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<http://hdl.handle.net/2324/3194>

出版情報 : RIFIS Technical Report. 104, 1995-02. Research Institute of Fundamental Information Science, Kyushu University

バージョン :

権利関係 :

RIFIS Technical Report

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February ,1995

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DUALITY IN DYNAMIC FUZZY SYSTEMS

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Abstract: This paper shows the resolvent equation, the maximum principle and the co-balayage theorem for a dynamic fuzzy system. We define a dual system for the dynamic fuzzy system, and gives a duality for Snell's optimal stopping problem by the dual system.

Keyword: Duality, dynamic fuzzy system, optimal stopping problem, balayage theorem, resolvent equation, maximum principle.

0. Introduction

In [4], we defined a dynamic fuzzy system using a fuzzy relation and gave limit theorems for the transition of fuzzy states of the system under the contractive properties of the fuzzy relation. Next, in [14, 15], we discussed potential theory of the system under the contractive and nonexpansive properties. Further, in Yoshida [11], we developed fundamental theory of the dynamic fuzzy system as a Markov chain with transition possibility measures. This paper analyses dual properties for the system in [11], and introduces a dual system of the dynamic fuzzy system as a dual Markov chain with transition possibility measures.

In Sections 2 and 3, we define various fuzzy transition operators and P -excessive possibility measures. For the dynamic fuzzy system, we show the resolvent equation, the maximum principle and the co-balayage theorem, which are well-known in Markov potential theory. In Section 4, we define a dual dynamic fuzzy system, on the basis of the results.

In this paper, we deal with Snell's optimal stopping problem for the dynamic fuzzy system (see [11, Section 4]), and discuss a duality for the problem. We give a dual representation by the dual dynamic fuzzy system. In Section 5, we also give a one-dimensional numerical example to illustrate our idea.

1. Dynamic fuzzy systems

We use the notations of dynamic fuzzy systems introduced by [11]. Let S be a metric space. We write a fuzzy set on S by its membership function $\tilde{s} : S \mapsto [0, 1]$ and an ordinary set $A(\subset S)$ by its indicator function $1_A : S \mapsto \{0, 1\}$. Refer to Zadeh [16] and Novák [6]

for the theory of fuzzy sets. We define operations $\bigwedge$ and $\bigvee$ for fuzzy sets as follows : Let Γ be an index set. For a family of fuzzy sets $\{\tilde{s}_n\}_{n \in \Gamma}$ on S , we put

$$\bigwedge_{n \in \Gamma} \tilde{s}_n(x) := \inf_{n \in \Gamma} \tilde{s}_n(x) \quad \text{and} \quad \bigvee_{n \in \Gamma} \tilde{s}_n(x) := \sup_{n \in \Gamma} \tilde{s}_n(x), \quad x \in S.$$

For a fuzzy set $\tilde{s}$ on S , its α -cut $\tilde{s}_\alpha$ is defined by

$$\tilde{s}_\alpha := \{x \in S \mid \tilde{s}(x) \geq \alpha\} \quad (\alpha \in (0, 1]) \quad \text{and} \quad \tilde{s}_0 := \text{cl}\{x \in S \mid \tilde{s}(x) > 0\},$$

where cl denotes the closure of a set. $\mathcal{E}(S)$ denotes the set of all countable unions of closed subsets of S , so called F_σ -sets (see [5] and [11]). $\mathcal{F}(S)$ denotes the set of all fuzzy sets on S satisfying the following conditions (F.i) and (F.ii) :

(F.i) $\tilde{s}_\alpha \in \mathcal{E}(S)$ for $\alpha \in [0, 1]$;

(F.ii) $\bigcap_{\alpha' < \alpha} \tilde{s}_{\alpha'} = \tilde{s}_\alpha$ for $\alpha \in (0, 1]$.

Then, we define

$$\mathcal{G}(S) := \left\{ \text{fuzzy sets } \tilde{s} \text{ on } S \mid \text{there exists a sequence } \{\tilde{s}_n\}_{n=0}^\infty \subset \mathcal{F}(S) \text{ such that } \bigvee_{n \geq 0} \tilde{s}_n = \tilde{s} \right\}.$$

Let a time space $\mathbf{N} := \{0, 1, 2, 3, \dots\}$. Let a state space E be a finite-dimensional Euclidean space. We put a path space by $\Omega := \prod_{k=0}^\infty E$ and we put a sample path $\omega = (\omega(0), \omega(1), \omega(2), \dots) \in \Omega$. Define a map $X_n(\omega) := \omega(n)$ and a shift $\theta_n(\omega) := (\omega(n), \omega(n+1), \omega(n+2), \dots)$ for $n \in \mathbf{N}$ and $\omega = (\omega(0), \omega(1), \omega(2), \dots) \in \Omega$. We put σ -fields by $\mathcal{M}_n := \sigma(X_0, X_1, \dots, X_n)$ ¹ for $n \in \mathbf{N}$ and $\mathcal{M} := \sigma(\bigcup_{n \in \mathbf{N}} \mathcal{M}_n)$ ². In this paper, we call $X := \{X_n\}_{n \in \mathbf{N}}$ a dynamic fuzzy system. The law of the transition is defined as follows. Let $\tilde{q}$ be time-invariant upper semicontinuous fuzzy relations on $E \times E$ satisfying the following normality condition :

$$\sup_{x \in E} \tilde{q}(x, y) = 1 \quad (y \in E) \quad \text{and} \quad \sup_{y \in E} \tilde{q}(x, y) = 1 \quad (x \in E).$$

Then $\tilde{q}$ means a transition fuzzy relation. Define $\tilde{P}(\Lambda) := \sup_{\omega \in \Lambda} \bigwedge_{n=0}^\infty \tilde{q}(X_n(\omega), X_{n+1}(\omega))$ for $\Lambda \in \mathcal{M}$. We define a fuzzy expectation by the possibility measure $\tilde{P}$: For an initial state $x \in E$,

$$E_x(h) := \int_{\{\omega \in \Omega : \omega(0) = x\}} h(\omega) \, d\tilde{P}(\omega) = \sup_{\omega \in \Omega : \omega(0) = x} h(\omega) \wedge \bigwedge_{n=0}^\infty \tilde{q}(X_n \omega, X_{n+1} \omega)$$

for all $\mathcal{M}$ -measurable fuzzy sets $h \in \mathcal{F}(\Omega)$, where $\int d\tilde{P}$ denotes Sugeno integral (Sugeno [10]).

¹It denotes the smallest σ -field on Ω relative to which $X_0, X_1, \dots, X_n$ are measurable.

²It denotes the smallest σ -field generated by $\bigcup_{n \in \mathbf{N}} \mathcal{M}_n$.

