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Geometry of polysymbols

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Abstract. We introduce a multiple generalization of the tame symbol, called *polysymbols*, associated to meromorphic functions on a Riemann surface as the Massey products in the Deligne cohomology, and also give a geometric construction of polysymbols using Chen's iterated integrals. We then show some basic properties of polysymbols, and show that trivializations of polysymbols give variations of mixed Hodge structure.

Introduction

Let f and g be two meromorphic functions on a closed Riemann surface $\overline{X}$. The *tame symbol* at $x \in \overline{X}$ defined by

$$\{f, g\}_x = (-1)^{\mathrm{ord}_x(f)\mathrm{ord}_x(g)} \frac{f^{\mathrm{ord}_x(g)}}{g^{\mathrm{ord}_x(f)}}(x), \tag{0.1}$$

where $\mathrm{ord}_x(f)$ stands for the order of f at x, is a function-theoretic analog of the Hilbert symbol and plays important roles in class field theory and algebraic K-theory. Among many works on the subject, there is a geometric interpretation of $\{f, g\}_x$, due to S. Bloch ([Bl]) and P. Deligne ([Dl]), as the holonomy of a line bundle with holomorphic connection $\langle f, g\rangle$ over $X = \overline{X} \setminus (\mathrm{supp}(f) \cup \mathrm{supp}(g))$ which is given by the cup product $f \cup g$ in the Deligne cohomology $H^2_D(X, \mathbb{Z}(2))$. We note that this construction may be regarded as a complex analytic analogue of the linking number of two knots (cf. [Mo]).

The purpose of this paper is to generalize the construction of Bloch and Deligne and introduce a multiple generalization $\{f_1, \ldots, f_n\}_x$ of the tame symbol associated to meromorphic functions $f_1, \ldots, f_n$ on $\overline{X}$, following after Milnor-Massey's construction of higher order linking numbers of a link ([Ms],[T]). In fact, associated to meromorphic functions $f_1, \ldots, f_n$ on $\overline{X}$, we introduce the set of isomorphism classes of line bundles with holomorphic connection $\langle f_1, \ldots, f_n\rangle$, called a *polysymbol*, over $X = \overline{X} \setminus \cup_{i=1}^n \mathrm{supp}(f_i)$ as the Massey product in the Deligne cohomology $H^2_D(X, \mathbb{Z}(n))$, and then define $\{f_1, \ldots, f_n\}_x$, *polysymbol at* x, by the map $\langle f_1, \ldots, f_n\rangle \to \mathbb{C}$ sending each cohomology class in $\langle f_1, \ldots, f_n\rangle$ to the holonomy of

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the associated line bundle with connection along a loop encircling x. Our theorem (cf. Theorem 2.5 below) then gives an explicit formula for this holonomy in terms of Chen's iterated integrals ([C2]). We also give a geometric construction of the polysymbol $\langle f_1, \ldots, f_n \rangle$ which may be regarded as a natural extension of Bloch-Ramakrishnan's ([Bl],[R]) and Hain's ([H]). Namely, using the iterated integrals, we define a map associated to $f_1, \ldots, f_n$ and a defining system A for $\langle f_1, \ldots, f_n \rangle$

$$T(f_1, \ldots, f_n)_A : X \longrightarrow N(\mathbb{Z})\backslash N(\mathbb{C})/C,$$

where $N(R)$ for a commutative ring R denotes the group of R-valued points of the Heisenberg group of degree $n+1$ and C is the center of $N(\mathbb{C})$. The manifold $P := N(\mathbb{Z})\backslash N(\mathbb{C})$ has a natural structure of a principal $\mathbb{C}^\times$-bundle over $N(\mathbb{Z})\backslash N(\mathbb{C})/C$ and carries a standard connection form θ. By comparing the holonomies, the polysymbol $\langle f_1, \ldots, f_n \rangle_A$ relative to A is interpreted as the isomorphism class of the pull-back of the bundle with connection (P, θ) under $T(f_1, \ldots, f_n)_A$. We then show, using our holonomy formula, some basic properties of polysymbols which generalize those of the tame symbol. Finally, we show that trivializations of polysymbols give variations of mixed Hodge structure.

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1. Deligne cohomology and tame symbols

In this section, for the convenience of readers, we recall the basic materials on the Deligne cohomology which will be used in the sequel. References are [Br] and [EV].

Let X be a complex manifold. The *Deligne cohomology*, denoted by $H^*_D(X, \mathbb{Z}(p))$ for an integer $p \geq 1$, is by definition the hypercohomology of the *Deligne complex* $\mathbb{Z}(p)_D$ defined by the complex of sheaves on X

$$\mathbb{Z}(p) := (2\pi\sqrt{-1})^p\mathbb{Z} \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_X$$

where $\mathbb{Z}(p)$ is in degree 0 and $\mathcal{O}_X$ and Ω^k_X $(1 \leq k \leq p-1)$ denote the sheaf of holomorphic functions and of holomorphic k-forms on X respectively. The map $\mathcal{O}_X \to \mathcal{O}_X^\times$ defined by $x \mapsto \exp(\frac{1}{(2\pi\sqrt{-1})^{p-1}}x)$ and the multiplication on Ω^k_X by $\frac{1}{(2\pi\sqrt{-1})^{p-1}}$ induce a quasi-isomorphism

$$\mathbb{Z}(p)_D \simeq (\mathcal{O}_X^\times \xrightarrow{d\log} \Omega^1_X \to \cdots \to \Omega^{p-1}_X)[-1]$$

and hence we have the isomorphisms

$$H^q_D(X,\mathbb{Z}(p)) \simeq \mathbb{H}^q(X, \mathcal{O}^\times_X \stackrel{d\log}{\to} \Omega^1_X \to \cdots \to \Omega^{p-1}_X), \quad q \geq 1.$$

For low values of p and q, the isomorphisms above are given as follows.

1.1. $p = q = 1$: An element of $H^1_D(X,\mathbb{Z}(1))$ is repsesented by a Čech 1-cocycle α for a suitable open cover $\{U_a\}$ of X

$$\alpha = (q_{ab}, F_a) \in C^1(\mathbb{Z}(1)) \oplus C^0(\mathcal{O}_X), \quad q_{ab} = \delta F_a$$

where δ stands for the boundary map for Čech cocycles with respect to open covers. Then we associate to α the function $f \in H^0(X, \mathcal{O}^\times_X)$ so that F_a is a single branch of $\log f$ restricted on U_a. This yields the isomorphism

$$H^1_D(X,\mathbb{Z}(1)) \simeq H^0(X, \mathcal{O}^\times_X).$$

In the following, for $f \in H^0(X, \mathcal{O}^\times_X)$, we denote by $\log_{a_1 \ldots a_r}(f)$ for a single branch of $\log f$ restricted on $U_{a_1\ldots a_r} := U_{a_1} \cap \cdots \cap U_{a_r}$.

1.2. $p = q = 2$: An element of $H^2_D(X,\mathbb{Z}(2))$ is represented by a Čech 2-cocycle

$$\beta = (q_{abc}, \log_{ab} f, \Omega_a) \in C^2(\mathbb{Z}(2)) \oplus C^1(\mathcal{O}_X) \oplus C^0(\Omega^1_X)$$

which is subject to the relations $q_{abc} = \delta \log_{ab} f,\ d\log_{ab} f = \delta\Omega_a$. We then associate to β the Čech 1-cocycle with value in the complex $\mathcal{O}^\times_X \stackrel{d\log}{\to} \Omega^1_X$

$$(\xi_{ab}, \omega_a) \in C^1(\mathcal{O}^\times_X) \oplus C^0(\Omega^1_X),\ \xi_{ab} = \exp(\frac{1}{2\pi\sqrt{-1}} \log_{ab} f), \omega_a = \frac{1}{2\pi\sqrt{-1}}\Omega_a$$

with $\delta\xi_{ab} = 1,\ d\log\xi_{ab} = \delta\omega_a$. This gives the isomorphism

$$H^2_D(X,\mathbb{Z}(2)) \simeq \mathbb{H}^1(X, \mathcal{O}^\times_X \stackrel{d\log}{\to} \Omega^1_X).$$

1.3. The hypercohomology $\mathbb{H}^1(X, \mathcal{O}^\times_X \stackrel{d\log}{\to} \Omega^1_X)$ is interpreted as the group of isomorphism classes of line bundles over X with holomorphic connection, where the group structure of the latter is induced by the tensor product of bundles with connection. Suppose we are given such a pair (L, ∇). As usual, we identify L with the associated principal $\mathbb{C}^\times$-bundle L^+. Take a suitable open cover $\{U_a\}$ of X such that L^+ has a section s_a over U_a. Then the pair (ξ_{ab}, ω_a) defined by $\xi_{ab} := \frac{s_b}{s_a}$ and $\omega_a := \frac{\nabla(s_a)}{s_a}$ gives a Čech 1-cocycle with value in the complex $\mathcal{O}^\times_X \stackrel{d\log}{\to} \Omega^1_X$ by the relation $d\log\xi_{ab} = \delta\omega_a$. Conversely, we easily see that any such cocycle comes from the pair (L,∇). This association preserves the group structures.

