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Koichi Niijima and Koichi Kuzume

Abstract— **An orthogonality condition of convolution type is derived for scaling functions satisfying a two-scale relation. In two spaces of the shifted scaling functions, one of which includes the other space, an inner product different from the L^2 inner product is introduced. The finer scaling function space is decomposed into the coarser one and its orthogonal complement. A wavelet function is constructed so that its shifted functions form an orthonormal basis in the orthogonal complement. Such wavelets contain the Daubechies' compactly supported wavelets as a special case. Also, a symmetric and almost compactly supported wavelet is obtained.**

I. Introduction

Since the discovery of wavelet by Moret [4], various wavelet theories have been developed. Among them, the multiresolution analysis formulated by Mallat [3] is a very important concept to construct wavelet bases. In the mutiresolution analysis, a scaling function satisfying a two-scale relation is first introduced and a wavelet basis is defined as a basis in the orthogonal complement of the scaling function space. Based on the multiresolution analysis, Daubechies has designed compactly supported orthogonal wavelets which give excellent filters for extracting high frequency components from signals. Unfortunately, however, these wavelets do not have linear phase characteristics except for the Haar wavelet. In the signal processing, the linear phase characteristics is more important than the orthogonality. One way for designing wavelet bases with linear phase characteristics is to relax orthogonality conditions. Unser and Aldroubi [5] have constructed biorthogonal wavelets using a two-scale relation of B-spline functions with odd order. These wavelets have linear phase characteristics since they are symmetric functions. However, the use of special functions such as spline functions does not allow us to have freedom to generate new wavelet bases.

In this paper, we develop a biorthogonal wavelet theory without starting from special functions such as B-spline functions. Our approach is rather similar to the Daubechies' approach. We first introduce a scaling function using a two-scale relation. Next, its dual scaling function is defined by another two-scale relation whose coefficients are given by a convolution of free parameters with the coefficients contained in the two-scale relation introduced first. By virtue of the freedom of parameters, a number of scaling functions can be constructed. We impose on the scaling function the condition that it is orthonormal to the shifted dual scaling functions. This condition can be expressed by the Fourier transform of convoluted coefficients. This is described in Section II.

We consider two spaces consisting of shifted scaling functions, one of which includes the other space. These spaces are necessary to be Hilbert spaces to obtain a direct sum decomposition of the finer scaling function space. In other papers such as [1], [2] and [3], the L^2 space equiped with the L^2 norm has been used as the whole space containing all the scaling function spaces. This paper introduces an inner product in the finer scaling function space using the dual scaling function. This inner product is introduced natularly in the coarser scaling function spaces. The norm is defined by the inner product. Thus, the finer scaling function space becomes a Hilbert space and it can be decomposed into the coarser scaling function space and its orthogonal complement. This is discussed in Section III. Since the complement space is a subspace of the finer scaling function space, an element of the complement space can be represented by a linear combination of the basis of the finer scaling function space. Coefficients of the linear combination are characterized by the orthogonality of the dual scaling functions with the wavelet function. It is shown that the wavelet function obtained has the property that its shifted functions form an orthonormal basis in the complement space. This is described in Section IV. In Section V, we derive decomposition and reconstruction formulas for signals using the results in Sections II and IV. Section VI is devoted to show some examples of our wavelets.

II. Orthogonality Condition for Scaling Functions

Let $\varphi(x)$ be a scaling function in $L^2(R)$ satisfying a two-scale relation

$$\varphi(x) = \sum_n \alpha_n \sqrt{2}\, \varphi(2x-n). \tag{1}$$

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Daubechies [?] has found an orthonormality condition for the shifted functions $\{\varphi(\cdot - k), \quad k \in Z\}$ as follows:

$$|u(\zeta)|^2 + |u(\zeta+\pi)|^2 = 1 \qquad \text{a.e.,} \tag{2}$$

where $u(\zeta)$ is given by

$$u(\zeta) = \frac{1}{\sqrt{2}} \sum_n \alpha_n e^{-in\zeta}. \tag{3}$$

In this paper, we extend this result to an orthonormality condition of convolution type by introducing a function related to $\varphi(x)$. We introduce a function $\varphi_p(x)$ in $L^2(R)$ by a two-scale relation

$$\varphi_p(x) = \sum_n (p * \alpha)_n \sqrt{2}\, \varphi_p(2x-n) \tag{4}$$

which contains $p = (p_m)_{m=-M,\dots,M}$ and $\alpha = (\alpha_n)_{n\in Z}$ appeared in (1), where the symbol * denotes the convolution

$$(p*\alpha)_n = \sum_{|n-\ell|\le M} p_{n-\ell}\, \alpha_\ell.$$

If $p_0 = 1$ and $p_m = 0$ for $m \ne 0$, then the function $\varphi_p(x)$ coincides with $\varphi(x)$.

We impose on the scaling function $\varphi(x)$ the orthonormality condition

$$\begin{aligned} (\varphi_p(\cdot - k), \varphi(\cdot)) &\equiv \int \varphi_p(x-k)\varphi(x)dx \\ &= \delta_{k0} \end{aligned} \tag{5}$$

for any $k \in Z$, where δ_{k0} denotes the Kronecker's delta symbol, and Z the set of integers.