Let $\mathcal{E} := \{A \mid A \in \mathcal{E}(E) \text{ and } A^c \in \mathcal{E}(E)\}$, where $A^c := E \setminus A$. We call a map $\tau : \Omega \mapsto \mathbf{N} \cup \{\infty\}$ an $\mathcal{E}$ -stopping time if

$$\{\tau = n\} \in \mathcal{M}_n \cap \mathcal{E}(\Omega) \quad \text{for all } n \in \mathbf{N}.$$

For example, a constant stopping time i.e. $\tau = n_0$ for some $n_0 \in \mathbf{N}$, is an $\mathcal{E}$ -stopping time. And, for $A \in \mathcal{E}$, its first entry time τ_A and its first hitting time σ_A are $\mathcal{E}$ -stopping times ([11, Lemma 1.5]) :

$$\tau_A(\omega) = \inf\{n \in \mathbf{N} \mid X_n(\omega) \in A\} \quad \omega \in \Omega;$$

$$\sigma_A(\omega) = \inf\{n \in \mathbf{N} \mid n \geq 1, X_n(\omega) \in A\} \quad \omega \in \Omega,$$

where the infima of the empty set are understood to be $+\infty$.

In this paper, we deal with fuzzy sets in $\mathcal{G}(E)$ and we call maps from $\mathcal{G}(E)$ to $\mathcal{G}(E)$ fuzzy transition operators. We define a fuzzy transition operator $P : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$P\tilde{s}(x) := E_x(\tilde{s}(X_1)) = \sup_{y \in E} \{\tilde{q}(x, y) \wedge \tilde{s}(y)\} \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E). \quad (1.1)$$

Next, we define fuzzy transition operators $P_n : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ ($n \in \mathbf{N}$) by

$$P_0 := I(\text{identity}) \quad \text{and} \quad P_{n+1} := P P_n \quad (n = 0, 1, 2, \dots).$$

Then, we have

$$P_n \tilde{s}(x) := E_x(\tilde{s}(X_n)) \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E).$$

Further, for an $\mathcal{E}$ -stopping time τ , we define a fuzzy transition operator $P_\tau : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$P_\tau \tilde{s}(x) := E_x(\tilde{s}(X_\tau)) \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E),$$

where we define $X_\tau := X_n$ on $\{\tau = n\}$, $n \in \mathbf{N} \cup \{\infty\}$.

2. Fuzzy transition operators

In this section, we define various fuzzy transition operators and investigate the relations. For $A \in \mathcal{E}(E)$, we define a map $I_A : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$I_A \tilde{s} := 1_A \wedge \tilde{s} \quad \text{for } \tilde{s} \in \mathcal{G}(E).$$

Then, the following lemma gives a decomposition of the hitting possibility P_{τ_A} by the fuzzy transition operator P .

Lemma 2.1. *Let $A \in \mathcal{E}$ and $\tilde{s} \in \mathcal{G}(E)$. Then*

$$P_{\tau_A} \tilde{s} = \bigvee_{n \in \mathbf{N}} (I_{A^c} P)^n I_A \tilde{s}.$$

Proof. From [11, Lemmas 1.1 and 3.1(i)], it is easily checked that $(I_{A^c}P)^n \tilde{s} \in \mathcal{G}(E)$ for $n \in \mathbf{N}$. By induction, we prove that for all $n \in \mathbf{N}$

$$E_x(\tilde{s}(X_{\tau_A}) \wedge 1_{\{\tau_A=n\}}) = E_x(\tilde{s}(X_n) \wedge 1_{\{\tau_A=n\}}) = (I_{A^c}P)^n I_A \tilde{s}(x) \quad x \in E. \quad (2.1)$$

When $n = 0$, (2.1) holds since

$$E_x(\tilde{s}(X_0) \wedge 1_{\{\tau_A=0\}}) = 1_A(x) \wedge \tilde{s}(x) = I_A \tilde{s}(x) = (I_{A^c}P)^0 I_A \tilde{s}(x) \quad x \in E.$$

Let $n \in \mathbf{N}$. We prove (2.1) for $n + 1$, assuming that (2.1) holds for n . By using Markov property ([11, Theorem 2.2]) and the property of the fuzzy transition operator P , we obtain

$$\begin{aligned} (I_{A^c}P)^{n+1} I_A \tilde{s}(x) &= (I_{A^c}P)(I_{A^c}P)^n I_A \tilde{s}(x) \\ &= 1_{A^c}(x) \wedge E_x(E_{X_1}(\tilde{s}(X_n) \wedge 1_{\{\tau_A=n\}})) \\ &= 1_{A^c}(x) \wedge E_x(\tilde{s}(X_{n+1}) \wedge 1_{\{\tau_A \circ \theta_1=n\}}) \\ &= E_x(\tilde{s}(X_{n+1}) \wedge 1_{\{\tau_A=n+1\}}) \quad x \in E. \end{aligned}$$

Thus, we get (2.1) inductively. We obtain this lemma from (2.1) and [11, Lemma 3.1(i)]. $\square$

Next, we discuss the other relations among important fuzzy transition operators in dynamic fuzzy systems. Let $A \in \mathcal{E}$. Then, we define fuzzy transition operators on $\mathcal{G}(E)$ as follows :

$$G := \bigvee_{n \in \mathbf{N}} P_n; \quad G_A := \bigvee_{n \in \mathbf{N}} (I_{A^c}P)^n; \quad G'_A := \bigvee_{n \in \mathbf{N}} (PI_{A^c})^n; \quad (2.2)$$

$$U_A := PG_A = G'_A P = \bigvee_{n \in \mathbf{N}} P(I_{A^c}P)^n. \quad (2.3)$$

Then, clearly

$$\begin{aligned} G_A &= I \vee I_{A^c}U_A; & G'_A &= I \vee U_A I_{A^c}; \\ G_\phi &= G'_\phi = G; & G_E &= G'_E = I; \quad U_\phi = PG; \quad U_E = P. \end{aligned} \quad (2.4)$$

From Lemma 2.1, we also have

$$G_A I_A = P_{\tau_A}; \quad U_A I_A = P_{\sigma_A}. \quad (2.5)$$

Further, we obtain the following resolvent equation (c.f. Revuz [9, Proposion 2.2.5]).

