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DUALITY IN DYNAMIC FUZZY SYSTEMS

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Abstract: This paper shows the resolvent equation, the maximum principle and the co-balayage theorem for a dynamic fuzzy system. We define a dual system for the dynamic fuzzy system, and gives a duality for Snell's optimal stopping problem by the dual system.

Keyword: Duality, dynamic fuzzy system, optimal stopping problem, balayage theorem, resolvent equation, maximum principle.

0. Introduction

In [4], we defined a dynamic fuzzy system using a fuzzy relation and gave limit theorems for the transition of fuzzy states of the system under the contractive properties of the fuzzy relation. Next, in [14, 15], we discussed potential theory of the system under the contractive and nonexpansive properties. Further, in Yoshida [11], we developed fundamental theory of the dynamic fuzzy system as a Markov chain with transition possibility measures. This paper analyses dual properties for the system in [11], and introduces a dual system of the dynamic fuzzy system as a dual Markov chain with transition possibility measures.

In Sections 2 and 3, we define various fuzzy transition operators and P-excessive possibility measures. For the dynamic fuzzy system, we show the resolvent equation, the maximum principle and the co-balayage theorem, which are well-known in Markov potential theory. In Section 4, we define a dual dynamic fuzzy system, on the basis of the results.

In this paper, we deal with Snell's optimal stopping problem for the dynamic fuzzy system (see [11, Section 4]), and discuss a duality for the problem. We give a dual representation by the dual dynamic fuzzy system. In Section 5, we also give a one-dimensional numerical example to illustrate our idea.

1. Dynamic fuzzy systems

We use the notations of dynamic fuzzy systems introduced by [11]. Let S be a metric space. We write a fuzzy set on S by its membership function $\tilde{s} : S \mapsto [0,1]$ and an ordinary set $A(\subset S)$ by its indicator function $1_A : S \mapsto \{0,1\}$. Refer to Zadeh [16] and Novàk [6] for the theory of fuzzy sets. We define operations $\bigwedge$ and $\bigvee$ for fuzzy sets as follows : Let Γ be an index set. For a family of fuzzy sets $\{\tilde{s}_n\}_{n\in\Gamma}$ on S, we put

$$\bigwedge_{n\in\Gamma} \tilde{s}_n(x) := \inf_{n\in\Gamma} \tilde{s}_n(x) \quad \text{and} \quad \bigvee_{n\in\Gamma} \tilde{s}_n(x) := \sup_{n\in\Gamma} \tilde{s}_n(x), \quad x \in S.$$

For a fuzzy set $\tilde{s}$ on S, its α-cut $\tilde{s}_\alpha$ is defined by

$$\tilde{s}_\alpha := \{x \in S \mid \tilde{s}(x) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{s}_0 := \mathrm{cl}\{x \in S \mid \tilde{s}(x) > 0\},$$

where cl denotes the closure of a set. $\mathcal{E}(S)$ denotes the set of all countable unions of closed subsets of S, so called F_σ-sets (see [5] and [11]). $\mathcal{F}(S)$ denotes the set of all fuzzy sets on S satisfying the following conditions (F.i) and (F.ii) :

(F.i) $\tilde{s}_\alpha \in \mathcal{E}(S)$ for $\alpha \in [0,1]$;

(F.ii) $\bigcap_{\alpha'<\alpha} \tilde{s}_{\alpha'} = \tilde{s}_\alpha$ for $\alpha \in (0,1]$.

Then, we define

$$\mathcal{G}(S) := \left\{ \text{fuzzy sets } \tilde{s} \text{ on } S \ \middle| \ \text{there exists a sequence } \{\tilde{s}_n\}_{n=0}^\infty \subset \mathcal{F}(S) \text{ such that } \bigvee_{n\geq 0} \tilde{s}_n = \tilde{s} \right\}.$$

Let a time space $\mathbf{N} := \{0,1,2,3,\cdots\}$. Let a state space E be a finite-dimensional Euclidean space. We put a path space by $\Omega := \prod_{k=0}^\infty E$ and we put a sample path $\omega = (\omega(0), \omega(1), \omega(2), \cdots) \in \Omega$. Define a map $X_n(\omega) := \omega(n)$ and a shift $\theta_n(\omega) := (\omega(n), \omega(n+1), \omega(n+2), \cdots)$ for $n \in \mathbf{N}$ and $\omega = (\omega(0), \omega(1), \omega(2), \cdots) \in \Omega$. We put σ-fields by $\mathcal{M}_n := \sigma(X_0, X_1, \cdots, X_n)$ [1] for $n \in \mathbf{N}$ and $\mathcal{M} := \sigma(\bigcup_{n\in\mathbf{N}} \mathcal{M}_n)$ [2]. In this paper, we call $X := \{X_n\}_{n\in\mathbf{N}}$ a dynamic fuzzy system. The law of the transition is defined as follows. Let $\tilde{q}$ be time-invariant upper semicontinuous fuzzy relations on $E \times E$ satisfying the following normality condition :

$$\sup_{x\in E} \tilde{q}(x,y) = 1 \ (y \in E) \quad \text{and} \quad \sup_{y\in E} \tilde{q}(x,y) = 1 \ (x \in E).$$

Then $\tilde{q}$ means a transition fuzzy relation. Define $\tilde{P}(\Lambda) := \sup_{\omega\in\Lambda} \bigwedge_{n=0}^\infty \tilde{q}(X_n(\omega), X_{n+1}(\omega))$ for $\Lambda \in \mathcal{M}$. We define a fuzzy expectation by the possibility measure $\tilde{P}$: For an initial state $x \in E$,

$$E_x(h) := \int\!\!\!\!\!-_{\{\omega\in\Omega:\omega(0)=x\}} h(\omega)\, \mathrm{d}\tilde{P}(\omega) = \sup_{\omega\in\Omega:\omega(0)=x} h(\omega) \wedge \bigwedge_{n=0}^{\infty} \tilde{q}(X_n\omega, X_{n+1}\omega)$$

for all $\mathcal{M}$-measurable fuzzy sets $h \in \mathcal{F}(\Omega)$, where $\int\!\!\!\!\!-\ \mathrm{d}\tilde{P}$ denotes Sugeno integral (Sugeno [10]).

[1]It denotes the smallest σ-field on Ω relative to which $X_0, X_1, \cdots, X_n$ are measurable.

[2]It denotes the smallest σ-field generated by $\bigcup_{n\in\mathbf{N}} \mathcal{M}_n$.