1.4. A crucial property of the Deligne cohomology for our purpose is that it is equipped with a cup product on the Deligne complexes

$$\mathbb{Z}(p)_D \otimes \mathbb{Z}(p')_D \to \mathbb{Z}(p+p')_D$$

given by

$$x \cup y = \begin{cases} x \cdot y & \deg(x) = 0, \\ x \wedge dy & \deg(x) \geq 0, \ \deg(y) = p', \\ 0, & \text{otherwise.} \end{cases}$$

The cup product is anti-commutative up to homotopy and so we have $\alpha \cup \beta = (-1)^{qr} \beta \cup \alpha$ for $\alpha \in H_D^q(X, \mathbb{Z}(p)), \beta \in H_D^r(X, \mathbb{Z}(p'))$.

1.5. As an application of the cup product, one can describe the tame symbol $\{f, g\}_x$ in terms of the Deligne cohomology. Suppose that f and g are meromorphic functions on a closed Riemann surface $\overline{X}$. Set $X = \overline{X} \setminus (\mathrm{supp}(f) \cup \mathrm{supp}(g))$ so that $f, g \in H^0(X, \mathcal{O}_X^\times)$. By 1.1, f and g are represented by the 1-cocycles

$$(q_{ab}, \log_a f) \text{ and } (q'_{ab}, \log_a g)$$

respectively with $q_{ab} = \delta \log_a f$, $q'_{ab} = \delta \log_a g$. By 1.4, the cup product $f \cup g$ is represented by the 2-cocycle

$$(q_{ab} q'_{bc}, q_{ab} \log_b g, \log_a f \frac{dg}{g})$$

which corresponds to the 1-cocycle with value in the complex $\mathcal{O}_X^\times \stackrel{d\log}{\to} \Omega_X^1$ via (1.2)

$$(g^{\frac{q_{ab}}{2\pi\sqrt{-1}}}, \frac{1}{2\pi\sqrt{-1}} \log_a f \frac{dg}{g}).$$

We write $\langle f, g \rangle$ for the corresponding isomorphism class of line bundles with holomorphic connection on X by 1.3. Noting that $\frac{q_{ab}}{2\pi\sqrt{-1}} = -(\log_a f - \log_b f)$, the holonomy of $\langle f, g \rangle$ along a loop l with base point x_0 in X is given by

$$\exp \frac{1}{2\pi\sqrt{-1}} \left(\int_l \log f \frac{dg}{g} - \log g(x_0) \int_l \frac{df}{f} \right)$$

which is precisely the tame symbol $\{f, g\}_x$ at $x \in X$ in (0.1) by Cauchy's theorem when l is a small loop encircling x.

2. Massey products and polysymbols

In this section, generalizing the construction in 1.5, we introduce *polysymbols* as the Massey products in the Deligne cohomology, and compute their holonomies explicitly in terms of Chen's iterated integrals. For Massey products and iterated integrals in general contexts, we refer to [My] and [C2].

Let $\overline{X}$ be a closed Riemann surface and let $f_1, \cdots, f_n$ be meromorphic functions on $\overline{X}$. Set $X = \overline{X} \setminus \cup_{i=1}^n \mathrm{supp}(f_i)$ so that $f_i \in H^0(X, \mathcal{O}_X^\times) = H_D^1(X, \mathbb{Z}(1))$ by 1.1.

In the following, we omit the subscripts $a, ab, \cdots$ in the Čech cochains, which stand for open subsets $U_a, U_{ab}, \cdots$, for the sake of simplicity.

Definition 2.1. The *Massey product* $\langle f_1, \ldots, f_n \rangle$ is said to be *defined* if there is an array A of Čech 1-cochains

$$A = \{\alpha_{i_1 \cdots i_k} = (q_{i_1 \cdots i_k}, \log f_{i_1 \cdots i_k}) \in C^1(\mathbb{Z}(k)) \oplus C^0(\mathcal{O}_X) \,|\, 1 \leq k \leq n-1, i_{p+1} = i_p + 1 (\forall p)\}$$

such that
(2.1.1) $\alpha_i = (q_i, \log f_i)$ is a 1-cocycle representing f_i for $1 \leq i \leq n$,
(2.1.2) $D\alpha_{i_1 \cdots i_k} = \alpha_{i_1 \cdots i_k - 1} \cup \alpha_{i_k} + \alpha_{i_1 \cdots i_k - 2} \cup \alpha_{i_k - 1 i_k} + \cdots + \alpha_{i_1} \cup \alpha_{i_2 \cdots i_k}$ for $1 \leq k \leq n-1, i_{p+1} = i_p + 1$ where D stands for the boundary operator of the Deligne cohomology.
An array A is called a *defining system* for $\langle f_1, \ldots, f_n \rangle$. The *value* of the Massey product relative to a defining system A, denoted by $\langle f_1, \ldots, f_n \rangle_A$, is then defined to be the cohomology class of $H_D^2(X, \mathbb{Z}(n))$ represented by the 2-cocycle

$$c(A) := \alpha_{1 \cdots n-1} \cup \alpha_n + \alpha_{1 \cdots n-2} \cup \alpha_{n-1 n} + \cdots + \alpha_1 \cup \alpha_{2 \cdots n}.$$

The Massey product $\langle f_1, \ldots, f_n \rangle$ itself is usually taken to be the subset of $H_D^2(X, \mathbb{Z}(n))$ consisting of $\langle f_1, \ldots, f_n \rangle_A$ for some defining system A.

Since $\dim X = 1$, the multiplication by $\frac{1}{(2\pi\sqrt{-1})^{n-2}}$ induces the isomorphism

$$H_D^2(X, \mathbb{Z}(n)) \simeq H_D^2(X, \mathbb{Z}(2)) \tag{2.2}$$

and hence each $\langle f_1, \ldots, f_n \rangle_A$ is identified with an isomorphism class of line bundles over X with holomorphic connection by 1.3 via (2.2). We also note that a line bundle over X with holomorphic connection is flat, namely its curvature is zero, since $\dim X = 1$.

Definition 2.3. We call $\langle f_1, \ldots, f_n \rangle_A$ and the associated isomorphism class of line bundles over X with holomorphic connection a *polysymbol* relative to A, and call $\langle f_1, \ldots, f_n \rangle$ simply the *polysymbol* for meromorphic functions $f_1, \ldots, f_n$.

For $x \in \overline{X}$, let $H_{l_x}(\langle f_1, \ldots, f_n \rangle_A)$ denote the holonomy of $\langle f_1, \ldots, f_n \rangle_A$ along a small loop l_x encircling x and we define the *polysymbol at* x, denoted by $\{f_1, \ldots, f_n\}_x$, by the map

$$\{f_1, \ldots, f_n\}_x \ : \ \langle f_1, \ldots, f_n \rangle \to \mathbb{C}; \ \langle f_1, \ldots, f_n \rangle_A \mapsto H_{l_x}(\langle f_1, \ldots, f_n \rangle_A).$$

It is easily seen by Stokes' theorem and the flatness of $\langle f_1, \ldots, f_n \rangle_A$ that the map $\{f_1, \ldots, f_n\}_x$ is well-defined, namely depends only on $f_1, \ldots, f_n$ and x.

Remark 2.4. For $n = 2$, $\langle f_1, f_2 \rangle$ consists of a single class $f_1 \cup f_2$ and so the map $\{f_1, f_2\}_x$ in Definition 2.3 is identified with the classical tame symbol by 1.5.