For latter use, we define $a(\zeta)$ by

$$a(\zeta) = \sum_{m=-M}^{M} p_m e^{-im\zeta}.$$

The following theorem holds.

Theorem 1: The orthonormality condition (5) is equivalent to

$$a(\zeta)\, |u(\zeta)|^2 + a(\zeta+\pi)\, |u(\zeta+\pi)|^2 = 1 \quad \text{a.e..} \tag{6}$$

Proof: We define the Fourier transform of $f(x)$ by

$$\widehat{f}(\xi) = \int f(x) e^{-i\xi x} dx.$$

The convolution theorem on frequency leads to

$$\begin{aligned} \delta_{k0} &= \int \varphi_p(x-k)\varphi(x)dx \\ &= \frac{1}{2\pi} \int \widehat{\varphi_p}(\xi)\, \overline{\widehat{\varphi}(\xi)}\, e^{ik\xi} d\xi. \end{aligned}$$

The Daubechies' trick yields

$$\delta_{k0} = \frac{1}{2\pi} \int_0^{2\pi} \sum_\ell \widehat{\varphi_p}(\xi + 2\pi\ell)\, \overline{\widehat{\varphi}(\xi+2\pi\ell)}\, e^{ik\xi} d\xi$$

from which we obtain

$$\sum_\ell \widehat{\varphi_p}(\xi+2\pi\ell)\, \overline{\widehat{\varphi}(\xi+2\pi\ell)} = 1 \qquad \text{a.e..} \tag{7}$$

By the Fourier transform of (1), we have

$$\widehat{\varphi}(\xi) = u(\xi/2)\, \widehat{\varphi}(\xi/2). \tag{8}$$

On the other hand, the Fourier transform of (4) gives

$$\widehat{\varphi_p}(\xi) = b(\xi/2)\, \widehat{\varphi_p}(\xi/2),$$

where

$$b(\zeta) = \frac{1}{\sqrt{2}} \sum_n (p*\alpha)_n\, e^{-in\zeta}.$$

Since we have $b(\zeta) = a(\zeta)u(\zeta)$ by an easy calculation, we obtain

$$\widehat{\varphi_p}(\xi) = a(\xi/2)\, u(\xi/2)\, \widehat{\varphi_p}(\xi/2). \tag{9}$$

Substituting (8) and (9) into (7) yields

$$\sum_\ell a(\xi/2+\pi\ell)\, |u(\xi/2+\pi\ell)|^2\, \widehat{\varphi_p}(\xi/2+\pi\ell)\, \overline{\widehat{\varphi}(\xi/2+\pi\ell)} = 1 \quad \text{a.e..} \tag{10}$$

Splitting the sum of (10) into even and odd ℓ, and using the $2\pi-$ periodicity of $a(\zeta)$ and $u(\zeta)$, we obtain (6), where we have put $\zeta = \xi/2$, and applied (7) again.

Remark: In case of $p_0 = 1$ and $p_m = 0$ for $m \ne 0$, we have $a(\zeta) \equiv 1$ and (6) coincides with (2).

From this theorem, we obtain the following result.

Corollary 1. The equation (6) is equivalent to

$$\sum_\ell \left(\sum_{m=-M}^{M} p_m \alpha_{\ell+2k-m} \right) \alpha_\ell = \delta_{k0}, \qquad k \in Z. \tag{11}$$

Proof. Using the definition of $a(\zeta)$ and $u(\zeta)$, the equation (6) can be written as

$$\sum_{m=-M}^{M} p_m \sum_n \sum_\ell \alpha_n\, \alpha_\ell \left(1 + (-1)^{m+n+\ell}\right)\, e^{-i(m+n-\ell)\zeta} = 2.$$

Since the terms in the parenthesis remain only for $m+n-\ell = 2k$, $k \in Z$, we have

$$\sum_k \sum_\ell \left(\sum_{m=-M}^{M} p_m\, \alpha_{\ell+2k-m} \right) \alpha_\ell\, e^{-2ik\zeta} = 1$$

which means (11).

III. Inner Product and Norm in Scaling Function Spaces

Let V_0 be a space spanned by the sifted scaling functions $\varphi(\cdot - k)$, $k \in Z$, that is,

$$V_0 = \{\sum_k c_k \varphi(\cdot - k), \quad \sum_k c_k^2 < +\infty\}.$$

We consider one more space on the scaling function $\varphi(x)$.

$$V_1 = \{\sum_k c_k \sqrt{2}\varphi(2\cdot - k), \quad \sum_k c_k^2 < +\infty\}.$$

By virtue of the two-scale relation (1), the inclusion $V_0 \subset V_1$ holds.

We shall introduce an inner product in the space V_1. The functions f and g in V_1 may be written as

$$f(x) = \sum_k c_k \sqrt{2}\ \varphi(2x - k),$$
$$g(x) = \sum_k d_k \sqrt{2}\ \varphi(2x - k).$$

Using the coefficients c_k of $f(x)$ and the parameters $p = (p_m)$ introduced previously, we define a function

$$f_p(x) = \sum_{k \in Z} (p * c)_k \sqrt{2}\ \varphi_p(2x - k). \qquad (12)$$

If c_n equals to α_n appeared in (1), then $f_p(x)$ gives $\varphi_p(x)$ defined in (4).

Let us define $(f, g)_p$ by

$$(f, g)_p \equiv (f_p, g) = \int f_p(x) g(x) dx. \qquad (13)$$

We assume that the parameters $p = (p_m)$ satisfy three conditions: (H1) $p_m = p_{-m}$, (H2) $\sum_{m=-M}^{M} p_m = 1$, (H3) $p_0 > \sum_{m \neq 0} |p_m|$.

Then we have the following result.

Proposition: $(f, g)_p$ in (13) is an inner product in V_1 and $\|f\|_p = \sqrt{(f, f)_p}$ is a norm in V_1.

Proof: We check the three conditions of inner product.

(i) $(f, g)_p = (g, f)_p$.

Substituting $f_p(x)$ and $g(x)$ into (f_p, g) and using the orthogonality condition $(\varphi_p(\cdot - k), \varphi(\cdot)) = \delta_{k0}$, we have

$$\begin{aligned}(f, g)_p &= \sum_k (p * c)_k d_k \\ &= \sum_m (p * d)_m c_m,\end{aligned}$$

where we have used the symmetry of p_m, that is, (H1).

(ii) $(f, g)_p$ is bilinear.

The proof is easy..

(iii) $(f, f)_p \geq 0$ and $(f, f)_p = 0$ implies $f = 0$.

It follows by the proof of (i) that

$$(f, f)_p = \sum_k (p * c)_k c_k.$$

The estimation of the right hand side from below gives

$$(f, f)_p \geq (p_0 - \sum_{m \neq 0} |p_m|) \sum_k c_k^2.$$

This inequality together with (H3) proves (iii).

It is easily shown that $\|f\|_p$ satisfies the conditions of norm (See [?]).

IV. Wavelet

By Proposition, the space $(V_1, \|\cdot\|_p)$ becomes a Hilbert space. Since the inclusion $V_0 \subset V_1$ holds by virtue of (1), V_1 can be decomposed as

$$V_1 = V_0 \oplus W_0, \qquad V_0 \perp W_0.$$

Since V_0 and W_0 are subspaces of V_1, the norm $\|\cdot\|_p$ in V_1 is naturally introduced both in V_0 and W_0. The space W_0 is called the orthogonal complement of V_0 in V_1. We shall construct an orthonormal basis of W_0 in the form $\{\psi(\cdot - k), \quad k \in Z\}$, where $\psi(x)$ is called a wavelet function. The space W_0 may be written as

$$W_0 = \{ \sum_k d_k \psi(\cdot - k), \quad \sum_k d_k^2 < +\infty \}.$$

By $V_0 \perp W_0$, the wavelet function $\psi(x)$ must satisfy

$$(\varphi(\cdot - k), \psi(\cdot))_p = 0, \quad k \in Z. \qquad (14)$$

From the definition of $(\cdot, \cdot)_p$, we have

$$\int \varphi_p(x - k)\ \psi(x) dx = 0. \qquad (15)$$

Since $\psi \in W_0 \subset V_1$, $\psi(x)$ can be expanded as

$$\psi(x) = \sum_n \beta_n \sqrt{2}\ \varphi(2x - n). \qquad (16)$$

We define $v(\zeta)$ by

$$v(\zeta) = \frac{1}{\sqrt{2}} \sum_n \beta_n\ e^{-in\zeta}.$$

The following theorem holds.

Theorem 2: The condition (15) is equivalent to

$$a(\zeta)\ u(\zeta)\ \overline{v(\zeta)} + a(\zeta + \pi)\ u(\zeta + \pi)\ \overline{v(\zeta + \pi)} = 0 \quad \text{a.e..} \qquad (17)$$

Proof: Since the proof is similar to that of Theorem 1, we describe only the outline. By (15) and using the convolution theorem on frequency and the Daubechies' trick again, we have

$$\begin{aligned} 0 &= \int \varphi_p(x-k)\ \psi(x)dx \\ &= \frac{1}{2\pi}\int \widehat{\varphi_p}(\xi)\ \overline{\widehat{\psi}(\xi)}\ e^{ik\xi}d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_\ell \widehat{\varphi_p}(\xi+2\pi\ell)\ \overline{\widehat{\psi}(\xi+2\pi\ell)}\ e^{ik\xi}d\xi \end{aligned}$$

and hence,

$$\sum_\ell \widehat{\varphi_p}(\xi+2\pi\ell)\ \overline{\widehat{\psi}(\xi+2\pi\ell)} = 0 \quad \text{a.e.} \tag{18}$$

The Fourier transform of (16) gives

$$\widehat{\psi}(\xi) = v(\xi/2)\ \widehat{\varphi}(\xi/2). \tag{19}$$

Regrouping the sums of (18) for odd and even ℓ, and substituting (9) and (19) lead to the result (17), where we used the 2π-periodicity of $a(\zeta)$, $u(\zeta)$ and $v(\zeta)$, and (7).

Using $a(\zeta) = \overline{a(\zeta)}$, (17) may be written as

$$a(\zeta)\ v(\zeta)\ \overline{u(\zeta)} + a(\zeta+\pi)\ v(\zeta+\pi)\ \overline{u(\zeta+\pi)} = 0 \quad \text{a.e..}$$

Since $u(\zeta)$ and $u(\zeta+\pi)$ cannot vanish together because of (6), there exists a 2π-periodic function $\lambda(\zeta)$ such that

$$a(\zeta)\ v(\zeta) = \lambda(\zeta)\ \overline{u(\zeta+\pi)} \quad \text{a.e.,} \tag{20}$$

and hence,

$$\lambda(\zeta) + \lambda(\zeta+\pi) = 0.$$

Consequently, $\lambda(\zeta)$ is represented as

$$\lambda(\zeta) = e^{i\zeta}\ \nu(2\zeta),$$

where $\nu(\xi)$ is a 2π-periodic function. Substituting this into (20) yields

$$a(\zeta)\ v(\zeta) = e^{i\zeta}\ \nu(2\zeta)\ \overline{u(\zeta+\pi)}.$$

It can be shown that $a(\zeta) > 0$ holds for any ζ. Indeed, we have, by Assumptions (H1) and (H3) on p_m,

$$\begin{aligned} a(\zeta) &= p_0 + 2\sum_{m=1}^{M} p_m \cos m\zeta \\ &\geq p_0 - \sum_{m\neq 0} |p_m| > 0. \end{aligned}$$

Therefore, we obtain

$$v(\zeta) = \frac{1}{a(\zeta)}\ e^{i\zeta}\ \nu(2\zeta)\ \overline{u(\zeta+\pi)}. \tag{21}$$

Substituting this into (19) gives

$$\widehat{\psi}(\xi) = \frac{1}{a(\xi/2)}\ e^{i\xi/2}\ \nu(\xi)\ \overline{u(\xi/2+\pi)}\ \widehat{\varphi}(\xi/2).$$

Using the arbitrariness of $\nu(\xi)$, we can obtain the following result.

Theorem 3: We choose

$$\nu(\xi) = -\sqrt{a(\xi/2)\ a(\xi/2+\pi)}, \tag{22}$$

that is,

$$\widehat{\psi}(\xi) = -e^{i\xi/2}\sqrt{\frac{a(\xi/2+\pi)}{a(\xi/2)}}\ \overline{u(\xi/2+\pi)}\ \widehat{\varphi}(\xi/2). \tag{23}$$

Then $\psi(x)$ satisfies the orthonormality condition

$$(\psi(\cdot-k), \psi(\cdot))_p = \delta_{k0}, \quad k \in Z. \tag{24}$$

Proof: It follows immediately that $\nu(\xi)$ is a 2π-periodic function.

By (12), we have for $\psi(x) = \sum_n \beta_n \sqrt{2}\ \varphi(2x-n)$,

$$\psi_p(x) = \sum_n (p*\beta)_n \sqrt{2}\ \varphi_p(2x-n).$$

The Fourier transform of both side yields

$$\widehat{\psi_p}(\xi) = a(\xi/2)\ v(\xi/2)\ \widehat{\varphi_p}(\xi/2).$$

Substituting (21) and (22) gives

$$\widehat{\psi_p}(\xi) = -e^{i\xi/2}\sqrt{a(\xi/2)\ a(\xi/2+\pi)}\ \overline{u(\xi/2+\pi)}\ \widehat{\varphi_p}(\xi/2). \tag{25}$$

By the definition of $(\cdot,\cdot)_p$ and the convolution theorem on frequency, we have

$$\begin{aligned} (\psi(\cdot-k), \psi(\cdot))_p &= \int \psi_p(x-k)\ \psi(x)dx \\ &= \frac{1}{2\pi}\int \widehat{\psi_p}(\xi)\ \overline{\widehat{\psi}(\xi)}\ e^{ik\xi}d\xi. \end{aligned} \tag{26}$$

Substituting (23) and (25) into (26) leads to

$$\begin{aligned} &(\psi(\cdot-k), \psi(\cdot))_p \\ &= \frac{1}{2\pi}\int a(\xi/2+\pi)\ |u(\xi/2+\pi)|^2\ \widehat{\varphi_p}(\xi/2)\overline{\widehat{\varphi}(\xi/2)}\ e^{ik\xi}d\xi. \end{aligned}$$

Using the Daubechies' trick and Theorem 1, we can obtain (24).

It is seen from Theorem 3 that $\{\psi(\cdot-k), \quad k \in Z\}$ is an orthonormal basis in W_0.

To find the wavelet function $\psi(x)$, it is necessary to determine the coefficients β_n.

Corollary 2: The coefficient β_n is given by

$$\beta_n = \sum_\ell (-1)^{\ell-1}\alpha_\ell\ r_{n+\ell+1},$$

where r_k denotes

$$r_k = \frac{1}{\pi}\int_0^{\pi}\sqrt{\frac{1+\sum_{m=1}^{M} q_m \cos m(\zeta+\pi)}{1+\sum_{m=1}^{M} q_m \cos m\zeta}}\cos k\zeta d\zeta$$

with $q_m = 2p_m/p_0$.

Proof: By the definition of $v(\zeta)$, β_n is sought as

$$\beta_n = \frac{1}{\sqrt{2}\pi}\int_0^{2\pi} v(\zeta)\, e^{in\zeta} d\zeta.$$

On the other hand, it follows from (21) and (22) that

$$v(\zeta) = -e^{i\zeta}\sqrt{\frac{a(\zeta+\pi)}{a(\zeta)}}\,\overline{u(\zeta+\pi)}.$$

Therefore, we have

$$\beta_n = -\frac{1}{\sqrt{2}\pi}\int_0^{2\pi} e^{i\zeta}\sqrt{\frac{a(\zeta+\pi)}{a(\zeta)}}\,\overline{u(\zeta+\pi)}\, e^{in\zeta} d\zeta.$$

Substituting $\overline{u(\zeta+\pi)} = \sum_\ell (-1)^\ell \alpha_\ell e^{i\ell\zeta}/\sqrt{2}$ and using the fact that $\sqrt{a(\zeta+\pi)/a(\zeta)}$ is an even and 2π-periodic function, we obtain

$$\beta_n = \sum_\ell (-1)^{\ell-1}\alpha_\ell \,\frac{1}{\pi}\int_0^{\pi}\sqrt{\frac{a(\zeta+\pi)}{a(\zeta)}}\,\cos(n+\ell+1)\zeta\, d\zeta.$$

The result of Corollary 2 follows by noticing the equation $a(\zeta) = p_0 + 2\sum_{m=1}^{M} p_m \cos m\zeta$.

V. Decomposition and Reconstruction Formulas

Using the results in Sections II and IV, we can obtain decomposition and reconstruction formulas for signals. Let $f(x)$ be a function in the space V_1. Then $f(x)$ can be expanded using the basis of V_1 as

$$f(x) = \sum_k c_k^1 \sqrt{2}\,\varphi(2x-k). \tag{27}$$

On the other hand, the direct sum $V_1 = V_0 \oplus W_0$ implies that f is in $V_0 \oplus W_0$ which yields

$$f(x) = \sum_k c_k^0\, \varphi(x-k) + \sum_k d_k^0\, \psi(x-k). \tag{28}$$

From (27) and (28), we have

$$\sum_k c_k^1 \sqrt{2}\,\varphi(2x-k) = \sum_k c_k^0\, \varphi(x-k) + \sum_k d_k^0\, \psi(x-k). \tag{29}$$

The decomposition formulas are derived as follows: Multiplying both side of (29) by the function $\varphi_p(x-i)$ and integrating the resulting equation, we have

$$\sum_k c_k^1 \int \sqrt{2}\,\varphi(2x-k)\,\varphi_p(x-i)\,dx$$
$$= \sum_k c_k^0 \int \varphi(x-k)\,\varphi_p(x-i)\,dx$$
$$+\sum_k d_k^0 \int \psi(x-k)\,\varphi_p(x-i)\,dx. \tag{30}$$

The left hand side is equal to $\sum_k (p*\alpha)_{k-2i}\, c_k^1$ because of $\varphi_p(x) = \sum_n (p*\alpha)_n \sqrt{2}\varphi_p(2x-n)$ and $(\varphi_p(\cdot-k), \varphi(\cdot)) = \delta_{k0}$. The right hand side is equal to c_i^0 by virtue of $(\varphi_p(\cdot-k), \varphi(\cdot)) = \delta_{k0}$ and $(\varphi_p(\cdot-k), \psi(\cdot)) = 0$. Therefore, we have

$$c_i^0 = \sum_k (p*\alpha)_{k-2i}\, c_k^1. \tag{31}$$

Next, multiplying both side of (29) by the function $\psi_p(x-i)$ and integrating the resulting equation give us to

$$d_i^0 = \sum_k (p*\beta)_{k-2i}\, c_k^1 \tag{32}$$

by the same reason as above.

The relations (31) and (32) are called the decomposition formulas.

Conversely, we can express c_i^1 using c_k^0 and d_k^0. We substitute (1) and (16) into (29) and compare the coefficients of both sides to get

$$c_k^1 = \sum_i \alpha_{k-2i}\, c_i^0 + \sum_i \beta_{k-2i}\, d_i^0 \tag{33}$$

which is called the reconstruction formula.

VI. Scaling Function and Wavelet Function for M=1

In case of $M = 1$, Corollary 1 means

$$\sum_\ell (p_{-1}\alpha_{\ell+2k+1} + p_0\alpha_{\ell+2k} + p_1\alpha_{\ell+2k-1})\,\alpha_\ell = \delta_{k0}, \quad k \in Z.$$

To seek a scaling function with the compact support $[-N, N]$, it suffices to solve

$$\sum_{\ell=-N}^{N} (p_{-1}\alpha_{\ell+2k+1} + p_0\alpha_{\ell+2k} + p_1\alpha_{\ell+2k-1})\,\alpha_\ell = \delta_{k0}, \quad k \in Z \tag{34}$$

with respect to α_n,$n = 0, \pm1, ..., \pm N$. In this section, we determine α_n in case of $N = 2$ by solving (34) under some regularity conditions on $\varphi(x)$.

Since $N = 2$, the α_n for $n = 0, \pm1, \pm2$ are unknown coefficients. The coefficients α_n must satisfy the equation (34) for $k = 0, \pm1, \pm2$. However, the equations for $k = 1, 2$

coincide with those for $k=-1,-2$, respectively. Hence, it suffices to consider (34) for $k=-2,-1,0$, that is,

$$p_{-1}(\alpha_{-2}\,\alpha_1+\alpha_{-1}\,\alpha_2)+p_0\alpha_{-2}\,\alpha_2=0, \tag{35}$$

$$p_{-1}(\alpha_{-1}+\alpha_1)(\alpha_{-2}+\alpha_0+\alpha_2) \\ +p_0\left((\alpha_{-2}+\alpha_2)\,\alpha_0+\alpha_{-1}\,\alpha_1\right)=0, \tag{36}$$

$$2p_{-1}\left(\alpha_{-1}\alpha_{-2}+\alpha_1\alpha_2+(\alpha_{-1}+\alpha_1)\,\alpha_0\right)+p_0\sum_{m=-2}^{2}\alpha_m^2=1, \tag{37}$$

where we have used $p_1=p_{-1}$.

In addition to these conditions, we put

$$\sum_{n=-2}^{2}\alpha_n=\sqrt{2} \tag{38}$$

which is derived by integrating both side of the two-scale relation (1). We further put

$$\sum_{n=-2}^{2}(-1)^n\alpha_n=0 \tag{39}$$

which follows from the condition of wavelet, $\int\psi(x)dx=0$, and put

$$\sum_{n=-2}^{2}(-1)^n n\,\alpha_n=0 \tag{40}$$

which is derived from the regularity condition $\int x\varphi(x)dx=0$.

There are six equations for determining five unknown coefficients. However, it is seen that (37) can be derived from (35), (36), (38), and (39). The proof is as follows: By (38) and (39), we have

$$\alpha_{-2}+\alpha_0+\alpha_2=1/\sqrt{2}, \tag{41}$$
$$\alpha_{-1}+\alpha_1=1/\sqrt{2}. \tag{42}$$

Substituting

$$\alpha_{-2}\,\alpha_1+\alpha_{-1}\,\alpha_2=1/2-(\alpha_{-1}\alpha_{-2}+\alpha_1\alpha_2+\alpha_0/\sqrt{2}),$$

which can be derived using (41) and (42), into (35) yields

$$p_{-1}(\alpha_{-1}\alpha_{-2}+\alpha_1\alpha_2+\alpha_0/\sqrt{2})=p_{-1}/2+p_0\alpha_{-2}\alpha_2. \tag{43}$$

The equation (36) may be rewritten, using (41) and (42), as

$$p_{-1}/2+p_0(\alpha_{-1}\alpha_1+\alpha_0/\sqrt{2}-\alpha_0^2)=0. \tag{44}$$

Transforming the left hand side of (37) and substituting (43) and (44) into the resulting expression, we finally obtain (37). Consequently, α_n, $n=0,\pm1,\pm2$, are sought by solving (40),(41),(42),(43) and (44), simultaneously. We use the Newton's method to compute α_n.

In the table below, we list the computed values α_n and β_n for various (p_{-1},p_0,p_1) satisfying (H1),(H2) and (H3) in Section III.

Table.
The filter coefficients α_n and β_n for various parameters $p_{-1}=p_1,\ p_0$.

$p_{-1}=p_1,\ p_0$	n	α_n	β_n
0, 1 Daubechies	-2	0.482962913145	-0.12940952255
	-1	0.836516303738	-0.22414386804
	0	0.224143868042	0.83651630374
	1	-0.12940952255	-0.48296291315
-0.125, 1.25	-4		0.000527425973
	-3		0.000898485540
	-2	0.431314493824	-0.113661559523
	-1	0.795735824325	-0.219961860269
	0	0.286660227270	0.730523960852
	1	-0.08862904314	-0.356452837731
	2	-0.01086793991	-0.039845540443
	3		-0.001821689760
	4		-0.000193812246
-0.25, 1.5	-4		0.000876016533
	-3		0.001130917276
	-2	0.394116067618	-0.099360965268
	-1	0.760884685302	-0.218521691591
	0	0.326205940658	0.657237691143
	1	-0.05377790411	-0.278386804747
	2	-0.01321522709	-0.057956752574
	3		-0.004121139438
	4		-0.000782010826
-0.375, 1.75	-6		-0.000021707623
	-5		-0.000104727290
	-4		0.000046025474
	-3		-0.001379056381
	-2	0.368835751123	-0.086442489646
	-1	0.731466933865	-0.211435860129
	0	0.347348822212	0.603002833160
	1	-0.02436015268	-0.228203656188
	2	-0.00907779215	-0.067786854193
	3		-0.005851944166
	4		-0.001525131643
	5		-0.000223344744
	6		-0.000055542443
-0.5, 2	-7		-0.000023342437
	-6		-0.000118322123
	-5		-0.000389002680
	-4		-0.002317266249
	-3		-0.007181850389
	-2	0.353553390593	-0.074595873421
	-1	0.707106781187	-0.196528378993
	0	0.353553390593	0.562325895710
	1		-0.196528378993
	2		-0.074595873421
	3		-0.007181850389
	4		-0.002317266249
	5		-0.000389002680
	6		-0.000118322123
	7		-0.000023342437

When $p_{-1}=p_1=-0.5$ and $p_0=2$, the symmetric wavelet function has been obtained.

The scaling and wavelet functions corresponding to the above filter coefficients are shown in Figures 1,2,3,4 and 5.

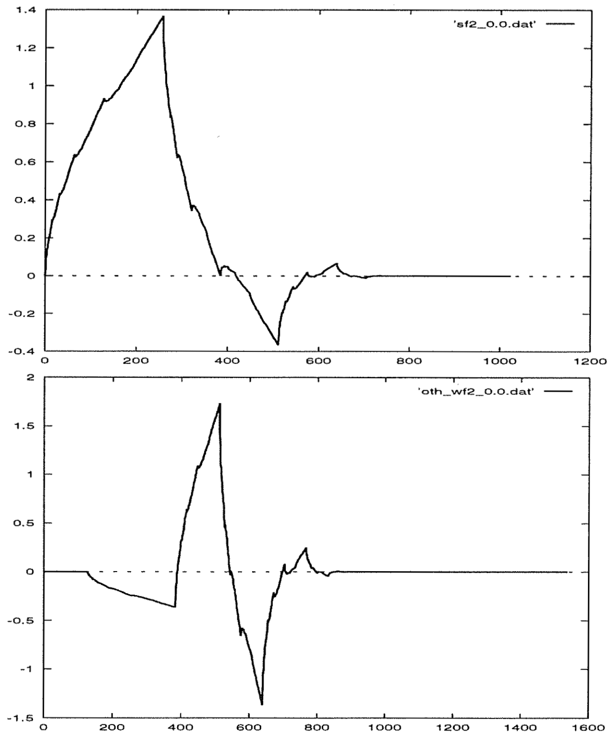

Fig. 1. The above shows the scaling function, and the below wavelet function for $p_{-1} = p_1 = 0.0$ and $p_0 = 1$. Daubechies' wavelet.

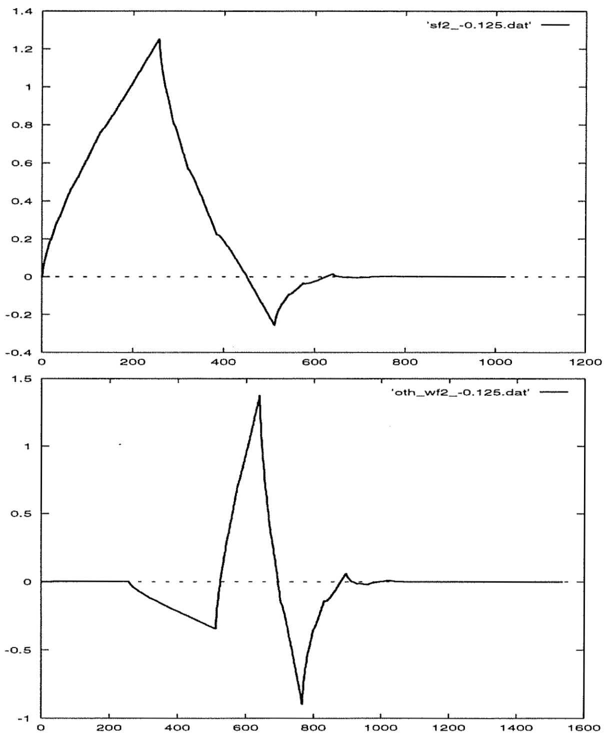

Fig. 2. The above shows the scaling function, and the below wavelet function for $p_{-1} = p_1 = -0.125$ and $p_0 = 1.25$.

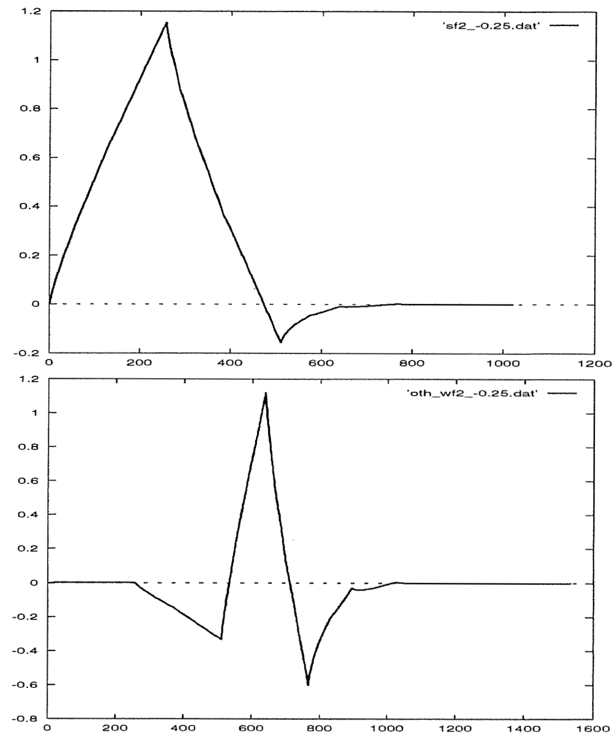

Fig. 3. The above shows the scaling function, and the below wavelet function for $p_{-1} = p_1 = -0.25$ and $p_0 = 1.5$.

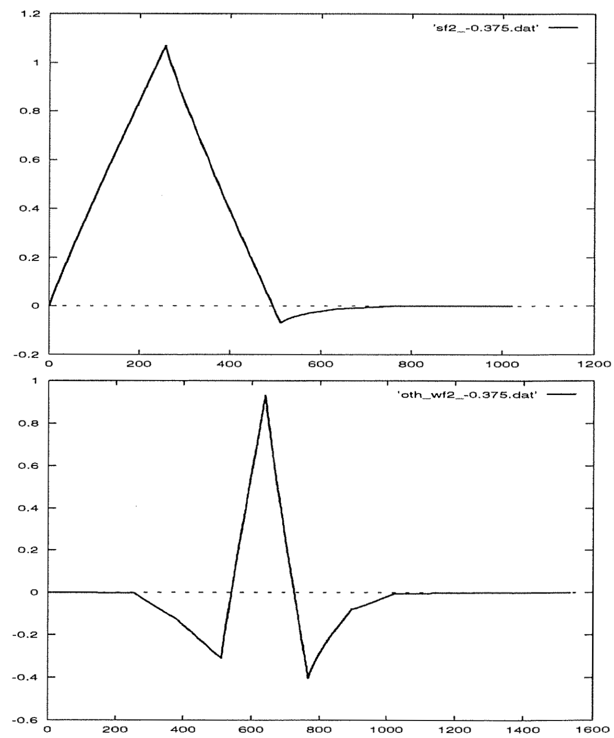

Fig. 4. The above shows the scaling function, and the below wavelet function for $p_{-1} = p_1 = -0.375$ and $p_0 = 1.75$.

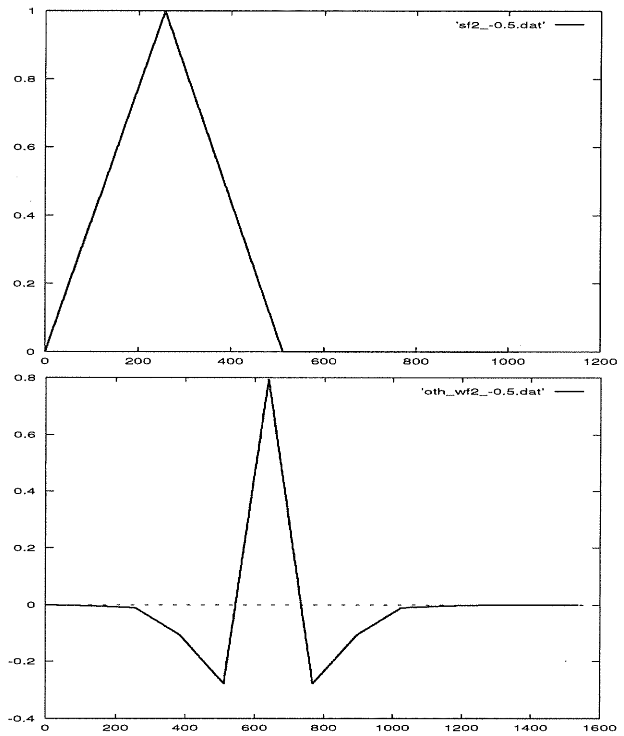

Fig. 5. The above shows the scaling function, and the below wavelet function for $p_{-1} = p_1 = -0.5$ and $p_0 = 2$.

VII. Conclusion

We developed a biorthogonal wavelet theory along the Daubechies' approach. Since our scaling and wavelet functions include free parameters, we can choose the desirable scaling and wavelet functions such as symmetric ones. In fact, we found the symmetric scaling and wavelet functions in case of $M = 1$ and $N = 2$, which have a short support length. The application of this wavelet to signal processing will be reported elsewhere. We also have the freedom of selecting the parameters N and M. The construction of scaling and wavelet functions in a general case and their applications are interesting problems remained in the future.

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