer ℓ such that $(\Phi^+)^{\ell-1}(\mathcal{S}) \neq 0$ and $(\Phi^+)^{\ell}(\mathcal{S}) = 0$. Combining the facts that indecomposable systems $\mathcal{T}$ with $\Phi^+(\mathcal{T}) = 0$ can be classified easily and that $\mathcal{S}$ is isomorphic to (and recovered as) $(\Phi^-)^{\ell-1}(\Phi^+)^{\ell-1}(\mathcal{S})$, they provided a complete classification. A similar argument holds for systems $\mathcal{S}$ with defect $\rho(\mathcal{S}) > 0$.

In their argument the finiteness of dimension is used crucially. In fact if an indecomposable system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ with $\dim H > 1$ satisfies that the defect $\rho(\mathcal{S}) < 0$, then $\Phi^+(\mathcal{S}) = (H^+; E_1^+, E_2^+, E_3^+, E_4^+)$ has the property that $\dim H^+ < \dim H$. The property guarantees the existence of a positive integer ℓ such that $(\Phi^+)^{\ell}(\mathcal{S}) = 0$. Although we can not expect such an argument anymore in the case of infinite-dimensional space, these functors Φ^+ and Φ^- are interesting on their own right. Therefore we shall extend these functors Φ^+ and Φ^- on infinite-dimensional Hilbert spaces and show that the Coxeter functors preserve the defect and indecomposability under certain conditions.

Definition.(Coxeter functor Φ^+) Let $\mathcal{S}ys^n$ be the category of the systems of n subspaces in Hilber spaces and homomorphisms. Let $\mathcal{S} = (H; E_1, \ldots, E_n)$ be a system of n subspaces in a Hilbert space H. Let $R := \oplus_{i=1}^n E_i$ and

$$\tau : R \ni x = (x_1, \ldots, x_n) \longmapsto \tau(x) = \sum_{i=1}^n x_i \in H.$$

Define $\mathcal{S}^+ = (H^+; E_1^+, \ldots, E_n^+)$ by

$$H^+ := \operatorname{Ker} \tau \text{ and } E_k^+ := \{(x_1, \ldots, x_n) \in H^+; x_k = 0\}.$$

Let $\mathcal{T} = (K; F_1, \ldots, F_n)$ be another system of n subspaces in a Hilbert space K and $\varphi : \mathcal{S} \to \mathcal{T}$ be a homomorphism. Since $\varphi : H \to K$ is a bounded linear operator with $\varphi(E_i) \subset F_i$, we can define a bounded linear operator $\varphi^+ : H^+ \to K^+$ by $\varphi^+(x_1, \ldots, x_n) = (\varphi(x_1), \ldots, \varphi(x_n))$. Since $\varphi^+(E_i^+) \subset F_i^+$, φ^+ define a homomorphism $\varphi^+ : \mathcal{S}^+ \to \mathcal{T}^+$. Thus we can introduce a covariant functor $\Phi^+ : \mathcal{S}ys^n \to \mathcal{S}ys^n$ by

$$\Phi^+(\mathcal{S}) = \mathcal{S}^+ \text{ and } \Phi^+(\varphi) = \varphi^+.$$

Example.If $\mathcal{S} = (\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C})$, then $\mathcal{S}^+ \cong (\mathbb{C}^2; \mathbb{C}(1,0), \mathbb{C}(0,1), \mathbb{C}(1,1))$.

Lemma 12.1. *Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces and consider $\mathcal{S}^+ = (H^+; E_1^+, E_2^+, E_3^+, E_4^+)$. Then*

$$E_1^+ \cap E_2^+ = \{(0, 0, a, -a) \in \oplus_{i=1}^4 E_i; a \in E_3 \cap E_4\}.$$

In particular, we have $\dim E_1^+ \cap E_2^+ = \dim E_3 \cap E_4$. Same formulae hold under permutation of subspaces.

Proof. Let $x = (x_1, x_2, x_3, x_4) \in E_1^+ \cap E_2^+$, then $x_1 = x_2 = 0$. Since $x \in H^+$, $\tau(x) = x_3 + x_4 = 0$. Thus $a := x_3 = -x_4 \in E_3 \cap E_4$ and $x = (0, 0, a, -a)$. The converse inclusion is clear. $\square$

Lemma 12.2. *Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces and consider $\mathcal{S}^+ = (H^+; E_1^+, E_2^+, E_3^+, E_4^+)$. If $E_3 \cap E_4 = 0$ and $E_3 + E_4 = H$, then $E_1^+ + E_2^+ = H^+$. Same formulae hold under permutation of subspaces.*

Proof. Let $z = (z_1, z_2, z_3, z_4) \in H^+$. Put $y_1 := z_1$ and $x_2 := z_2$. Since $E_3 + E_4 = H$, there exist $y_3 \in E_3$ and $y_4 \in E_4$ such that $-y_1 = y_3 + y_4$. Since $y_1 + y_3 + y_4 = 0$, $y := (y_1, 0, y_3, y_4) \in H^+$. Similarly there exist $x_3 \in E_3$ and $x_4 \in E_4$ such that $-x_2 = x_3 + x_4$, so that $x := (0, x_2, x_3, x_4) \in H^+$.

Since $z \in H^+$, $z_1 + z_2 + z_3 + z_4 = 0$. Hence

$$\begin{aligned} z_3 + z_4 &= -z_1 - z_2 = -y_1 - y_2 \\ &= (y_3 + y_4) + (x_3 + x_4) = (x_3 + y_3) + (x_4 + y_4) \in E_3 + E_4. \end{aligned}$$

Because $E_3 \cap E_4 = 0$, we have $z_3 = x_3 + y_3$ and $z_4 = x_4 + y_4$. Therefore $z = x + y \in E_3^+ + E_4^+$. $\square$

Example. Let $\mathcal{S}_{S,T} = (H; E_1, E_2, E_3, E_4)$ be a bounded operator system. Combining the preceding two lemmas Lemma 12.1 and Lemma 12.2 with a characterization of bounded operator systems in Corollary 8.11, we have that $\mathcal{S}^+ = (H^+; E_1^+, E_2^+, E_3^+, E_4^+)$ is a bounded operator system up to permutation of subspaces. More precisely, $(H^+; E_3^+, E_4^+, E_1^+, E_2^+)$ is a bounded operator system.