Let $\mathcal{E} := \{A \mid A \in \mathcal{E}(E) \text{ and } A^c \in \mathcal{E}(E)\}$, where $A^c := E \setminus A$. We call a map $\tau : \Omega \mapsto \mathbf{N} \cup \{\infty\}$ an $\mathcal{E}$-stopping time if

$$\{\tau = n\} \in \mathcal{M}_n \cap \mathcal{E}(\Omega) \quad \text{for all } n \in \mathbf{N}.$$

For example, a constant stopping time i.e. $\tau = n_0$ for some $n_0 \in \mathbf{N}$, is an $\mathcal{E}$-stopping time. And, for $A \in \mathcal{E}$, its first entry time τ_A and its first hitting time σ_A are $\mathcal{E}$-stopping times ([11, Lemma 1.5]) :

$$\tau_A(\omega) = \inf\{n \in \mathbf{N} \mid X_n(\omega) \in A\} \quad \omega \in \Omega;$$

$$\sigma_A(\omega) = \inf\{n \in \mathbf{N} \mid n \geq 1, X_n(\omega) \in A\} \quad \omega \in \Omega,$$

where the infima of the empty set are understood to be $+\infty$.

In this paper, we deal with fuzzy sets in $\mathcal{G}(E)$ and we call maps from $\mathcal{G}(E)$ to $\mathcal{G}(E)$ fuzzy transition operators. We define a fuzzy transition operator $P : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$P\tilde{s}(x) := E_x(\tilde{s}(X_1)) = \sup_{y \in E}\{\tilde{q}(x,y) \wedge \tilde{s}(y)\} \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E). \tag{1.1}$$

Next, we define fuzzy transition operators $P_n : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ $(n \in \mathbf{N})$ by

$$P_0 := I(\text{identity}) \quad \text{and} \quad P_{n+1} := PP_n \ (n = 0, 1, 2, \cdots).$$

Then, we have

$$P_n\tilde{s}(x) := E_x(\tilde{s}(X_n)) \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E).$$

Further, for an $\mathcal{E}$-stopping time τ, we define a fuzzy transition operator $P_\tau : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$P_\tau\tilde{s}(x) := E_x(\tilde{s}(X_\tau)) \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E),$$

where we define $X_\tau := X_n$ on $\{\tau = n\}$, $n \in \mathbf{N} \cup \{\infty\}$.

2. Fuzzy transition operators

In this section, we define various fuzzy transition operators and investigate the relations. For $A \in \mathcal{E}(E)$, we define a map $I_A : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$I_A\tilde{s} := 1_A \wedge \tilde{s} \quad \text{for } \tilde{s} \in \mathcal{G}(E).$$

Then, the following lemma gives a decomposition of the hitting possibility P_{τ_A} by the fuzzy transition operator P.

Lemma 2.1. *Let $A \in \mathcal{E}$ and $\tilde{s} \in \mathcal{G}(E)$. Then*

$$P_{\tau_A}\tilde{s} = \bigvee_{n \in \mathbf{N}} (I_{A^c}P)^n I_A\tilde{s}.$$

Proof. From [11, Lemmas 1.1 and 3.1(i)], it is easily checked that $(I_{A^c}P)^n\tilde{s} \in \mathcal{G}(E)$ for $n \in \mathbf{N}$. By induction, we prove that for all $n \in \mathbf{N}$

$$E_x(\tilde{s}(X_{\tau_A}) \wedge 1_{\{\tau_A=n\}}) = E_x(\tilde{s}(X_n) \wedge 1_{\{\tau_A=n\}}) = (I_{A^c}P)^n I_A\tilde{s}(x) \quad x \in E. \tag{2.1}$$

When $n = 0$, (2.1) holds since

$$E_x(\tilde{s}(X_0) \wedge 1_{\{\tau_A=0\}}) = 1_A(x) \wedge \tilde{s}(x) = I_A\tilde{s}(x) = (I_{A^c}P)^0 I_A\tilde{s}(x) \quad x \in E.$$

Let $n \in \mathbf{N}$. We prove (2.1) for $n+1$, assuming that (2.1) holds for n. By using Markov property ([11, Theorem 2.2]) and the property of the fuzzy transition operator P, we obtain

$$\begin{aligned} (I_{A^c}P)^{n+1}I_A\tilde{s}(x) &= (I_{A^c}P)(I_{A^c}P)^n I_A\tilde{s}(x) \\ &= 1_{A^c}(x) \wedge E_x(E_{X_1}(\tilde{s}(X_n) \wedge 1_{\{\tau_A=n\}})) \\ &= 1_{A^c}(x) \wedge E_x(\tilde{s}(X_{n+1}) \wedge 1_{\{\tau_A\circ\theta_1=n\}}) \\ &= E_x(\tilde{s}(X_{n+1}) \wedge 1_{\{\tau_A=n+1\}}) \qquad x \in E. \end{aligned}$$

Thus, we get (2.1) inductively. We obtain this lemma from (2.1) and [11, Lemma 3.1(i)]. □

Next, we discuss the other relations among important fuzzy transition operators in dynamic fuzzy systems. Let $A \in \mathcal{E}$. Then, we define fuzzy transition operators on $\mathcal{G}(E)$ as follows :

$$G := \bigvee_{n\in\mathbf{N}} P_n; \quad G_A := \bigvee_{n\in\mathbf{N}} (I_{A^c}P)^n; \qquad G'_A := \bigvee_{n\in\mathbf{N}} (PI_{A^c})^n; \tag{2.2}$$

$$U_A := PG_A = G'_AP = \bigvee_{n\in\mathbf{N}} P(I_{A^c}P)^n. \tag{2.3}$$

Then, clearly

$$G_A = I \vee I_{A^c}U_A; \qquad G'_A = I \vee U_AI_{A^c}; \tag{2.4}$$

$$G_\phi = G'_\phi = G; \quad G_E = G'_E = I; \quad U_\phi = PG; \quad U_E = P.$$

From Lemma 2.1, we also have

$$G_AI_A = P_{\tau_A}; \qquad U_AI_A = P_{\sigma_A}. \tag{2.5}$$

Further, we obtain the following resolvent equation (c.f. Revuz [9, Proposion 2.2.5]).

Theorem 2.1 (resolvent equation). *Let $A, B \in \mathcal{E}$ satisfy $A \subset B$. Then :*

$$U_A = \bigvee_{n\in\mathbf{N}} (U_BI_{B\setminus A})^nU_B = \bigvee_{n\in\mathbf{N}} U_B(I_{B\setminus A}U_B)^n; \tag{2.6}$$

$$U_A = U_B \vee U_B I_{B\setminus A} U_A = U_B \vee U_A I_{B\setminus A} U_B. \tag{2.7}$$

Proof. Generally, for fuzzy transition operators $L, M, N : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ and $n \in \mathbf{N}$, we can easily check

$$(N(L \vee M))^n N = \bigvee_{\substack{k,n_0,n_1,\cdots,n_k \in \mathbf{N} \\ k+n_0+n_1+\cdots+n_k=n}} (NL)^{n_0} NM(NL)^{n_1} NM \cdots M(NL)^{n_k} N.$$

Therefore,

$$\bigvee_{n\in\mathbf{N}} (N(L \vee M))^n N = \bigvee_{k\in\mathbf{N}} \bigvee_{k,n_0,n_1,\cdots,n_k\in\mathbf{N}} ((NL)^{n_0} N)M((NL)^{n_1} N)M \cdots M((NL)^{n_k} N).$$

Putting $R := \bigvee_{n\in\mathbf{N}} (NL)^n N$, we obtain

$$\bigvee_{n\in\mathbf{N}} (N(L \vee M))^n N = \bigvee_{k\in\mathbf{N}} (RM)^k R. \tag{2.8}$$

Let $A, B \in \mathcal{E}$ satisfy $A \subset B$. Here, taking $N := P, L := I_{B^c}, M := I_{B\setminus A}$ in (2.8), we obtain $R = U_B$ and

$$U_A = \bigvee_{n\in\mathbf{N}} (PI_{A^c})^n P = \bigvee_{n\in\mathbf{N}} (P(I_{B^c} \vee I_{B\setminus A}))^n P = \bigvee_{n\in\mathbf{N}} (U_B I_{B\setminus A})^n U_B.$$

Therefore we obtain (2.6). We can easily check (2.7) from (2.6). □

In the rest of this section, we prove the maximum principle for dynamic fuzzy systems (c.f. [9, Theorem 2.1.11]).