Now, we compute the holonomy of a polysymbol $\langle f_1, \ldots, f_n \rangle_A$ relative to a defining system A in terms of Chen's iterated integrals ([C2]). Recall that for 1-forms $w_1, \ldots, w_n$ on X and a path $\gamma : [0,1] \to X$ with $\gamma^* w_i = F_i(t)dt$ $(1 \leq i \leq n)$, the iterated integral $\displaystyle\int_\gamma w_1 \cdots w_n$ is defined by

$$\int_\gamma w_1 \cdots w_n := \int_{0 \leq t_1 < \cdots < t_n \leq 1} F_1(t_1) \cdots F_n(t_n) dt_1 \cdots dt_n.$$

Theorem 2.5. *For a loop l with base point x_0 in X, the holonomy of the polysymbol $\langle f_1, \ldots, f_n \rangle_A$ relative to a defining system A (Definition 2.1) along l, denoted by $H_l(\langle f_1, \ldots, f_n \rangle_A)$, is given by*

$$H_l(\langle f_1, \ldots, f_n \rangle_A) = \exp\left(\frac{1}{(2\pi\sqrt{-1})^{n-1}} M_{12\cdots n}(l)\right)$$

where $M_{12\cdots n}(l)$ is given by

$$\begin{aligned}
&\int_l \frac{df_1}{f_1} \cdots \frac{df_n}{f_n} + \log f_1(x_0) \int_l \frac{df_2}{f_2} \cdots \frac{df_n}{f_n} + \cdots + \log f_{1\cdots n-1}(x_0) \int_l \frac{df_n}{f_n} \\
&\quad + \sum_{\substack{i+j+k=n \\ i \geq 0, j \geq 1, k \geq 1}} \sum_{k_1 + \cdots + k_p = k} (-1)^p \log f_{1\cdots i}(x_0) \int_l \frac{df_{i+1}}{f_{i+1}} \cdots \frac{df_{i+j}}{f_{i+j}} \\
&\qquad \times \log f_{i+j+1\cdots i+j+k_1}(x_0) \log f_{i+j+k_1+1\cdots i+j+k_1+k_2}(x_0) \ldots \log f_{i+j+k_1+\cdots+k_{p-1}+1\cdots n}(x_0).
\end{aligned}$$

For example, we have

$$\begin{aligned}
M_{12}(l) &= \int_l \frac{df_1}{f_1} \frac{df_2}{f_2} + \log f_1(x_0) \int_l \frac{df_2}{f_2} - \log f_2(x_0) \int_l \frac{df_1}{f_1}, \\
M_{123}(l) &= \int_l \frac{df_1}{f_1} \frac{df_2}{f_2} \frac{df_3}{f_3} + \log f_1(x_0) \int_l \frac{df_2}{f_2} \frac{df_3}{f_3} + \log f_{12}(x_0) \int_l \frac{df_3}{f_3} - \int_l \frac{df_1}{f_1} \log f_{23}(x_0) \\
&\quad - \int_l \frac{df_1}{f_1} \frac{df_2}{f_2} \log f_3(x_0) - \log f_1(x_0) \int_l \frac{df_2}{f_2} \log f_3(x_0) + \int_l \frac{df_1}{f_1} \log f_2(x_0) \log f_3(x_0).
\end{aligned}$$

Proof. First, note that the condition (2.1.2) is expressed in terms of cocycles by

$$(2.6)\quad \left(\delta q_{i_1\cdot\cdot i_k}, -q_{i_1\cdot\cdot i_k}+\delta\log f_{i_1\cdot\cdot i_k}, \frac{df_{i_1\cdot\cdot i_k}}{f_{i_1\cdot\cdot i_k}}\right)$$
$$= \begin{pmatrix} q_{i_1\cdot\cdot i_k-1}q_{i_k}+q_{i_1\cdot\cdot i_k-2}q_{i_k-1 i_k}+\cdots+q_{i_1}q_{i_2\cdot\cdot i_k}, \\ q_{i_1\cdot\cdot i_k-1}\log f_{i_k}+q_{i_1\cdot\cdot i_k-2}\log f_{i_k-1 i_k}+\cdots+q_{i_1}\log f_{i_2\cdot\cdot i_k}, \log f_{i_1\cdot\cdot i_k-1}\frac{df_{i_k}}{f_{i_k}} \end{pmatrix}$$

In particular, we have

$$(2.7)\qquad q_{1\cdot\cdot k}=\delta\log f_{1\cdot\cdot k}-(q_{1\cdot\cdot k-1}\log f_k+q_{1\cdot\cdot k-2}\log f_{k-1k}+\cdots+q_1\log f_{2\cdot\cdot k}).$$

The cocycle $c(A)$ is expressed by

$$(2.8)\quad c(A)=\left(*, q_{1\cdot\cdot n-1}\log f_n+q_{1\cdot\cdot n-2}\log f_{n-1n}+\cdots+q_1\log f_{2\cdot\cdot n}, \log f_{1\cdot\cdot n-1}\tfrac{df_n}{f_n}\right).$$

By (2.7) and (2.8), we have

$$c(A)$$
$$=\begin{pmatrix} *, \\ \displaystyle\sum_{\substack{l+k=n\\ l\ge 1,k\ge 1}}\sum_{k_1+\cdots+k_p=k}(-1)^{p-1}\delta\log f_{1\cdot\cdot l}\log f_{l+1\cdot\cdot l+k_1}\log f_{l+k_1+1\cdot\cdot l+k_1+k_2}\ldots\log f_{l+k_1+\cdot\cdot+k_{p-1}+1\cdot\cdot n}, \\ \log f_{1\cdot\cdot n-1}\frac{df_n}{f_n} \end{pmatrix}.$$

Using the relation $\frac{df_{i_1\cdot\cdot i_k}}{f_{i_1\cdot\cdot i_k}}=\log f_{i_1\cdot\cdot i_k-1}\frac{df_{i_k}}{f_{i_k}}$ from (2.6), the contribution of the term $\log f_{1\cdot\cdot n}\frac{df_n}{f_n}$ to the holonomy is exp of

$$\begin{aligned}
\int_l \log f_{1\cdot\cdot n-1}\frac{df_n}{f_n} &= \log f_{1\cdot\cdot n-1}(x_0)\int_l\frac{df_n}{f_n}+\int_l\frac{df_{1\cdot\cdot n-1}}{f_{1\cdot\cdot n-1}}\frac{df_n}{f_n}\\
&= \log f_{1\cdot\cdot n-1}(x_0)\int_l\frac{df_n}{f_n}+\int_l\left(\log f_{1\cdot\cdot n-2}\frac{df_{n-1}}{f_{n-1}}\right)\frac{df_n}{f_n}\\
&= \log f_{1\cdot\cdot n-1}(x_0)\int_l\frac{df_n}{f_n}+\log f_{1\cdot\cdot n-2}(x_0)\int_l\frac{df_{n-1}}{f_{n-1}}\frac{df_n}{f_n}+\int_l\frac{df_{1\cdot\cdot n-2}}{f_{1\cdot\cdot n-2}}\frac{df_{n-1}}{f_{n-1}}\frac{df_n}{f_n}\\
&\quad\ldots\\
&= \log f_{1\cdot\cdot n-1}(x_0)\int_l\frac{df_n}{f_n}+\log f_{1\cdot\cdot n-2}(x_0)\int_l\frac{df_{n-1}}{f_{n-1}}\frac{df_n}{f_n}+\ldots\\
&\quad+\log f_1(x_0)\int_l\frac{df_2}{f_2}\ldots\frac{df_n}{f_n}+\int_l\frac{df_1}{f_1}\ldots\frac{df_n}{f_n}.
\end{aligned}$$

The contribution of the term

$$(-\delta\log f_{1\cdot\cdot l})\log f_{l+1\cdot\cdot l+k_1}\log f_{l+k_1+1\cdot\cdot l+k_1+k_2}\ldots\log f_{l+k_1+\cdot\cdot+k_{p-1}+1\cdot\cdot n}$$

to the holonomy is exp of

$$
\begin{aligned}
&\left(\int_l \frac{df_{1\cdot\cdot l}}{f_{1\cdot\cdot l}}\right) \log f_{l+1\cdot\cdot l+k_1}(x_0) \log f_{l+k_1+1\cdot\cdot l+k_1+k_2}(x_0) \ldots \log f_{l+k_1+\cdot\cdot+k_{p-1}+1\cdot\cdot n}(x_0) \\
&\quad = \left(\int_l \log f_{1\cdot\cdot l-1} \frac{df_l}{f_l}\right) \log f_{l+1\cdot\cdot l+k_1}(x_0) \log f_{l+k_1+1\cdot\cdot l+k_1+k_2}(x_0) \ldots \log f_{l+k_1+\cdot\cdot+k_{p-1}+1\cdot\cdot n}(x_0) \\
&\quad = \Bigg(\log f_{1\cdot\cdot l-1}(x_0) \int_l \frac{df_l}{f_l} + \log f_{1\cdot\cdot l-2}(x_0) \int_l \frac{df_{l-1}}{f_{l-1}} \frac{df_l}{f_l} + \cdots + \log f_1(x_0) \int_l \frac{df_2}{f_2} \ldots \frac{df_l}{f_l} \\
&\qquad\quad + \int_l \frac{df_1}{f_1} \ldots \frac{df_l}{f_l}\Bigg) \log f_{l+1\cdot\cdot l+k_1}(x_0) \log f_{l+k_1+1\cdot\cdot l+k_1+k_2}(x_0) \ldots \log f_{l+k_1+\cdot\cdot+k_{p-1}+1\cdot\cdot n}(x_0) \\
&\quad = \sum_{\substack{i+j=l \\ i\geq 0, j\geq 1}} \log f_{1\cdot\cdot i}(x_0) \int_l \frac{df_{i+1}}{f_{i+1}} \ldots \frac{df_{i+j}}{f_{i+j}} \\
&\qquad\qquad \times \log f_{i+j+1\cdot\cdot i+j+k_1}(x_0) \log f_{i+j+k_1+1\cdot\cdot i+j+k_1+k_2}(x_0) \ldots \log f_{i+j+k_1+\cdot\cdot+k_{p-1}+1\cdot\cdot n}(x_0).
\end{aligned}
$$

Since $\langle f_1, \ldots, f_n \rangle_A$ corresponds to the class $[\frac{1}{(2\pi\sqrt{-1})^{n-1}} c(A)] \in H^2(X, \mathbb{Z}(2))$, getting all these together, we obtain the desired formula. $\square$

3. Geometirc construction of polysymbols

In this section, we give a geometric construction of the isomorphism class of line bundles with holomorphic connection represented by the polysymbol $\langle f_1, \cdots, f_n \rangle_A$ relative to a defining system A introduced in Section 2. Our method may be regarded as a natural generalization of Bloch-Ramakrishnan ([Bl],[R]) and Hain's ([H]). We keep the same notations as in Section 2.

Let $N = N_{n+1}$ be the Heisenberg group of degree $n+1$ so that the set of R-valued points of N for any commutative ring R is given by

$$
N(R) := \left\{ \begin{pmatrix}
1 & x_1 & x_{12} & \ldots & x_{12\ldots n-1} & x_{12\ldots n} \\
0 & 1 & x_2 & x_{23} & \ldots & x_{23\ldots n} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & x_{n-1} & x_{n-1n} \\
 & & & & 1 & x_n \\
0 & \ldots & & & 0 & 1
\end{pmatrix} \mid x_{i_1\cdot\cdot i_k} \in R \right\}.
$$

We set

$$C := \{ \begin{pmatrix} 1 & 0 & & \dots & 0 & z \\ 0 & 1 & 0 & & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \\ \vdots & & & \ddots & 1 & 0 \\ 0 & \dots & & \dots & 0 & 1 \end{pmatrix} \mid z \in \mathbb{C}\} = \text{the center of } N(\mathbb{C}) \simeq \mathbb{C},$$

$$B := N(\mathbb{Z})\backslash N(\mathbb{C})/C.$$

We fix base points $x_0 \in X$ and $N(\mathbb{Z})AC \in B$ where the matrix

$$A := \begin{pmatrix} 1 & a_1 & a_{12} & \dots & a_{12\dots n-1} & 0 \\ & 1 & a_2 & a_{23} & \dots & a_{23\dots n} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & a_{n-1} & a_{n-1n} \\ & & & & 1 & a_n \\ & & & & & 1 \end{pmatrix}.$$

For simplicity, we set

$$\omega_i := \frac{1}{2\pi\sqrt{-1}} \frac{df_i}{f_i}.$$

We then define a holomorphic map

$$T(f_1, \dots, f_n)_A \ : \ X \longrightarrow B$$

by

$$T(f_1, \dots, f_n)_A(x) := N(\mathbb{Z})A \begin{pmatrix} 1 & \int_{\gamma_x} \omega_1 & \int_{\gamma_x} \omega_1\omega_2 & \dots & \int_{\gamma_x} \omega_1 \dots \omega_{n-1} & 0 \\ & 1 & \int_{\gamma_x} \omega_2 & \int_{\gamma_x} \omega_2\omega_3 & \dots & \int_{\gamma_x} \omega_2 \dots \omega_n \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & \int_{\gamma_x} \omega_{n-1} & \int_{\gamma_x} \omega_{n-1}\omega_n \\ & & & & 1 & \int_{\gamma_x} \omega_n \\ & & & & & 1 \end{pmatrix} C$$

where γ_x is a path from x_0 to x in X.

For a loop l based at x_0, we define $m'_{i_1\dots i_k}(l)$ and $m_{i_1\dots i_k}(l) \in \mathbb{C}$ inductively as follows:

$$(3.1) \qquad m'_i(l) := \int_l \omega_i =: m_i(l), \quad m'_{i_1\dots i_k}(l) := \int_l \omega_{i_1}\dots\omega_{i_k},$$
$$m_{i_1\dots i_k}(l) := m'_{i_1\dots i_k}(l) + a_{i_1}m'_{i_2\dots i_k}(l) + a_{i_1i_2}m'_{i_3\dots i_k}(l) + \dots + a_{i_1\dots i_{k-1}}m'_{i_k}(l)$$
$$- (m_{i_1\dots i_{k-1}}(l)a_{i_k} + m_{i_1\dots i_{k-2}}(l)a_{i_{k-1}i_k} + \dots + m_{i_1}(l)a_{i_2\dots i_k}).$$

For example, we have

$$\begin{aligned}
m_{12}(l) &= m'_{12}(l) + a_1m'_2(l) - m_1(l)a_2 \\
&= \int_l \omega_1\omega_2 + a_1\int_l \omega_2 - \int_l \omega_1 a_2, \\
m_{123}(l) &= m'_{123}(l) + a_1m'_{23}(l) + a_{12}m'_3(l) - (m_1(l)a_{23} + m_{12}(l)a_3) \\
&= \int_l \omega_1\omega_2\omega_3 + a_1\int_l \omega_2\omega_3 + a_{12}\int_l \omega_3 - \left(\int_l \omega_1 a_{23} + \int_l \omega_1\omega_2 a_3 + a_1\int_l \omega_2 a_3 - \int_l \omega_1 a_2 a_3\right), \\
&\dots \\
m_{12\dots n}(l) &= m'_{12\dots n}(l) + a_1 m'_{2\dots n}(l) + \dots + a_{1\dots n-1}m'_n(l) \\
&\quad - (m_{1\dots n-1}(l)a_n + m_{1\dots n-2}(l)a_{n-1n} + \dots + m_1(l)a_{2\dots n}) \\
&= \int_l \omega_1\dots\omega_n + a_1\int_l \omega_2\dots\omega_n + \dots + a_{1\cdot\cdot n-1}\int_l \omega_n \\
&\quad - \sum_{\substack{i+j+k=n \\ i\geq 0, j\geq 1, k\geq 1}} \sum_{k_1+\dots+k_p=k} (-1)^{p-1} a_{1\cdot\cdot i}\int_l \omega_{i+1}\dots\omega_{i+j} \\
&\qquad \times a_{i+j+1\cdot\cdot i+j+k_1} a_{i+j+k_1+1\cdot\cdot i+j+k_1+k_2}\dots a_{i+j+k_1+\cdot\cdot+k_{p-1}+1\cdot\cdot n}.
\end{aligned}$$

Lemma 3.2. *Under the assumption that*

$$m_{i_1\dots i_k}(l) \in \mathbb{Z} \ \ (1 \leq k \leq n-1, i_{p+1} = i_p + 1, [l] \in \pi_1(X, x_0)),$$

the map $T(f_1, \cdots, f_n)_A$ *does not depend on the choice of a path* γ_x.

Proof. Take another path γ'_x and set $l = \gamma'_x \vee \gamma_x^{-1}$. By the formula (1.6.1) of [C2], we have

$$\begin{aligned}
\int_{\gamma'_x} \omega_{i_1}\cdots\omega_{i_k} &= \int_{l\gamma_x} \omega_{i_1}\cdots\omega_{i_k} \\
&= \sum_{0\leq p\leq k} \int_l \omega_{i_1}\cdots\omega_{i_p}\int_{\gamma_x} \omega_{i_{p+1}}\cdots\omega_{i_k} \\
&= \sum_{0\leq p\leq k} m'_{i_1\dots i_{k-p}}(l)\int_{\gamma_x} \omega_{i_{k-p+1}}\dots\omega_{i_k}.
\end{aligned}$$

Writing simply $m_{i_1 \ldots i_k}, m'_{i_1 \ldots i_k}$ for $m_{i_1 \ldots i_k}(l), m'_{i_1 \ldots i_k}(l)$ respectively, we have

$$
\begin{aligned}
& A \begin{pmatrix}
1 & \int_{\gamma'_x} \omega_1 & \int_{\gamma'_x} \omega_1\omega_2 & \ldots & \int_{\gamma'_x} \omega_1 \ldots \omega_{n-1} & 0 \\
 & 1 & \int_{\gamma'_x} \omega_2 & \int_{\gamma'_x} \omega_2\omega_3 & \ldots & \int_{\gamma'_x} \omega_2 \ldots \omega_n \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & \int_{\gamma'_x} \omega_{n-1} & \int_{\gamma'_x} \omega_{n-1}\omega_n \\
 & & & & 1 & \int_{\gamma'_x} \omega_n \\
 & & & & & 1
\end{pmatrix} \\
= {} & A \begin{pmatrix}
1 & m'_1 + \int_{\gamma_x} \omega_1 & m'_{12} + m'_1 \int_{\gamma_x} \omega_2 & \ldots & \sum_{0 \le p \le n-1} m'_{1 \ldots n-p-1} \int_{\gamma_x} \omega_{n-p} \cdots \omega_{n-1} & 0 \\
 & 1 & m'_2 + \int_{\gamma_x} \omega_2 & \ldots & \ldots & \sum_{0 \le p \le n-1} m'_{2 \ldots n-p} \int_{\gamma_x} \omega_{n-p+1} \cdots \omega_n \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & m'_{n-1} \int_{\gamma_x} \omega_{n-1} & m'_{n-1n} + m'_{n-1} \int_{\gamma_x} \omega_n \\
 & & & & 1 & m'_n \int_{\gamma_x} \omega_n \\
 & & & & & 1
\end{pmatrix} \\
= {} & A \begin{pmatrix}
1 & m'_1 & m'_{12} & \ldots & m'_{12 \ldots n-1} & 0 \\
 & 1 & m'_2 & m'_{23} & \ldots & m'_{23 \ldots n} \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & m'_{n-1} & m'_{n-1n} \\
 & & & & 1 & m'_n \\
 & & & & & 1
\end{pmatrix}
\begin{pmatrix}
1 & \int_{\gamma_x} \omega_1 & \int_{\gamma_x} \omega_1\omega_2 & \ldots & \int_{\gamma_x} \omega_1 \ldots \omega_{n-1} & 0 \\
 & 1 & \int_{\gamma_x} \omega_2 & \int_{\gamma_x} \omega_2\omega_3 & \ldots & \int_{\gamma_x} \omega_2 \ldots \omega_n \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & \int_{\gamma_x} \omega_{n-1} & \int_{\gamma_x} \omega_{n-1}\omega_n \\
 & & & & 1 & \int_{\gamma_x} \omega_n \\
 & & & & & 1
\end{pmatrix} \\
= {} & \begin{pmatrix}
1 & m_1 & m_{12} & \ldots & m_{12 \ldots n-1} & 0 \\
 & 1 & m_2 & m_{23} & \ldots & m_{23 \ldots n} \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & m_{n-1} & m_{n-1n} \\
 & & & & 1 & m_n \\
 & & & & & 1
\end{pmatrix}
A
\begin{pmatrix}
1 & \int_{\gamma_x} \omega_1 & \int_{\gamma_x} \omega_1\omega_2 & \ldots & \int_{\gamma_x} \omega_1 \ldots \omega_{n-1} & 0 \\
 & 1 & \int_{\gamma_x} \omega_2 & \int_{\gamma_x} \omega_2\omega_3 & \ldots & \int_{\gamma_x} \omega_2 \ldots \omega_n \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & \int_{\gamma_x} \omega_{n-1} & \int_{\gamma_x} \omega_{n-1}\omega_n \\
 & & & & 1 & \int_{\gamma_x} \omega_n \\
 & & & & & 1
\end{pmatrix}
\end{aligned}
$$

where the third equality follows from (3.1). By definition of $T(f_1, \ldots, f_n)_A$, the assumption then implies the conclusion. $\square$

Proposition 3.3. *If the matrix A is given by a defining system for $\langle f_1, \ldots, f_n \rangle$ in Definition 2.1, namely*

$$
a_{i_1 \ldots i_k} = \frac{1}{(2\pi\sqrt{-1})^k} \log f_{i_1 \ldots i_k}(x_0) \ \ (1 \le k \le n-1, i_{p+1} = i_p + 1),
$$

then we have

$$
m_{i_1 \ldots i_k}(l) \in \mathbb{Z} \ \ ([l] \in \pi_1(X, x_0)).
$$

Proof. A defining system A for $\langle f_1, \ldots, f_n \rangle$ also provides a defining system, say A again, for $\langle f_{i_1}, \ldots, f_{i_k} \rangle$ in an obvious manner and its value $\langle f_{i_1}, \ldots, f_{i_k} \rangle_A = 0$ for $1 \le k \le n-1, i_{p+1} = i_p + 1$. Since the holonomy of $\langle f_{i_1}, \ldots, f_{i_k} \rangle_A$ along l is $\exp(2\pi\sqrt{-1} m_{i_1 \ldots i_k}(l))$ by Theorem 2.5 and (3.1), we have $m_{i_1 \ldots i_k}(l) \in \mathbb{Z}$. $\square$

Next, we let $P := N(\mathbb{Z}) \backslash N(\mathbb{C})$ and consider a line bundle

$$
\pi \ : \ P \longrightarrow B
$$

induced by the natural projection. As usual, we identify P with the associated principal $\mathbb{C}^\times$-bundle where $\exp(2\pi\sqrt{-1}\lambda) \in \mathbb{C}^\times$ acts on a fiber $z \in \mathbb{C} \simeq C$ by $z+\lambda$. Let θ be the 1-form on $N(\mathbb{C})$ defined by

$$\begin{aligned}\theta \;&:= \sum_{k=0}^{n-1} \sum_{1\le i_1<i_2<\cdots<i_k\le n-1} (-1)^k x_{1\ldots i_1} x_{i_1+1\ldots i_2} \ldots x_{i_{k-1}+1\ldots i_k} dx_{i_k+1\ldots n} \\ &= \text{the } (1,n+1)\text{-component of } x^{-1}dx.\end{aligned}$$

Proposition 3.4. *The 1-form θ gives a connection form on the bundle P.*

Proof. Since $(yx)^{-1}d(yx) = x^{-1}dx$ for $y \in N(\mathbb{Z})$, θ is left $N(\mathbb{Z})$-invariant and hence boils down to a 1-form on P. To show that θ is a connection form on P, we need to check that (i) θ is a right $\mathbb{C}^\times$-invariant and (ii) θ is a Maurer-Cartan form along fibers ([KN, Ch.II,1]). (i) is, as above, obvious by the definition of θ and (ii) also follows from that θ is of the form

$$\theta = dx_{1\ldots n-1} + (\text{terms without } x_{1\ldots n-1}). \qquad \square$$

Definition 3.5. Under the assumption that

$$m_{i_1\ldots i_k}(l) \in \mathbb{Z} \;\; (1 \le k \le n-1, i_{p+1} = i_p + 1, [l] \in \pi_1(X, x_0)),$$

we define $\langle\langle f_1, \ldots, f_n\rangle\rangle_A$ by the isomorphism class of the pull-back of (P, θ) under $T(f_1, \ldots, f_n)_A$:

$$\langle\langle f_1, \ldots, f_n\rangle\rangle_A := \text{ isom. class of } T(f_1, \ldots, f_n)_A^*(P, \theta).$$

In the following, we shall compute the holonomy $H_l(\langle\langle f_1, \ldots, f_n\rangle\rangle_A)$ of $\langle\langle f_1, \ldots, f_n\rangle\rangle_A$ along $l \in \pi_1(X)$ and show that it coincides with $H_l(\langle f_1, \ldots, f_n\rangle_A)$.

For this, let us consider the map for a path $\gamma : I \to X$

$$s_\gamma \;:\; I := [0,1] \longrightarrow P$$

defined by

$$s_\gamma(t) := AZ(t),$$

$$Z(t) := \begin{pmatrix} 1 & \int_{\gamma_t}\omega_1 & \int_{\gamma_t}\omega_1\omega_2 & \ldots & \int_{\gamma_t}\omega_1\ldots\omega_{n-1} & \int_{\gamma_t}\omega_1\ldots\omega_n \\ & 1 & \int_{\gamma_t}\omega_2 & \int_{\gamma_t}\omega_2\omega_3 & \ldots & \int_{\gamma_t}\omega_2\ldots\omega_n \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & \int_{\gamma_t}\omega_{n-1} & \int_{\gamma_t}\omega_{n-1}\omega_n \\ & & & & 1 & \int_{\gamma_t}\omega_n \\ & & & & & 1 \end{pmatrix}$$

where a path $\gamma_t : I \to X$ is defined by $\gamma_t(t') := \gamma(tt')$, and we let

$$\tilde{s}_\gamma \;:\; I \longrightarrow \langle\langle f_1, \ldots, f_n \rangle\rangle_A$$

be the map defined by $\tilde{s}_\gamma(t) := (\gamma(t), s_\gamma(t))$:

$$\begin{tikzcd} & \langle\langle f_1, \ldots, f_n \rangle\rangle_A \arrow[d] \\ I \arrow[ur, "\tilde{s}_\gamma"] \arrow[r, "\gamma"] & X \end{tikzcd}$$

Theorem 3.6. *The map $\tilde{s}_\gamma$ is a parallel displacement of γ in $\langle\langle f_1, \ldots, f_n \rangle\rangle_A$.*

Proof. Let PX be the space of all paths in X. We regard $\int \omega_{i_1} \ldots \omega_{i_k}$ as a function on PX by

$$\left(\int \omega_{i_1} \ldots \omega_{i_k} \right)(\gamma) := \int_\gamma \omega_{i_1} \ldots \omega_{i_k}.$$

For $\gamma \in PX$, we define $p_\gamma \;:\; I \longrightarrow PX$ by

$$p_\gamma(t) := \gamma_t.$$

Then we have

$$\left(p_\gamma^* \int \omega_{i_1} \ldots \omega_{i_k} \right)(t) = \int_{\gamma_t} \omega_{i_1} \ldots \omega_{i_k}.$$

By Proposition 1.5.2 of [C2], we have

$$\begin{aligned} d_{PX} \int \omega_{i_1} \ldots \omega_{i_k} = & - \sum_{p=1}^{k} \int \omega_{i_1} \ldots d_X\omega_{i_p} \ldots \omega_{i_k} - \sum_{p=1}^{k-1} \int \omega_{i_1} \ldots (\omega_{i_p} \wedge \omega_{i_{p+1}}) \ldots \omega_{i_k} \\ & - \mathrm{ev}_0^* \omega_{i_1} \wedge \int \omega_{i_2} \ldots \omega_{i_k} + \int \omega_{i_1} \ldots \omega_{i_{k-1}} \wedge \mathrm{ev}_1^* \omega_{i_k} \end{aligned}$$

where $\mathrm{ev}_t \;:\; PX \to X$ is defined by $\mathrm{ev}_t(\gamma) := \gamma(t)$. Hence we have

$$\begin{aligned} d_I \int_{\gamma_t} \omega_{i_1} \ldots \omega_{i_k} &= d_I p_\gamma^* \int \omega_{i_1} \ldots \omega_{i_k} \\ &= p_\gamma^* d_{PX} \int \omega_{i_1} \ldots \omega_{i_k} \\ &= p_\gamma^* \left(-\mathrm{ev}_0^* \omega_{i_1} \wedge \int \omega_{i_2} \ldots \omega_{i_k} + \int \omega_{i_1} \ldots \omega_{i_{k-1}} \wedge \mathrm{ev}_1^* \omega_{i_k} \right) \\ &= -\gamma_0^* \omega_{i_1} \wedge \int_{\gamma_t} \omega_{i_2} \ldots \omega_{i_k} + \int_{\gamma_t} \omega_{i_1} \ldots \omega_{i_{k-1}} \wedge \gamma_1^* \omega_{i_k} \\ &= \int_{\gamma_t} \omega_{i_1} \ldots \omega_{i_{k-1}} \wedge \gamma^* \omega_{i_k}. \end{aligned}$$

Since the connection form θ is the $(1,n+1)$-component of $x^{-1}dx$ ($x \in N(\mathbb{C})$) and $s_\gamma(t)^{-1}d_I s_\gamma(t) = (AZ(t))^{-1}d_I(AZ(t)) = Z(t)^{-1}d_I Z(t)$, it suffices to show that the $(1,n+1)$-component of $Z(t)^{-1}d_I Z(t) = 0$. In fact, we have

$$\begin{aligned}
&(1,n+1)\text{-entry of } Z(t)d_I Z(t) \\
&= \sum_{k=0}^{n-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n-1} (-1)^k \int_{\gamma_t} \omega_1 \ldots \omega_{i_1} \int_{g_t} \omega_{i_1+1} \ldots \omega_{i_2} \ldots \\
&\qquad\qquad \times \int_{\gamma_t} \omega_{i_{k-1}+1} \ldots \omega_{i_k} d_I \int_{\gamma_t} \omega_{i_k+1} \ldots \omega_n \\
&= \sum_{k=0}^{n-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n-1} (-1)^k \int_{\gamma_t} \omega_1 \ldots \omega_{i_1} \int_{g_t} \omega_{i_1+1} \ldots \omega_{i_2} \ldots \\
&\qquad\qquad \times \int_{\gamma_t} \omega_{i_{k-1}+1} \ldots \omega_{i_k} d_I \int_{\gamma_t} \omega_{i_k+1} \ldots \omega_{n-1} \wedge \gamma^* \omega_n
\end{aligned}$$

and here the term

$$(-1)^k \int_{\gamma_t} \omega_1 \ldots \omega_{i_1} \int_{g_t} \omega_{i_1+1} \ldots \omega_{i_2} \ldots \int_{\gamma_t} \omega_{i_{k-1}+1} \ldots \omega_{i_k} d_I \int_{\gamma_t} \omega_{i_k+1} \ldots \omega_{n-1} \wedge \gamma^* \omega_n$$

is cancelled out by the term

$$(-1)^{k+1} \int_{\gamma_t} \omega_1 \ldots \omega_{i_1} \int_{g_t} \omega_{i_1+1} \ldots \omega_{i_2} \ldots \int_{\gamma_t} \omega_{i_{k-1}+1} \ldots \omega_{i_k} d_I \int_{\gamma_t} \omega_{i_k+1} \ldots \omega_{n-1} d_I \int_{\gamma_t} \omega_n$$

and therefore the above sum $= 0$. $\square$

By Theorem 3.6, we can compute the holonomy of $\langle\langle f_1, \ldots, f_n \rangle\rangle_A$ as follows.

Theorem 3.7. *Assume that*

$$m_{i_1 \ldots i_k}(l) \in \mathbb{Z} \;\; (1 \le k \le n-1, i_{p+1} = i_p + 1, [l] \in \pi_1(X, x_0)).$$

Then the holonomy $H_l(\langle\langle f_1, \ldots, f_n \rangle\rangle_A)$ of $\langle\langle f_1, \ldots, f_n \rangle\rangle_A$ along l is given by

$$H_l(\langle\langle f_1, \ldots, f_n \rangle\rangle_A) = \exp\left(2\pi\sqrt{-1} m_{12\ldots n}(l)\right).$$

Proof. The initial point of $\tilde{s}_l$ of l based at x_0 is

$$(x_0, s_l(0) = N(\mathbb{Z})AC).$$

The terminal point of $\tilde{s}_l$ is $(x_0, s_l(1))$, where

$$\begin{aligned}
s_l(1) &= N(\mathbb{Z})A\begin{pmatrix}
1 & \int_l \omega_1 & \int_l \omega_1\omega_2 & \dots & \int_l \omega_1\dots\omega_{n-1} & \int_l \omega_1\dots\omega_n \\
 & 1 & \int_l \omega_2 & \int_l \omega_2\omega_3 & \dots & \int_l \omega_2\dots\omega_n \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & \int_l \omega_{n-1} & \int_l \omega_{n-1}\omega_n \\
 & & & & 1 & \int_l \omega_n \\
 & & & & & 1
\end{pmatrix} \\
&= N(\mathbb{Z})A\begin{pmatrix}
1 & m'_1 & m'_{12} & \dots & m'_{12\dots n-1} & m'_{12\dots n} \\
 & 1 & m'_2 & m'_{23} & \dots & m'_{23\dots n} \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & m'_{n-1} & m'_{n-1n} \\
 & & & & 1 & m'_n \\
 & & & & & 1
\end{pmatrix} \\
&= N(\mathbb{Z})\begin{pmatrix}
1 & m_1 & m_{12} & \dots & m_{12\dots n-1} & m_{12\dots n} \\
 & 1 & m_2 & m_{23} & \dots & m_{23\dots n} \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & m_{n-1} & m_{n-1n} \\
 & & & & 1 & m_n \\
 & & & & & 1
\end{pmatrix} A \qquad \text{(by (3.1))} \\
&= N(\mathbb{Z})\begin{pmatrix}
1 & m_1 & m_{12} & \dots & m_{12\dots n-1} & 0 \\
 & 1 & m_2 & m_{23} & \dots & m_{23\dots n} \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & m_{n-1} & m_{n-1n} \\
 & & & & 1 & m_n \\
 & & & & & 1
\end{pmatrix}\begin{pmatrix}
1 & a_1 & a_{12} & \dots & a_{12\dots n-1} & a_{12\dots n} + m_{12\dots n} \\
 & 1 & a_2 & a_{23} & \dots & a_{23\dots n} \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & a_{n-1} & a_{n-1n} \\
 & & & & 1 & a_n \\
 & & & & & 1
\end{pmatrix} \\
&= N(\mathbb{Z})\begin{pmatrix}
1 & a_1 & a_{12} & \dots & a_{12\dots n-1} & a_{12\dots n} + m_{12\dots n} \\
 & 1 & a_2 & a_{23} & \dots & a_{23\dots n} \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & 1 & a_{n-1} & a_{n-1n} \\
 & & & & 1 & a_n \\
 & & & & & 1
\end{pmatrix} \qquad \text{(by assumption)}
\end{aligned}$$

where we write simply $m_{i_1\dots i_k}, m'_{i_1\dots i_k}$ for $m_{i_1\dots i_k}(l), m'_{i_1\dots i_k}(l)$ respectively. Hence the holonomy $H_l(\langle\langle f_1, \dots, f_n\rangle\rangle)$ is given by $\exp(2\pi\sqrt{-1}m_{12\dots n})$. □

Theorem 3.8. *If the matrix A is given by a defining system for $\langle f_1, \dots, f_n\rangle$ in Definition 2.1, namely*

$$a_{i_1\dots i_k} = \frac{1}{(2\pi\sqrt{-1})^k}\log f_{i_1\dots i_k}(x_0) \ \ (1 \le k \le n-1, i_{p+1} = i_p + 1),$$

then we have

$$\langle f_1, \dots, f_n\rangle_A = \langle\langle f_1, \dots, f_n\rangle\rangle_A.$$

Proof. By Proposition 3.3, the assumption of Theorem 3.7 is satisfied and hence

$H_l(\langle\langle f_1, \ldots, f_n\rangle\rangle_A) = \exp\left(2\pi\sqrt{-1}m_{12\ldots n}(l)\right)$. Since $M_{12\ldots n}(l) = (2\pi\sqrt{-1})^n m_{12\ldots n}(l)$ by Theorem 2.5 and (3.1), we have $H_l(\langle f_1, \ldots, f_n\rangle_A) = H_l(\langle\langle f_1, \ldots, f_n\rangle\rangle_A)$ for all $[l] \in \pi_1(X, x_0)$. Since the isomorphism class of a flat line bundle is determined by its holonomy representation, $\langle f_1, \cdots, f_n\rangle_A$ coincides with $\langle\langle f_1, \cdots, f_n\rangle\rangle_A$. $\square$

4. Properties of polysymbols

In this section, using our holonomy formula, Theorem 2.5, we show some basic properties of polysymbols which generalize those of the classical tame symbol. We keep the same notations as in Sections 2 and 3.

Proposition 4.1 (multiplicativity). *Assume that $f_j = f'_j \cdot f''_j$ for meromorphic functions f'_j, f''_j on $\overline{X}$. Suppose that $A' = \{(q'_{i_1\ldots i_k}, \log f'_{i_1\ldots i_k})\}$ and $A'' = \{(q''_{i_1\ldots i_k}, f''_{i_1\ldots i_k})\}$ are defining systems for $\langle f_1, \ldots, f'_j, \ldots, f_n\rangle$ and $\langle f_1, \ldots, f''_j, \ldots f_n\rangle$ respectively as in Definition 2.1 such that $f'_{i_1\ldots i_k} = f''_{i_1\ldots i_k}$ if $j \notin \{i_1, \ldots, i_k\}$. Then an array $A = \{(q_{i_1\ldots i_k}, \log f_{i_1\ldots i_k})\}$ defined by*

$$q_{i_1\ldots i_k} := \begin{cases} q'_{i_1\ldots i_k} = q''_{i_1\ldots i_k} & j \notin \{i_1, \ldots, i_k\}, \\ q'_{i_1\ldots i_k} \cdot q''_{i_1\ldots i_k}, & j \in \{i_1, \ldots, i_k\} \end{cases} \quad f_{i_1\ldots i_k} := \begin{cases} f'_{i_1\ldots i_k} = f''_{i_1\ldots i_k} & j \notin \{i_1, \ldots, i_k\}, \\ f'_{i_1\ldots i_k} \cdot f''_{i_1\ldots i_k}, & j \in \{i_1, \ldots, i_k\} \end{cases}$$

gives a defining system for $\langle f_1, \ldots, f'_j f''_j, \ldots, f_n\rangle$ and we have

$$\langle f_1, \ldots, f'_j f''_j, \ldots, f_n\rangle_A = \langle f_1, \ldots, f'_j, \ldots, f_n\rangle_{A'} + \langle f_1, \ldots, f''_j, \ldots, f_n\rangle_{A''}.$$

Proof. It is easy to see that A is a defining system for $\langle f_1, \ldots, f'_j f''_j, \ldots, f_n\rangle$ under the assumption. By Theorem 2.5 and by the general formulas

$$\log(fg)(x_0) = \log f(x_0) + \log g(x_0), \frac{d(fg)}{fg} = \frac{df}{f} + \frac{dg}{g},$$

we have

$$H_l(\langle f_1, \ldots, f'_j f''_j, \ldots, f_n\rangle_A) = H_l(\langle f_1, \ldots, f'_j, \ldots, f_n\rangle_{A'}) \cdot H_l(\langle f_1, \ldots, f''_j, \ldots, f_n\rangle_{A''})$$

for any $[l] \in \pi_1(X, x_0)$. This proves the assertion. $\square$

Proposition 4.2 (symmetric relation). *Let $\mathfrak{S}_n$ be the symmetric group on $\{1, \ldots, n\}$. For $\sigma \in \mathfrak{S}_n$, let $\sigma(A) = \{(q_{\sigma(i_1)\ldots\sigma(i_k)}, f_{\sigma(i_1)\ldots\sigma(i_k)})\}$ be a defining system for $\langle f_{\sigma(1)}, \ldots, f_{\sigma(n)}\rangle$. Then we have*

$$\sum_{\sigma \in \mathfrak{S}_n} \langle f_{\sigma(1)}, \ldots, f_{\sigma(n)}\rangle_{\sigma(A)} = 0.$$

Proof. By Theorem 2.5 and cancellation in pairs, we have

$$\prod_{\sigma\in\mathfrak{S}_n} H_l(\langle f_{\sigma(1)},\dots,f_{\sigma(n)}\rangle_{\sigma(A)}) = \exp\left(\frac{1}{(2\pi\sqrt{-1})^{n-1}}\sum_{\sigma\in\mathfrak{S}_n}\int_l \frac{df_{\sigma(1)}}{f_{\sigma(1)}}\cdots\frac{df_{\sigma(n)}}{f_{\sigma(n)}}\right)$$

for any $[l]\in\pi_1(X,x_0)$. Using the general formula (1.5.1) of [C1]

$$\int_l w_1\cdots w_r\int_l w_{r+1}\cdots w_{r+s} = \sum_{\sigma\in SH}\int_l w_{\sigma(1)}\cdots w_{\sigma(r+s)}$$

where SH denotes the the set of all (r,s)-shuffles, i.e. permutations σ with $\sigma^{-1}(1)<\cdots<\sigma^{-1}(r),\sigma^{-1}(r+1)<\cdots<\sigma^{-1}(r+s)$, we have

$$\sum_{\sigma\in\mathfrak{S}_n}\int_l \frac{df_{\sigma(1)}}{f_{\sigma(1)}}\cdots\frac{df_{\sigma(n)}}{f_{\sigma(n)}} = \int_l\frac{df_1}{f_1}\cdots\int_l\frac{df_n}{f_n}\in(2\pi\sqrt{-1})^n\mathbb{Z}.$$

Hence we have

$$\prod_{\sigma\in\mathfrak{S}_n} H_l(\langle f_{\sigma(1)},\dots,f_{\sigma(n)}\rangle_{\sigma(A)}) = 1$$

for any $[l]\in\pi_1(X,x_0)$. This proves the assertion. $\square$

The following theorem is regarded as a generalization of the classical reciprocity law of Tate and Weil (Ch.III,4 of [Se]).

Theorem 4.3 (reciprocity law). *Assume that $\langle f_1,\dots,f_n\rangle$ is defined. Then we have the following product formula*

$$\prod_{x\in\overline{X}}\{f_1,\dots,f_n\}_x = 1.$$

Proof. Let Y be the surface obtained by removing from $\overline{X}$ small open disks centered at points in $\cup_{i=1}^n\mathrm{supp}(f_i)$ and let $\partial Y = l_1\cup\cdots\cup l_N$ (disjoint union) be the boundary of Y. Then for any defining system A for $\langle f_1,\dots,f_n\rangle$, we have

$$\begin{aligned}\prod_{x\in\overline{X}} H_{l_x}(\langle f_1,\dots,f_n\rangle_A) &= \prod_{i=1}^N H_{l_i}(\langle f_1,\dots,f_n\rangle_A)\\ &= \exp\left(\int_{\mathrm{Int}(Y)} -\text{curv. of } \langle f_1,\dots,f_n\rangle_A\right) \text{ ([Br, Prop.2.4.6, 6.1.1])}\\ &= 1,\end{aligned}$$

since the curvature of $\langle f_1,\dots,f_n\rangle_A$ is zero. By Definition 2.3, the assertion is proved. $\square$

5. Variation of mixed Hodge structure

In this section, we show that trivializations of polysymbols give variations of mixed Hodge structure (cf. Section 7 of [H]).

First, recall that a *variation of mixed Hodge structure* on a complex manifold X consists of a triple $(V_{\mathbb{Z}}, W_\bullet, F^\bullet)$

(i) a local system $V_{\mathbb{Z}}$ of finitely generated $\mathbb{Z}$-modules on X;

(ii) an increasing filtration $W_\bullet$ of $V_{\mathbb{Z}}$ by local systems of finitely generated $\mathbb{Z}$-modules;

(iii) a decreasing filtration $F^\bullet$ of $V_{\mathbb{Z}} \otimes \mathcal{O}_X$ by holomorphic subbundles

which are required to satisfy

(1) (Griffiths' transversality) $\nabla F^i \subset \Omega^1 \otimes F^{i-1}$

where ∇ is the canonical flat connection on $V_{\mathbb{Z}} \otimes \mathcal{O}_X$;

(2) for each point in X, $W_\bullet$ and $F^\bullet$ define a mixed Hodge structure on each fiber.

Now, let us go back to our previous setting and keep the same notations as in Section 2,3. So, $f_1, \ldots, f_n$ are meromorphic functions on a closed Riemann surface $\overline{X}$ and $X = \overline{X} \setminus \cup_{i=1}^n \mathrm{supp}(f_i)$.

Definition 5.1. A *trivialization* of a polysymbol $\langle f_1, \ldots, f_n \rangle_A$ relative to a defining system $A = \{\alpha_{i_1 \ldots i_k}\} = \{(q_{i_1 \ldots i_k}, \log f_{i_1 \ldots i_k})\}$ is a 1-cochain $\alpha_{1 \ldots n} = (q_{1 \ldots n}, \log f_{1 \ldots n})$ satisfying the relation

$$d\alpha_{1 \ldots n} = \alpha_{1 \ldots n-1} \cup \alpha_n + \cdots + \alpha_1 \cup \alpha_{2 \ldots n}.$$

Assume in the following that we have a trivialization of $\langle f_1, \ldots, f_n \rangle_A$ as in Definition 5.1, which yields $\langle f_1, \ldots, f_n \rangle_A = 0$. We set

$$a_{i_0 \ldots i_k} = \frac{1}{(2\pi\sqrt{-1})^k} \log f_{i_1 \ldots i_k}(x_0), \quad \omega_{i_0 \ldots i_k} = \frac{1}{(2\pi\sqrt{-1})^k} \frac{df_{i_1 \ldots i_k}}{f_{i_1 \ldots i_k}}.$$

For the standard basis $\{e_0, \ldots, e_n\}$ of $\mathbb{C}^{n+1}$, we consider the vectors $v_0, \ldots, v_n$ defined by

$$\begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 1 & a_1 & \ldots & a_{1 \ldots n} \\ & \ddots & \ddots & \vdots \\ & & 1 & a_n \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & \int_{\gamma_x} \omega_1 & \ldots & \int_{\gamma_x} \omega_1 \ldots \omega_n \\ & \ddots & \ddots & \vdots \\ & & 1 & \int_{\gamma_x} \omega_n \\ & & & 1 \end{pmatrix} \begin{pmatrix} e_0 \\ (2\pi\sqrt{-1}) e_1 \\ \vdots \\ (2\pi\sqrt{-1})^n e_n. \end{pmatrix}$$

The proof of Lemma 3.2 shows that the map $F : X \to N(\mathbb{C})$ defined by

$$F(x) = \begin{pmatrix} 1 & a_1 & \dots & a_{1\dots n} \\ & \ddots & \ddots & \vdots \\ & & 1 & a_n \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & \int_{\gamma_x} \omega_1 & \dots & \int_{\gamma_x} \omega_1 \dots \omega_n \\ & \ddots & \ddots & \vdots \\ & & 1 & \int_{\gamma_x} \omega_n \\ & & & 1 \end{pmatrix}$$

modulo $N(\mathbb{Z})$ does not depend on the choice of a path γ_x. Therefore, the $\mathbb{Z}$-span $V_{\mathbb{Z}}(x)$ of the vectors $v_0, \dots, v_n$ is well-defined. These vectors induce an increasing filtration of $V_{\mathbb{Z}}(x)$ defined by

$$W_0 = \mathrm{span}_{\mathbb{Z}}\{v_0, \dots, v_n\}, W_{-1} = \mathrm{span}_{\mathbb{Z}}\{v_1, \dots, v_n\}, \dots, W_{-n} = \mathrm{span}_{\mathbb{Z}}\{v_n\}.$$

In addition, we have a decreasing filtration on $\mathbb{C}^{n+1}$ defined by

$$F^0 = \mathrm{span}_{\mathbb{C}}\{e_0\}, F^{-1} = \mathrm{span}_{\mathbb{C}}\{e_0, e_1\}, \dots, F^{-n} = \mathrm{span}_{\mathbb{C}}\{e_0, \dots e_n\}.$$

Theorem 5.2. *The triple $(V_{\mathbb{Z}}, W_\bullet, F^\bullet)$ defined as above is a variation of mixed Hodge structure on X with $V_{\mathbb{Z}} \otimes \mathcal{O}_X = X \times \mathbb{C}^{n+1}$ whose graded quotients of $W_\bullet$ are $\mathbb{Z}(0), \mathbb{Z}(1), \dots, \mathbb{Z}(n)$.*

Proof. First, we consider a connection ∇ on $X \times \mathbb{C}^{n+1} \to X$ defined by

$$\nabla v = dv - v\omega$$

for a section $v : X \to \mathbb{C}^{n+1}$, where

$$\omega = 2\pi\sqrt{-1} \begin{pmatrix} 0 & \omega_1 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & 0 & \omega_n \\ & & & 0 \end{pmatrix}$$

The connection ∇ is flat because

$$d\omega = 0 \text{ and } \omega \wedge \omega = 0$$

by the fact that each component of ω is a closed and holomorphic 1-form on a 1-dimensional complex manifold X. By the definition of the (multi-valued) map $F : X \to N(\mathbb{C})$, we find that the vectors $v_1, \dots, v_n$ as sections satisfy $\nabla v_i = 0$. Therefore, $W_0, W_{-1}, \dots, W_{-n}$ are local systems on X because the monodromy representation has values in $N(\mathbb{Z})$. The Griffiths' transversality follows from the fact that the connection matrix ω is a strictly upper triangular matrix. $\square$

This theorem means that polysymbols are obstructions to getting variations of mixed Hodge structure.

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