Let $0 \oplus E_i \oplus 0 := 0 \oplus \cdots \oplus 0 \oplus E_i \oplus 0 \oplus \cdots \oplus 0 \subset R$ and $q_i \in B(R)$ be the projection onto $0 \oplus E_i \oplus 0$. Let $\imath_+ : H^+ \to R$ be a canonical embedding. Then we have an exact sequence:

$$0 \longrightarrow H^+ \xrightarrow{\imath_+} R \xrightarrow{\tau} H$$

Furthermore we have

$$\operatorname{Ker} \tau q_i = \operatorname{Ker} q_i, \;\; E_i = \operatorname{Im} \tau q_i = \overline{\operatorname{Im} \tau q_i} \;\; \text{and} \;\; E_i^+ = \operatorname{Ker} q_i \imath_+.$$

These properties characterize $\mathcal{S}^+ = (H^+; E_1^+, E_2^+, E_3^+, E_4^+)$.

Proposition 12.3. *Let X, Y and Z be Hilbert spaces and $T : X \to Y$ and $S : Y \to Z$ be bounded linear maps. Suppose that a sequence*

$$0 \longrightarrow X \xrightarrow{T} Y \xrightarrow{S} Z.$$

is exact. Let $p_1, ..., p_n \in B(Y)$ be projections with $\sum_i p_i = I$ and $p_i p_j = 0$ for $i \neq j$. Furthermore we assume that

$$\operatorname{Ker} Sp_i = \operatorname{Ker} p_i \textit{ and } \operatorname{Im} Sp_i \textit{ is closed in } Z.$$

Let $E_i := \operatorname{Im} Sp_i \subset Z$ and $E_i' := \operatorname{Ker} p_i T \subset X$. Define $\mathcal{S} = (Z; E_1, \ldots, E_n)$ and $\mathcal{S}' = (X; E_1', \ldots, E_n')$. Then $\mathcal{S}' \cong \Phi^+(\mathcal{S})$

Proof. Consider the restriction $S_i := S|_{\operatorname{Im} p_i} : \operatorname{Im} p_i \to \operatorname{Im} Sp_i$. Since $\operatorname{Ker} Sp_i = \operatorname{Ker} p_i$, S_i is one to one. Because $\operatorname{Im} Sp_i$ is closed, $\operatorname{Im} Sp_i$ is complete. Therefore S_i is an invertible operator by open mapping theorem. Define $\varphi : Y = \oplus_{i=1}^n \operatorname{Im} p_i \to \oplus_{i=1}^n E_i$ by $\varphi((y_i)_i) = (S_i(y_i))_i$ for $(y_i)_i \in \oplus_{i=1}^n \operatorname{Im} p_i$.Then φ is an invertible operator. Consider $\tau : \oplus_{i=1}^n E_i \to Z$ given $\tau((z_i)_i) = \sum_{i=1}^n z_i$. Let $Z^+ = \operatorname{Ker} \tau$ and $\imath_+ : Z^+ \to \oplus_{i=1}^n E_i$ be a canonical embedding. Then $\tau\varphi = S$. Define $\psi : X \to Z^+$ by $\psi(x) = \varphi T(x)$ for $x \in X$. The map ψ is well-defined, because $\tau(\psi(x)) = \tau(\varphi T(x)) = ST(x) = 0$. Then the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & X & \xrightarrow{T} & Y & \xrightarrow{S} & Z \\
 & & {\scriptstyle\psi}\downarrow & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle id_Z}\downarrow \\
0 & \longrightarrow & Z^+ & \xrightarrow{\imath_+} & \oplus_{i=1}^n E_i & \xrightarrow{\tau} & Z
\end{array}$$

is commutative. Furthermore maps ψ and φ are invertible operators. Let $q_i \in B(\oplus_{i=1}^n E_i)$ be a projection onto $0 \oplus E_i \oplus 0$. Then $q_i = \varphi p_i \varphi^{-1}$, $E_i^+ = \operatorname{Ker}(q_i \imath_+)$ and $E_i' = \operatorname{Ker}(p_i T)$. Therefore $\psi(E_i') = E_i^+$. Thus $\psi : \mathcal{S}' \to \Phi^+(\mathcal{S})$ is a desired isomorphism. $\square$

Definition.(Coxeter functor Φ^-) In [GP] Gelfand and Ponomarev introduced a dual functor Φ^- using quotients of vector spaces. If H is a Hilbert space and K a subspace of H, then it is convenient to identify the quotient space H/K with the orthogonal complement $K^\perp$. Therefore we shall generalize their functor Φ^- in terms of orthogonal complements instead of quotients in our case of Hilbert spaces. Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Let $e_i^\perp \in B(H)$ be the projection onto $E_i^\perp \subset H$. Let $Q := \oplus_{i=1}^n E_i^\perp$ and

$$\mu : H \ni x \longmapsto \mu(x) = (e_1^\perp x, \dots, e_n^\perp x) \in Q.$$

Then $\mu^* : Q \to H$ is given by $\mu^*(y_1, \dots, y_n) = \sum_{i=1}^n y_i$. Define $H^- := \operatorname{Ker} \mu^* \subset Q$. Let $\imath_- : H^- \to Q$ be a canonical embedding. Then $q_- := \imath_-^* : Q \to H^-$ is the projection. Let $0 \oplus E_i^\perp \oplus 0 := 0 \oplus \dots 0 \oplus E_i^\perp \oplus 0 \dots \oplus 0 \subset Q$ and $r_i \in B(Q)$ be the projection onto $0 \oplus E_i^\perp \oplus 0$. Define $\mathcal{S}^- = (H^-; E_1^-, \dots, E_n^-)$ by

$$E_i^- := \overline{q_-(0 \oplus E_i^\perp \oplus 0)} = \overline{\operatorname{Im} q_- r_i} \subset H^-.$$

We note that

$$H^- := \operatorname{Ker} \mu^* = Q \cap (\operatorname{Im} \mu)^\perp \cong Q/\overline{\operatorname{Im} \mu}.$$

We have an exact sequence

$$0 \longrightarrow H^- \xrightarrow{\imath_-} Q \xrightarrow{\mu^*} H$$

and a sequence

$$H \xrightarrow{\mu} Q \xrightarrow{q_-} H^- \longrightarrow 0,$$

satisfying that $\overline{\operatorname{Im} \mu} = \operatorname{Ker} q_-$ and q_- is onto. Thus it is easy to see that our definition of $\mathcal{S}^- = (H^-; E_1^-, \dots, E_n^-)$ coincides with the original one by Gelfand and Ponomarev up to isomorphism in the case of finite-dimensional spaces.