Lemma 2.2. *Let τ be an $\mathcal{E}$-stopping time and $\tilde{s} \in \mathcal{G}(E)$. Then*

$$G\tilde{s}(x) = E_x\left(\bigvee_{n<\tau} \tilde{s}(X_n)\right) \vee P_\tau G\tilde{s}(x) \quad \text{for } x \in E.$$

Proof. Using strong Markov property ([11, Theorem 2.2]), we have

$$\begin{aligned}
P_\tau G\tilde{s}(x) &= E_x(G\tilde{s}(X_\tau)) \\
&= E_x\left(E_{X_\tau}\left(\bigvee_{n\geq 0} \tilde{s}(X_n)\right)\right) \\
&= E_x\left(\bigvee_{n\geq 0} \tilde{s}(X_{\tau+n})\right) \\
&= E_x\left(\bigvee_{n\geq \tau} \tilde{s}(X_n)\right) \quad x \in E.
\end{aligned}$$

Therefore we obtain this lemma. □

We define a partial order $\leq$ on $\mathcal{G}(E)$ as follows : For $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$,

$$\tilde{s} \leq \tilde{r} \quad \text{means that} \quad \tilde{s}(x) \leq \tilde{r}(x) \text{ for all } x \in E.$$

Theorem 2.2 (maximum principle). *Let $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$. Assume $\{\tilde{s} = 0\} \in \mathcal{E}(E)$. If*

$$G\tilde{s} \leq G\tilde{r} \quad \text{on } \{\tilde{s} > 0\}, \tag{2.9}$$

then

$$G\tilde{s} \leq G\tilde{r} \quad \text{on } E. \tag{2.10}$$

Proof. First, we assume $\tilde{s} \in \mathcal{F}(E)$. Let $A := \{\tilde{s} > 0\}$. We have $A = \bigcup_{n=1}^{\infty}\{\tilde{s} \geq 1/n\} \in \mathcal{E}(E)$ from [11, Lemma 1.1]. We also have $A^c = \{\tilde{s} = 0\} \in \mathcal{E}(E)$. Therefore τ_A is an $\mathcal{E}$-stopping time. From (2.9), we have

$$P_{\tau_A} G\tilde{s} \leq P_{\tau_A} G\tilde{r}. \tag{2.11}$$

Then $E_x(\bigvee_{n<\tau_A}(\tilde{s}(X_n))) = 0$ for all $x \in E$, where we put $\bigvee_{n<0}(\tilde{s}(X_n\omega)) := 0$ for $\omega \in \Omega$. Therefore, by (2.11) and Lemma 2.2, we obtain

$$\begin{aligned} G\tilde{s}(x) &= E_x\left(\bigvee_{n<\tau_A} \tilde{s}(X_n)\right) \vee P_{\tau_A} G\tilde{s}(x) \\ &\leq E_x\left(\bigvee_{n<\tau_A} \tilde{r}(X_n)\right) \vee P_{\tau_A} G\tilde{r}(x) \\ &= G\tilde{s}(x) \qquad x \in E. \end{aligned}$$

Thus, we obtain (2.10) for $\tilde{s} \in \mathcal{F}(E)$. Next, we get the result for $\tilde{s} \in \mathcal{G}(E)$ by applying [11, Lemma 3.1(i)]. □

3. P-excessive possibility measures

First, we deal with possibility measures and introduce P-excessive property for them as a duality of the following P-superharmonic fuzzy sets in [11, Section 4]

Definition 3.1. A fuzzy set $\tilde{s} \in \mathcal{G}(E)$ is called P-superharmonic if

$$\tilde{s}(x) \geq P\tilde{s}(x) \quad \text{for all } x \in E.$$

Definition 3.2 ([7, 11]). A map $\mu : \mathcal{E}(E) \mapsto [0, 1]$ is called a possibility measure on E if it satisfies (P.i) — (P.iii) :

(P.i) $\mu(\phi) = 0$ and $\mu(E) = 1$;

(P.ii) If $A, B \in \mathcal{E}(E)$ satisfy $A \subset B$, then $\mu(A) \leq \mu(B)$;

(P.iii) If $\{A_n\}_{n=0}^{\infty} \subset \mathcal{E}(E)$, then $\mu(\bigcup_{n=0}^{\infty} A_n) = \bigvee_{n=0}^{\infty} \mu(A_n)$.

We define a fuzzy integration of a fuzzy set $\tilde{s} \in \mathcal{G}(E)$ by the possibility measure μ :

$$\mu(\tilde{s}) := \int_E \tilde{s}(x) \, \mathrm{d}\mu(x) = \sup_{\alpha \in [0,1]} (\alpha \wedge \mu(\tilde{s}_\alpha)).$$

Then, (P.i) — (P.iii) are equivalent to the following (I.i) — (I.iii) (see Ralescu and Adams [8]) :

(I.i) $\mu(1_\phi) = 0$ and $\mu(1_E) = 1$;

(I.ii) If $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$ satisfy $\tilde{s} \leq \tilde{r}$, then $\mu(\tilde{s}) \leq \mu(\tilde{r})$;

(I.iii) If $\{\tilde{s}_n\}_{n=0}^{\infty} \subset \mathcal{G}(E)$, then $\mu(\bigvee_{n=0}^{\infty} \tilde{s}_n) = \bigvee_{n=0}^{\infty} \mu(\tilde{s}_n)$.

Let $\mathcal{P}$ be the set of all possibility measures on E. We introduce a partial order on $\mathcal{P}$ as follows : For $\mu, \nu \in \mathcal{P}$,

$$\mu \geq \nu \quad \text{means that} \quad \mu(A) \geq \nu(A) \text{ for all } A \in \mathcal{E}(E). \tag{3.1}$$

By [8], (3.1) is equivalent to

$$\mu(\tilde{s}) \geq \nu(\tilde{s}) \text{ for all } \tilde{s} \in \mathcal{G}(E). \tag{3.2}$$

We define $\mu L(\tilde{s}) := \mu(L\tilde{s})$ for $\tilde{s} \in \mathcal{G}(E)$ and fuzzy transition operators $L : \mathcal{G}(E) \mapsto \mathcal{G}(E)$.