Theorem 2.1 (resolvent equation). *Let $A, B \in \mathcal{E}$ satisfy $A \subset B$. Then :*

$$U_A = \bigvee_{n \in \mathbf{N}} (U_B I_{B \setminus A})^n U_B = \bigvee_{n \in \mathbf{N}} U_B (I_{B \setminus A} U_B)^n; \quad (2.6)$$

$$U_A = U_B \vee U_B I_{B \setminus A} U_A = U_B \vee U_A I_{B \setminus A} U_B. \quad (2.7)$$

Proof. Generally, for fuzzy transition operators $L, M, N : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ and $n \in \mathbf{N}$, we can easily check

$$(N(L \vee M))^n N = \bigvee_{\substack{k, n_0, n_1, \dots, n_k \in \mathbf{N} \\ k + n_0 + n_1 + \dots + n_k = n}} (NL)^{n_0} NM (NL)^{n_1} NM \dots M (NL)^{n_k} N.$$

Therefore,

$$\bigvee_{n \in \mathbf{N}} (N(L \vee M))^n N = \bigvee_{k \in \mathbf{N}} \bigvee_{n_0, n_1, \dots, n_k \in \mathbf{N}} ((NL)^{n_0} N) M ((NL)^{n_1} N) M \dots M ((NL)^{n_k} N).$$

Putting $R := \bigvee_{n \in \mathbf{N}} (NL)^n N$, we obtain

$$\bigvee_{n \in \mathbf{N}} (N(L \vee M))^n N = \bigvee_{k \in \mathbf{N}} (RM)^k R. \quad (2.8)$$

Let $A, B \in \mathcal{E}$ satisfy $A \subset B$. Here, taking $N := P, L := I_{B^c}, M := I_{B \setminus A}$ in (2.8), we obtain $R = U_B$ and

$$U_A = \bigvee_{n \in \mathbf{N}} (P I_{A^c})^n P = \bigvee_{n \in \mathbf{N}} (P(I_{B^c} \vee I_{B \setminus A}))^n P = \bigvee_{n \in \mathbf{N}} (U_B I_{B \setminus A})^n U_B.$$

Therefore we obtain (2.6). We can easily check (2.7) from (2.6). $\square$

In the rest of this section, we prove the maximum principle for dynamic fuzzy systems (c.f. [9, Theorem 2.1.11]).

Lemma 2.2. *Let τ be an $\mathcal{E}$ -stopping time and $\tilde{s} \in \mathcal{G}(E)$. Then*

$$G\tilde{s}(x) = E_x \left(\bigvee_{n < \tau} \tilde{s}(X_n) \right) \vee P_\tau G\tilde{s}(x) \quad \text{for } x \in E.$$

Proof. Using strong Markov property ([11, Theorem 2.2]), we have

$$\begin{aligned} P_\tau G\tilde{s}(x) &= E_x(G\tilde{s}(X_\tau)) \\ &= E_x \left(E_{X_\tau} \left(\bigvee_{n \geq 0} \tilde{s}(X_n) \right) \right) \\ &= E_x \left(\bigvee_{n \geq 0} \tilde{s}(X_{\tau+n}) \right) \\ &= E_x \left(\bigvee_{n \geq \tau} \tilde{s}(X_n) \right) \quad x \in E. \end{aligned}$$

Therefore we obtain this lemma. $\square$

We define a partial order $\leq$ on $\mathcal{G}(E)$ as follows : For $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$,

$$\tilde{s} \leq \tilde{r} \quad \text{means that} \quad \tilde{s}(x) \leq \tilde{r}(x) \text{ for all } x \in E.$$

Theorem 2.2 (maximum principle). *Let $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$. Assume $\{\tilde{s} = 0\} \in \mathcal{E}(E)$. If*

$$G\tilde{s} \leq G\tilde{r} \quad \text{on } \{\tilde{s} > 0\}, \quad (2.9)$$

then

$$G\tilde{s} \leq G\tilde{r} \quad \text{on } E. \quad (2.10)$$

Proof. First, we assume $\tilde{s} \in \mathcal{F}(E)$. Let $A := \{\tilde{s} > 0\}$. We have $A = \bigcup_{n=1}^{\infty} \{\tilde{s} \geq 1/n\} \in \mathcal{E}(E)$ from [11, Lemma 1.1]. We also have $A^c = \{\tilde{s} = 0\} \in \mathcal{E}(E)$. Therefore τ_A is an $\mathcal{E}$ -stopping time. From (2.9), we have

$$P_{\tau_A} G\tilde{s} \leq P_{\tau_A} G\tilde{r}. \quad (2.11)$$

Then $E_x(\bigvee_{n < \tau_A} (\tilde{s}(X_n))) = 0$ for all $x \in E$, where we put $\bigvee_{n < 0} (\tilde{s}(X_n\omega)) := 0$ for $\omega \in \Omega$. Therefore, by (2.11) and Lemma 2.2, we obtain

$$\begin{aligned} G\tilde{s}(x) &= E_x \left(\bigvee_{n < \tau_A} \tilde{s}(X_n) \right) \vee P_{\tau_A} G\tilde{s}(x) \\ &\leq E_x \left(\bigvee_{n < \tau_A} \tilde{r}(X_n) \right) \vee P_{\tau_A} G\tilde{r}(x) \\ &= G\tilde{s}(x) \quad x \in E. \end{aligned}$$

Thus, we obtain (2.10) for $\tilde{s} \in \mathcal{F}(E)$. Next, we get the result for $\tilde{s} \in \mathcal{G}(E)$ by applying [11, Lemma 3.1(i)]. $\square$

3. P -excessive possibility measures

First, we deal with possibility measures and introduce P -excessive property for them as a duality of the following P -superharmonic fuzzy sets in [11, Section 4]

Definition 3.1. A fuzzy set $\tilde{s} \in \mathcal{G}(E)$ is called P -superharmonic if

$$\tilde{s}(x) \geq P\tilde{s}(x) \quad \text{for all } x \in E.$$

Definition 3.2 ([7, 11]). A map $\mu : \mathcal{E}(E) \mapsto [0, 1]$ is called a possibility measure on E if it satisfies (P.i) — (P.iii) :

(P.i) $\mu(\phi) = 0$ and $\mu(E) = 1$;

(P.ii) If $A, B \in \mathcal{E}(E)$ satisfy $A \subset B$, then $\mu(A) \leq \mu(B)$;

(P.iii) If $\{A_n\}_{n=0}^\infty \subset \mathcal{E}(E)$, then $\mu(\bigcup_{n=0}^\infty A_n) = \bigvee_{n=0}^\infty \mu(A_n)$.

We define a fuzzy integration of a fuzzy set $\tilde{s} \in \mathcal{G}(E)$ by the possibility measure μ :

$$\mu(\tilde{s}) := \int_E \tilde{s}(x) \, d\mu(x) = \sup_{\alpha \in [0,1]} (\alpha \wedge \mu(\tilde{s}_\alpha)).$$

Then, (P.i) — (P.iii) are equivalent to the following (I.i) — (I.iii) (see Ralescu and Adams [8]) :

(I.i) $\mu(1_\phi) = 0$ and $\mu(1_E) = 1$;

(I.ii) If $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$ satisfy $\tilde{s} \leq \tilde{r}$, then $\mu(\tilde{s}) \leq \mu(\tilde{r})$;

(I.iii) If $\{\tilde{s}_n\}_{n=0}^\infty \subset \mathcal{G}(E)$, then $\mu(\bigvee_{n=0}^\infty \tilde{s}_n) = \bigvee_{n=0}^\infty \mu(\tilde{s}_n)$.

Let $\mathcal{P}$ be the set of all possibility measures on E . We introduce a partial order on $\mathcal{P}$ as follows : For $\mu, \nu \in \mathcal{P}$,

$$\mu \geq \nu \quad \text{means that} \quad \mu(A) \geq \nu(A) \text{ for all } A \in \mathcal{E}(E). \quad (3.1)$$

By [8], (3.1) is equivalent to

$$\mu(\tilde{s}) \geq \nu(\tilde{s}) \text{ for all } \tilde{s} \in \mathcal{G}(E). \quad (3.2)$$

We define $\mu L(\tilde{s}) := \mu(L\tilde{s})$ for $\tilde{s} \in \mathcal{G}(E)$ and fuzzy transition operators $L : \mathcal{G}(E) \mapsto \mathcal{G}(E)$.

Definition 3.3. A possibility measure $\mu \in \mathcal{P}$ is called P -excessive if

$$\mu \geq \mu P.$$

We investigate properties of P -excessive possibility measures.

Lemma 3.1. Let $\mu \in \mathcal{P}$. μ is P -excessive if and only if

$$\mu = \mu G. \quad (3.3)$$

Proof. Let μ be P -excessive. Then

$$\mu = \mu \vee \mu P \vee \mu P^2 \vee \cdots \vee \mu P^n \quad \text{for all } n \in \mathbf{N}.$$

So we obtain (3.3). The converse proof is trivial. $\square$

We obtain the following lemma by Theorem 2.1.

Lemma 3.2 (c.f. [9, Proposition 2.2.6]). *Let μ be P -excessive. Then :*

$$\mu I_A U_A \leq \mu I_B U_B \leq \mu \quad \text{for } A, B \in \mathcal{E} : A \subset B \quad (3.4)$$

Proof. Let $A, B \in \mathcal{E}$ satisfy $A \subset B$. First we assume $\mu I_B U_B \leq \mu$. By induction, we prove

$$\bigvee_{k=0}^{n-1} \mu I_A U_B (I_{B \setminus A} U_B)^k \vee \mu I_B U_B (I_{B \setminus A} U_B)^n \leq \mu I_B U_B \leq \mu \quad \text{for } n \in \mathbb{N}. \quad (3.5)$$

We have (3.5) for $n = 1$ since

$$\mu I_A U_B \vee \mu I_B U_B I_{B \setminus A} U_B \leq \mu I_A U_B \vee \mu I_{B \setminus A} U_B \leq \mu I_B U_B \leq \mu.$$

Let $n \in \mathbb{N}$. We suppose (3.5) holds for n . Then, by multiplying of $I_{B \setminus A} U_B$ from the right side of (3.5) and taking its $\vee$ -operation with $\mu I_A U_B$, we obtain

$$\bigvee_{k=0}^n \mu I_A U_B (I_{B \setminus A} U_B)^k \vee \mu I_B U_B (I_{B \setminus A} U_B)^{n+1} \leq \mu I_{B \setminus A} U_B \vee \mu I_A U_B = \mu I_B U_B \leq \mu.$$

Thus, we obtain (3.5) for all $n \in \mathbb{N}$. From (3.5), (2.6) and the property (I.iii) for μ , we obtain

$$\mu I_A U_A = \bigvee_{k=0}^{\infty} \mu I_A U_B (I_{B \setminus A} U_B)^k \leq \mu I_B U_B \leq \mu.$$

Consequently, we get $\mu I_A U_A \leq \mu$ if $A, B \in \mathcal{E}$ ($A \subset B$) and $\mu I_B U_B \leq \mu$. Finally, since $\mu I_E U_E = \mu P \leq \mu$, the proof is completed. $\square$

Next, we give a duality for Snell's optimal stopping problem [11, Section 4]. The duality in Markov processes are found in [9, Propositions 2.5.6] and [2]. Let $\mu \in \mathcal{P}$ be a fixed initial possibility measure. Let $\tilde{s} \in \mathcal{G}(E)$ be a fuzzy goal. We consider a problem : Maximize $\mu P_\tau(\tilde{s})$ with respect to finite $\mathcal{E}$ -stopping times. For simplicity, we deal with a case $\tilde{c} = 1_E$ in [11, Section 4]. Then, we define a possibility measure

$$\nu(\tilde{s}) := \sup_{\tau : \text{finite } \mathcal{E}\text{-stopping times}} \mu P_\tau(\tilde{s}), \quad \tilde{s} \in \mathcal{G}(E). \quad (3.6)$$

Lemma 3.3. *ν has the following properties :*

- (i) $\nu = \mu G$;
- (ii) $\nu = \mu \vee \nu P$.

Proof. (i) We have $\mu P_\tau \leq \mu G$ for all finite $\mathcal{E}$ -stopping times τ . Therefore $\nu \leq \mu G$. While, $\mu P_n \leq \nu$ for all $n \in \mathbb{N}$. By the property (I.iii) for μ , we have $\nu \geq \mu G$. Therefore we obtain $\nu = \mu G$.

(ii) From (i) and the property (I.iii), we obtain

$$\nu = \mu G = \bigvee_{n \in \mathbb{N}} \mu P_n = \mu \vee \bigvee_{n \in \mathbb{N}} \mu P_n P = \mu \vee \mu G P = \mu \vee \nu P.$$

Therefore this lemma holds. $\square$

Theorem 3.1 (Duality in Snell's optimal stopping problem). *The optimal stopping problem (3.6) has the following dual representation :*

(i) ν is the smallest P -excessive possibility measure dominating μ . Namely

$$\nu(\tilde{s}) = \min_{\mu' \in \mathcal{P}: \mu' \geq \mu, \mu' \leq \mu' P} \mu'(\tilde{s}) \quad \text{for } \tilde{s} \in \mathcal{G}(E). \quad (3.7)$$

(ii) Further, it holds that

$$\nu(\tilde{s}) = \min_{\tilde{r}: P\text{-superharmonic}, \tilde{r} \geq \tilde{s}} \mu(\tilde{r}) \quad \text{for } \tilde{s} \in \mathcal{G}(E). \quad (3.8)$$

Proof. (i) From Lemma 3.3(ii), we have $\nu \geq \mu$ and $\nu \geq \nu P$. Let $\mu' \in \mathcal{P}$ be P -excessive such that $\mu' \geq \mu$. Then, by Lemma 3.1,

$$\mu'(\tilde{s}) = \mu' G(\tilde{s}) = \mu'(G\tilde{s}) \geq \mu(G\tilde{s}) = \mu G(\tilde{s}) = \nu(\tilde{s}) \quad \text{for } \tilde{s} \in \mathcal{G}(E).$$

Therefore, we obtain (i).

(ii) From Lemma 3.3(ii) and [11, Lemma 4.1(ii)], we have

$$\min\{\mu(\tilde{r}) \mid \tilde{r} \text{ is } P\text{-superharmonic and } \tilde{r} \geq \tilde{s}\} = \mu(G\tilde{s}) = \mu G(\tilde{s}) = \nu(\tilde{s}) \quad \text{for } \tilde{s} \in \mathcal{G}(E). \quad (3.9)$$

Therefore, we obtain (ii) and the proof is completed. $\square$

Finally, by using Theorem 3.1, we obtain the following balayage theorem regarding P -excessive measures (c.f. [9, Theorem 2.4.5]).

Theorem 3.2 (co-balayage theorem). *Let μ be P -excessive and let $A \in \mathcal{E}$. Then*

$$\mu H_A = \mu I_A G, \quad (3.10)$$

where $H_A := I_A G' A = I_A \vee I_A U_A I_{A^c}$. Then, (3.10) is the smallest P -excessive possibility measure which dominates μ on A .

Proof. First, we prove that μH_A is the smallest P -excessive possibility measure which dominates μ on A . We have

$$\mu H_A I_A = \mu(I_A \vee I_A U_A I_{A^c}) I_A = \mu I_A.$$

Therefore $\mu H_A = \mu$ on A . Further, from (2.3) and Lemma 3.2, we have

$$\mu H_A P = \mu I_A G'_A P = \mu I_A U_A = \mu I_A U_A I_A \vee \mu I_A U_A I_{A^c} \leq \mu I_A \vee \mu I_A U_A I_{A^c} = \mu H_A.$$

Thus μH_A is P -excessive. While, let $\mu' \in \mathcal{P}$ be P -excessive dominating μ on A . Using Lemma 3.2, we have

$$\mu' = \mu' I_A \vee \mu' I_{A^c} \geq \mu' I_A \vee \mu' I_A U_A I_{A^c} \geq \mu I_A \vee \mu I_A U_A I_{A^c} = \mu H_A.$$

Thus, μH_A is the smallest P -excessive dominating μ on A . Therefore, by Lemma 3.1(i) and Theorem 3.1(i), we obtain $\mu H_A = \mu I_A G$. Thus we obtain this theorem. $\square$

Corollary 3.1. *Let μ be P -excessive. Then*

$$\mu I_A U_A = \mu I_A G P \quad \text{for } A \in \mathcal{E}.$$

Proof. Trivial from Theorem 3.2 and (2.3). $\square$

4. Dual dynamic fuzzy systems

In this section, we deal with a dual system for dynamic fuzzy systems $X = \{X_n\}_{n \in \mathbf{N}}$ and we give an explicit representation for the duality in Theorem 3.1. We define a binary relation $\langle \cdot, \cdot \rangle$ on $\mathcal{G}(E) \times \mathcal{G}(E)$:

$$\langle \tilde{s}, \tilde{r} \rangle := \int_E \tilde{s}(x) \wedge \tilde{r}(x) \, dx = \sup_{x \in E} \{ \tilde{s}(x) \wedge \tilde{r}(x) \} \quad \text{for } \tilde{s}, \tilde{r} \in \mathcal{G}(E). \quad (4.1)$$

Define $\hat{q}(x, y) := \tilde{q}(y, x)$ for $x, y \in E$. Then, the fuzzy relation $\hat{q}$ is time-invariant and also satisfies the same normality condition. We may define a dual dynamic fuzzy system $\hat{X} = \{\hat{X}_n\}_{n \in \mathbf{N}}$ with a fuzzy transition operator $\hat{P}$:

$$\hat{P}\tilde{s}(x) := \sup_{y \in E} \{ \hat{q}(x, y) \wedge \tilde{s}(y) \} = \sup_{y \in E} \{ \tilde{s}(y) \wedge \tilde{q}(y, x) \} \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E). \quad (4.2)$$

For the system, we use the notations ‘ $\hat{\cdot}$ ’ to distinguish X and $\hat{X}$. The path space is $\hat{\Omega} := \Omega$ and a path is $\hat{\omega} = (\hat{\omega}(0), \hat{\omega}(1), \hat{\omega}(2), \dots) \in \hat{\Omega}$. We define $\hat{X}_n(\hat{\omega}) := \hat{\omega}(n)$ ($\hat{\omega} \in \hat{\Omega}, n \in \mathbf{N}$) and we put σ -fields by $\hat{\mathcal{M}}_n := \sigma(\hat{X}_0, \hat{X}_1, \dots, \hat{X}_n)$ ($n \in \mathbf{N}$) and $\hat{\mathcal{M}} := \sigma(\bigcup_{n \in \mathbf{N}} \hat{\mathcal{M}}_n)$. Then, we also define a possibility measure $\hat{P}(\Lambda) := \sup_{\hat{\omega} \in \Lambda} \bigwedge_{n=0}^{\infty} \hat{q}(\hat{X}_n(\hat{\omega}), \hat{X}_{n+1}(\hat{\omega}))$ for $\Lambda \in \hat{\mathcal{M}}$. For $h \in \mathcal{F}(\hat{\mathcal{M}})$, we define an expectation by the possibility measure $\hat{P}$:

$$\hat{E}_x(h) := \int_{\{\hat{\omega} \in \hat{\Omega} : \hat{\omega}(0)=x\}} h(\hat{\omega}) \, d\hat{P}(\hat{\omega}) = \sup_{\hat{\omega} \in \hat{\Omega} : \hat{\omega}(0)=x} h(\hat{\omega}) \wedge \bigwedge_{n=0}^{\infty} \hat{q}(\hat{X}_n \hat{\omega}, \hat{X}_{n+1} \hat{\omega}) \quad \text{for } x \in E.$$

Then, we have the following lemma.

Lemma 4.1. *Let $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$ and $\{\tilde{s}_n\}_{n=0}^{\infty} \subset \mathcal{G}(E)$. Then :*

- (i) $\langle P\tilde{s}, \tilde{r} \rangle = \langle \tilde{s}, \hat{P}\tilde{r} \rangle;$
- (ii) $\langle I_A\tilde{s}, \tilde{r} \rangle = \langle \tilde{s}, I_A\tilde{r} \rangle \quad \text{for } A \in \mathcal{E}(E);$
- (iii) $\langle \tilde{s}, \tilde{r} \rangle = \langle \tilde{r}, \tilde{s} \rangle;$
- (iv) $\langle \tilde{s}, \tilde{s} \rangle = \langle \tilde{s}, 1 \rangle = \sup_{x \in E} \tilde{s}(x);$
- (v) $\langle 0, \tilde{s} \rangle = \langle \tilde{s}, 0 \rangle = 0;$
- (vi) $\langle \bigvee_{n \geq 0} \tilde{s}_n, \tilde{r} \rangle = \sup_{n \geq 0} \langle \tilde{s}_n, \tilde{r} \rangle.$

Proof. (i) We have

$$\langle P\tilde{s}, \tilde{r} \rangle = \sup_{x \in E} \{ \sup_{y \in E} \tilde{q}(x, y) \wedge \tilde{s}(y) \} \wedge \tilde{r}(x) = \sup_{y \in E} \tilde{s}(y) \wedge \{ \sup_{x \in E} \tilde{r}(x) \wedge \tilde{q}(x, y) \} = \langle \tilde{s}, \hat{P}\tilde{r} \rangle.$$

(ii) — (vi) are trivial from the definitions. $\square$

Define the first entry time $\hat{\tau}_A$ and the first hitting time $\hat{\sigma}_A$ for $A \in \mathcal{E}$ for the dual system $\hat{X}$ similarly to Section 1, and define possibilities $\hat{P}_{\hat{\tau}_A}$ and $\hat{P}_{\hat{\sigma}_A}$ similarly to Section 2. Then we have the following results.

Proposition 4.1. *Let $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$ and $A \in \mathcal{E}$. Then :*

- (i) $\langle U_A\tilde{s}, \tilde{r} \rangle = \langle \tilde{s}, \hat{U}_A\tilde{r} \rangle$, where $\hat{U}_A := \bigvee_{n \in \mathbb{N}} \hat{P}(I_{A^c}\hat{P})^n$;
- (ii) $\langle G\tilde{s}, \tilde{r} \rangle = \langle \tilde{s}, \hat{G}\tilde{r} \rangle$, where $\hat{G} := \bigvee_{n \in \mathbb{N}} \hat{P}_n$;
- (iii) $\langle H_A\tilde{s}, \tilde{r} \rangle = \langle \tilde{s}, \hat{P}_{\hat{\tau}_A}\tilde{r} \rangle;$
- (iv) $\langle I_A U_A\tilde{s}, \tilde{r} \rangle = \langle \tilde{s}, \hat{P}_{\hat{\sigma}_A}\tilde{r} \rangle;$
- (v) $\langle I_A U_A I_A\tilde{s}, \tilde{r} \rangle = \langle \tilde{s}, I_A \hat{U}_A I_A\tilde{r} \rangle.$

Proof. (i) From (2.3) and Lemma 4.1, we have

$$\langle U_A\tilde{s}, \tilde{r} \rangle = \left\langle \bigvee_{n \in \mathbb{N}} (P I_{A^c})^n P\tilde{s}, \tilde{r} \right\rangle = \sup_{n \in \mathbb{N}} \langle (P I_{A^c})^n P\tilde{s}, \tilde{r} \rangle = \sup_{n \in \mathbb{N}} \langle \tilde{s}, \hat{P}(I_{A^c}\hat{P})^n \tilde{r} \rangle = \langle \tilde{s}, \hat{U}_A\tilde{r} \rangle.$$

(ii) and (v) are similar. (iii) By (2.3) and Lemma 4.1(ii)(vi), similarly to (i) we have

$$\langle H_A\tilde{s}, \tilde{r} \rangle = \langle I_A\tilde{s}, \tilde{r} \rangle \vee \langle I_A U_A I_{A^c}\tilde{s}, \tilde{r} \rangle = \langle \tilde{s}, I_A\tilde{r} \rangle \vee \langle \tilde{s}, I_{A^c}\hat{U}_A I_A\tilde{r} \rangle = \left\langle \tilde{s}, \bigvee_{n \in \mathbb{N}} (I_{A^c}\hat{P})^n I_A\tilde{r} \right\rangle = \langle \tilde{s}, \hat{P}_{\hat{\tau}_A}\tilde{r} \rangle.$$

Thus we obtain (iii). Finally, (iv) follows from

$$\bigvee_{n \in \mathbb{N}} (I_{A^c}\hat{P})^n I_A\tilde{r} = \hat{P}_{\hat{\tau}_A}\tilde{r} \quad \text{for } \tilde{r} \in \mathcal{G}(E).$$

Therefore we obtain this proposition. $\square$

The following lemma characterizes densities of P -excessive possibility measures.

Lemma 4.2. *Let $\tilde{s} \in \mathcal{G}(E)$ satisfy $\sup_{x \in E} \tilde{s}(x) = 1$. Put $\mu \in \mathcal{P}$ by*

$$\mu(A) := \begin{cases} \sup_{x \in A} \tilde{s}(x) & \text{for } A \in \mathcal{E}(E) \ (A \neq \emptyset), \\ 0 & \text{for } A = \emptyset. \end{cases}$$

Then, the possibility measure μ is P -excessive if and only if $\tilde{s}$ is $\hat{P}$ -superharmonic.

Proof. Suppose that $\tilde{s}$ is $\hat{P}$ -superharmonic. Then we have

$$\begin{aligned} \mu P(A) &= \mu(P1_A) \\ &= \sup_{x \in E} \{\tilde{s}(x) \wedge P1_A(x)\} \\ &= \sup_{x \in E} \{\tilde{s}(x) \wedge \sup_{y \in A} \tilde{q}(x, y)\} \\ &= \sup_{y \in A} \sup_{x \in E} \{\tilde{s}(x) \wedge \tilde{q}(x, y)\} \\ &= \sup_{y \in A} \hat{P}\tilde{s}(y) \\ &\leq \sup_{y \in A} \tilde{s}(y) \\ &= \mu(A), \quad A \in \mathcal{E}(E). \end{aligned}$$

Therefore we have μ is P -excessive. Conversely, if μ is P -excessive, then similarly we obtain

$$\sup_{y \in A} \hat{P}\tilde{s}(y) \leq \sup_{y \in A} \tilde{s}(y), \quad A \in \mathcal{E}(E).$$

Taking $A := \{y\}$ ($y \in E$), we have that $\tilde{s}$ is $\hat{P}$ -superharmonic. Therefore, the proof is completed. $\square$

Next, we investigate a possibility measure ξ relative to which the dynamic fuzzy systems X and $\hat{X}$ are dual. Let $\tilde{m} \in \mathcal{G}(E)$ satisfy $\sup_{x \in E} \tilde{m}(x) = 1$. Let ξ be a fixed possibility measure with the density $\tilde{m}$:

$$\xi(A) := \begin{cases} \sup_{x \in A} \tilde{m}(x) & \text{for } A \in \mathcal{E}(E) \ (A \neq \emptyset), \\ 0 & \text{for } A = \emptyset. \end{cases} \quad (4.3)$$

We define a binary relation $\langle \cdot, \cdot \rangle_\xi$ on $\mathcal{G}(E) \times \mathcal{G}(E)$:

$$\langle \tilde{s}, \tilde{r} \rangle_\xi := \int_E \tilde{s}(x) \wedge \tilde{r}(x) \, d\xi(x) = \sup_{x \in E} \{\tilde{s}(x) \wedge \tilde{r}(x) \wedge \tilde{m}(x)\} \quad \text{for } \tilde{s}, \tilde{r} \in \mathcal{G}(E). \quad (4.4)$$

Theorem 4.1. *The dynamic fuzzy systems X and $\hat{X}$ are dual relative to the possibility measure ξ if and only if ξ is P -excessive and $\hat{P}$ -excessive. Then, for $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$ and $\{\tilde{s}_n\}_{n=0}^\infty \subset \mathcal{G}(E)$, we have :*

- (i) $\langle P\tilde{s}, \tilde{r} \rangle_\xi = \langle \tilde{s}, \hat{P}\tilde{r} \rangle_\xi$;
- (ii) $\langle I_A\tilde{s}, \tilde{r} \rangle_\xi = \langle \tilde{s}, I_A\tilde{r} \rangle_\xi$ for $A \in \mathcal{E}(E)$;
- (iii) $\langle \tilde{s}, \tilde{r} \rangle_\xi = \langle \tilde{r}, \tilde{s} \rangle_\xi$;
- (iv) $\langle \tilde{s}, \tilde{s} \rangle_\xi = \langle \tilde{s}, 1 \rangle_\xi = \sup_{x \in E} \{ \tilde{s}(x) \wedge \tilde{m}(x) \}$;
- (v) $\langle 0, \tilde{s} \rangle_\xi = \langle \tilde{s}, 0 \rangle_\xi = 0$;
- (vi) $\langle \bigvee_{n \geq 0} \tilde{s}_n, \tilde{r} \rangle_\xi = \sup_{n \geq 0} \langle \tilde{s}_n, \tilde{r} \rangle_\xi$;
- (vii) $\langle U_A\tilde{s}, \tilde{r} \rangle_\xi = \langle \tilde{s}, \hat{U}_A\tilde{r} \rangle_\xi$, where $\hat{U}_A := \bigvee_{n \in \mathbb{N}} \hat{P}(I_{A^c}\hat{P})^n$.
- (viii) $\langle G\tilde{s}, \tilde{r} \rangle_\xi = \langle \tilde{s}, \hat{G}\tilde{r} \rangle_\xi$, where $\hat{G} := \bigvee_{n \in \mathbb{N}} \hat{P}_n$;
- (ix) $\langle H_A\tilde{s}, \tilde{r} \rangle_\xi = \langle \tilde{s}, \hat{P}_{\hat{\tau}_A}\tilde{r} \rangle_\xi$;
- (x) $\langle I_A U_A \tilde{s}, \tilde{r} \rangle_\xi = \langle \tilde{s}, \hat{P}_{\hat{\sigma}_A}\tilde{r} \rangle_\xi$;
- (xi) $\langle I_A U_A I_A \tilde{s}, \tilde{r} \rangle_\xi = \langle \tilde{s}, I_A \hat{U}_A I_A \tilde{r} \rangle_\xi$.

Proof. If ξ is P -excessive, then we have

$$\begin{aligned}
\tilde{m}(y) &= \mu(1_{\{y\}}) \\
&\geq \mu(P1_{\{y\}}) \\
&= \mu \left(\sup_{y' \in E} \{ \tilde{q}(\cdot, y') \wedge 1_{\{y\}}(y') \} \right) \\
&= \mu(\tilde{q}(\cdot, y)) \\
&= \sup_{x \in E} \{ \tilde{q}(x, y) \wedge \tilde{m}(x) \} \\
&\geq \tilde{q}(x, y) \wedge \tilde{m}(x) \quad \text{for all } x, y \in E.
\end{aligned}$$

Therefore,

$$\tilde{q}(x, y) \wedge \tilde{m}(x) \leq \tilde{q}(x, y) \wedge \tilde{m}(y) \quad \text{for all } x, y \in E. \quad (4.5)$$

Further, if ξ is $\hat{P}$ -excessive, in the same way, we have the reverse inequality in (4.5). Therefore

$$\tilde{q}(x, y) \wedge \tilde{m}(x) = \tilde{q}(x, y) \wedge \tilde{m}(y) \quad \text{for all } x, y \in E. \quad (4.6)$$

Similarly, we can easily derive that ξ is P -excessive and $\hat{P}$ -excessive from (4.6). Therefore (4.6) is equivalent to the condition that ξ is P -excessive and $\hat{P}$ -excessive. While, from (4.6), we obtain

$$\begin{aligned}
\langle P\tilde{s}, \tilde{r} \rangle_\xi &= \sup_{x \in E} \{ \{ \sup_{y \in E} \tilde{q}(x, y) \wedge \tilde{s}(y) \} \wedge \tilde{r}(x) \wedge \tilde{m}(x) \} \\
&= \sup_{y \in E} \{ \tilde{s}(y) \wedge \{ \sup_{x \in E} \tilde{r}(x) \wedge \tilde{q}(x, y) \} \wedge \tilde{m}(x) \} \\
&= \langle \tilde{s}, \hat{P}\tilde{r} \rangle_\xi.
\end{aligned} \quad (4.7)$$

Conversely, in (4.7) we take $\tilde{s} := 1_{\{y\}}$ and $\tilde{r} := 1_{\{x\}}$ for $x, y \in E$. Then (4.7) is reduced to (4.6). Therefore, X and $\hat{X}$ are dual relative to the possibility measure ξ if and only if ξ is P -excessive and $\hat{P}$ -excessive. The proof of (i) – (xi) are similar to the proofs of Lemma 4.1 and Proposition 4.1. $\square$

In the rest of this section, we discuss Snell's optimal stopping problem in the duality of dynamic fuzzy systems. We use the binary relation $\langle \cdot, \cdot \rangle$ on $\mathcal{G}(E) \times \mathcal{G}(E)$. In the problem (3.6), we consider a case when an initial possibility measure μ is represented by

$$\mu(A) := \sup_{x \in A} \tilde{u}(x), \quad A \in \mathcal{E}(E). \quad (4.8)$$

Then, from (3.6), we have

$$\nu(\tilde{s}) := \sup_{\tau: \text{finite } \mathcal{E}\text{-stopping times}} \langle \tilde{u}, P_\tau \tilde{s} \rangle, \quad \tilde{s} \in \mathcal{G}(E). \quad (4.9)$$

Theorem 4.2. *The optimal stopping problem (4.9) has the following dual representation:*

$$\nu(\tilde{s}) = \min_{\tilde{v}: \hat{P}\text{-superharmonic}, \tilde{v} \geq \tilde{u}} \langle \tilde{v}, \tilde{s} \rangle = \min_{\tilde{r}: P\text{-superharmonic}, \tilde{r} \geq \tilde{s}} \langle \tilde{u}, \tilde{r} \rangle. \quad (4.10)$$

Proof. From (4.8) and (3.8), it is trivial that

$$\nu(\tilde{s}) = \min_{\tilde{r}: P\text{-superharmonic}, \tilde{r} \geq \tilde{s}} \langle \tilde{u}, \tilde{r} \rangle. \quad (4.11)$$

It is sufficient to prove that

$$\nu(\tilde{s}) = \min_{\tilde{v}: \hat{P}\text{-superharmonic}, \tilde{v} \geq \tilde{u}} \langle \tilde{v}, \tilde{s} \rangle. \quad (4.12)$$

Let $\tilde{v}$ be $\hat{P}$ -superharmonic and $\tilde{v} \geq \tilde{u}$, and put a possibility measure μ' by

$$\mu'(A) := \sup_{x \in A} \tilde{v}(x), \quad A \in \mathcal{E}(E).$$

From Lemma 4.2, μ' is P -excessive and $\mu' \geq \mu$. By (3.7), we obtain

$$\nu(\tilde{s}) \leq \mu'(\tilde{s}) = \langle \tilde{v}, \tilde{s} \rangle. \quad (4.13)$$

Especially, $\hat{G}\tilde{u}$ is $\hat{P}$ -superharmonic and $\hat{G}\tilde{u} \geq \tilde{u}$ since

$$\hat{G}\tilde{u} = \bigvee_{n \geq 0} \hat{P}^n \tilde{u} = \tilde{u} \vee \bigvee_{n \geq 1} \hat{P}^n \tilde{u} = \tilde{u} \vee \hat{P}\hat{G}\tilde{u}.$$

Then, from (3.9) and Proposition 4.1(ii), we obtain

$$\nu(\tilde{s}) = \mu G(\tilde{s}) = \mu(G\tilde{s}) = \langle \tilde{u}, G\tilde{s} \rangle = \langle \hat{G}\tilde{u}, \tilde{s} \rangle.$$

Together with (4.13), we obtain (4.12). Therefore, the proof is completed. $\square$

5. A numerical example

We consider a numerical example with a one-dimensional state space $E = \mathbf{R}$, where $\mathbf{R}$ is the set of all real numbers. We calculate the optimal value $\nu(\tilde{s})$ for Snell's problem (4.9).

Example 5.1. We give fuzzy sets $\tilde{s}, \tilde{u}$ and a fuzzy relation $\tilde{q}$ by

$$\tilde{s}(x) = (0.8 - 0.1|1 - x|) \vee 0, \quad x \in \mathbf{R}, \quad (5.1)$$

$$\tilde{u}(x) = (0.8 - 0.1|1 + x|) \vee 0, \quad x \in \mathbf{R}, \quad (5.2)$$

$$\tilde{q}(x, y) = (1 - |y - x^3|) \vee 0, \quad x, y \in \mathbf{R}. \quad (5.3)$$

then, we give a possibility measure μ by (4.8). Let $\tilde{s}^*$ be the smallest $\hat{P}$ -superharmonic fuzzy set dominating $\tilde{s}$. By [13, Example 4.1], we have

$$\tilde{s}^*(x) = \begin{cases} \tilde{s}(x) & \text{for } x \in (-\infty, x_1) & \approx (-\infty, -1.1651) \\ \tilde{q}(x, x) & \text{for } x \in [x_1, x_2) & \approx [-1.1651, -1.1147) \\ \alpha^* \approx 0.7296 & \text{for } x \in [x_2, x_3) & \approx [-1.1147, 0.2964) \\ \tilde{s}(x) & \text{for } x \in [x_3, x_4) & \approx [0.2964, 0.8687) \\ \tilde{q}(x, x) & \text{for } x \in [x_4, x_5) & \approx [0.8687, 0.8789) \\ 0.8 & \text{for } x \in [x_5, x_6) & \approx [0.8789, 1.0880) \\ \tilde{q}(x, x) & \text{for } x \in [x_6, x_7) & \approx [1.0880, 1.0916) \\ \tilde{s}(x) & \text{for } x \in [x_7, +\infty) & \approx [1.0916, +\infty). \end{cases}$$

Next, let $\tilde{u}^*$ be the smallest $\hat{P}$ -superharmonic fuzzy set dominating $\tilde{u}$. Since $\tilde{q}(x, y) = \tilde{q}(y, x)$ ($x, y \in \mathbf{R}$), in a method similar to [13, Example 4.1], we can easily check

$$\tilde{u}^*(x) = \begin{cases} 0.8 & \text{for } x \in (-\infty, x_8) & \approx (-\infty, 0.2091) \\ \tilde{q}(x, x) & \text{for } x \in [x_8, x_9) & \approx [0.2091, 0.4098) \\ \tilde{u}(x) & \text{for } x \in [x_9, x_{10}) & \approx [0.4098, 0.6749) \\ 0.6325 & \text{for } x \in [x_{10}, +\infty) & \approx [0.6749, +\infty). \end{cases}$$

Fig. 5.1 shows the fuzzy sets $\tilde{s}$ and $\tilde{u}^*$, and Fig. 5.2 shows the fuzzy sets $\tilde{u}$ and $\tilde{s}^*$.

Fig. 5.1. The fuzzy sets $\tilde{s}(x)$ and $\tilde{u}(x)$ and a fuzzy set $\tilde{q}(x, x)$.

Fig. 5.2. The fuzzy sets $\tilde{u}(x)$ and $\tilde{s}(x)$ and a fuzzy set $\tilde{q}(x, x)$.

From Theorem 4.2 and Figs. 5.1 and 5.2, we obtain

$$\nu(\tilde{s}) = \langle \tilde{u}^*, \tilde{s} \rangle = \langle \tilde{u}, \tilde{s}^* \rangle = \alpha^* \approx 0.7296.$$

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