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A MINIMAX THEOREM FOR ZERO-SUM STOPPING GAMES IN DYNAMIC FUZZY SYSTEMS

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Abstract : This paper deals with an optimal stopping game in dynamic fuzzy systems with fuzzy rewards. We give a fuzzy relational equation, whose unique solution is the optimal discounted fuzzy reward. This paper estimates discounted fuzzy rewards, by introducing a fuzzy expectation with a density given by fuzzy goals. We show the existence of the value of the game, by giving a minimax theorem for fuzzy expected values.

Keyword : Minimax theorem; dynamic fuzzy system; optimal stopping; fuzzy expected value.

1 Introduction

We, in the previous papers [8, 14, 15, 16], defined a dynamic fuzzy system using a fuzzy relation and gave limit theorems for the transition of fuzzy states of the system under the contractive and non-expansive properties of of the fuzzy relation. Recently, Kurano et al. [9] also introduced fuzzy rewards for the system and discussed fuzzy decision processes for total fuzzy rewards with respect to a partial order, which is called the fuzzy max order. In [9], we defined fuzzy rewards by maps from fuzzy states to fuzzy numbers. The definition is a natural extension of the classical rewards which are given by real valued functions on a crisp state space (see [5]).

Non-cooperative two-person zero-sum fuzzy games was studied by [3, 12] in the framwork of fuzzy matrix games. This paper deals with a zero-sum stopping game in the dynamic fuzzy system with fuzzy rewards. We estimate discounted fuzzy rewards by a fuzzy expectation with a density given by fuzzy goals on the basis of the concept of decision making in Bellman and Zadeh [1]. In Section 3, this paper gives a minimax theorem regarding fuzzy expected values in the game. In Section 2, we prove that the optimal fuzzy reward is a unique solution of a fuzzy relational equation, and, in Section 4, we give the both player's optimal stopping time and show that it is a saddle point in the class of finite stopping times. In Section 5, a numerical example is given to illustrate our theoretical idea.

Let E be a metric space. Let $\mathcal{F}(E)$ be the set of all fuzzy sets $\tilde{s} : E \mapsto [0,1]$ which are upper semi-continuous and satisfy $\sup_{x \in E} \tilde{s}(x) = 1$. Let $\tilde{q} : E \times E \mapsto [0,1]$ be a continuous fuzzy relation satisfying $\tilde{q}(x, \cdot) \in \mathcal{F}(E)$ $(x \in E)$. Let $\tilde{s} \in \mathcal{F}(E)$. In this paper, we deal with a sequence of fuzzy states $\{\tilde{s}_n\}_{n=0}^{\infty}$ defined by the following dynamic fuzzy system (see Kurano et al. [8]) :

$$\tilde{s}_0 := \tilde{s} \quad \text{and} \quad \tilde{s}_{n+1}(y) := \sup_{x \in E} \min\{\tilde{s}_n(x), \tilde{q}(x,y)\}, \quad y \in E, \ n = 0,1,2,\cdots. \tag{1.1}$$

For simplicity, we define a map $\tilde{q} : \mathcal{F}(E) \mapsto \mathcal{F}(E)$ as follows. For any $\tilde{s} \in \mathcal{F}(E)$,

$$\tilde{q}(\tilde{s})(y) := \sup_{x \in E} \min\{\tilde{s}(x), \tilde{q}(x,y)\}, \quad y \in E. \tag{1.2}$$

Then, (1.1) is represented by

$$\tilde{q}^0(\tilde{s}) := \tilde{s} \quad \text{and} \quad \tilde{q}^n(\tilde{s}) := \tilde{q}(\tilde{q}^{n-1}(\tilde{s})), \quad n = 1,2,\cdots. \tag{1.3}$$

Firstly, we describe fuzzy numbers in order to define fuzzy rewards. Let $\mathbf{R}$ be the set of all real numbers. For a fuzzy set $\tilde{a}$ on $\mathbf{R}$ and $\alpha \in [0,1]$, the α-cut $\tilde{a}_\alpha$ is defined by

$$\tilde{a}_\alpha := \{z \in \mathbf{R} \mid \tilde{a}(z) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{a}_0 := \mathrm{cl}\{z \in \mathbf{R} \mid \tilde{a}(z) > 0\},$$

where cl denotes the closure of a set (for the details, refer to Novàk [10] and Zadeh [17]). Then, a fuzzy set $\tilde{a}$ on $\mathbf{R}$ is called a fuzzy number if $\tilde{a}$ satisfies the following conditions (N1) — (N3) :

(N1) The α-cut $\tilde{a}_\alpha$ is a bounded closed subinterval of $\mathbf{R}$ for $\alpha \in [0,1]$. We represent it by $[\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$.

(N2) $\bigcap_{\alpha' < \alpha} \tilde{a}_{\alpha'} = \tilde{a}_\alpha$ for $\alpha > 0$.

(N3) $\tilde{a}$ is normal, i.e., $\sup_{z \in \mathbf{R}} \tilde{a}(z) = 1$.

We denote the set of all fuzzy numbers by $\mathcal{F}_n(\mathbf{R})$, and denote the set of all bounded closed subintervals of $\mathbf{R}$ by $\mathcal{C}(\mathbf{R})$.

An addition and a scalar multiplication for nonnegative fuzzy numbers are defined as follows (for example, see [6]) : For $\tilde{a}, \tilde{b} \in \mathcal{F}_n(\mathbf{R}_+)$ and $\lambda \geq 0$, the addition $\tilde{a} + \tilde{b}$ of $\tilde{a}$ and $\tilde{b}$ and the scalar multiplication $\lambda\tilde{a}$ of λ and $\tilde{a}$ are fuzzy numbers given by

$$(\tilde{a} + \tilde{b})_\alpha = [\tilde{a}_\alpha^- + \tilde{b}_\alpha^-, \tilde{a}_\alpha^+ + \tilde{b}_\alpha^+], \quad \alpha \in [0,1],$$

$$(\lambda\tilde{a})_\alpha = [\lambda\tilde{a}_\alpha^-, \lambda\tilde{a}_\alpha^+], \quad \alpha \in [0,1].$$

We define a partial order $\succeq$ on $\mathcal{F}_n(\mathbf{R}_+)$: Let $\tilde{a}, \tilde{b} \in \mathcal{F}_n(\mathbf{R}_+)$.

$$\tilde{a} \succeq \tilde{b} \quad \text{means that} \quad \tilde{a}_\alpha^- \geq \tilde{b}_\alpha^- \quad \text{and} \quad \tilde{a}_\alpha^+ \geq \tilde{b}_\alpha^+ \quad \text{for all } \alpha \in [0,1].$$

Then $(\mathcal{F}_n(\mathbf{R}_+), \succeq)$ becomes a lattice ([2]), and $\succeq$ is called the fuzzy max order. Further, for $\tilde{a}, \tilde{b} \in \mathcal{F}_n(\mathbf{R}_+)$, we define the maximum $\tilde{a} \vee \tilde{b}$ of $\tilde{a}$ and $\tilde{b}$ with respect to the order $\succeq$ by a fuzzy number such that

$$(\tilde{a} \vee \tilde{b})_\alpha = [\max\{\tilde{a}_\alpha^-, \tilde{b}_\alpha^-\}, \max\{\tilde{a}_\alpha^+, \tilde{b}_\alpha^+\}], \quad \alpha \in [0,1].$$