Definition 3.3. A possibility measure $\mu \in \mathcal{P}$ is called P-excessive if

$$\mu \geq \mu P.$$

We investigate properties of P-excessive possibility measures.

Lemma 3.1. *Let $\mu \in \mathcal{P}$. μ is P-excessive if and only if*

$$\mu = \mu G. \tag{3.3}$$

Proof. Let μ be P-excessive. Then

$$\mu = \mu \vee \mu P \vee \mu P_2 \vee \cdots \vee \mu P_n \quad \text{for all } n \in \mathbf{N}.$$

So we obtain (3.3). The converse proof is trivial. □

We obtain the following lemma by Theorem 2.1.

Lemma 3.2 (c.f. [9, Proposition 2.2.6]). *Let μ be P-excessive. Then :*

$$\mu I_A U_A \leq \mu I_B U_B \leq \mu \quad \text{for } A, B \in \mathcal{E} : A \subset B \tag{3.4}$$

Proof. Let $A, B \in \mathcal{E}$ satisfy $A \subset B$. First we assume $\mu I_B U_B \leq \mu$. By induction, we prove

$$\bigvee_{k=0}^{n-1} \mu I_A U_B (I_{B\setminus A} U_B)^k \vee \mu I_B U_B (I_{B\setminus A} U_B)^n \leq \mu I_B U_B \leq \mu \quad \text{for } n \in \mathbf{N}. \tag{3.5}$$

We have (3.5) for $n = 1$ since

$$\mu I_A U_B \vee \mu I_B U_B I_{B\setminus A} U_B \leq \mu I_A U_B \vee \mu I_{B\setminus A} U_B \leq \mu I_B U_B \leq \mu.$$

Let $n \in \mathbf{N}$. We suppose (3.5) holds for n. Then, by multiplying of $I_{B\setminus A} U_B$ from the right side of (3.5) and taking its $\vee$-operation with $\mu I_A U_B$, we obtain

$$\bigvee_{k=0}^{n} \mu I_A U_B (I_{B\setminus A} U_B)^k \vee \mu I_B U_B (I_{B\setminus A} U_B)^{n+1} \leq \mu I_{B\setminus A} U_B \vee \mu I_A U_B = \mu I_B U_B \leq \mu.$$

Thus, we obtain (3.5) for all $n \in \mathbf{N}$. From (3.5), (2.6) and the property (I.iii) for μ, we obtain

$$\mu I_A U_A = \bigvee_{k=0}^{\infty} \mu I_A U_B (I_{B\setminus A} U_B)^k \leq \mu I_B U_B \leq \mu.$$

Consequently, we get $\mu I_A U_A \leq \mu$ if $A, B \in \mathcal{E}$ $(A \subset B)$ and $\mu I_B U_B \leq \mu$. Finally, since $\mu I_E U_E = \mu P \leq \mu$, the proof is completed. □

Next, we give a duality for Snell's optimal stopping problem [11, Section 4]. The duality in Markov processes are found in [9, Propositions 2.5.6] and [2]. Let $\mu \in \mathcal{P}$ be a fixed initial possibility measure. Let $\tilde{s} \in \mathcal{G}(E)$ be a fuzzy goal. We consider a problem : Maximize $\mu P_\tau(\tilde{s})$ with respect to finite $\mathcal{E}$-stopping times. For simplicity, we deal with a case $\tilde{c} = 1_E$ in [11, Section 4]. Then, we define a possibility measure

$$\nu(\tilde{s}) := \sup_{\tau\,:\,\text{finite}\,\mathcal{E}-\text{stopping times}} \mu P_\tau(\tilde{s}), \quad \tilde{s} \in \mathcal{G}(E). \tag{3.6}$$

Lemma 3.3. *ν has the following properties :*

(i) $\nu = \mu G$;

(ii) $\nu = \mu \vee \nu P$.

Proof. (i) We have $\mu P_\tau \leq \mu G$ for all finite $\mathcal{E}$-stopping times τ. Therefore $\nu \leq \mu G$. While, $\mu P_n \leq \nu$ for all $n \in \mathbf{N}$. By the property (I.iii) for μ, we have $\nu \geq \mu G$. Therefore we obtain $\nu = \mu G$.

(ii) From (i) and the property (I.iii), we obtain

$$\nu = \mu G = \bigvee_{n \in \mathbf{N}} \mu P_n = \mu \vee \bigvee_{n \in \mathbf{N}} \mu P_n P = \mu \vee \mu G P = \mu \vee \nu P.$$

Therefore this lemma holds. □

Theorem 3.1 (Duality in Snell's optimal stopping problem). *The optimal stopping problem (3.6) has the following dual representation :*

(i) *ν is the smallest P-excessive possibility measure dominating μ. Namely*

$$\nu(\tilde{s}) = \min_{\mu' \in \mathcal{P}:\, \mu' \geq \mu,\, \mu' \leq \mu' P} \mu'(\tilde{s}) \quad \text{for } \tilde{s} \in \mathcal{G}(E). \tag{3.7}$$

(ii) *Further, it holds that*

$$\nu(\tilde{s}) = \min_{\tilde{r}:\, P-\text{superharmonic},\, \tilde{r} \geq \tilde{s}} \mu(\tilde{r}) \quad \text{for } \tilde{s} \in \mathcal{G}(E). \tag{3.8}$$

Proof. (i) From Lemma 3.3(ii), we have $\nu \geq \mu$ and $\nu \geq \nu P$. Let $\mu' \in \mathcal{P}$ be P-excessive such that $\mu' \geq \mu$. Then, by Lemma 3.1,

$$\mu'(\tilde{s}) = \mu' G(\tilde{s}) = \mu'(G\tilde{s}) \geq \mu(G\tilde{s}) = \mu G(\tilde{s}) = \nu(\tilde{s}) \quad \text{for } \tilde{s} \in \mathcal{G}(E).$$

Therefore, we obtain (i).

(ii) From Lemma 3.3(ii) and [11, Lemma 4.1(ii)], we have

$$\min\{\mu(\tilde{r}) \mid \tilde{r} \text{ is } P\text{-superharmonic and } \tilde{r} \geq \tilde{s}\} = \mu(G\tilde{s}) = \mu G(\tilde{s}) = \nu(\tilde{s}) \quad \text{for } \tilde{s} \in \mathcal{G}(E). \tag{3.9}$$

Therefore, we obtain (ii) and the proof is completed. □

Finally, by using Theorem 3.1, we obtain the following balayage theorem regarding P-excessive measures (c.f. [9, Theorem 2.4.5]).

Theorem 3.2 (co-balayage theorem). *Let μ be P-excessive and let $A \in \mathcal{E}$. Then*

$$\mu H_A = \mu I_A G, \tag{3.10}$$

where $H_A := I_A G'_A = I_A \vee I_A U_A I_{A^c}$. Then, (3.10) is the smallest P-excessive possibility measure which dominates μ on A.

Proof. First, we prove that μH_A is the smallest P-excessive possibility measure which dominates μ on A. We have

$$\mu H_A I_A = \mu(I_A \vee I_A U_A I_{A^c}) I_A = \mu I_A.$$

Therefore $\mu H_A = \mu$ on A. Further, from (2.3) and Lemma 3.2, we have

$$\mu H_A P = \mu I_A G'_A P = \mu I_A U_A = \mu I_A U_A I_A \vee \mu I_A U_A I_{A^c} \leq \mu I_A \vee \mu I_A U_A I_{A^c} = \mu H_A.$$

Thus μH_A is P-excessive. While, let $\mu' \in \mathcal{P}$ be P-excessive dominating μ on A. Using Lemma 3.2, we have

$$\mu' = \mu' I_A \vee \mu' I_{A^c} \geq \mu' I_A \vee \mu' I_A U_A I_{A^c} \geq \mu I_A \vee \mu I_A U_A I_{A^c} = \mu H_A.$$

Thus, μH_A is the smallest P-excessive dominating μ on A. Therefore, by Lemma 3.1(i) and Theorem 3.1(i), we obtain $\mu H_A = \mu I_A G$. Thus we obtain this theorem. □

Corollary 3.1. *Let μ be P-excessive. Then*

$$\mu I_A U_A = \mu I_A G P \quad \text{for } A \in \mathcal{E}.$$

Proof. Trivial from Theorem 3.2 and (2.3). □

4. Dual dynamic fuzzy systems

In this section, we deal with a dual system for dynamic fuzzy systems $X = \{X_n\}_{n \in \mathbf{N}}$ and we give an explicit representation for the duality in Theorem 3.1. We define a binary relation $\langle \cdot, \cdot \rangle$ on $\mathcal{G}(E) \times \mathcal{G}(E)$:

$$\langle \tilde{s}, \tilde{r} \rangle := \int_E \tilde{s}(x) \wedge \tilde{r}(x) \, \mathrm{d}x = \sup_{x \in E} \{\tilde{s}(x) \wedge \tilde{r}(x)\} \quad \text{for } \tilde{s}, \tilde{r} \in \mathcal{G}(E). \tag{4.1}$$

Define $\hat{q}(x,y) := \tilde{q}(y,x)$ for $x, y \in E$. Then, the fuzzy relation $\hat{q}$ is time-invariant and also satisfies the same normality condition. We may define a dual dynamic fuzzy system $\hat{X} = \{\hat{X}_n\}_{n \in \mathbf{N}}$ with a fuzzy transition operator $\hat{P}$:

$$\hat{P}\tilde{s}(x) := \sup_{y \in E} \{\hat{q}(x,y) \wedge \tilde{s}(y)\} = \sup_{y \in E} \{\tilde{s}(y) \wedge \tilde{q}(y,x)\} \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E). \tag{4.2}$$

For the system, we use the notations ' ^ ' to distinguish X and $\hat{X}$. The path space is $\hat{\Omega} := \Omega$ and a path is $\hat{\omega} = (\hat{\omega}(0), \hat{\omega}(1), \hat{\omega}(2), \cdots) \in \hat{\Omega}$. We define $\hat{X}_n(\hat{\omega}) := \hat{\omega}(n)$ ($\hat{\omega} \in \hat{\Omega}, n \in \mathbf{N}$) and we put σ-fields by $\hat{\mathcal{M}}_n := \sigma(\hat{X}_0, \hat{X}_1, \cdots, \hat{X}_n)$ ($n \in \mathbf{N}$) and $\hat{\mathcal{M}} := \sigma(\bigcup_{n \in \mathbf{N}} \hat{\mathcal{M}}_n)$. Then, we also define a possibility measure $\hat{P}(\Lambda) := \sup_{\hat{\omega} \in \Lambda} \bigwedge_{n=0}^{\infty} \hat{q}(\hat{X}_n(\hat{\omega}), \hat{X}_{n+1}(\hat{\omega}))$ for $\Lambda \in \hat{\mathcal{M}}$. For $h \in \mathcal{F}(\hat{\mathcal{M}})$, we define an expectation by the possibility measure $\hat{P}$:

$$\hat{E}_x(h) := \int_{\{\hat{\omega} \in \hat{\Omega} : \hat{\omega}(0) = x\}} h(\hat{\omega}) \, \mathrm{d}\hat{P}(\hat{\omega}) = \sup_{\hat{\omega} \in \hat{\Omega} : \hat{\omega}(0) = x} h(\hat{\omega}) \wedge \bigwedge_{n=0}^{\infty} \hat{q}(\hat{X}_n \hat{\omega}, \hat{X}_{n+1} \hat{\omega}) \quad \text{for } x \in E.$$

Then, we have the following lemma.

Lemma 4.1. *Let $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$ and $\{\tilde{s}_n\}_{n=0}^{\infty} \subset \mathcal{G}(E)$. Then :*

(i) $\langle P\tilde{s}, \tilde{r}\rangle = \langle \tilde{s}, \hat{P}\tilde{r}\rangle$;

(ii) $\langle I_A\tilde{s}, \tilde{r}\rangle = \langle \tilde{s}, I_A\tilde{r}\rangle$ *for* $A \in \mathcal{E}(E)$;

(iii) $\langle \tilde{s}, \tilde{r}\rangle = \langle \tilde{r}, \tilde{s}\rangle$;

(iv) $\langle \tilde{s}, \tilde{s}\rangle = \langle \tilde{s}, 1\rangle = \sup_{x\in E} \tilde{s}(x)$;

(v) $\langle 0, \tilde{s}\rangle = \langle \tilde{s}, 0\rangle = 0$;

(vi) $\langle \bigvee_{n\geq 0} \tilde{s}_n, \tilde{r}\rangle = \sup_{n\geq 0}\langle \tilde{s}_n, \tilde{r}\rangle$.

Proof. (i) We have

$$\langle P\tilde{s}, \tilde{r}\rangle = \sup_{x\in E}\{\{\sup_{y\in E} \tilde{q}(x,y) \wedge \tilde{s}(y)\} \wedge \tilde{r}(x)\} = \sup_{y\in E} \tilde{s}(y) \wedge \{\sup_{x\in E} \tilde{r}(x) \wedge \tilde{q}(x,y)\}\} = \langle \tilde{s}, \hat{P}\tilde{r}\rangle.$$

(ii) — (vi) are trivial from the definitions. □

Define the first entry time $\hat{\tau}_A$ and the first hitting time $\hat{\sigma}_A$ for $A \in \mathcal{E}$ for the dual system $\hat{X}$ similarly to Section 1, and define possibilities $\hat{P}_{\hat{\tau}_A}$ and $\hat{P}_{\hat{\sigma}_A}$ similarly to Section 2. Then we have the following results.

Proposition 4.1. *Let* $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$ *and* $A \in \mathcal{E}$. *Then :*

(i) $\langle U_A\tilde{s}, \tilde{r}\rangle = \langle \tilde{s}, \hat{U}_A\tilde{r}\rangle$, *where* $\hat{U}_A := \bigvee_{n\in\mathbf{N}} \hat{P}(I_{A^c}\hat{P})^n$;

(ii) $\langle G\tilde{s}, \tilde{r}\rangle = \langle \tilde{s}, \hat{G}\tilde{r}\rangle$, *where* $\hat{G} := \bigvee_{n\in\mathbf{N}} \hat{P}_n$;

(iii) $\langle H_A\tilde{s}, \tilde{r}\rangle = \langle \tilde{s}, \hat{P}_{\hat{\tau}_A}\tilde{r}\rangle$;

(iv) $\langle I_A U_A\tilde{s}, \tilde{r}\rangle = \langle \tilde{s}, \hat{P}_{\hat{\sigma}_A}\tilde{r}\rangle$;

(v) $\langle I_A U_A I_A\tilde{s}, \tilde{r}\rangle = \langle \tilde{s}, I_A\hat{U}_A I_A\tilde{r}\rangle$.

Proof. (i) From (2.3) and Lemma 4.1, we have

$$\langle U_A\tilde{s}, \tilde{r}\rangle = \left\langle \bigvee_{n\in\mathbf{N}} (PI_{A^c})^n P\tilde{s}, \tilde{r}\right\rangle = \sup_{n\in\mathbf{N}}\langle (PI_{A^c})^n P\tilde{s}, \tilde{r}\rangle = \sup_{n\in\mathbf{N}}\langle \tilde{s}, \hat{P}(I_{A^c}\hat{P})^n\tilde{r}\rangle = \langle \tilde{s}, \hat{U}_A\tilde{r}\rangle.$$

(ii) and (v) are similar. (iii) By (2.3) and Lemma 4.1(ii)(vi), similarly to (i) we have

$$\langle H_A\tilde{s}, \tilde{r}\rangle = \langle I_A\tilde{s}, \tilde{r}\rangle \vee \langle I_A U_A I_{A^c}\tilde{s}, \tilde{r}\rangle = \langle \tilde{s}, I_A\tilde{r}\rangle \vee \langle \tilde{s}, I_{A^c}\hat{U}_A I_A\tilde{r}\rangle = \left\langle \tilde{s}, \bigvee_{n\in\mathbf{N}} (I_{A^c}\hat{P})^n I_A\tilde{r}\right\rangle = \langle \tilde{s}, \hat{P}_{\hat{\tau}_A}\tilde{r}\rangle.$$

Thus we obtain (iii). Finally, (iv) follows from

$$\bigvee_{n\in\mathbf{N}} (I_{A^c}\hat{P})^n I_A\tilde{r} = \hat{P}_{\hat{\tau}_A}\tilde{r} \quad \text{for } \tilde{r} \in \mathcal{G}(E).$$

Therefore we obtain this proposition. □

The following lemma characterizes densities of P-excessive possibility measures.

Lemma 4.2. *Let $\tilde{s} \in \mathcal{G}(E)$ satisfy $\sup_{x\in E} \tilde{s}(x) = 1$. Put $\mu \in \mathcal{P}$ by*

$$\mu(A) := \begin{cases} \sup_{x\in A} \tilde{s}(x) & \text{for } A \in \mathcal{E}(E)\ (A \neq \emptyset), \\ 0 & \text{for } A = \emptyset. \end{cases}$$

Then, the possibility measure μ is P-excessive if and only if $\tilde{s}$ is $\hat{P}$-superharmonic.

Proof. Suppose that $\tilde{s}$ is $\hat{P}$-superharmonic. Then we have

$$\begin{aligned} \mu P(A) &= \mu(P1_A) \\ &= \sup_{x\in E}\{\tilde{s}(x) \wedge P1_A(x)\} \\ &= \sup_{x\in E}\{\tilde{s}(x) \wedge \sup_{y\in A} \tilde{q}(x,y)\} \\ &= \sup_{y\in A}\sup_{x\in E}\{\tilde{s}(x) \wedge \tilde{q}(x,y)\} \\ &= \sup_{y\in A} \hat{P}\tilde{s}(y) \\ &\leq \sup_{y\in A} \tilde{s}(y) \\ &= \mu(A), \quad A \in \mathcal{E}(E). \end{aligned}$$

Therefore we have μ is P-excessive. Conversely, if μ is P-excessive, then similarly we obtain

$$\sup_{y\in A} \hat{P}\tilde{s}(y) \leq \sup_{y\in A} \tilde{s}(y), \quad A \in \mathcal{E}(E).$$

Taking $A := \{y\}$ $(y \in E)$, we have that $\tilde{s}$ is $\hat{P}$-superharmonic. Therefore, the proof is completed. □

Next, we investigate a possibility measure ξ relative to which the dynamic fuzzy systems X and $\hat{X}$ are dual. Let $\tilde{m} \in \mathcal{G}(E)$ satisfy $\sup_{x\in E} \tilde{m}(x) = 1$. Let ξ be a fixed possibility measure with the density $\tilde{m}$:

$$\xi(A) := \begin{cases} \sup_{x\in A} \tilde{m}(x) & \text{for } A \in \mathcal{E}(E)\ (A \neq \emptyset), \\ 0 & \text{for } A = \emptyset. \end{cases} \tag{4.3}$$

We define a binary relation $\langle \cdot, \cdot \rangle_\xi$ on $\mathcal{G}(E) \times \mathcal{G}(E)$:

$$\langle \tilde{s}, \tilde{r} \rangle_\xi := \int_E \tilde{s}(x) \wedge \tilde{r}(x)\ \mathrm{d}\xi(x) = \sup_{x\in E}\{\tilde{s}(x) \wedge \tilde{r}(x) \wedge \tilde{m}(x)\} \quad \text{for } \tilde{s}, \tilde{r} \in \mathcal{G}(E). \tag{4.4}$$

Theorem 4.1. *The dynamic fuzzy systems X and $\hat{X}$ are dual relative to the possibility measure ξ if and only if ξ is P-excessive and $\hat{P}$-excessive. Then, for $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$ and $\{\tilde{s}_n\}_{n=0}^{\infty} \subset \mathcal{G}(E)$, we have :*

(i) $\langle P\tilde{s}, \tilde{r}\rangle_\xi = \langle \tilde{s}, \hat{P}\tilde{r}\rangle_\xi$;

(ii) $\langle I_A\tilde{s}, \tilde{r}\rangle_\xi = \langle \tilde{s}, I_A\tilde{r}\rangle_\xi$ for $A \in \mathcal{E}(E)$;

(iii) $\langle \tilde{s}, \tilde{r}\rangle_\xi = \langle \tilde{r}, \tilde{s}\rangle_\xi$;

(iv) $\langle \tilde{s}, \tilde{s}\rangle_\xi = \langle \tilde{s}, 1\rangle_\xi = \sup_{x\in E}\{\tilde{s}(x) \wedge \tilde{m}(x)\}$;

(v) $\langle 0, \tilde{s}\rangle_\xi = \langle \tilde{s}, 0\rangle_\xi = 0$;

(vi) $\langle \bigvee_{n\geq 0} \tilde{s}_n, \tilde{r}\rangle_\xi = \sup_{n\geq 0}\langle \tilde{s}_n, \tilde{r}\rangle_\xi$;

(vii) $\langle U_A\tilde{s}, \tilde{r}\rangle_\xi = \langle \tilde{s}, \hat{U}_A\tilde{r}\rangle_\xi$, where $\hat{U}_A := \bigvee_{n\in \mathbf{N}} \hat{P}(I_{A^c}\hat{P})^n$.

(viii) $\langle G\tilde{s}, \tilde{r}\rangle_\xi = \langle \tilde{s}, \hat{G}\tilde{r}\rangle_\xi$, where $\hat{G} := \bigvee_{n\in \mathbf{N}} \hat{P}_n$;

(ix) $\langle H_A\tilde{s}, \tilde{r}\rangle_\xi = \langle \tilde{s}, \hat{P}_{\hat{\tau}_A}\tilde{r}\rangle_\xi$;

(x) $\langle I_AU_A\tilde{s}, \tilde{r}\rangle_\xi = \langle \tilde{s}, \hat{P}_{\hat{\sigma}_A}\tilde{r}\rangle_\xi$;

(xi) $\langle I_AU_AI_A\tilde{s}, \tilde{r}\rangle_\xi = \langle \tilde{s}, I_A\hat{U}_AI_A\tilde{r}\rangle_\xi$.

Proof. If ξ is P-excessive, then we have

$$\begin{aligned}\tilde{m}(y) &= \mu(1_{\{y\}}) \\ &\geq \mu(P1_{\{y\}}) \\ &= \mu\left(\sup_{y'\in E}\{\tilde{q}(\cdot, y') \wedge 1_{\{y\}}(y')\}\right) \\ &= \mu\left(\tilde{q}(\cdot, y)\right) \\ &= \sup_{x\in E}\{\tilde{q}(x,y) \wedge \tilde{m}(x)\} \\ &\geq \tilde{q}(x,y) \wedge \tilde{m}(x) \quad \text{for all } x, y \in E.\end{aligned}$$

Therefore,

$$\tilde{q}(x,y) \wedge \tilde{m}(x) \leq \tilde{q}(x,y) \wedge \tilde{m}(y) \quad \text{for all } x,y \in E. \tag{4.5}$$

Further, if ξ is $\hat{P}$-excessive, in the same way, we have the reverse inequality in (4.5). Therefore

$$\tilde{q}(x,y) \wedge \tilde{m}(x) = \tilde{q}(x,y) \wedge \tilde{m}(y) \quad \text{for all } x,y \in E. \tag{4.6}$$

Similarly, we can easily derive that ξ is P-excessive and $\hat{P}$-excessive from (4.6). Therefore (4.6) is equivalent to the condition that ξ is P-excessive and $\hat{P}$-excessive. While, from (4.6), we obtain

$$\begin{aligned}\langle P\tilde{s}, \tilde{r}\rangle_\xi &= \sup_{x\in E}\{\{\sup_{y\in E} \tilde{q}(x,y) \wedge \tilde{s}(y)\} \wedge \tilde{r}(x) \wedge \tilde{m}(x)\} \\ &= \sup_{y\in E}\{\tilde{s}(y) \wedge \{\sup_{x\in E} \tilde{r}(x) \wedge \tilde{q}(x,y)\} \wedge \tilde{m}(x)\} \\ &= \langle \tilde{s}, \hat{P}\tilde{r}\rangle_\xi.\end{aligned} \tag{4.7}$$

Conversely, in (4.7) we take $\tilde{s} := 1_{\{y\}}$ and $\tilde{r} := 1_{\{x\}}$ for $x, y \in E$. Then (4.7) is reduced to (4.6). Therefore, X and $\hat{X}$ are dual relative to the possibility measure ξ if and only if ξ is P-excessive and $\hat{P}$-excessive. The proof of (i) – (xi) are similar to the proofs of Lemma 4.1 and Proposition 4.1. □

In the rest of this section, we discuss Snell's optimal stopping problem in the duality of dynamic fuzzy systems. We use the binary relation $\langle \cdot, \cdot \rangle$ on $\mathcal{G}(E) \times \mathcal{G}(E)$. In the problem (3.6), we consider a case when an initial possibility measure μ is represented by

$$\mu(A) := \sup_{x \in A} \tilde{u}(x), \quad A \in \mathcal{E}(E). \tag{4.8}$$

Then, from (3.6), we have

$$\nu(\tilde{s}) := \sup_{\tau : \text{finite } \mathcal{E}-\text{stopping times}} \langle \tilde{u}, P_\tau \tilde{s} \rangle, \quad \tilde{s} \in \mathcal{G}(E). \tag{4.9}$$

Theorem 4.2. *The optimal stopping problem (4.9) has the following dual representation:*

$$\nu(\tilde{s}) = \min_{\tilde{v} : \hat{P}-\text{superharmonic}, \tilde{v} \geq \tilde{u}} \langle \tilde{v}, \tilde{s} \rangle = \min_{\tilde{r} : P-\text{superharmonic}, \tilde{r} \geq \tilde{s}} \langle \tilde{u}, \tilde{r} \rangle. \tag{4.10}$$

Proof. From (4.8) and (3.8), it is trivial that

$$\nu(\tilde{s}) = \min_{\tilde{r} : P-\text{superharmonic}, \tilde{r} \geq \tilde{s}} \langle \tilde{u}, \tilde{r} \rangle. \tag{4.11}$$

It is sufficient to prove that

$$\nu(\tilde{s}) = \min_{\tilde{v} : \hat{P}-\text{superharmonic}, \tilde{v} \geq \tilde{u}} \langle \tilde{v}, \tilde{s} \rangle. \tag{4.12}$$

Let $\tilde{v}$ be $\hat{P}$-superharmonic and $\tilde{v} \geq \tilde{u}$, and put a possibility measure μ' by

$$\mu'(A) := \sup_{x \in A} \tilde{v}(x), \quad A \in \mathcal{E}(E).$$

From Lemma 4.2, μ' is P-excessive and $\mu' \geq \mu$. By (3.7), we obtain

$$\nu(\tilde{s}) \leq \mu'(\tilde{s}) = \langle \tilde{v}, \tilde{s} \rangle. \tag{4.13}$$

Especially, $\hat{G}\tilde{u}$ is $\hat{P}$-superharmonic and $\hat{G}\tilde{u} \geq \tilde{u}$ since

$$\hat{G}\tilde{u} = \bigvee_{n \geq 0} \hat{P}^n \tilde{u} = \tilde{u} \vee \bigvee_{n \geq 1} \hat{P}^n \tilde{u} = \tilde{u} \vee \hat{P}\hat{G}\tilde{u}.$$

Then, from (3.9) and Proposition 4.1(ii), we obtain

$$\nu(\tilde{s}) = \mu G(\tilde{s}) = \mu(G\tilde{s}) = \langle \tilde{u}, G\tilde{s} \rangle = \langle \hat{G}\tilde{u}, \tilde{s} \rangle.$$

Together with (4.13), we obtain (4.12). Therefore, the proof is completed. □

5. A numerical example

We consider a numerical example with a one-dimensional state space $E = \mathbf{R}$, where $\mathbf{R}$ is the set of all real numbers. We calculate the optimal value $\nu(\tilde{s})$ for Snell's problem (4.9).

Example 5.1. We give fuzzy sets $\tilde{s}, \tilde{u}$ and a fuzzy relation $\tilde{q}$ by

$$\tilde{s}(x) = (0.8 - 0.1|1 - x|) \vee 0, \quad x \in \mathbf{R}, \tag{5.1}$$

$$\tilde{u}(x) = (0.8 - 0.1|1 + x|) \vee 0, \quad x \in \mathbf{R}, \tag{5.2}$$

$$\tilde{q}(x, y) = (1 - |y - x^3|) \vee 0, \quad x, y \in \mathbf{R}. \tag{5.3}$$

then, we give a possibility measure μ by (4.8). Let $\tilde{s}^*$ be the smallest $\hat{P}$-superharmonic fuzzy set dominating $\tilde{s}$. By [13, Example 4.1], we have

$$\tilde{s}^*(x) = \begin{cases} \tilde{s}(x) & \text{for} \quad x \in (-\infty, x_1) & \approx & (-\infty, -1.1651) \\ \tilde{q}(x, x) & \text{for} \quad x \in [x_1, x_2) & \approx & [-1.1651, -1.1147) \\ \alpha^* \approx 0.7296 & \text{for} \quad x \in [x_2, x_3) & \approx & [-1.1147, 0.2964) \\ \tilde{s}(x) & \text{for} \quad x \in [x_3, x_4) & \approx & [0.2964, 0.8687) \\ \tilde{q}(x, x) & \text{for} \quad x \in [x_4, x_5) & \approx & [0.8687, 0.8789) \\ 0.8 & \text{for} \quad x \in [x_5, x_6) & \approx & [0.8789, 1.0880) \\ \tilde{q}(x, x) & \text{for} \quad x \in [x_6, x_7) & \approx & [1.0880, 1.0916) \\ \tilde{s}(x) & \text{for} \quad x \in [x_7, +\infty) & \approx & [1.0916, +\infty). \end{cases}$$

Next, let $\tilde{u}^*$ be the smallest $\hat{P}$-superharmonic fuzzy set dominating $\tilde{u}$. Since $\hat{q}(x, y) = \tilde{q}(y, x)$ $(x, y \in \mathbf{R})$, in a method similar to [13, Example 4.1], we can easily check

$$\tilde{u}^*(x) = \begin{cases} 0.8 & \text{for} \quad x \in (-\infty, x_8) & \approx & (-\infty, 0.2091) \\ \tilde{q}(x, x) & \text{for} \quad x \in [x_8, x_9) & \approx & [0.2091, 0.4098) \\ \tilde{u}(x) & \text{for} \quad x \in [x_9, x_{10}) & \approx & [0.4098, 0.6749) \\ 0.6325 & \text{for} \quad x \in [x_{10}, +\infty) & \approx & [0.6749, +\infty). \end{cases}$$

Fig. 5.1 shows the fuzzy sets $\tilde{s}$ and $\tilde{u}^*$, and Fig. 5.2 shows the fuzzy sets $\tilde{u}$ and $\tilde{s}^*$.

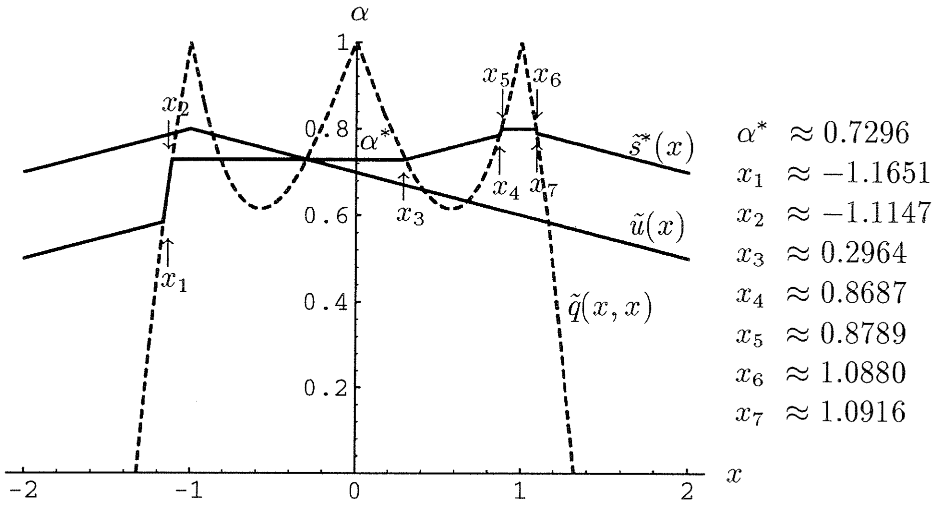

Fig. 5.1. The fuzzy sets $\tilde{s}(x)$ and $\tilde{u}^*(x)$ and a fuzzy set $\tilde{q}(x,x)$.

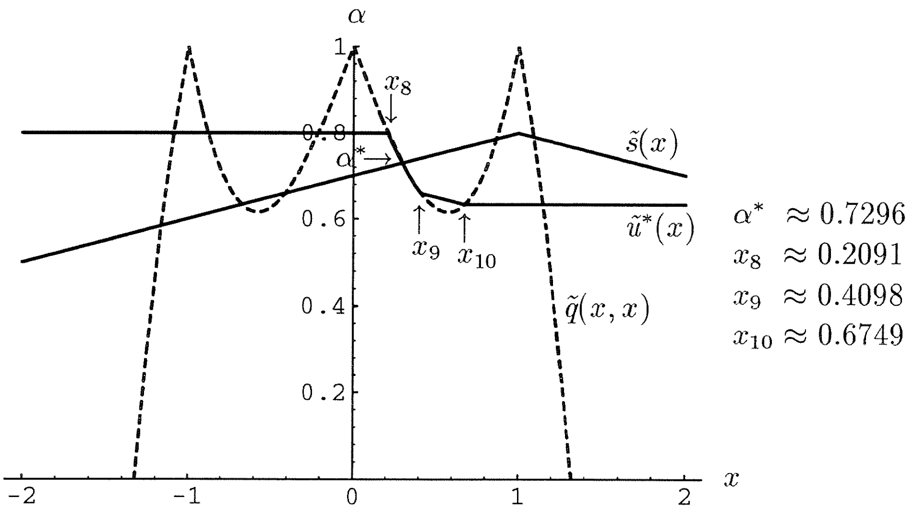

Fig. 5.2. The fuzzy sets $\tilde{u}(x)$ and $\tilde{s}^*(x)$ and a fuzzy set $\tilde{q}(x,x)$.

From Theorem 4.2 and Figs. 5.1 and 5.2, we obtain

$$\nu(\tilde{s}) = \langle \tilde{u}^*, \tilde{s} \rangle = \langle \tilde{u}, \tilde{s}^* \rangle = \alpha^* \approx 0.7296.$$

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