Define $\Phi^-(\mathcal{S}) := \mathcal{S}^- = (H^-; E_1^-, \dots, E_n^-)$. Then there is a relation between $\mathcal{S}^+$ and $\mathcal{S}^-$. We recall some elementary facts first.

Lemma 12.4. *Let H and K be Hilbert spaces and M a closed subspace of H. Let $T : H \to K$ be a bounded operator. Consider $T^* : K \to H$. Then $\overline{T(M^\perp)} = ((T^*)^{-1}(M))^\perp \subset K$.*

Lemma 12.5. *Let L be a Hilbert space and M, K closed subspaces of L. Let $P_K \in B(L)$ be the projection onto K. Then $\overline{P_K(M^\perp)} = K \cap (K \cap M)^\perp$.*

Proof. By the preceding lemma,

$$(\overline{P_K(M^\perp)})^\perp = P_K^{-1}(M) = \{x \in L; P_K x \in M\}.$$

Decompose $x \in L$ such that $x = x_1 + x_2$ with $x_1 \in K$, $x_2 \in K^\perp$. Then $P_K x \in M$ if and only if $x_1 \in M$. Therefore $(\overline{P_K(M^\perp)})^\perp = (K \cap M) + K^\perp$. Thus $\overline{P_K(M^\perp)} = K \cap (K \cap M)^\perp$. $\square$

Proposition 12.6. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Then we have*

$$\Phi^-(\mathcal{S}) = \Phi^\perp\Phi^+\Phi^\perp(\mathcal{S}).$$

Proof. Since $\Phi^\perp(\mathcal{S}) = (H; E_1^\perp, \dots, E_n^\perp)$, we have

$$\Phi^+\Phi^\perp(\mathcal{S}) = (H'; (E_1^\perp)^+, \dots, (E_n^\perp)^+),$$

where $H' = \{(y_1, \dots, y_n) \in \oplus_{i=1}^n E_i^\perp; y_1 + \dots + y_n = 0\}$. Therefore we have $H' = H^-$.

Applying the preceding Lemma by putting $L = \oplus_{i=1}^n E_i^\perp$, $M = \{(y_1, \dots, y_n) \in L; y_k = 0\}$ and $K = H^- \subset L$, we have

$$E_k^- = \overline{q_-(0 \oplus E_k^\perp \oplus 0)} = \overline{P_K(M^\perp)} = K \cap (K \cap M)^\perp = H^- \cap ((E_k^\perp)^+)^\perp.$$

Therefore $(E_k^-)^\perp = (E_k^\perp)^+$ in H^-. Hence $\Phi^\perp\Phi^-(\mathcal{S}) = \Phi^+\Phi^\perp(\mathcal{S})$. This implies the conclusion. $\square$

Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H and $\mathcal{T} = (K; F_1, \dots, F_n)$ be another system of n subspaces in a Hilbert space K. Let $\varphi : \mathcal{S} \to \mathcal{T}$ be a homomorphism, i.e., $\varphi : H \to K$ is a bounded linear operator with $\varphi(E_i) \subset F_i$. Define $\varphi^- : \Phi^-(\mathcal{S}) \to \Phi^-(\mathcal{T})$ by

$$\varphi^- := \Phi^\perp\Phi^+\Phi^\perp(\varphi).$$

Thus we can introduce a covariant functor $\Phi^- : \mathcal{S}ys^n \to \mathcal{S}ys^n$ by

$$\Phi^-(\mathcal{S}) = \mathcal{S}^- \text{ and } \Phi^-(\varphi) = \varphi^-.$$

Remark. Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Let $R := \oplus_{i=1}^n E_i$ and $\tau : R \to H$ is given by $\tau(x) = \sum_{i=1}^n x_i$. Let $H^0 := \operatorname{Ker}\tau$ and $q_0 : R \to H^0$ be the canonical projection. Define $E_k^0 := \overline{q_0(0 \oplus E_k \oplus 0)}$. Let $\mathcal{S}^0 := (H^0; E_1^0, \dots, E_n^0)$ and $\Phi^0(\mathcal{S}) = \mathcal{S}^0$. Then we have

$$\Phi^+(\mathcal{S}) = \Phi^\perp\Phi^0(\mathcal{S}) \text{ and } \Phi^-(\mathcal{S}) = \Phi^0\Phi^\perp(\mathcal{S}).$$

Furthermore

$$\Phi^-\Phi^+(\mathcal{S}) = (\Phi^0)^2(\mathcal{S}) \text{ and } \Phi^+\Phi^-(\mathcal{S}) = \Phi^\perp(\Phi^0)^2\Phi^\perp(\mathcal{S}).$$

Suppose that H is finite-dimensional. Then

$$\dim H^0 = \dim \operatorname{Ker}\tau = \dim R - \dim \operatorname{Im}\tau = \sum_i \dim E_i - \dim(\sum_i E_i)$$

In particular, if $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ is an indecomposable system of four subspaces with $\dim H \geq 2$, then $\dim H^0 = \sum_i \dim E_i - \dim H$ and the defect

$$\rho(\mathcal{S}) = \sum_i \dim E_i - 2\dim H = \dim H^0 - \dim H.$$

We shall characterize $\Phi^-(\mathcal{S})$. The following fact is useful: Let H and K be Hilbert spaces and $T : H \to K$ be a bounded linear operator. Then $\operatorname{Im} T$ is closed in K if and only if $\operatorname{Im} T^*$ is closed in H.