Next, we denote by $\mathcal{F}(E:\mathbf{R})$ the family of all maps $\tilde{f}:\mathcal{F}(E)\mapsto\mathcal{F}_n(\mathbf{R})$. This paper calls $\tilde{f}\in\mathcal{F}(E:\mathbf{R})$ a fuzzy-number-valued function on $\mathcal{F}(E)$. We introduce an addition, a scalar multiplication and a maximum on $\mathcal{F}(E:\mathbf{R})$ as follows : For $\tilde{f},\tilde{h}\in\mathcal{F}(E:\mathbf{R})$ and $\lambda\geq 0$, the addition $\tilde{f}+\tilde{h}$ of $\tilde{f}$ and $\tilde{h}$, the scalar multiplication $\lambda\tilde{f}$ of λ and $\tilde{f}$, and the maximum $\tilde{f}\vee\tilde{h}$ of $\tilde{f}$ and $\tilde{h}$ are given by

$$(\tilde{f}+\tilde{h})(\tilde{s}):=\tilde{f}(\tilde{s})+\tilde{h}(\tilde{s}),\quad \tilde{s}\in\mathcal{F}(E);$$

$$(\lambda\tilde{f})(\tilde{s}):=\lambda\tilde{f}(\tilde{s}),\quad \lambda\geq 0,\ \tilde{s}\in\mathcal{F}(E);$$

$$(\tilde{f}\vee\tilde{h})(\tilde{s}):=\tilde{f}(\tilde{s})\vee\tilde{h}(\tilde{s}),\quad \tilde{s}\in\mathcal{F}(E).$$

Let $\mathbf{N}:=\{0,1,2,\cdots\}$ be a time space. Let β be a constant satisfying $0<\beta<1$, where β means a discount rate. Let $\tilde{r},\tilde{c}^1,\tilde{c}^0,\tilde{c}^2\in\mathcal{F}(E:\mathbf{R})$ be bounded in the sense that $\|\tilde{r}\|$, $\|\tilde{c}^1\|$ and $\|\tilde{c}^2\|$ are finite, where the norm $\|\cdot\|$ is given by (2.2) in Section 2. We assume that $\tilde{c}^1(\tilde{s})\preceq\tilde{c}^0(\tilde{s})\preceq\tilde{c}^2(\tilde{s})$ for all $\tilde{s}\in\mathcal{F}(E)$. For a sequence of fuzzy states $\{\tilde{s}_n\}_{n=0}^{\infty}$ defined by (1.1), $\tilde{r}(\tilde{s}_n)$ means a running fuzzy reward at a state $\tilde{s}_n$ and $\tilde{c}^i(\tilde{s}_n)$ mean terminal fuzzy rewards for player $i(=1,2)$ respectively. Then, for player 1's stopping times m^1 and player 2's m^2, we define discounted fuzzy rewards for player 1, which are losses for player 2, by

$$\tilde{u}(\tilde{s},m^1,m^2):=\sum_{n=0}^{\min\{m^1,m^2\}-1}\beta^n\tilde{r}(\tilde{s}_n)+\beta^{\min\{m^1,m^2\}}\tilde{c}(\tilde{s},m^1,m^2)\quad\text{for }\tilde{s}\in\mathcal{F}(E),\tag{1.4}$$

where $\tilde{c}(\tilde{s},m^1,m^2)$ is defined by

$$\tilde{c}(\tilde{s},m^1,m^2):=\begin{cases}\tilde{c}^1(\tilde{s}_{m^1}) & \text{if } m^1<m^2,\\ \tilde{c}^0(\tilde{s}_{m^1}) & \text{if } m^1=m^2,\\ \tilde{c}^2(\tilde{s}_{m^2}) & \text{if } m^1>m^2,\end{cases}\tag{1.5}$$

we put the sum $\sum_{n=0}^{0-1}\beta^n\tilde{r}(\tilde{s}_n):=1_{\{0\}}\in\mathcal{F}_n(\mathbf{R})$. Then, $\tilde{u}(\cdot,m^1,m^2)\in\mathcal{F}(E:\mathbf{R})$ is trivial when m^1 or m^2 is finite, and further we can check it by Yoshida et al. [15, Theorem 3.2] when $m^1=m^2=\infty$. This type of rewards in Markov chains are well-known and first studied by Dynkin [5]. Kurano et al. [9] also studied this type of fuzzy rewards in fuzzy decision processes. This paper discusses the optimal stopping game for (1.4) in dynamic fuzzy systems (1.1). Put a fuzzy goal by a fuzzy number $\tilde{g}:\mathbf{R}\mapsto[0,1]$ which is an upper semi-continuous and nondecreasing function with $\tilde{g}(0)=0$ and $\lim_{z\to\infty}\tilde{g}(z)=1$. Then we note that $\tilde{g}_\alpha=[\tilde{g}_\alpha^-,\infty)$ for $\alpha\in[0,1]$. We define fuzzy expected values by

$$\tilde{E}\left(\tilde{u}(\tilde{s},m^1,m^2)\right):=\int\!\!\!\!-_{\mathbf{R}}\tilde{u}(\tilde{s},m^1,m^2)(z)\,\mathrm{d}\tilde{P}(z)=\sup_{z\in\mathbf{R}}\min\{\tilde{u}(\tilde{s},m^1,m^2)(z),\tilde{g}(z)\}\tag{1.6}$$

for $m^1,m^2\in\mathbf{N}\cup\{\infty\}$, where $\tilde{P}$ is the possibility measure generated by the density $\tilde{g}$ and $\int\!\!\!\!-\,\mathrm{d}\tilde{P}$ denotes Sugeno integral ([11, 13]). In this paper, we consider the following optimal stopping game.

Problem 1.1. Maximize (1.6) with respect to player 1's stopping times m^1 and minimize (1.6) with respect to player 2's stopping times m^2.

The fuzzy expectation implies the degree of satisfaction of discounted fuzzy rewards for player 1, and the fuzzy goal $\tilde{g}(z)$ means a kind of utility function for fuzzy payoffs z in (1.6) (see Sakawa and Nishizaki [12]).

From (1.4), we can define an optimal fuzzy reward with respect to the fuzzy max order $\succeq$ as follows :

$$\tilde{u}(\tilde{s}, *, m^2) := \bigvee_{m^1 \geq 0} \tilde{u}(\tilde{s}, m^1, m^2) \quad \text{for } \tilde{s} \in \mathcal{F}(E), \tag{1.7}$$

$$\tilde{u}(\tilde{s}, m^1, *) := \bigwedge_{m^2 \geq 0} \tilde{u}(\tilde{s}, m^1, m^2) \quad \text{for } \tilde{s} \in \mathcal{F}(E), \tag{1.8}$$

where $\vee$ and $\wedge$ mean the supremum and the infimum with respect to the fuzzy max order $\succeq$ respectively. Then, we note that $\tilde{u}(\tilde{s}, *, m^2), \tilde{u}(\tilde{s}, m^1, *) \in \mathcal{F}(E : \mathbf{R})$ for finite m^1, m^2. Further we put

$$\tilde{v}^*(\tilde{s}) := \bigwedge_{m^2 \geq 0} \tilde{u}(\tilde{s}, *, m^2) \quad \text{for } \tilde{s} \in \mathcal{F}(E), \tag{1.9}$$

$$\tilde{v}_*(\tilde{s}) := \bigvee_{m^1 \geq 0} \tilde{u}(\tilde{s}, m^1, *) \quad \text{for } \tilde{s} \in \mathcal{F}(E). \tag{1.10}$$

Then, it is trivial that

$$\tilde{c}^1(\tilde{s}) \preceq \tilde{v}_*(\tilde{s}) \preceq \tilde{v}^*(\tilde{s}) \preceq \tilde{c}^2(\tilde{s}) \tag{1.11}$$

since, by taking $m^2 = 0$ in (1.9) and $m^1 = 0$ in (1.10), we have

$$\tilde{v}^*(\tilde{s}) \preceq \tilde{u}(\tilde{s}, *, 0) = \tilde{c}^0(\tilde{s}) \vee \tilde{c}^2(\tilde{s}) = \tilde{c}^2(\tilde{s}),$$

$$\tilde{v}_*(\tilde{s}) \succeq \tilde{u}(\tilde{s}, 0, *) = \tilde{c}^1(\tilde{s}) \wedge \tilde{c}^0(\tilde{s}) = \tilde{c}^1(\tilde{s}).$$

2 Optimal fuzzy rewards

In this section, we give a fuzzy relational equation to characterize the optimal fuzzy rewards $\tilde{v}_*$ and $\tilde{v}^*$.

Lemma 2.1. *Let $[a_n, b_n], [a, b] \in \mathcal{C}(\mathbf{R})$ $(n = 0, 1, 2, \cdots)$. Then*

(i)

$$\beta\left(\bigvee_{n \geq 0} [a_n, b_n]\right) = \bigvee_{n \geq 0} \beta[a_n, b_n] \qquad \textit{if } \sup_{n \geq 0} b_n < \infty,$$

$$\beta\left(\bigwedge_{n \geq 0} [a_n, b_n]\right) = \bigwedge_{n \geq 0} \beta[a_n, b_n] \qquad \textit{if } \inf_{n \geq 0} a_n > -\infty;$$

(ii)

$$\beta\left(\sum_{n\geq 0}[a_n,b_n]\right)=\sum_{n\geq 0}\beta[a_n,b_n] \qquad \textit{if } \sum_{n\geq 0} a_n > -\infty \textit{ and } \sum_{n\geq 0} b_n < \infty;$$

(iii)

$$\bigvee_{n\geq 0}([a_n,b_n]+[a,b]) = \left(\bigvee_{n\geq 0}[a_n,b_n]\right)+[a,b] \qquad \textit{if } \sup_{n\geq 0} b_n < \infty,$$

$$\bigwedge_{n\geq 0}([a_n,b_n]+[a,b]) = \left(\bigwedge_{n\geq 0}[a_n,b_n]\right)+[a,b] \qquad \textit{if } \inf_{n\geq 0} a_n > -\infty,$$

where we define $\bigvee_{n\geq 0}[a_n,b_n] := [\sup_{n\geq 0} a_n, \sup_{n\geq 0} b_n]$, $\bigwedge_{n\geq 0}[a_n,b_n] := [\inf_{n\geq 0} a_n, \inf_{n\geq 0} b_n]$ *and* $\sum_{n\geq 0}[a_n,b_n] := [\sum_{n\geq 0} a_n, \sum_{n\geq 0} b_n]$.

Proof. They are trivial. □

Lemma 2.2. *Let* $\tilde{f}_n, \tilde{h} \in \mathcal{F}(E:\mathbf{R})$ $(n=0,1,2,\cdots)$. *Then, for* $\tilde{s} \in \mathcal{F}(E)$,

(i)

$$\beta\left(\bigvee_{n\geq 0}\tilde{f}_n(\tilde{s})\right) = \bigvee_{n\geq 0}\beta\tilde{f}_n(\tilde{s}) \qquad \textit{if } \sup_{n\geq 0}\tilde{f}_n(\tilde{s})_0^+ < \infty;$$

$$\beta\left(\bigwedge_{n\geq 0}\tilde{f}_n(\tilde{s})\right) = \bigwedge_{n\geq 0}\beta\tilde{f}_n(\tilde{s}) \qquad \textit{if } \inf_{n\geq 0}\tilde{f}_n(\tilde{s})_0^- > -\infty;$$

(ii)

$$\beta\left(\sum_{n\geq 0}\tilde{f}_n(\tilde{s})\right) = \sum_{n\geq 0}\beta\tilde{f}_n(\tilde{s}) \qquad \textit{if } \sum_{n\geq 0}\tilde{f}_n(\tilde{s})_0^- > -\infty \textit{ and } \sum_{n\geq 0}\tilde{f}_n(\tilde{s})_0^+ < \infty;$$

(iii)

$$\bigvee_{n\geq 0}\left(\tilde{f}_n(\tilde{s})+\tilde{h}(\tilde{s})\right) = \left(\bigvee_{n\geq 0}\tilde{f}_n(\tilde{s})\right)+\tilde{h}(\tilde{s}) \qquad \textit{if } \sup_{n\geq 0}\tilde{f}_n(\tilde{s})_0^+ < \infty;$$

$$\bigwedge_{n\geq 0}\left(\tilde{f}_n(\tilde{s})+\tilde{h}(\tilde{s})\right) = \left(\bigwedge_{n\geq 0}\tilde{f}_n(\tilde{s})\right)+\tilde{h}(\tilde{s}) \qquad \textit{if } \inf_{n\geq 0}\tilde{f}_n(\tilde{s})_0^- > -\infty.$$

Proof. They are trivial, by applying Lemma 2.1 for their α-cuts. □

Next, we introduce a distance between fuzzy-number-valued functions on $\mathcal{F}(E)$. We denote the Hausdorff metric on $\mathcal{C}(\mathbf{R})$ by δ (see [7]):

$$\delta([a_1,b_1],[a_2,b_2]) := \max\{|a_1-a_2|,|b_1-b_2|\} \qquad \text{for } [a_1,b_1],[a_2,b_2]\in\mathcal{C}(\mathbf{R}).$$

Then, we define a metric on $\mathcal{F}(E:\mathbf{R})$ as follows :

$$d(\tilde{f},\tilde{h}) := \sup_{\alpha\in[0,1],\tilde{s}\in\mathcal{F}(E)} \delta(\tilde{f}(\tilde{s})_\alpha, \tilde{h}(\tilde{s})_\alpha) \quad \text{for } \tilde{f},\tilde{h}\in\mathcal{F}(E:\mathbf{R}). \tag{2.1}$$

Further, we define a norm $\|\cdot\|$ on $\mathcal{F}(E:\mathbf{R})$ by

$$\|\tilde{f}\| := d(\tilde{f}, I_{\{0\}}) = \sup_{\alpha\in[0,1],\tilde{s}\in\mathcal{F}(E)} \delta(\tilde{f}(\tilde{s})_\alpha, \{0\}) \quad \text{for } \tilde{f}\in\mathcal{F}(E:\mathbf{R}), \tag{2.2}$$

where we put $I_{\{0\}}\in\mathcal{F}(E:\mathbf{R})$ by

$$I_{\{0\}}(\tilde{s}) := \tilde{0} \text{ (the crisp number zero)} \quad \text{for } \tilde{s}\in\mathcal{F}(E).$$

Then the following elementary results can be easily checked (c.f. [4]).

Lemma 2.3. *Let* $[a_1,b_1],[c_1,d_1],[a_2,b_2],[c_2,d_2]\in\mathcal{C}(\mathbf{R})$. *Then*

(i) $\delta([a_1,b_1]\vee[c_1,d_1],[a_2,b_2]\vee[c_2,d_2]) \le \max(\delta([a_1,b_1],[a_2,b_2]),\delta([c_1,d_1],[c_2,d_2]))$;
$\delta([a_1,b_1]\wedge[c_1,d_1],[a_2,b_2]\wedge[c_2,d_2]) \le \max(\delta([a_1,b_1],[a_2,b_2]),\delta([c_1,d_1],[c_2,d_2]))$;

(ii) $\delta([a_1,b_1]+[c_1,d_1],[a_2,b_2]+[c_2,d_2]) \le \delta([a_1,b_1],[a_2,b_2])+\delta([c_1,d_1],[c_2,d_2])$;

(iii) $\delta(\beta[a_1,b_1],\beta[c_1,d_1]) = \beta\delta([a_1,b_1],[c_1,d_1])$.

Lemma 2.4. *Let* $\tilde{f}_1,\tilde{f}_2,\tilde{h}_1,\tilde{h}_2\in\mathcal{F}(E:\mathbf{R})$. *Then*

(i) $d(\tilde{f}_1\vee\tilde{h}_1,\tilde{f}_2\vee\tilde{h}_2) \le \max(d(\tilde{f}_1,\tilde{f}_2),d(\tilde{h}_1,\tilde{h}_2))$;
$d(\tilde{f}_1\wedge\tilde{h}_1,\tilde{f}_2\wedge\tilde{h}_2) \le \max(d(\tilde{f}_1,\tilde{f}_2),d(\tilde{h}_1,\tilde{h}_2))$;

(ii) $d(\tilde{f}_1+\tilde{h}_1,\tilde{f}_2+\tilde{h}_2) \le d(\tilde{f}_1,\tilde{f}_2)+d(\tilde{h}_1,\tilde{h}_2)$;

(iii) $d(\beta\tilde{f}_1,\beta\tilde{f}_2) = \beta d(\tilde{f}_1,\tilde{f}_2)$;

(iv) $d(\tilde{f}_1(\tilde{q}),\tilde{f}_2(\tilde{q})) \le d(\tilde{f}_1,\tilde{f}_2)$.

Proof. (i) — (iii) are trivial from the definition (2.1) and Lemma 2.3. (iv) Let $\tilde{s}\in\mathcal{F}(E)$. Since $\tilde{q}(\tilde{s})\in\mathcal{F}(E)$, we have

$$\begin{aligned}\delta(\tilde{f}_1(\tilde{q}(\tilde{s}))_\alpha,\tilde{f}_2(\tilde{q}(\tilde{s}))_\alpha) &\le \sup_{\alpha\in[0,1],\tilde{s}'\in\mathcal{F}(E)} \delta(\tilde{f}_1(\tilde{s}')_\alpha,\tilde{f}_2(\tilde{s}')_\alpha) \\ &= d(\tilde{f}_1,\tilde{f}_2) \quad \text{for all } \alpha\in[0,1],\ \tilde{s}\in\mathcal{F}(E).\end{aligned}$$

This yields (iv). $\square$

We obtain the following theorems for the optimal fuzzy rewards in (1.9) and (1.10).

Theorem 2.1. *It holds that* $\tilde{v}_*,\tilde{v}^*\in\mathcal{F}(E:\mathbf{R})$.

Proof. Let $\tilde{s} \in \mathcal{F}(E)$. From Lemmas 2.1 and 2.3 and (1.5), for $m' < m''$ and $\alpha \in [0,1]$, we have

$$
\begin{aligned}
&\delta\left(\bigvee_{0\le m\le m''} \tilde{u}(\tilde{s},m,*)_\alpha,\ \bigvee_{0\le m\le m'} \tilde{u}(\tilde{s},m,*)_\alpha\right) \\
&\le \delta\left(\bigvee_{m'\le m\le m''} \tilde{u}(\tilde{s},m,*)_\alpha,\ \tilde{u}(\tilde{s},m',*)_\alpha\right) \\
&\le \max_{m'\le m\le m''} \delta\left(\bigwedge_{m^2\ge 0} \tilde{u}(\tilde{s},m,m^2)_\alpha,\ \bigwedge_{m^2\ge 0} \tilde{u}(\tilde{s},m',m^2)_\alpha\right) \\
&\le \max_{m'\le m\le m''}\ \max_{0\le m^2\le m''+1} \delta\left(\tilde{u}(\tilde{s},m,m^2)_\alpha,\ \tilde{u}(\tilde{s},m',m^2)_\alpha\right) \\
&\le \max_{m'\le m\le m''}\ \max_{0\le m^2\le m''+1} \delta\left(\sum_{n=0}^{\min\{m,m^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_\alpha,\ \sum_{n=0}^{\min\{m',m^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_\alpha\right) \\
&\quad + \max_{m'\le m\le m''}\ \max_{0\le m^2\le m''+1} \delta\left(\beta^{\min\{m,m^2\}}\tilde{c}(\tilde{s},m,m^2)_\alpha,\ \beta^{\min\{m',m^2\}}\tilde{c}(\tilde{s},m',m^2)_\alpha\right) \\
&\le \max_{m'\le m\le m''}\ \max_{0\le m^2\le m''+1} \delta\left(\sum_{n=\min\{m',m^2\}}^{\min\{m,m^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_\alpha,\ \{0\}\right) \\
&\quad + \max_{m'\le m\le m''}\ \max_{0\le m^2\le m''+1} \delta\left(\beta^{\min\{m,m^2\}}\tilde{c}(\tilde{s},m,m^2)_\alpha,\ \beta^{\min\{m',m^2\}}\tilde{c}(\tilde{s},m',m^2)_\alpha\right) \\
&\le \sum_{n=m'}^{\infty} \beta^n \delta\left(\tilde{r}(\tilde{q}^n(\tilde{s}))_\alpha, \{0\}\right) + 2 \sup_{m\ge m'}\ \sup_{m^2\ge m'} \beta^{\min\{m,m^2\}} \delta(\tilde{c}(\tilde{s},m,m^2)_\alpha, \{0\}) \\
&\le \sum_{n=m'}^{\infty} \beta^n \|\tilde{r}\| + 2\beta^{m'}\|\tilde{c}\|,
\end{aligned}
$$

where $\|\tilde{c}\| := \max\{\|\tilde{c}^0\|, \|\tilde{c}^1\|, \|\tilde{c}^2\|\}$. By letting $m'' \to \infty$, we obtain

$$
\delta\left(\tilde{v}_*(\tilde{s})_\alpha,\ \bigvee_{0\le m\le m'} \tilde{u}(\tilde{s},m,*)_\alpha\right) \le \epsilon(m') \quad \text{for all } m' \ge 1,\ \alpha \in [0,1],
$$

where

$$
\epsilon(m') := \sum_{n=m'}^{\infty} \beta^n \|\tilde{r}\| + 2\beta^{m'}\|\tilde{c}\| \quad \text{for } m' \ge 1.
$$

By Lemma 2.3(i), we have

$$
\begin{aligned}
&\delta\left(\tilde{v}_*(\tilde{s})_\alpha, \tilde{v}_*(\tilde{s})_{\alpha'}\right) \\
&\le \delta\left(\tilde{v}_*(\tilde{s})_\alpha,\ \bigvee_{0\le m\le m'} \tilde{u}(\tilde{s},m,*)_\alpha\right) + \delta\left(\bigvee_{0\le m\le m'} \tilde{u}(\tilde{s},m,*)_\alpha,\ \bigvee_{0\le m\le m'} \tilde{u}(\tilde{s},m,*)_{\alpha'}\right) \\
&\quad + \delta\left(\bigvee_{0\le m\le m'} \tilde{u}(\tilde{s},m,*)_{\alpha'},\ \tilde{v}_*(\tilde{s})_{\alpha'}\right) \\
&\le 2\epsilon(m') + \max_{0\le m\le m'} \delta\left(\tilde{u}(\tilde{s},m,*)_\alpha, \tilde{u}(\tilde{s},m,*)_{\alpha'}\right) \quad \text{for } m' \ge 1,\ \alpha' < \alpha.
\end{aligned}
$$

Since $\epsilon(m')$ is independent of α and α' and we have $\tilde{u}(\tilde{s}, m, *) \in \mathcal{F}_n(\mathbf{R})$ for finite m, this yields

$$\lim_{\alpha' \uparrow \alpha} \delta(\tilde{v}_*(\tilde{s})_\alpha, \tilde{v}_*(\tilde{s})_{\alpha'}) \leq 2\epsilon(m') \quad \text{for all } m' \geq 1, \alpha > 0.$$

Therefore

$$\lim_{\alpha' \uparrow \alpha} \delta(\tilde{v}_*(\tilde{s})_\alpha, \tilde{v}_*(\tilde{s})_{\alpha'}) = 0.$$

Since $\tilde{v}_*(\tilde{s})_\alpha \subset \tilde{v}_*(\tilde{s})_{\alpha'}$ holds trivially for $\alpha' < \alpha$, we obtain $\tilde{v}_*(\tilde{s}) \in \mathcal{F}_n(\mathbf{R})$ for all $\tilde{s} \in \mathcal{F}(E)$, using [8, Lemma 3]. Thus we get $\tilde{v}_* \in \mathcal{F}(E : \mathbf{R})$. Similarly we can check $\tilde{v}^* \in \mathcal{F}(E : \mathbf{R})$. Therefore we obtain this theorem. □

For $\tilde{a}, \tilde{b}_1, \tilde{b}_2 \in \mathcal{F}_n(\mathbf{R})$ such that $\tilde{b}_1 \preceq \tilde{b}_2$, it is trivial that

$$\{\tilde{b}_1 \vee \tilde{a}\} \wedge \tilde{b}_2 = \tilde{b}_1 \vee \{\tilde{a} \wedge \tilde{b}_2\}.$$

Then, we write it simply by

$$\tilde{b}_1 \vee \tilde{a} \wedge \tilde{b}_2.$$

Theorem 2.2. *It holds that $\tilde{v}_* = \tilde{v}^*$. We write it by $\tilde{v}$. Then, $\tilde{v}$ is a unique solution of the following fuzzy relational equation :*

$$\tilde{v}(\tilde{s}) = \tilde{c}^1(\tilde{s}) \vee \{\tilde{r}(\tilde{s}) + \beta \tilde{v}(\tilde{q}(\tilde{s}))\} \wedge \tilde{c}^2(\tilde{s}) \quad \textit{for } \tilde{s} \in \mathcal{F}(E). \tag{2.3}$$

Proof. Let $\tilde{s} \in \mathcal{F}(E)$. From (1.1) and Lemma 2.2, we have

$$\begin{aligned}
\tilde{v}_*(\tilde{s}) &= \bigvee_{m^1 \geq 0} \bigwedge_{m^2 \geq 0} \left\{ \sum_{n=0}^{\min\{m^1,m^2\}-1} \beta^n \tilde{r}(\tilde{s}_n) + \beta^{\min\{m^1,m^2\}} \tilde{c}(\tilde{s}, m^1, m^2) \right\} \\
&= \bigwedge_{m^2 \geq 0} \tilde{c}(\tilde{s}, 0, m^2) \vee \bigvee_{m^1 \geq 1} \bigwedge_{m^2 \geq 0} \left\{ \sum_{n=0}^{\min\{m^1,m^2\}-1} \beta^n \tilde{r}(\tilde{s}_n) + \beta^{\min\{m^1,m^2\}} \tilde{c}(\tilde{s}, m^1, m^2) \right\} \\
&= \tilde{c}^1(\tilde{s}) \vee \bigvee_{m^1 \geq 1} \bigwedge_{m^2 \geq 0} \left\{ \sum_{n=0}^{\min\{m^1,m^2\}-1} \beta^n \tilde{r}(\tilde{s}_n) + \beta^{\min\{m^1,m^2\}} \tilde{c}(\tilde{s}, m^1, m^2) \right\} \\
&= \tilde{c}^1(\tilde{s}) \vee \bigvee_{m^1 \geq 1} \left\{ \bigwedge_{m^2 \geq 1} \left\{ \sum_{n=0}^{\min\{m^1,m^2\}-1} \beta^n \tilde{r}(\tilde{s}_n) + \beta^{\min\{m^1,m^2\}} \tilde{c}(\tilde{s}, m^1, m^2) \right\} \wedge \tilde{c}(\tilde{s}, m^1, 0) \right\} \\
&= \tilde{c}^1(\tilde{s}) \vee \bigvee_{m^1 \geq 1} \bigwedge_{m^2 \geq 1} \left\{ \sum_{n=0}^{\min\{m^1,m^2\}-1} \beta^n \tilde{r}(\tilde{s}_n) + \beta^{\min\{m^1,m^2\}} \tilde{c}(\tilde{s}, m^1, m^2) \right\} \wedge \tilde{c}^2(\tilde{s}) \\
&= \tilde{c}^1(\tilde{s}) \vee \left\{ \tilde{r}(\tilde{s}) + \beta \bigvee_{m^1 \geq 0} \bigwedge_{m^2 \geq 0} \left\{ \sum_{n=0}^{\min\{m^1,m^2\}-1} \beta^n \tilde{r}(\tilde{s}_n) + \beta^{\min\{m^1,m^2\}} \tilde{c}(\tilde{q}(\tilde{s}), m^1, m^2) \right\} \right\} \wedge \tilde{c}^2(\tilde{s}) \\
&= \tilde{c}^1(\tilde{s}) \vee \{\tilde{r}(\tilde{s}) + \beta \tilde{v}_*(\tilde{q}(\tilde{s}))\} \wedge \tilde{c}^2(\tilde{s}) \quad \text{for } \tilde{s} \in \mathcal{F}(E).
\end{aligned}$$

Therefore $\tilde{v}_*$ satisfies (2.3). Similarly, we can check that $\tilde{v}^*$ is a solution of (2.3).

If $\tilde{v}, \tilde{w} \in \mathcal{F}(E : \mathbf{R})$ are solutions of (2.2), then by Lemma 2.4 we have

$$\begin{aligned} d(\tilde{v}, \tilde{w}) &= d(\tilde{c}^1 \vee \{\tilde{r} + \beta \tilde{v}(\tilde{q})\} \wedge \tilde{c}^2, \tilde{c}^1 \vee \{\tilde{r} + \beta \tilde{w}(\tilde{q})\} \wedge \tilde{c}^2) \\ &\leq d(\tilde{r} + \beta \tilde{v}(\tilde{q}), \tilde{r} + \beta \tilde{w}(\tilde{q})) \\ &\leq d(\beta \tilde{v}(\tilde{q}), \beta \tilde{w}(\tilde{q})) \\ &= \beta d(\tilde{v}(\tilde{q}), \tilde{w}(\tilde{q})) \\ &\leq \beta d(\tilde{v}, \tilde{w}). \end{aligned}$$

So we obtain $\tilde{v} = \tilde{w}$ since $0 < \beta < 1$. Thus (2.3) has a unique solution. Therefore we get $\tilde{v}_* = \tilde{v}^*$, and it is a unique solution of (2.3). □

3 A Minmax Theorem

In this section, we discuss the fuzzy expectation of fuzzy rewards. From now on, we fix an initial fuzzy state $\tilde{s} \in \mathcal{F}(E)$. Define a level α^* by

$$\alpha^* := \sup\{\alpha \in [0,1] | \ \tilde{g}^-_\alpha \leq \tilde{v}(\tilde{s})^+_\alpha\}. \tag{3.1}$$

Then, from the definition (N2) of fuzzy numbers, we have

$$\tilde{g}^-_{\alpha^*} \leq \tilde{v}(\tilde{s})^+_{\alpha^*}. \tag{3.2}$$

We prove that α^* equals to the optimal expected value.

Lemma 3.1. *Let $\tilde{a} \in \mathcal{F}_n(\mathbf{R})$ and $\alpha \in [0,1]$.*

(i) *If $\tilde{E}(\tilde{a}) > \alpha$, then $\tilde{g}^-_\alpha \leq \tilde{a}^+_\alpha$.*

(ii) *If $\tilde{g}^-_\alpha \leq \tilde{a}^+_\alpha$, then $\tilde{E}(\tilde{a}) \geq \alpha$.*

Proof. (i) Let α satisfy $\tilde{E}(\tilde{a}) > \alpha$. By the definition of the fuzzy expectation $\tilde{E}(\tilde{a})$, we have

$$\tilde{g}_\alpha \cap \tilde{a}_\alpha \neq \emptyset.$$

Since $\tilde{g}$ is nondecreasing and $\tilde{a} \in \mathcal{F}_n(\mathbf{R})$, it is equivalent to

$$\tilde{g}^-_\alpha \leq \tilde{a}^+_\alpha.$$

(ii) Let

$$\tilde{g}^-_\alpha \leq \tilde{v}(\tilde{s})^+_\alpha.$$

This is equivalent to

$$\tilde{g}_\alpha \cap \tilde{v}(\tilde{s})_\alpha \neq \emptyset.$$

Therefore $\tilde{E}(\tilde{a}) \geq \alpha$. Thus we get this lemma. □

Theorem 3.1. *It holds that*

$$\alpha^* = \tilde{E}(\tilde{v}(\tilde{s})). \tag{3.3}$$

Proof. Let α satisfy $0 \le \alpha < \tilde{E}(\tilde{v}(\tilde{s}))$. By Lemma 3.1(i), we have

$$\tilde{g}_\alpha^- \le \tilde{v}(\tilde{s})_\alpha^+.$$

Therefore $\alpha \le \alpha^*$. Thus, we get $\alpha^* \ge \tilde{E}(\tilde{v}(\tilde{s}))$.

Next, let $\alpha < \alpha^*$. Then we have

$$\tilde{g}_\alpha^- \le \tilde{v}(\tilde{s})_\alpha^+.$$

By Lemma 3.1(ii), $\alpha \le \tilde{E}(\tilde{v}(\tilde{s}))$. Thus we get $\alpha^* \le \tilde{E}(\tilde{v}(\tilde{s}))$. Therefore we obtain (3.3). □

We obtain the following minmax theorem regrading fuzzy expected values.

Theorem 3.2. *It holds that*

$$\tilde{E}(\tilde{v}(\tilde{s})) = \sup_{m^1 \ge 0} \inf_{m^2 \ge 0} \tilde{E}(\tilde{u}(\tilde{s}, m^1, m^2)) = \inf_{m^2 \ge 0} \sup_{m^1 \ge 0} \tilde{E}(\tilde{u}(\tilde{s}, m^1, m^2)). \tag{3.4}$$

$$\tilde{E}(\tilde{v}(\tilde{s})) = \sup_{m^1 \ge 0} \tilde{E}(\tilde{u}(\tilde{s}, m^1, *)) = \inf_{m^2 \ge 0} \tilde{E}(\tilde{u}(\tilde{s}, *, m^2)). \tag{3.5}$$

Proof. Firstly, we prove that

$$\tilde{E}(\tilde{v}(\tilde{s})) = \sup_{m^1 \ge 0} \tilde{E}(\tilde{u}(\tilde{s}, m^1, *)). \tag{3.6}$$

From Theorem 2.2, we have

$$\tilde{u}(\tilde{s}, m, *)_\alpha^+ \le \tilde{v}_*(\tilde{s})_\alpha^+ = \tilde{v}(\tilde{s})_\alpha^+. \quad \text{for all } \alpha \in [0,1] \text{ and } m \ge 0.$$

Since $\tilde{g}$ is nondecreasing, by the definition of the fuzzy expectations $\tilde{E}(\tilde{v}(\tilde{s}))$ and $\tilde{E}(\tilde{u}(\tilde{s}, m, *))$, we get

$$\tilde{E}(\tilde{v}(\tilde{s})) \ge \tilde{E}(\tilde{u}(\tilde{s}, m, *)). \quad \text{for all } m \ge 0. \tag{3.7}$$

If $\alpha^* = 0$, then (3.6) is trivial from (3.7) and Theorem 3.1. We assume $\alpha^* > 0$. Let ϵ be an arbitrary real number such that $0 < \epsilon < \alpha^*$. Let α satisfy $\alpha^* - \epsilon < \alpha < \alpha^*$. From the definition of $\tilde{v}_*(\tilde{s})$, there exists a subsequence $\{m'\}$ such that

$$\tilde{u}(\tilde{s}, m', *)_\alpha^+ \uparrow \tilde{v}_*(\tilde{s})_\alpha^+ = \tilde{v}(\tilde{s})_\alpha^+ \quad \text{as } m' \to \infty.$$

Since

$$\tilde{g}_\alpha^- < \tilde{g}_{\alpha^*}^- \le \tilde{v}(\tilde{s})_{\alpha^*}^+ \le \tilde{v}(\tilde{s})_\alpha^+,$$

there exists m_0' such that

$$\tilde{g}_\alpha^- < \tilde{u}(\tilde{s}, m_0', *)_\alpha^+.$$

From Lemma 3.1 (ii), we have

$$\tilde{E}(\tilde{u}(\tilde{s}, m_0', *)) \geq \alpha > \alpha^* - \epsilon.$$

Since ϵ is arbitrary, we get

$$\sup_{m \geq 0} \tilde{E}(\tilde{u}(\tilde{s}, m, *)) \geq \alpha^*.$$

Together with (3.7) and Theorem 3.1, we obtain (3.6). This implies the left equality in (3.5).

Next we prove that

$$\tilde{E}(\tilde{u}(\tilde{s}, m^1, *)) = \inf_{m^2 \geq 0} \tilde{E}(\tilde{u}(\tilde{s}, m^1, m^2)) = \min_{m^2 : 0 \leq m^2 \leq m^1 + 1} \tilde{E}(\tilde{u}(\tilde{s}, m^1, m^2)) \quad \text{for finite } m^1 \geq 0. \tag{3.8}$$

Fix any $m^1 \geq 0$. We have

$$\tilde{u}(\tilde{s}, m^1, *)^+_\alpha \leq \tilde{u}(\tilde{s}, m^1, m)^+_\alpha. \quad \text{for all } \alpha \in [0,1] \text{ and } m \geq 0.$$

Since $\tilde{g}$ is nondecreasing, by the definition of the fuzzy expectation $\tilde{E}(\tilde{u}(\tilde{s}, m^1, *))$ and $\tilde{E}(\tilde{u}(\tilde{s}, m^1, m))$, we get

$$\tilde{E}(\tilde{u}(\tilde{s}, m^1, *)) \leq \tilde{E}(\tilde{u}(\tilde{s}, m^1, m)). \quad \text{for all } m \geq 0. \tag{3.9}$$

If $\tilde{E}(\tilde{u}(\tilde{s}, m^1, *)) = 1$, then (3.8) is trivial. We assume $\tilde{E}(\tilde{u}(\tilde{s}, m^1, *)) < 1$. Let ϵ be an arbitrary real number such that $0 < \epsilon < 1 - \tilde{E}(\tilde{u}(\tilde{s}, m^1, *))$. Let α satisfy $\tilde{E}(\tilde{u}(\tilde{s}, m^1, *)) < \alpha < \tilde{E}(\tilde{u}(\tilde{s}, m^1, *)) + \epsilon$. From the definition of $\tilde{u}(\tilde{s}, m^1, *)$, there exists m' such that

$$\tilde{u}(\tilde{s}, m^1, m')^+_\alpha = \tilde{u}(\tilde{s}, m^1, *)^+_\alpha.$$

Then we have

$$\tilde{u}(\tilde{s}, m^1, m')^+_\alpha = \tilde{u}(\tilde{s}, m^1, *)^+_\alpha < \tilde{g}^-_\alpha.$$

By Lemma 3.1(i),

$$\tilde{E}(\tilde{u}(\tilde{s}, m^1, m')) \leq \alpha < \tilde{E}(\tilde{u}(\tilde{s}, m^1, *)) + \epsilon.$$

Since ϵ is arbitrary, we get

$$\min_{m : 0 \leq m \leq m^1 + 1} \tilde{E}(\tilde{u}(\tilde{s}, m^1, m)) \leq \tilde{E}(\tilde{u}(\tilde{s}, m^1, *)).$$

Together with (3.9) and Theorem 3.1, we obtain (3.8). From (3.6) and (3.8), we get the left equalities in (3.4) and (3.5). We can also check the right equalities in (3.4) and (3.5) similarly. Thus we get this theorem. □

4 Optimal stopping times

In this section, we give optimal stopping times for Problem 1.1. The following lemma is trivial from the definitions.

Lemma 4.1. *Let* $\tilde{a}_n, \tilde{a} \in \mathcal{F}_n(\mathbf{R})$ $(n = 0, 1, 2, \cdots)$. *Then, for* $\alpha \in [0, 1]$,

(i)

$$\left(\sum_{n \geq 0} \tilde{a}_n \right)_\alpha^+ = \sum_{n \geq 0} \tilde{a}_{n,\alpha}^+;$$

(ii)

$$\left(\bigvee_{n \geq 0} \tilde{a}_n \right)_\alpha^+ = \sup_{n \geq 0} \tilde{a}_{n,\alpha}^+;$$

(iii)

$$(\beta \tilde{a})_\alpha^+ = \beta \tilde{a}_\alpha^+,$$

where $\tilde{a}_{n,\alpha}$ *denotes the* α*-cut of* $\tilde{a}_n$.

We define times

$$\tau^1 := \inf\{m \in \mathbf{N} \mid \tilde{v}(\tilde{s}_m)_{\alpha^*}^+ = \tilde{c}^1(\tilde{s}_m)_{\alpha^*}^+\}, \tag{4.1}$$

$$\tau^2 := \inf\{m \in \mathbf{N} \mid \tilde{v}(\tilde{s}_m)_{\alpha^*}^+ = \tilde{c}^2(\tilde{s}_m)_{\alpha^*}^+\}, \tag{4.2}$$

where the infimum of the empty set is understood to be $+\infty$. We check the following proposition by the standard method in the theory of zero-sum sequential games.

Proposition 4.1. *If* τ^1 *and* τ^2 *are finite, then*

$$\tilde{v}(\tilde{s})_{\alpha^*}^+ = \tilde{u}(\tilde{s}, \tau^1, \tau^2)_{\alpha^*}^+ = \tilde{u}(\tilde{s}, \tau^1, *)_{\alpha^*}^+ = \tilde{u}(\tilde{s}, *, \tau^2)_{\alpha^*}^+. \tag{4.3}$$

Proof. From Theorem 2.2 and Lemmas 2.2 and 4.1, for $m \geq 0$, we have

$$\begin{aligned}
&\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \beta^m \tilde{v}(\tilde{q}^m(\tilde{s}))_{\alpha^*}^+ \\
&= \sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \beta^m \{\tilde{c}^1(\tilde{q}^m(\tilde{s})) \vee \{\tilde{r}(\tilde{q}^m(\tilde{s})) + \beta \tilde{v}(\tilde{q}^{m+1}(\tilde{s}))\} \wedge \tilde{c}^2(\tilde{q}^m(\tilde{s}))\}_{\alpha^*}^+ \\
&= \sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ \\
&\quad + \min\{\max\{\beta^m \tilde{c}^1(\tilde{q}^m(\tilde{s}))_{\alpha^*}^+, \{\beta^m \tilde{r}(\tilde{q}^m(\tilde{s}))_{\alpha^*}^+ + \beta^{m+1} \tilde{v}(\tilde{q}^{m+1}(\tilde{s}))_{\alpha^*}^+\}\}, \beta^m \tilde{c}^2(\tilde{q}^m(\tilde{s}))_{\alpha^*}^+\}
\end{aligned}$$

This yields

$$\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{v}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*}$$

$$\begin{aligned}
&= \max\left\{\left\{\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{c}^1(\tilde{q}^m(\tilde{s}))^+_{\alpha^*}\right\}, \left\{\sum_{n=0}^{m} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{m+1} \tilde{v}(\tilde{q}^{m+1}(\tilde{s}))^+_{\alpha^*}\right\}\right\} \\
&\geq \sum_{n=0}^{m} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{m+1} \tilde{v}(\tilde{q}^{m+1}(\tilde{s}))^+_{\alpha^*} \quad \text{for } m < \tau^2, \qquad (4.4)
\end{aligned}$$

and

$$\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{v}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*}$$

$$\begin{aligned}
&= \min\left\{\left\{\sum_{n=0}^{m} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{m+1} \tilde{v}(\tilde{q}^{m+1}(\tilde{s}))^+_{\alpha^*}\right\}, \left\{\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{c}^2(\tilde{q}^m(\tilde{s}))^+_{\alpha^*}\right\}\right\} \\
&\leq \sum_{n=0}^{m} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{m+1} \tilde{v}(\tilde{q}^{m+1}(\tilde{s}))^+_{\alpha^*} \quad \text{for } m < \tau^1. \qquad (4.5)
\end{aligned}$$

Therefore, for all $m < \min\{\tau^1, \tau^2\}$,

$$\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{v}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*} = \sum_{n=0}^{m} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{m+1} \tilde{v}(\tilde{q}^{m+1}(\tilde{s}))^+_{\alpha^*}. \qquad (4.6)$$

Therefore, we obtain

$$\begin{aligned}
\tilde{v}(\tilde{s})^+_{\alpha^*} &= \sum_{n=0}^{\min\{\tau^1,\tau^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{\min\{\tau^1,\tau^2\}} \tilde{v}(\tilde{q}^{\min\{\tau^1,\tau^2\}}(\tilde{s}))^+_{\alpha^*} \\
&= \sum_{n=0}^{\min\{\tau^1,\tau^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{\min\{\tau^1,\tau^2\}} \tilde{c}(\tilde{s}, \tau^1, \tau^2)^+_{\alpha^*} \\
&= \tilde{u}(\tilde{s}, \tau^1, \tau^2)^+_{\alpha^*}. \qquad (4.7)
\end{aligned}$$

Next we consider the two cases $\tau^1 < \tau^2$ and $\tau^1 \geq \tau^2$.
Case $\tau^1 < \tau^2$: If $m \geq \tau^2$, then, by (4.4) and (4.7) we have

$$\begin{aligned}
\tilde{u}(\tilde{s}, \tau^1, \tau^2)^+_{\alpha^*} &= \sum_{n=0}^{\min\{\tau^1,\tau^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{\min\{\tau^1,\tau^2\}} \tilde{v}(\tilde{q}^{\min\{\tau^1,\tau^2\}}(\tilde{s}))^+_{\alpha^*} \\
&\geq \sum_{n=0}^{\min\{m,\tau^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{\min\{m,\tau^2\}} \tilde{v}(\tilde{q}^{\min\{m,\tau^2\}}(\tilde{s}))^+_{\alpha^*} \\
&= \sum_{n=0}^{\min\{m,\tau^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{\min\{m,\tau^2\}} \tilde{c}^2(\tilde{q}^{\min\{m,\tau^2\}}(\tilde{s}))^+_{\alpha^*} \\
&\geq \sum_{n=0}^{\min\{m,\tau^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{\min\{m,\tau^2\}} \tilde{c}(\tilde{s}, m, \tau^2)^+_{\alpha^*} \\
&= \tilde{u}(\tilde{s}, m, \tau^2)^+_{\alpha^*}.
\end{aligned}$$

If $m < \tau^2$, then, by (4.4) and (4.7) we have

$$\begin{aligned}
\tilde{u}(\tilde{s}, \tau^1, \tau^2)^+_{\alpha^*} &= \sum_{n=0}^{\min\{\tau^1,\tau^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{\min\{\tau^1,\tau^2\}} \tilde{v}(\tilde{q}^{\min\{\tau^1,\tau^2\}}(\tilde{s}))^+_{\alpha^*} \\
&\geq \sum_{n=0}^{\min\{m,\tau^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{\min\{m,\tau^2\}} \tilde{v}(\tilde{q}^{\min\{m,\tau^2\}}(\tilde{s}))^+_{\alpha^*} \\
&\geq \sum_{n=0}^{\min\{m,\tau^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{\min\{m,\tau^2\}} \tilde{c}^1(\tilde{q}^{\min\{m,\tau^2\}}(\tilde{s}))^+_{\alpha^*} \\
&= \sum_{n=0}^{\min\{m,\tau^2\}-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{\min\{m,\tau^2\}} \tilde{c}(\tilde{s}, m, \tau^2)^+_{\alpha^*} \\
&= \tilde{u}(\tilde{s}, m, \tau^2)^+_{\alpha^*}.
\end{aligned}$$

Therefore we get

$$\tilde{u}(\tilde{s}, \tau^1, \tau^2)^+_{\alpha^*} = \tilde{u}(\tilde{s}, *, \tau^2)^+_{\alpha^*} \quad \text{if } \tau^1 < \tau^2. \tag{4.8}$$

Case $\tau^1 \geq \tau^2$: Checking the case $\tau^1 \geq \tau^2$ similarly, we also have

$$\tilde{u}(\tilde{s}, \tau^1, \tau^2)^+_{\alpha^*} = \tilde{u}(\tilde{s}, *, \tau^2)^+_{\alpha^*} \quad \text{if } \tau^1 \geq \tau^2. \tag{4.9}$$

From (4.8) and (4.9), we obtain

$$\tilde{u}(\tilde{s}, \tau^1, \tau^2)^+_{\alpha^*} = \tilde{u}(\tilde{s}, *, \tau^2)^+_{\alpha^*}. \tag{4.10}$$

Using (4.5) and (4.7), similarly we have

$$\tilde{u}(\tilde{s}, \tau^1, \tau^2)^+_{\alpha^*} = \tilde{u}(\tilde{s}, \tau^1, *)^+_{\alpha^*}. \tag{4.11}$$

Therefore, (4.6), (4.10) and (4.11) complete the proof. □

Theorem 4.1. *If τ^1 and τ^2 are finite, then τ^1 is player 1's optimal stopping time and τ^2 is player 2's optimal stopping time for Problem 1.1. