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Period Lengths of Cellular Automata $cam - 90$ with Memory

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Abstract

Cellular automata $ca - 90$ have states 0 and 1, and their dynamics, driven by the local transition rule 90, can be simply represented with Laurent polynomials over a finite field $F_2 = \{0, 1\}$. Cellular automata $cam - 90$ with memory, whose configurations are pairs of those of $ca - 90$, are introduced as a useful machinery to solve certain equations on configurations, in particular, to compute fixed or kernel configurations of $ca - 90$. This paper defines a notion of linear dynamical systems with memory, states their basic properties, and then studies some period lengths of one-dimensional and two-dimensional cellular automata $cam - 90$ with memory.

1 Introduction

As is well-known cellular automata have been initially introduced by J. von Neuamnn as a model of self-reproducing systems and S. Wolfram [1, 2] has recognized as a mathematical model of complex systems. Many authors [1, 2, 3] have investigated dynamical behaviors of finite additive cellular automata. In their pioneer work [1] Martin, Oldlyzko and Wolfram studied a lot of fundamental properties of additive cellular automata with cells arranged around a circle, by using Laurent polynomials. Explicit formulas for period lengths of limit cycles of additive cellular automata of such type were found by Guan and He [3]. Manna and Stauffer [4] analyzed phase transitions of all nearest neighbor cellular automata on square lattices without memory, and da Silva [5] studied critical behavior at the transition to chaos of several binary mixtures of cellular automata and fractal dimensions associated with the damage spreading and the propagation time of damage. By means of an extensive numerical study of elementary cellular automata Binder [6] proposed a topological classification of cellular automata, complementary to that of Wolfram derived from the attractor globality.

Kawahara et al. [7] studied the period lengths of cellular automata on square lattices with rule 90, by an algebraic formalization of configurations and transition functions with Laurent polynomials [1, 2]. In this paper we introduce a notion of cellular automata $cam - 90$ with memory associated to one- and two-dimensional cellular automata $ca - 90$ with rule 90 and then study period lengths of cellular automata $cam - 90$ with memory. The main results will be stated in Theorem 4.2, 4.4 and 4.5.

Now we will explain a motivation why we introduce a notion of cellular automata with memory. A two-dimensional cellular automaton $ca - 90(m, n)$ is basically defined as follows (where m and n are integers > 1): A configuration $c = (c_{i,j})$ of ca–$90(m, n)$ is an $(m-1) \times (n-1)$

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matrix over a prime field $F_2 = \{0,1\}(= \mathbf{Z}/2\mathbf{Z})$, that is, $c_{i,j} = 0$ or 1. The next configuration $\delta(c) = (\delta(c)_{i,j})$ of c is given by a formula:

$$\delta(c)_{i,j} = c_{i-1,j} + c_{i,j-1} + c_{i,j} + c_{i,j+1} + c_{i+1,j} \bmod 2$$

for $i = 1, 2, \cdots, m-1$ and $j = 1, 2, \cdots, n-1$. Here we assume the null boundary condition $c_{0,j} = c_{m,j} = c_{i,0} = c_{i,n} = 0$. That is, $ca-90(m,n)$ is regarded as a linear dynamical system $(F_2^{(m-1)(n-1)}, \delta)$. Let us consider to find kernel configurations of $ca-90(m,n)$. (A kernel configuration c is a configuration such that $\delta(c) = 0$, where 0 means a null matrix). To find kernel configurations, one has to solve the following system of equations:

$$(*) \quad c_{i-1,j} + c_{i,j-1} + c_{i,j+1} + c_{i+1,j} = 0 \bmod 2$$

for $i = 1, 2, \cdots, m-1$ and $j = 1, 2, \cdots, n-1$. Let c_i denote the i-th row vector of c for $i = 1, 2, \cdots, m-1$, that is, $c_i = (c_{i,1}, c_{i,2}, \cdots, c_{i,n-1})$. Then the above equation $(*)$ is equivalent to

$$c_{i+1} = \tau(c_i) + c_{i-1} \bmod 2 \quad (i = 1, 2, \cdots, m-1),$$

where τ is the transition function of $ca-90(n)$, and $c_0 = c_m = 0$. Given the first row c_1 of a solution $c = (c_{i,j})$ of $(*)$, one can calculate all other rows $c_2, c_3, \cdots, c_{m-1}$ in order and then c_m must be a zero vector. This motivates an idea of a new linear system $\hat{\tau}(c,d) = (\tau(c) + d, c)$, which will be called a linear system with memory associated to $ca-90(n) = (F_2^{n-1}, \tau)$. A more general notion of of linear systems with memory will be defined in Sec. II. An important property of a linear system with memory is a fact that its transition function is injective. Hence, whenever the configuration space is finite, all configurations lie on some limit cycles and so the main interest of its dynamical behavior will be focused on period lengths of limit cycles, which are one of important topological indexes of dynamical systems.

The paper is organized as follows. In Sec. II we first introduce a notion of linear systems with memory associated to linear systems over a general field and define the characteristic polynomials deeply related with transition functions of cellular automata $ca-90(m)$. Then the fundamental properties on iterations of transition functions of linear systems with memory are stated. Also some well-known explicit formulas of the characteristic polynomials over F_2 (or modulo 2) are recalled. In Sec. III we review some fundamentals on one-dimensional cellular automata $ca-90(m)$ and two-dimensional cellular automata $ca-90(m,n)$ needed in Sec. IV. Almost all results stated in Sec. III have been already showed in [8, 7] but the proofs are simpler and self-contained. The Sec. IV proves the main theorems on period lengths of particular configurations of cellular automata $cam-90(m)$ and $cam-90(m,n)$ with memory, by using results obtained in the previous sections.

2 Linear Systems with Memory

A *linear system* over a field F is a pair (X, τ) of a vector space X over F and a linear transformation $\tau : X \to X$, which will be called the transition function (or dynamics) of (X, τ). A vector of X will be called a configuration of (X, τ).

As stated in the introduction the notion of linear systems with memory is motivated as a useful machinery to solve certain kinds of equations on configurations in higher dimensional linear systems.

We now define linear systems with memory associated to linear systems over general fields.

Definition 2.1 *Let (X, τ) be a linear system over a field F. The linear system $(X, \hat{\tau})$ with memory associated to (X, τ) is constructed as follows. A configuration of $(X, \hat{\tau})$ is a pair (c, d)*

of vectors c and d of X. The transition function $\hat{\tau}$ of $(X, \hat{\tau})$ is defined by

$$\hat{\tau}(c, d) = (-\tau(c) - d, c)$$

for all configurations (c, d) of $(X, \hat{\tau})$. □

It is trivial that the transition function $\hat{\tau}$ of $(X, \hat{\tau})$ is a linear transformation on $X \oplus X$ and is injective. Thus, whenever X is finite, $\hat{\tau}$ is bijective and all configurations (c, d) of $(X, \hat{\tau})$ are always on limit cycles.

In the rest of this section we assume that (X, τ) is a linear system over a field F. For a polynomial $\xi(z) = a_0 + a_1 z + \cdots + a_k z^k$ over F with an indeterminate z we define the extended linear transformation $\xi(\tau) : X \to X$ of τ by $\xi(\tau)c = a_0 c + a_1 \tau(c) + \cdots + a_k \tau^k(c)$ for all $c \in X$. It is trivial that $\xi(\tau)$ is a linear transformation on X. The main interest of the paper is concerned with period lengths of limit cycles in $(X, \hat{\tau})$. The period lengths are generally dominated by the property of iterations of $\hat{\tau}$. In order to explicitly compute the iterations the characteristic polynomials $\varphi_0(z), \varphi_1(z), \varphi_2(z), \cdots$ will be introduced by induction as follows:

$$\varphi_0(z) = 0, \varphi_1(z) = 1 \text{ and } \varphi_{k+2}(z) = -z\varphi_{k+1}(z) - \varphi_k(z) \quad (k \geq 0).$$

For example, $\varphi_2(z) = -z, \varphi_3(z) = z^2 - 1, \varphi_4(z) = -z^3 + 2z, \varphi_5(z) = z^4 - 3z^2 + 1$ and so on.

The following lemma shows how iterations of transition functions of linear systems with memory can be computed by the characteristic polynomials.

Lemma 2.2 *Let (c, d) be a configuration of $(X, \hat{\tau})$. Then the equality*

$$\hat{\tau}^k(c, d) = (\varphi_{k+1}(\tau)c - \varphi_k(\tau)d, \varphi_k(\tau)c - \varphi_{k-1}(\tau)d)$$

holds for all positive integers k. In particular, $\hat{\tau}^k(c, 0) = (\varphi_{k+1}(\tau)c, \varphi_k(\tau)c)$, where 0 denotes the zero vector of X.

Proof. For $k = 1$ the equality is trivial from

$$\hat{\tau}(c, d) = (-\tau(c) - d, c) = (\varphi_2(\tau)c - \varphi_1(\tau)d, \varphi_1(\tau)c - \varphi_0(\tau)d)$$

by $\varphi_0(\tau)c = 0, \varphi_1(\tau)c = c$ and $\varphi_2(\tau)c = -\tau(c)$. Assume that the equality is valid for k with $k \geq 1$. Then

$$\begin{aligned}
\hat{\tau}^{k+1}(c, d) &= \hat{\tau}(\varphi_{k+1}(\tau)c - \varphi_k(\tau)d, \varphi_k(\tau)c - \varphi_{k-1}(\tau)d) \\
&= (-\tau(\varphi_{k+1}(\tau)c - \varphi_k(\tau)d) - \{\varphi_k(\tau)c - \varphi_{k-1}(\tau)d\}, \varphi_{k+1}(\tau)c - \varphi_k(\tau)d) \\
&= (\{-\tau\varphi_{k+1}(\tau)c - \varphi_k(\tau)c\} - \{-\tau\varphi_k(\tau)d - \varphi_{k-1}(\tau)d\}, \varphi_{k+1}(\tau)c - \varphi_k(\tau)d) \\
&= (\varphi_{k+2}(\tau)c - \varphi_{k+1}(\tau)d, \varphi_{k+1}(\tau)c - \varphi_k(\tau)d). \quad \Box
\end{aligned}$$

As the first application of the last lemma we can easily prove how the period length of a configuration with zero memory is related with a simple property expressed by an extended linear transformation of transition functions. That is, we claim the following

Proposition 2.3 *Let c be a configuration of (X, τ) and k a positive integer.*

(a) *If $\varphi_k(\tau)c = 0$, then $\varphi_{k+i}(\tau)c = -\varphi_{k-i}(\tau)c$ for all $i = 0, 1, \cdots, k$.*

(b) *If $\varphi_k(\tau)c = 0$, then $\hat{\tau}^{2k}(c, 0) = (c, 0)$.*

(c) *If $\hat{\tau}^{2k}(c, 0) = (c, 0)$, then $\tau\varphi_k(\tau)c = 0$.* □

Note that the characteristic polynomial $\varphi_k(z)$ is the characteristic polynomial of a $(k-1) \times (k-1)$ matrix, that is,

$$\varphi_k(z) = \begin{vmatrix} -z & -1 & & & \\ -1 & -z & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & -z & -1 \\ & & & -1 & -z \end{vmatrix}$$

for all integers $k > 1$.

Cellular automata $ca-90$, which will be discussed later, are linear systems over a prime field $F_2 = \{0, 1\}$ of characteristic 2. It is well-known that the characteristic polynomials $\varphi_0(z), \varphi_1(z), \varphi_2(z), \cdots$ over F_2 (or modulo 2) satisfy some specific formulas, which will be useful to compute period lengths of configurations of cellular automata with memory associated with $ca-90$.

Proposition 2.4 *Let i and u be positive integers and k a nonnegative integer. Then for the characteristic polynomials $\varphi_0(z), \varphi_1(z), \cdots$ over F_2 the following holds:*

(a) $\varphi_{2i-1}(z) = \{\varphi_{i-1}(z)\}^2 + \{\varphi_i(z)\}^2$ *and* $\varphi_{2i}(z) = z\{\varphi_i(z)\}^2$,

(b) $\varphi_{2^k i}(z) = z^{2^k-1}\{\varphi_i(z)\}^{2^k}$ *and* $\varphi_{2^k i-1}(z) = \{\varphi_{i-1}(z)\}^{2^k} + \{\varphi_i(z)\}^{2^k} \sum_{j=1}^{k} z^{2^k-2^j}$,

(c) $\varphi_{2^k}(z) = z^{2^k-1}$ *and* $\varphi_{2^k-1}(z) = \sum_{j=1}^{k} z^{2^k-2^j}$,

(d) $\varphi_{2^k(2^u-1)}(z) = z^{2^k-1} \sum_{j=1}^{u} z^{2^k(2^u-2^j)}$. $\square$

3 Cellular Automata $ca-90$

In this section we recall some fundamentals on one-dimensional cellular automata $ca-90(m)$ and two-dimensional cellular automata $ca-90(m, n)$ for the later study of the paper. Almost all results stated in the section have been showed in [8, 7] but this section will give them simpler and self-contained proofs.

In what follows we assume that m is an integer > 1. Let $F_2[x]$ be the polynomial ring over a prime field $F_2 = \{0, 1\}$ of characteristic 2 with an indeterminate x, and $F_2[x]/(x^{2m} - 1)$ a quotient ring of $F_2[x]$ by an ideal $(x^{2m} - 1)$ generated by a polynomial $x^{2m} - 1$. A polynomial in the quotient ring $F_2[x]/(x^{2m} - 1)$ is sometimes called a *Laurent* polynomial (Cf. [1]). Define Laurent polynomials $t_m(i) = x^i + x^{-i}$ in $F_2[x]/(x^{2m} - 1)$ for all integers i. In particular, we set $t_m = t_m(1)(= x + x^{-1})$. (We will omit a suffix m in $t_m(i)$ and t_m unless confusion occurs.)

The following is the basic properties of Laurent polynomials $t(i)$ in the quotient ring $F_2[x]/(x^{2m} - 1)$.

Proposition 3.1 *In the quotient ring $F_2[x]/(x^{2m} - 1)$ the following holds for integers i, j and a nonnegative integer k:*

(a) $t(0) = t(m) = 0$,

(b) $t(-i) = t(i)$,

(c) $t^{2^k} = t(2^k)$, $\{t(i)\}^{2^k} = t(2^k i)$,

(d) $t(i)t(j) = t(i - j) + t(i + j)$,

(e) $t(2m+i) = t(i)$,

(f) $t(m+i) = t(m-i)$. □

Making use of Laurent polynomials $t_m(i)$ we define one-dimensional cellular automata *ca*–90(m). The algebraic formulation of *ca*–90(m) enables us to apply the multiplicative structure of the quotient ring $F_2[x]/(x^{2m}-1)$ to theoretical study of cellular automata.

Definition 3.2 *A configuration c of a cellular automaton $ca-90(m)$ is a Laurent polynomial*

$$c = \sum_{i=1}^{m-1} c_i t(i)$$

in the quotient ring $F_2[x]/(x^{2m}-1)$, where $c_i = 0$ or 1 for all integers $i = 1, 2, \cdots, m-1$. The transition function τ of $ca-90(m)$ is defined by $\tau(c) = tc$ for every configuration c. A configuration a of $ca-90(m)$ is a particular configuration whose all cells have the state 1, that is, $a = \sum_{i=1}^{m-1} t(i)$. □

The following proposition states some basic behaviors of the characteristic polynomials $\varphi_0(t), \varphi_1(t), \varphi_2(t), \cdots$ and Laurent polynomials $t(i)$ in cellular automata $ca-90$.

Proposition 3.3 *In a cellular automaton $ca-90(m)$ the following holds for a positive integer i and a nonnegative integer k:*

(a) $t\{t(1) + t(2) + \cdots + t(i)\} = t(1) + t(i) + t(i+1)$,

(b) $ta = t(1) + t(m-1)$,

(c) $t(i)a = t(i) + t(m-i)$,

(d) $\varphi_i(t)t = t(i)$,

(e) $\varphi_m(t)c = 0$ *for all configurations* c,

(f) $\varphi_{m-1}(t)t(i) = t(m-i)$,

(g) $\{\varphi_i(t)\}^{2^k} t(m-2^k) = t(m-2^k i)$. *In particular* $\varphi_i(t)t(m-1) = t(m-i)$. □

For an odd integer m the least positive integer u satisfying $2^u = \pm 1(\text{mod} m)$ is called the *multiplicative suborder modulo* m of 2, which will be denoted by $sord(2; m)$. The existence of positive integers u such that $2^u = \pm 1(\text{mod} m)$ follows from Fermat-Euler theorem on fundamental number theory.

The period length of a particular configuration $a = \sum_{j=1}^{m-1} t(j)$ of *ca*–90(m) is roughly determined by the multiplicative suborder of m modulo 2. Using the results of the last proposition this fact will be proved in the following

Proposition 3.4 *Let m be an odd integer ≥ 3, k a positive integer, and $u = sord(2; m)$. Then the following holds:*

(a) $t^{2^u - 1} a = a$ *in* $ca-90(m)$,

(b) $t^{2^k(2^u-1)} t^{2^{k-1}} a = t^{2^{k-1}} a$ *in* $ca-90(2^k m)$,

(c) $\{t(m)\}^{2^{k-1}} a = 0$ *and* $\{t(m)\}^{2^k} = 0$ *in* $ca-90(2^k m)$.

(Note that $t = t_{2^k m} = t_{2^k m}(1)$ in (b) and $t(m) = t_{2^k m}(m)$ in (c).)

Proof. (a) As $2^u = m(2r+1) \pm 1$ for an integer r, it follows from 3.1(c),(e) and (f) that

$$t^{2^k} = t(2^k) = t(m(2r+1) \pm 1) = t(m \pm 1) = t(m-1)$$

in $ca-90(m)$. Hence using 3.3(d) and (f) we have

$$t^{2^u-1}t(j) = t^{2^u-1}\varphi_j(t)t = \varphi_j(t)t^{2^u} = \varphi_j(t)t(m-1) = t(m-j)$$

and so

$$t^{2^u-1}a = \sum_{j=1}^{m-1} t^{2^u-1}t(j) = \sum_{j=1}^{m-1} t(m-j) = a.$$

(b) First note that

$$t^{2^{k+u}} = t(2^{k+u}) = t(2^k m(2r+1) \pm 2^k) = t(2^k m - 2^k)$$

and

$$\begin{aligned} t^{2^{k-1}} \textstyle\sum_{j=1}^{m-1} t(2^k j) &= \textstyle\sum_{j=1}^{m-1}\{t(2^k j - 2^{k-1}) + t(2^k j + 2^{k-1})\} \quad \text{(by 3.1(d))} \\ &= t(2^{k-1}) + t(2^k m - 2^{k-1}) \\ &= t^{2^{k-1}} a \quad \text{(by 3.3(c))}. \end{aligned}$$

in $ca-90(2^k m)$. Hence we have

$$\begin{aligned} t^{2^k(2^u-1)}t(2^k j) &= t^{2^k(2^u-1)}\{t(j)\}^{2^k} && \text{(by 3.1(c))} \\ &= t^{2^k(2^u-1)}\{\varphi_j(t)t\}^{2^k} && \text{(by 3.3(d))} \\ &= \{\varphi_j(t)\}^{2^k} t^{2^{k+u}} \\ &= \{\varphi_j(t)\}^{2^k} t(2^k m - 2^k) \\ &= t(2^k m - 2^k j) && \text{(by 3.3(g))} \end{aligned}$$

and so

$$\begin{aligned} t^{2^k(2^u-1)}t^{2^{k-1}}a &= t^{2^k(2^u-1)}t^{2^{k-1}} \textstyle\sum_{j=1}^{m-1} t(2^k j) \\ &= t^{2^{k-1}} \textstyle\sum_{j=1}^{m-1} t^{2^k(2^u-1)}t(2^k j) \\ &= t^{2^{k-1}} \textstyle\sum_{j=1}^{m-1} t(2^k m - 2^k j) \\ &= t^{2^{k-1}} a. \end{aligned}$$

(c) It is immediate from 3.1(a),(c) and 3.3(c) that $\{t(m)\}^{2^k} = t(2^k m) = 0$ and $\{t(m)\}^{2^{k-1}} a = t(2^{k-1}m)a = t(2^{k-1}m) + t(2^k m - 2^{k-1}m) = 0$. □

Recall that the assertion (a) and (b) of the last proposition was proved by using the injectivity of transition functions and the substitution operators in [8, 7]. However the assertions (a) and (b) have been showed only from the basic properties Laurent polynomials $t(i)$ and the characteristic polynomials $\varphi_j(t)$.

In what follows we assume that m and n are integers > 1. Let $F_2[x,y]$ be the polynomial ring over a prime field $F_2 = \{0,1\}$ with two indeterminates x and y, and $F_2[x,y]/(x^{2m}-1, y^{2n}-1)$ a quotient ring of $F_2[x,y]$ by an ideal $(x^{2m}-1, y^{2n}-1)$ generated by two polynomials $x^{2m}-1$ and $y^{2n}-1$. A polynomial in the quotientring $F_2[x,y]/(x^{2m}-1, y^{2n}-1)$ is sometimes called a *Laurent* polynomial. Define Laurent polynomials $t_m(i) = x^i + x^{-i}$ and $s_n(j) = y^j + y^{-j}$ for all integers i and j. In particular, we set $t_m = t_m(1) (= x + x^{-1})$ and $s_n = s_n(1) (= y + y^{-1})$. Further we set $a_m = \sum_{i=1}^{m-1} t_m(i)$ and $b_n = \sum_{j=1}^{n-1} s_n(j)$. The following proposition gives elementary formulae on Laurent polynomials $t_m(i)$ and $s_n(j)$. (We will omit suffixes m and n in $t_m(i)$, t_m, a_m, $s_n(j)$, s_n, and b_n unless confusion occurs.)

Proposition 3.5 *In the quotient ring $F_2[x,y]/(x^{2m}-1, y^{2n}-1)$ the following holds for integers i, j and a nonnegative integer k:*

(a) $t(0) = t(m) = 0$; $s(0) = s(n) = 0$,

(b) $t(-i) = t(i)$; $s(-j) = s(j)$,

(c) $t^{2^k} = t(2^k)$, $\{t(i)\}^{2^k} = t(2^k i)$; $s^{2^k} = s(2^k)$, $\{s(j)\}^{2^k} = s(2^k j)$,

(d) $t(i)t(j) = t(i-j) + t(i+j)$; $s(i)s(j) = s(i-j) + s(i+j)$,

(e) $t(2m+i) = t(i)$; $s(2n+j) = s(j)$,

(f) $t(m+i) = t(m-i)$; $s(n+j) = s(n-j)$. $\square$

Making use of Laurent polynomials $t(i)$ and $s(j)$ we reformulate two-dimensional cellular automata $ca-90(m,n)$, which is essentially the same as the combinatorial one stated in the introduction. The algebraic formulation of $ca-90(m,n)$ enables us to apply the multiplicative structure of the quotient ring $F_2[x,y]/(x^{2m}-1, y^{2n}-1)$ to a further study of cellular automata.

Definition 3.6 *A configuration c of a cellular automaton ca–$90(m,n)$ is a Laurent polynomial*

$$c = \sum_{i=1,j=1}^{m-1,n-1} c_{i,j} t(i)s(j)$$

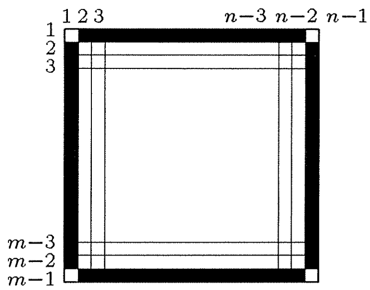
in the quotient ring $F_2[x,y]/(x^{2m}-1, y^{2n}-1)$, where $c_{i,j} \in F_2$ for all i and j with $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. The transition function $\delta (= \delta_{m,n})$ of ca–$90(m,n)$ is defined by $\delta(c) = (t+s)c$ for every configuration c. A configuration ab of ca–$90(m,n)$ is a particular configuration whose all cells have the state 1, that is,

$$ab = \sum_{i=1,j=1}^{m-1,n-1} t(i)s(j).$$

The next configuration to ab is denoted by $\alpha (= \alpha_{m,n})$, that is, $\alpha = (t+s)ab$. $\square$

Note that the configuration space $ca-90(m,n)$ consisting of all configurations is an $(m-1)(n-1)$-dimensional vector space over F_2 with a basis $\{t(i)s(j) : i = 1, 2, \cdots, m-1, j = 1, 2, \cdots, n-1\}$ (Cf. [8]).

The configuration $\alpha = (t+s)ab = t(1)b + t(m-1)b + as(1) + as(n-1)$ of $ca-90(m,n)$ is illustrated by the following figure:



The period length of a particular configuration $\alpha = (t+s)a$ of $ca-90(m,n)$ is roughly determined by the multiplicative suborders of m and n modulo 2. Using the results 3.4 for $ca-90(m)$ this fact will be proved in the following

Corollary 3.7 *Let m and n be odd integers ≥ 3, and k and l positive integers. Set $h = \max(k, l)$ and w the least common multiple of $u = sord(2; m)$ and $v = sord(2, n)$. Then the following holds:*

(a) $(t+s)^{2^w-1}\alpha = \alpha$ *in* $ca-90(m, n)$,

(b) $(t+s)^{2^h(2^w-1)}(t+s)^{2^h-1}\alpha = (t+s)^{2^h-1}\alpha$ *in* $ca-90(2^k m, 2^l n)$,

Proof. (a) Recall that $t^{2^u-1}a = a$ in $ca-90(m)$ and $s^{2^v-1}b = b$ in $ca-90(n)$ by 4.4(a). Then as $2^w - 1$ is a multiple of $2^u - 1$ and $2^v - 1$ we have

$$\begin{aligned}(t+s)^{2^w-1}\alpha &= (t^{2^w} + s^{2^w})ab \\ &= tt^{2^w-1}ab + ass^{2^w-1}b \\ &= tab + asb \\ &= \alpha\end{aligned}$$

in $ca-90(m, n)$.

(b) As $t^{2^h(2^w-1)}t^{2^h}a = t^{2^h}a$ in $ca-90(2^k m)$ and $s^{2^h(2^w-1)}s^{2^h}b = s^{2^h}b$ in $ca-90(2^l n)$ from Prop.2.3(b) it follows that

$$\begin{aligned}(t+s)^{2^h(2^w-1)}(t+s)^{2^h-1}\alpha &= (t+s)^{2^{w+h}}ab \\ &= (t^{2^{w+h}} + s^{2^{w+h}})ab \\ &= (t^{2^h} + s^{2^h})ab \\ &= (t+s)^{2^h-1}\alpha. \quad \square\end{aligned}$$

4 Cellular Automata $cam-90$ with Memory

In this section we discuss one- and two-dimensional cellular automata $cam-90$ with memory associated to $ca-90$. As stated in the section 2 all configurations of $cam-90$ lie on limit cycles and their period lengths is one of important topological parameters concerned with the dynamical property of $cam-90$. After reviewing a few basic formulas of the characteristic polynomials $\varphi_n(z)$ mod 2, we study some properties of period lengths of limit cycles on which certain particular configurations lie.

First we recall one-dimensional cellular automata $cam-90(m)$ with memory.

Definition 4.1 *A configuration of one-dimensional cellular automaton cam–90(m) with memory is a pair (c, d) of configurations c and d of $ca-90(m)$. The transition function $\hat{\tau}$ of $cam-90(m)$ is defined by $\hat{\tau}(c, d) = (tc + d, c)$ for all configurations (c, d) of $cam-90(m)$.*

For example it is easy to see that $a = t(1)$, $ta = 0$ and $\hat{\tau}(ta, 0) = (ta, 0)$ in $cam-90(2)$, that $a = ta = t(1)+t(2)$, $\hat{\tau}(ta, 0) = (ta, ta)$, $\hat{\tau}^2(ta, 0) = (0, ta)$ and $\hat{\tau}^3(ta, 0) = (ta, 0)$ in $cam-90(3)$, and that $a = t(1) + t(2) + t(3)$, $ta = t(1) + t(3)$, $\hat{\tau}(ta, 0) = (0, ta)$ and $\hat{\tau}^2(ta, 0) = (ta, 0)$ in $cam-90(4)$.

The following theorem is a basic result on period lengths of configurations of one-dimensional cellular automata $cam-90(m)$ with memory.

Theorem 4.2 *Let (c, d) be a configuration of $cam-90(m)$. Then*

(a) $\hat{\tau}^{2m}(c, d) = (c, d)$,

(b) $\hat{\tau}^m(c, d) = (c, d)$ *if and only if both of c and d are symmetric.*

(c) *If $m > 4$, then the period length $K(m)$ of a limit cycle on which a particular configuration $(ta, 0)$ of $cam{-}90(m)$ lies is equal to m.*

Proof. (a) Note that $\varphi_m(t)c = 0$ for all c of ca–$90(m)$ by 3.3(e). Hence $\varphi_{2m}(t)c = t\{\varphi_m(t)\}^2 c = 0$ and

$$\varphi_{2m+1}(t)c = \varphi_{2m-1}(t)c = \varphi_{m+m-1}(t)c = \varphi_{m-m+1}(t)c = \varphi_1(t)c = c$$

by 2.3(a), which claims $\hat{\tau}^{2m}(c, d) = (c, d)$. (This means that the period lengths of all limit cycles are divisors of $2m$.)
(b) First note that $\varphi_m(t)c = 0$ and $\varphi_m(t)d = 0$ from 3.3(e). Also from 2.3(a) it follows that $\varphi_{m+1}(t)c = \varphi_{m-1}(t)c$. Hence by 2.2 we have

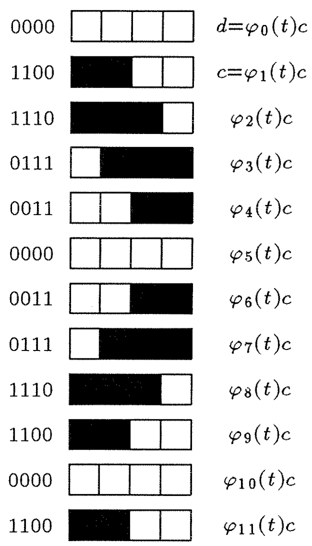
$$\hat{\tau}^m(c, d) = (\varphi_{m-1}(t)c, \varphi_{m-1}(t)d).$$

But 3.3(f) means that $\varphi_{m-1}(t)$ is a reverse operator on configurations of $ca{-}90(m)$.
(c) Recall from 3.3(d) that $\varphi_i(t)t = t(i)$ for $i \geq 0$. Then using 2.2 and 3.3(c) we have

$$\begin{aligned} \hat{\tau}^i(ta, 0) &= (\varphi_{i+1}(t)ta, \varphi_i(t)ta) \\ &= (t(i+1)a, t(i)a) \\ &= (t(i+1) + t(m-i-1), t(i) + t(m-i)) \end{aligned}$$

for all $i \geq 0$. From this one can observe that the least positive integer i such that $\hat{\tau}^i(ta, 0) = (ta, 0)$ is equal to m. (Note that if m is even then $t(m/2) + t(m - m/2) = 0$ but $t(m/2 + 1) + t(m - m/2 - 1) \neq t(1) + t(m-1) = ta$.) □

The following figure indicates the the period length of a limit cycle on which a configuration $(t(1) + t(2), 0)$ of $cam{-}90(5)$ lies is equal to 10.



Now we recall two-dimensional cellular automata $cam{-}90(m, n)$ with memory.

Definition 4.3 *A configuration of two-dimensional cellular automaton $cam - 90(m, n)$ with memory is a pair (c, d) of configurations c and d of $ca - 90(m, n)$. The transition function $\hat{\delta}$ of $cam - 90(m, n)$ is defined by $\hat{\delta}(c, d) = ((t + s)c + d, c)$ for all configurations (c, d) of $cam{-}90(m, n)$.* □

Theorem 4.4 *Let m and n be odd integers ≥ 3, and let k and l be positive integers. Then the following statements hold for the period length $K(m, n)$ of a limit cycle on which a particular configuration $(\alpha, 0)$ of $cam{-}90(m, n)$ lies.*

(a) $K(m,n) \mid 2^{2w}-1$,

(b) $K(2^k m, 2^l n) \mid 2^{h+1}(2^{2w}-1)$,

(c) $K(2^k m, 2^l) \mid 2^h m$ *for* $l \geq 2$,

(d) $K(2^k, 2^l) = 2^h$ *for* $k, l \geq 2$ *and* $(k,l) \neq (2,2)$,

where $h = \max(k,l)$, *u is the multiplicative suborder of 2 modulo m, v the multiplicative suborder of 2 modulo n, and w the least common multiple of u and v.*

Proof. (a) It suffices to see that $\varphi_{2^{2w}}(t+s)\alpha = \alpha$ and $\varphi_{2^{2w}-1}(t+s)\alpha = 0$ by 2.2. The former follows from

$$\varphi_{2^{2w}}(t+s)\alpha = (t+s)^{2^{2w}-1}\alpha = \alpha$$

by 3.7(a) since $2^w - 1 \mid 2^{2w}-1$. The latter is obtained by the following computation

$$\begin{aligned}
\varphi_{2^{2w}-1}(t+s)\alpha &= \textstyle\sum_{j=1}^{2w}(t+s)^{2^{2w}-2^j}\alpha \\
&= \textstyle\sum_{j=1}^{w}\{(t+s)^{2^{2w}-2^j} + (t+s)^{2^{2w}-2^{j+w}}\}\alpha \\
&= \textstyle\sum_{j=1}^{w}(t+s)^{2^{2w}-2^{j+w}}\{(t+s)^{2^j(2^w-1)}\alpha + \alpha\} \\
&= 0
\end{aligned}$$

since $(t+s)^{2^w-1}\alpha = \alpha$ by 3.7(a).
(b) It suffices to see that $\varphi_{2^h(2^{2w}-1)}(t+s)\alpha = 0$ from 2.3(b). Note that

$$\begin{aligned}
\varphi_{2^h(2^{2w}-1)}(t+s)\alpha &= (t+s)^{2^h-1}\textstyle\sum_{j=1}^{2w}(t+s)^{2^h(2^{2w}-2^j)}\alpha \\
&= (t+s)^{2^h-1}\textstyle\sum_{j=1}^{w}\{(t+s)^{2^h(2^{2w}-2^j)} + (t+s)^{2^h(2^{2w}-2^{j+w})}\}\alpha \\
&= \textstyle\sum_{j=1}^{w}(t+s)^{2^h(2^{2w}-2^{j+w})}\{(t+s)^{2^j 2^h(2^w-1)} + 1\}(t+s)^{2^h-1}\alpha \\
&= 0
\end{aligned}$$

since $(t+s)^{2^h(2^w-1)}(t+s)^{2^h-1}\alpha = (t+s)^{2^h-1}\alpha$ by 3.7(b).
(c) It suffices to see that $\varphi_{2^{h-1}m}(t+s)\alpha = 0$ from 2.3(b). Note that

$$\begin{aligned}
\varphi_{2^{h-1}m}(t+s)\alpha &= (t+s)^{2^{h-1}-1}\varphi_m((t+s)^{2^{h-1}})\alpha \\
&= (t+s)^{2^{h-1}}\varphi_m((t+s)^{2^{h-1}})ab \\
&= t^{2^{h-1}}\varphi_m(t^{2^{h-1}})ab \\
&= t^{2^{h-1}-1}\varphi_m(t^{2^{h-1}})tab \\
&= \varphi_{2^{h-1}m}(t)tab \\
&= 0
\end{aligned}$$

since $(t+s)^{2^{h-1}}ab = t^{2^{h-1}}ab$ by $s^{2^{l-1}}b = 0$ and $\varphi_{2^{k-1}m}(t)ta = t(2^{k-1}m)a = 0$.
(d) It is immediate from 3.4(c) that

$$\varphi_{2^{h-1}}(t+s)\alpha = (t+s)^{2^{h-1}}ab = (t^{2^{h-1}} + s^{2^{h-1}})ab = 0.$$

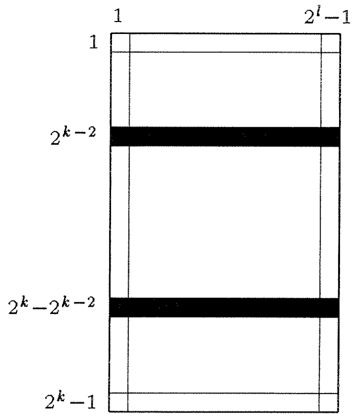
Hence by Lemma 2.3(b) we have $\hat{\delta}^{2^h}(\alpha,0) = (\alpha,0)$, which proves $K(2^k,2^l) \mid 2^h$. To see $K(2^k,2^l) = 2^h$ it suffices to show $(t+s)\varphi_{2^{h-2}}(t+s)\alpha \neq 0$ by the virtue of 2.3(c). Now one may assume $h = k$ without loss of generality.

$$\begin{aligned}
(t+s)\varphi_{2^{k-2}}(t+s)\alpha &= (t+s)(t+s)^{2^{k-2}-1}(t+s)ab \\
&= (t+s)(t^{2^{k-2}} + s^{2^{k-2}})ab.
\end{aligned}$$

(i) In the case of $k > l \geq 2$. As $s^{2^{k-2}}b = 0$ we have

$$\begin{aligned}(t+s)\varphi_{2^{k-2}}(t+s)\alpha &= (t+s)t(2^{k-2})ab \\ &= (t+s)\{t(2^{k-2})b + t(2^k - 2^{k-2})b\} \\ &\neq 0.\end{aligned}$$

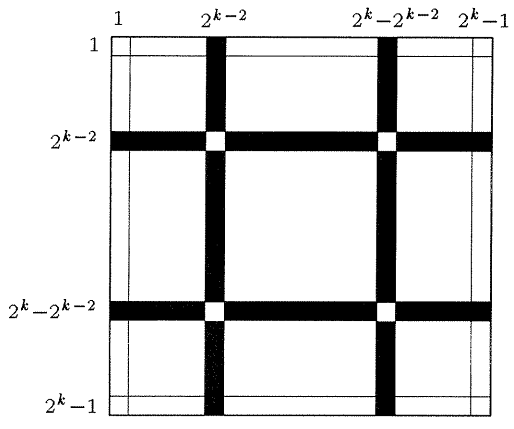
The configuration $t(2^{k-2})b + t(2^k - 2^{k-2})b$ of ca–$90(2^k, 2^l)$ is illustrated by the following figure:



(ii) In the case of $k = l \geq 3$. We have

$$\begin{aligned}(t+s)\varphi_{2^{k-2}}(t+s)\alpha &= (t+s)\{t(2^{k-2})b + t(2^k - 2^{k-2})b + as(2^{k-2}) + as(2^k - 2^{k-2})\} \\ &\neq 0.\end{aligned}$$

The configuration $t(2^{k-2})b+t(2^k-2^{k-2})b+as(2^{k-2})+as(2^k-2^{k-2})$ of ca–$90(2^k, 2^k)$ is illustrated by the following figure:



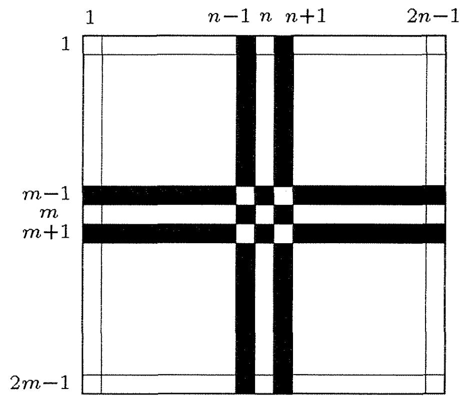
Theorem 4.5 *Let m and n be odd integers ≥ 3, and let k and l be positive integers. Then the period length $K(2^k m, 2^l n)$ of the configuration $(\alpha, 0)$ in $cam-90(2^k m, 2^l n)$ is even.*

Proof. Assume that $K(2^k m, 2^l n)$ is odd, that is, $K(2^k m, 2^l n) \mid 2^{2w} - 1$. Then $\hat{\delta}^{2^{2w}-1}(\alpha, 0) = (\alpha, 0)$ and so $\alpha = \varphi_{2^{2w}}(t+s)\alpha = (t+s)^{2^{2w}-1}\alpha$ in ca–$90(2^k m, 2^l n)$. However by the result in [7] it is impossible if at least one of k and l is greater than 1. Thus it suffices to inspect the case of $k = l = 1$. Recall that $t^{2(2^u-1)}ta = ta$ (by 3.4(b)) and $t^{2^u-1}ta = t(m-1)+t(m+1)$ in ca–$90(2m)$. (For $2^u = (2r+1)m \pm 1$ for some integer r. If r is even then $t^{2^u} = t(2^u)a = t(m \pm 1)$, and if r is odd then $t^{2^u}t(2m + m \pm 1) = t(m \mp 1)$. Hence $t^{2^u-1}ta = t(m \mp 1)a = t(m-1) + t(m+1)$

in $ca-90(2m)$.) Thus $t^{2^w-1}ta = t^{2^u-1}ta = t(m-1)+t(m+1)$ in $ca-90(2m)$. Similarly $s^{2^w-1}sb = s(n-1)+s(n+1)$ in $ca-90(2n)$. Therefore we have

$$\begin{aligned}
\varphi_{2^{2w}}(t+s)\alpha &= (t+s)^{2^{2w}-1}\alpha \\
&= (t+s)^{2^w-1}(t+s)^{2^{w-1}2(2^w-1)}\alpha \\
&= (t+s)^{2^w-1}\alpha \\
&= (t^{2^{2w}}+s^{2^{2w}})ab \\
&= t(m-1)b+t(m+1)b+as(n-1)+as(n+1) \\
&\neq t(1)b+t(2m-1)b+as(1)+as(2n-1)(=\alpha)
\end{aligned}$$

in $ca-90(2m,2n)$. This proves that $K(2^k m, 2^l n)$ is even. The configuration $t(m-1)b+t(m+1)b+as(n-1)+as(n+1)$ of $ca-90(2m,2n)$ is illustrated by the following figure:



References

[1] O. Martin, A.M. Oldlyzko and S. Wolfram, *Algebraic properties of cellular automata*, Comm. Math. Physics **93** (1984) 219–258.

[2] S. Wolfram, *Theory and applications of cellular automata* (World Scientific, Singapore, 1986).

[3] P. Guan and Y. He, *Exact results for deterministic cellular automata with additive rules*, J. Statist. Phys. **43** (1986) 463–478.

[4] S.S. Manna and D. Stauffer, *Systematics of transitions of square-lattice cellular automata*, Physica A **162** (1990), 176–186.

[5] L.R. da Silva, *Fractal dimension at the phase transition of inhomogeneous cellular automata*, J. Statist. Phys. **53** (1988), 985–990.

[6] P.M. Binder, *Topological classification of cellular automata*, J. Phys. A: Math. Gen. **24** (1991), L31–L34.

[7] Y. Kawahara, S. Kumamoto, Y. Mizoguchi, M. Nohmi, H. Ohtsuka and T. Shoudai, *Period lengths of cellular automata on square lattices with rule 90*, J. Math. Physics **36(3)** (1995), 1435 – 1456.

[8] Y. Kawahara, *Existence of the characteristic numbers associated with cellular automata with local transition rule 90*, Bull. Inform. Cybernet. **24(3-4)** (1991) 121–136.

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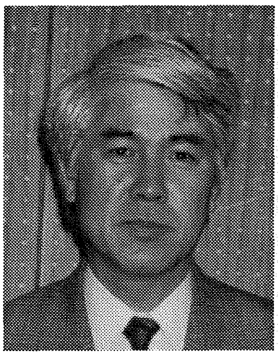

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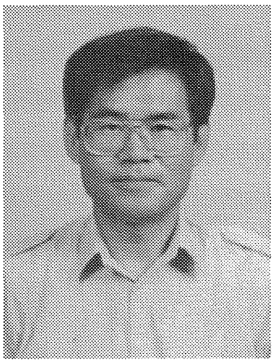

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