Proposition 12.7. *Let U, V and W be Hilbert spaces and $A : U \to V$ and $B : V \to W$ be bounded linear operators. Suppose that a sequence*

$$U \xrightarrow{A} V \xrightarrow{B} W \longrightarrow 0$$

is exact. Let $p_1, ..., p_n \in B(V)$ be projections with $\sum_i p_i = I$ and $p_i p_j = 0$ for $i \neq j$. Furthermore we assume that

$$\operatorname{Im} p_i A \text{ is closed in } V \text{ and } \operatorname{Im} p_i A = \operatorname{Im} p_i.$$

Let $L_i' := \overline{\operatorname{Im} B p_i} \subset W$ and $L_i := \operatorname{Ker} p_i A \subset U$. Define $\mathcal{S} = (U; L_1, \dots, L_n)$ and $\mathcal{S}' = (W; L_1', \dots, L_n')$. Then $\mathcal{S}' \cong \Phi^-(\mathcal{S})$

Proof. Since $\operatorname{Im} B = W$ is closed, $\operatorname{Im} B^* \subset V$ is also closed. Then

$$\operatorname{Im} B^* = (\operatorname{Ker} B)^\perp = (\operatorname{Im} A)^\perp = \operatorname{Ker} A^*$$

and $\operatorname{Ker} B^* = (\operatorname{Im} B)^\perp = W^\perp = 0$. Hence the dual sequence

$$0 \longrightarrow W \xrightarrow{B^*} V \xrightarrow{A^*} U$$

is exact. We shall apply Proposition 12.3 by putting $X = W$, $Y = V$, $Z = U$, $T = B^*$ and $S = A^*$. We can check the assumption of the Proposition. In fact,

$$\operatorname{Ker} S p_i = \operatorname{Ker} A^* p_i = (\operatorname{Im} p_i A)^\perp = (\operatorname{Im} p_i)^\perp = \operatorname{Ker} p_i,$$

and $\operatorname{Im} S p_i = \operatorname{Im} A^* p_i = \operatorname{Im}(p_i A)^*$ is closed, because $\operatorname{Im}(p_i A)$ is closed. Let

$$E_i := \operatorname{Im} S p_i = \operatorname{Im}(p_i A)^* = (\operatorname{Ker} p_i A)^\perp = (L_i)^\perp \subset U$$

and

$$E_i' := \operatorname{Ker} p_i T = \operatorname{Ker} p_i B^* = (\operatorname{Im} B p_i)^\perp = (L_i')^\perp \subset W.$$

Then $(X; E_1', \dots, E_n') \cong \Phi^+(Z; E_1, \dots, E_n)$, that is, we have

$$(W; (L_1')^\perp, \dots, (L_n')^\perp) \cong \Phi^+(U; (L_1)^\perp, \dots, (L_n)^\perp).$$

Thus $(\mathcal{S}')^\perp \cong \Phi^+(\mathcal{S}^\perp)$. Hence

$$\mathcal{S}' \cong \Phi^\perp \Phi^+ \Phi^\perp (\mathcal{S}) = \Phi^-(\mathcal{S}).$$

□

Proposition 12.8. *Let $\mathcal{S}$ and $\mathcal{T}$ be systems of n subspaces in a Hilbert space H. Then we have $\Phi^+(\mathcal{S} \oplus \mathcal{T}) \cong \Phi^+(\mathcal{S}) \oplus \Phi^+(\mathcal{T})$,*
$\Phi^-(\mathcal{S} \oplus \mathcal{T}) \cong \Phi^-(\mathcal{S}) \oplus \Phi^-(\mathcal{T})$, and $\Phi^\perp(\mathcal{S} \oplus \mathcal{T}) \cong \Phi^\perp(\mathcal{S}) \oplus \Phi^\perp(\mathcal{T})$.

Proof. It is straightforward to prove them. □

Definition. Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Then $\mathcal{S}$ is said to be *reduced from above* if for any $k = 1, \dots, n$

$$\sum_{i \neq k} E_i = H.$$

In particular we have $E_k \subset \sum_{i \neq k} E_i$. Similarly $\mathcal{S}$ is said to be *reduced from below* if for any $k = 1, \dots, n$

$$\sum_{i \neq k} E_i^{\perp} = H.$$

In particular we have $E_k^{\perp} \subset \sum_{i \neq k} E_i^{\perp}$ and $\cap_{i \neq k} E_i = 0$

It is evident taht $\mathcal{S} \oplus \mathcal{T}$ is reduced from above if and only if both $\mathcal{S}$ and $\mathcal{T}$ are reduced from above. Similarly $\mathcal{S} \oplus \mathcal{T}$ is reduced from below if and only if both $\mathcal{S}$ and $\mathcal{T}$ are reduced from below.

Example.(1) Any bounded operator system is reduced from above and reduced from below. In fact $E_1 + E_2 = H$, $E_1 + E_4 = H$, $E_2 + E_4 = H$ and $E_1^{\perp} + E_2^{\perp} = H$, $E_1^{\perp} + E_4^{\perp} = H$, $E_2^{\perp} + E_4^{\perp} = H$.
(2)The exotic examples in section 10 are reduced from above and reduced from below.

We shall show a duality theorem between Coxeter functors Φ^+ and Φ^-.

Theorem 12.9. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that $\mathcal{S}$ is reduced from above. Then we have*

$$\Phi^- \Phi^+(\mathcal{S}) \cong \mathcal{S}.$$

Proof. Let $R = \oplus_{i=1}^n E_i$. Consider a sequence

$$H^+ \xrightarrow{\imath_+} R \xrightarrow{\tau} H \longrightarrow 0.$$

Since $\mathcal{S}$ is reduced from above, $\operatorname{Im} \tau = \sum_{i=1}^n E_i = H$. Thus the above sequence is exact. Let $p_i \in B(R)$ be the projection onto $0 \oplus E_i \oplus 0$. We shall apply Proposition 12.7 by putting $U = H^+$, $V = R$, $W = H$, $A = \imath_+$ and $B = \tau$. We can check the assumption of the proposition. In fact, since $\mathcal{S}$ is reduced from above, for any $x_k \in E_k$, there exist $x_i \in E_i$ for $i \neq k$ such that $x_k = \sum_{i \neq k} -x_i$. Then $\sum_{i=1}^n x_i = 0$, that is, $x := (x_i)_i \in H^+$. Then

$$p_k A(x) = 0 \oplus x_k \oplus 0 \in 0 \oplus E_k \oplus 0.$$

Thus $\operatorname{Im} p_k A = 0 \oplus E_k \oplus 0 = \operatorname{Im} p_k$ and $\operatorname{Im} p_k A$ is closed. Therefore $(W; L_1', \dots, L_n') \cong \Phi^-(U; L_1, \dots, L_n)$. Since

$$L_k' = \overline{\operatorname{Im} B p_k} = \overline{\operatorname{Im} \tau p_k} = E_k$$

and
$$L_k = \operatorname{Ker} p_k A = \operatorname{Ker} p_k \imath_+ = E_k^+,$$
we have
$$\mathcal{S} = (H; E_1, \dots, E_n) \cong \Phi^-(H^+; E_1^+, \dots, E_n^+) = \Phi^-\Phi^+(\mathcal{S}).$$
□

Similarly we have the follwoing:

Theorem 12.10. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that $\mathcal{S}$ is reduced from below. Then we have*
$$\Phi^+\Phi^-(\mathcal{S}) \cong \mathcal{S}.$$

Proof. If $\mathcal{S}$ is reduced from below, then $\mathcal{S}^\perp$ is reduced from above. Hence $\Phi^-\Phi^+(\mathcal{S}^\perp) \cong \mathcal{S}^\perp$. Then
$$\mathcal{S} \cong \Phi^\perp\Phi^-\Phi^+\Phi^\perp(\mathcal{S}) = \Phi^\perp\Phi^-\Phi^\perp\Phi^\perp\Phi^+\Phi^\perp(\mathcal{S}) = \Phi^+\Phi^-(\mathcal{S}).$$
□

! !

Proposition 12.11. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Then $\Phi^+(\mathcal{S}) = 0$ if and only if for any $k = 1, \dots, n$*
$$E_k \cap (\sum_{i \neq k} E_i) = 0.$$

Proof. It is easy to see that $\Phi^+(\mathcal{S}) = 0$ if and only if for any $x_i \in E_i$ with $i = 1, \dots, n$ $\sum_i x_i = 0$ imples $x_1 = \cdots = x_n = 0$. The latter condition is equal to that $E_k \cap (\sum_{i \neq k} E_i) = 0$ for any $k = 1, \dots, n$. □

The above conditon $E_k \cap (\sum_{i \neq k} E_i) = 0$ for any $k = 1, \dots, n$ is something like an opposite of that $\mathcal{S}$ is reduced from above.

Proposition 12.12. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Then $\Phi^+(\mathcal{S}) = 0$ and $\sum_{i=1}^n E_i$ is closed in H if and only if $(H; E_1, \dots, E_n, (\sum_{i=1}^n E_i)^\perp)$ is isomorphic to a system of direct sum decomposition, that is, there is an orthogonal direct sum decomposition $K = \oplus_{i=1}^{n+1} K_i$ of a Hilbert space K and $(H; E_1, \dots, E_n, (\sum_{i=1}^n E_i)^\perp)$ is isomorphic to a system $(K; K_1, \dots, K_{n+1})$, in particular $\mathcal{S}$ is isomorphic to a commutative system.*

Proof. Assume that $\Phi^+(\mathcal{S}) = 0$ and $\sum_{i=1}^n E_i$ is closed in H. Let $E_{n+1} = (\sum_{i=1}^n E_i)^\perp$. Let $R := \oplus_{i=1}^{n+1} E_i$ and $K_i := 0 \oplus \cdots \oplus 0 \oplus E_i \oplus 0 \oplus \cdots \oplus 0 \subset R$. Define $\varphi : K \to H$ by $\varphi((x_i)_i) = \sum_i x_i$. Then the bounded operator φ is onto, because $\sum_{i=1}^n E_i$ is closed in H. Since $\Phi^+(\mathcal{S}) = 0$, φ is one to one by the preceding proposition. It is clear that $\varphi(K_i) = E_i$. Hence $(H; E_1, \dots, E_{n+1})$ is isomorphic to $(K; K_1, \dots, K_{n+1})$. The converse and the rest are trivial. □

Example. Let $T \in B(K)$ be a positive operator with dense range and $\operatorname{Im} T \neq K$. Let $H = K \oplus K$, $E_1 = K \oplus 0$ and $E_2 = \operatorname{graph} T$. Put $\mathcal{S} = (H; E_1, E_2)$. Then $\Phi^+(\mathcal{S}) = 0$ and $(E_1 + E_2)^\perp = 0$. But $(H; E_1, E_2, 0)$ is not isomorphic to a system of direct sum decomposition. In fact $E_1 + E_2 = K \oplus \operatorname{Im} T$ is not closed.

We also have the following:

Proposition 12.13. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H Then $\Phi^-(\mathcal{S}) = 0$ if and only if for any $k = 1, \dots, n$*

$$E_k^\perp \cap (\sum_{i \neq k} E_i^\perp) = 0.$$

Proposition 12.14. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. If $\mathcal{S}$ is reduced from above and $\mathcal{S} \neq 0$, then $\Phi^+(\mathcal{S}) \neq 0$. Similarly if $\mathcal{S}$ is reduced from below and $\mathcal{S} \neq 0$, then $\Phi^-(\mathcal{S}) \neq 0$.*

Proof. Suppose that $E_i = 0$ for any $i = 1, \dots, n$. Then $H = \sum_{i=1}^{n-1} E_i = 0$. This contradicts to that $\mathcal{S} \neq 0$. Therefore $E_k \neq 0$ for some k. Since $\sum_{i \neq k} E_i = H$, for a non-zero $x_k \in E_k$, there exist $x_i \in E_k$ for $i \neq k$ such that $-x_k = \sum_{i \neq 0} x_i$. Therefore $x := (x_1, \dots, x_n) \in H^+$ is non-zero, that is, $\Phi^+(\mathcal{S}) \neq 0$. The other is similarly proved. $\square$

Remark. By Proposition 6.1, if a system of n subspaces $\mathcal{S} = (H; E_1, \dots, E_n)$ is indecomposable and $\dim H \geq 2$, then for any distinct n-1 subspaces $E_{i_1}, \dots, E_{i_{n-1}}$, we have that

$$\bigcap_{k=1}^{n-1} E_{i_k} = 0 \text{ and } \bigvee_{k=1}^{n-1} E_{i_k} = H,$$

that is,

$$\overline{\sum_{k=1}^{n-1} E_{i_k}^\perp} = H \text{ and } \overline{\sum_{k=1}^{n-1} E_{i_k}} = H,$$

Unless H is finite-dimensional, these conditions seems to be weaker than that $\mathcal{S}$ is reduced from below and above.

Remark. Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H and consider $\mathcal{S}^+ = (H^+; E_1^+, \dots, E_n^+)$. Then for any distinct n-1 subspaces $E_{i_1}^+, \dots, E_{i_{n-1}}^+$, we have that

$$\bigcap_{k=1}^{n-1} E_{i_k}^+ = 0.$$

In fact, for example, let $(x_1, \dots, x_n) \in \cap_{k=1}^{n-1} E_k^+$. Then $x_1 = x_2 = \dots = x_{n-1} = 0$. Since $(x_1, \dots, x_n) \in H^+$, we have $\sum_{i=1}^{n} x_k = 0$. Hence $x_n = 0$. Thus $\cap_{k=1}^{n-1} E_k^+ = 0$.

On the other hand the above condition implies that

$$\overline{\sum_{k=1}^{n-1} (E_{i_k}^{+})^{\perp}} = H^{+}.$$

This condition is a little weaker than that $\mathcal{S}^{+}$ is reduced from below unless H is finite dimensional.

Conider $\mathcal{S}^{-} = \Phi^{\perp}\Phi^{+}\Phi^{\perp}(\mathcal{S})$ similarly. Then we have

$$\overline{\sum_{k=1}^{n-1} E_{i_k}^{-}} = H^{-}.$$

The condition is a little weaker than that $\mathcal{S}^{-}$ is reduced from above unless H is finite dimensional.

Theorem 12.15. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that $\mathcal{S}$ is reduced from above and $\mathcal{S}^{+} = \Phi^{+}(\mathcal{S})$ is reduced from below. If $\mathcal{S}$ is indecomposable, then $\Phi^{+}(\mathcal{S})$ is also indecomposable.*

Proof. On the contrary suppose that $\mathcal{S}^{+}$ were decomposable. Then there exist non-zero systems $\mathcal{T}_1$ and $\mathcal{T}_2$ of n subspaces such that $\mathcal{S}^{+} = \mathcal{T}_1 \oplus \mathcal{T}_2$. Since $\mathcal{S}$ is reduced from above,

$$\mathcal{S} \cong \Phi^{-}\Phi^{+}(\mathcal{S}) = \Phi^{-}(\mathcal{T}_1) \oplus \Phi^{-}(\mathcal{T}_1),$$

by a duality Theorem 12.9. Since $\mathcal{S}^{+} = \Phi^{+}(\mathcal{S})$ is reduced from below, $\mathcal{T}_1$ and $\mathcal{T}_2$ are also reduced from below. By another duality Theorem 12.10, $\Phi^{+}\Phi^{-}(\mathcal{T}_i) \cong \mathcal{T}_i$ for $i = 1, 2$. Since $\mathcal{T}_i \neq 0$, we have $\Phi^{-}(\mathcal{T}_i) \neq 0$. (We could use Propsition 12 instead.) This implies that $\mathcal{S}$ is decomposable. This is a contradiction. Therefore $\mathcal{S}^{+}$ is indecomposable. $\square$

Example. Let $\mathcal{S}_\gamma = (H; E_1, E_2, E_3, E_4)$ be an exotic example in section 10. Since $E_i + E_j = H$ and $E_i \cap E_j = 0$ for distinct $i, j \in \{1, 2, 4\}$, we have $E_k^{+} + E_m^{+} = H$ and $E_k^{+} \cap E_m^{+} = 0$ for distinct $k, m \in \{3, 4\}$ or $k, m \in \{1, 3\}$ or $k, m \in \{2, 3\}$ by Lemma 12.1 and Lemma 12.2. Since $E_k^{+} + E_m^{+} = H$ is closed, $(E_k^{+})^{\perp} + (E_m^{+})^{\perp}$ is closed. Hence $(E_k^{+})^{\perp} + (E_m^{+})^{\perp} = H$ Therefore $\mathcal{S}_\gamma$ is reduced from above and $\Phi^{+}(\mathcal{S}_\gamma)$ is reduced from below. Since $\mathcal{S}_\gamma$ is indecomposable, $\Phi^{+}(\mathcal{S}_\gamma)$ is also indecomposable.

Similarly we have the following:

Theorem 12.16. *Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Suppose that $\mathcal{S}$ is reduced from below and $\mathcal{S}^{-} = \Phi^{-}(\mathcal{S})$ is reduced from above. If $\mathcal{S}$ is indecomposable, then $\Phi^{-}(\mathcal{S})$ is also indecomposable.*

We shall show that the Coxeter functors Φ^{+} and Φ^{-} preserve the defect under certain conditions.

Let $\mathcal{S} = (H; E_1, \dots, E_n)$ be a system of n subspaces in a Hilbert space H. Consider $\mathcal{S}^+ = (H^+; E_1^+, \dots, E_n^+)$. Let $R = \oplus_{i=1}^n E_i$ and $p_0 \in B(R)$ be the projection of R onto H^+. Let $e_i \in B(H)$ be the projection of H onto E_i. Recall that $\tau : R \to H$ is given by $\tau(a) = \sum_{i=1}^n a_i$ for $a = (a_1, \dots, a_n) \in R$.

Lemma 12.17. *Suppose that $\sum_{i=1}^n e_i$ is invertible. Then for $a = (a_1, \dots, a_n) \in R$ we have*

$$p_0(a) = (a_k - e_k(\sum_{i=1}^n e_i)^{-1}(\tau(a)))_k \in H^+$$

Proof. Recall that $\tau^* : H \to R$ is given by $\tau^*(y) = (e_1 y, \dots, e_n y)$ for $y \in H$. Consider the orthogonal decomposition $R = H^+ \oplus (H^+)^\perp$. Since $H^+ = \operatorname{Ker} \tau$, $(H^+)^\perp = \overline{\operatorname{Im} \tau^*}$ in R. Define

$$x = (x_k)_k := (a_k - e_k(\sum_{i=1}^n e_i)^{-1}(\tau(a)))_k \in R.$$

Then

$$\tau(x) = \sum_{k=1}^n (a_k - e_k(\sum_{i=1}^n e_i)^{-1}(\tau(a))) = \tau(a) - (\sum_{k=1}^n e_k)(\sum_{i=1}^n e_i)^{-1}(\tau(a)) = 0.$$

Therefore $x \in H^+$. Put $y := (\sum_{i=1}^n e_i)^{-1}(\tau(a)) \in H$. Then $\tau^*(y) = (e_1 y, \dots, e_n y) \in (H^+)^\perp$. Since $a = x + \tau^*(y) \in H^+ \oplus (H^+)^\perp$, we have $p_0(a) = x$. □

Corollary 12.18. *Suppose that $\sum_{i=1}^n e_i$ is invertible. Then $\operatorname{Im} \tau^*$ is closed and*

$$(H^+)^\perp = \operatorname{Im} \tau^* = \{(e_1 y, \dots, e_n y) \in R; y \in H\}.$$

Proof. By the above lemma, we have

$$(H^+)^\perp = \operatorname{Im}(I - p_0) = \{(e_1 y, \dots, e_n y) \in R; y \in H\} = \operatorname{Im} \tau^*.$$

□

Lemma 12.19. *Suppose that $\mathcal{S}$ is reduced from above and $\sum_{i=1}^n e_i$ is invertible. Then for $k = 1, \dots, n$*

$$(E_k^+)^\perp = \{(\delta_{jk} a_j - e_j(\sum_{i=1}^n e_i)^{-1}(a_k))_j \in H^+; a_k \in E_k\}.$$

Proof. Since $\mathcal{S}$ is reduced from above, we have $\operatorname{Im} p_k p_0 = 0 \oplus E_k \oplus$. In fact, for any $a_k \in E_k$, there exist $a_i \in E_i, (i \neq k)$ such that $-a_k = \sum_{i \neq k} a_i$. Then $(a_1, \dots, a_n) \in H^+$ and

$$p_k p_0(a_1, \dots, a_n) = (0, \dots, 0, a_k, 0, \dots, 0) \in 0 \oplus E_k \oplus 0.$$

The converse inclusion is trivial. Since $\operatorname{Im} p_k p_0 = 0 \oplus E_k \oplus$ is closed, $(\operatorname{Im} p_k p_0)^* = \operatorname{Im} p_0 p_k$ is also closed. Hence

$$(E_k^+)^\perp = E_k^0 = \operatorname{Im} p_0 p_k = \{p_0(0, \dots, 0, a_k, 0, \dots, 0); a_k \in E_k\}$$

Therefore the conclusion follows from Lemma 12.17 . $\square$

Proposition 12.20. *Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces and $\mathcal{S}^+ = (H^+; E_1^+, E_2^+, E_3^+, E_4^+)$. Suppose that $\mathcal{S}$ is reduced from above and $f := e_1 + e_2 + e_3 + e_4$ is invertible. Then*

$(E_1^+)^\perp \cap (E_2^+)^\perp$

$= \{(e_1 u - e_1 f^{-1} e_1 u, -e_2 f^{-1} e_1 u, -e_3 f^{-1} e_1 u, -e_4 f^{-1} e_1 u); u \in E_3^\perp \cap E_4^\perp\}.$

Moreover we have

$$\dim((E_1^+)^\perp \cap (E_2^+)^\perp) = \dim(E_3^\perp \cap E_4^\perp).$$

The same formulae hold under permutation of subspaces.

Proof. Let $x = (x_1, x_2, x_3, x_4) \in (E_1^+)^\perp \cap (E_2^+)^\perp$. Then by the preceding lemma, there exist $a_1 \in E_1$ and $a_2 \in E_2$ such that

$$\begin{aligned} x &= (x_1, x_2, x_3, x_4) \\ &= (a_1 - e_1 f^{-1} a_1, -e_2 f^{-1} a_1, -e_3 f^{-1} a_1 - e_4 f^{-1} a_1) \\ &= (-e_1 f^{-1} a_2, a_2 - e_2 f^{-1} a_2, -e_3 f^{-1} a_2 - e_4 f^{-1} a_2). \end{aligned}$$

Put $u := f^{-1}(a_1 - a_2) \in H$. Then $a_1 = e_1 u$, $a_2 = -e_2 u$, $e_3 u = 0$ and $e_4 u = 0$. Therefore $u \in E_3^\perp \cap E_4^\perp$ and

$$x = (e_1 u - e_1 f^{-1} e_1 u, -e_2 f^{-1} e_1 u, -e_3 f^{-1} e_1 u, -e_4 f^{-1} e_1 u).$$

Conversely suppose that

$$x = (e_1 u - e_1 f^{-1} e_1 u, -e_2 f^{-1} e_1 u, -e_3 f^{-1} e_1 u, -e_4 f^{-1} e_1 u),$$

for some $u \in E_3^\perp \cap E_4^\perp$. Put $a_1 := e_1 u \in E_1$ and $a_2 := -e_2 u \in E_2$. Since $e_3 u = 0$ and $e_4 u = 0$, we have

$$a_1 - a_2 = e_1 u + e_2 u = e_1 u + e_2 u + e_3 u + e_4 u = fu.$$

Because f is invertible, $u = f^{-1}(a_1 - a_2)$. Therefore

$$x = (a_1 - e_1 f^{-1} a_1, -e_2 f^{-1} a_1, -e_3 f^{-1} a_1 - e_4 f^{-1} a_1) \in (E_1^+)^\perp.$$

On the other hand, $a_1 = e_1 u = e_1 f^{-1}(a_1 - a_2)$. Hence

$$a_1 - e_1 f^{-1} a_1 = -e_1 f^{-1} a_2.$$

Since $a_2 = -e_2 u = -e_2 f^{-1}(a_1 - a_2)$, we have

$$-e_2 f^{-1} a_1 = a_2 - e_2 f^{-1} a_2.$$

Since $e_3 f^{-1}(a_1 - a_2) = e_3 u = 0$, we have $e_3 f^{-1} a_1 = e_3 f^{-1} a_2$. Similarly $e_4 f^{-1} a_1 = e_4 f^{-1} a_2$. Therefore

$$x = (-e_1 f^{-1} a_2, a_2 - e_2 f^{-1} a_2, -e_3 f^{-1} a_2 - e_4 f^{-1} a_2) \in (E_2^+)^\perp.$$

Thus $x \in (E_1^+)^\perp \cap (E_2^+)^\perp$.

Moreover define $T : E_3^\perp \cap E_4^\perp \to (E_1^+)^\perp \cap (E_2^+)^\perp$ by

$$Tu = (e_1u - e_1f^{-1}e_1u, -e_2f^{-1}e_1u, -e_3f^{-1}e_1u, -e_4f^{-1}e_1u)$$

for $u \in E_3^\perp \cap E_4^\perp$. Then T is a bounded, surjective operator. We shall show that T is one to one. Suppose that $Tu = 0$. Since $e_2f^{-1}e_1u = 0$, $f^{-1}e_1u \in E_2^\perp$. Similarly $f^{-1}e_1u \in E_3^\perp$ and $f^{-1}e_1u \in E_4^\perp$. Since $\mathcal{S}$ is reduced from above,

$$f^{-1}e_1u \in E_2^\perp \cap E_3^\perp \cap E_4^\perp = (E_2 + E_3 + E_4)^\perp = H^\perp = 0.$$

Hence $e_1u = 0$. Similary we have $e_2u = 0$. Therefore $fu = e_1u + e_2u + e_3u + e_4u = 0$. Since f is invertible, $u = 0$. Thus T is an invertible operator. Therefore $\dim((E_1^+)^\perp \cap (E_2^+)^\perp) = \dim(E_3^\perp \cap E_4^\perp)$. □

Theorem 12.21. *Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. Suppose that $\mathcal{S}$ is reduced from above and $f := e_1 + e_2 + e_3 + e_4$ is invertible. If $\mathcal{S}$ is a quasi-Fredholm system, then $\Phi^+(\mathcal{S})$ is also a quasi-Fredholm system and*

$$\rho(\Phi^+(\mathcal{S})) = \rho(\mathcal{S}).$$

Proof. It follows from Lemma 12.1 and Proposition 12.20 . □

Theorem 12.22. *Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces. Suppose that $\mathcal{S}$ is reduced from below and $g := e_1^\perp + e_2^\perp + e_3^\perp + e_4^\perp$ is invertible. If $\mathcal{S}$ is a quasi-Fredholm system, then $\Phi^-(\mathcal{S})$ is also a quasi-Fredholm system and*

$$\rho(\Phi^-(\mathcal{S})) = \rho(\mathcal{S}).$$

Proof. Recall that $\mathcal{S}$ is reduced from below if and only if $\Phi^\perp(\mathcal{S})$ is reduced from above, and $\mathcal{S}$ is a quasi-Fredholm system if and only if $\Phi^\perp(\mathcal{S})$ is a quasi-Fredholm system. Applying the preceding theorem, $\Phi^-(\mathcal{S}) = \Phi^\perp\Phi^+\Phi^\perp(\mathcal{S})$ a quasi-Fredholm system and

$$\rho(\Phi^-(\mathcal{S})) = -\rho(\Phi^+\Phi^\perp(\mathcal{S})) = -\rho(\Phi^\perp(\mathcal{S})) = \rho(\mathcal{S}).$$

□

Remark. Suppose that $\mathcal{S}$ is reduced from above and H is finite-dimesional, then $f := \sum_{i=1}^n e_i$ is automatically invertible. If fact, Let $x \in \operatorname{Ker} f$. Then $(e_ix|x) = 0$ so that $e_ix = 0$. Since $\mathcal{S}$ is reduced from above, $x \in \cap_i E_i^\perp = 0$ Thus $\operatorname{Ker} f = 0$. Then $\overline{\operatorname{Im} f} = (\operatorname{Ker} f)^\perp = H$. Since H is finite-dimensional, $\operatorname{Im} f = H$. Therefore f is invertible.

Example. Let $\mathcal{S}$ be an operator system. Since $E_1 = K \oplus 0, E_2 = 0 \oplus K$, we have that $f = \sum_{i=1}^4 e_i \geq I$ is invertible. Moreover if $\mathcal{S} = \mathcal{S}_T$ is associated with a single bounded operator T, then $E_4 = \{(x, x) \in H; x \in K\}$. Thus $E_i + E_j = H$ for $(i, j) = (1, 2), (1, 4), (2, 4)$ and $\mathcal{S}$ is reduced from above. Therefore, if $\mathcal{S}_T$ is a quasi-Fredholm system, then $\Phi^+(\mathcal{S}_T)$ is also a quasi-Fredholm system and $\rho(\Phi^+(\mathcal{S}_T)) = \rho(\mathcal{S}_T)$.

Similarly, let $\mathcal{S}_\gamma$ be an exotic example in section 10. Then $\mathcal{S}_\gamma$ is reduced from above and f is invertible. Since $\mathcal{S}_\gamma$ is a quasi-Fredholm system, $\Phi^+(\mathcal{S}_\gamma)$ is also a quasi-Fredholm system and $\rho(\Phi^+(\mathcal{S}_\gamma)) = \rho(\mathcal{S}_\gamma)$.

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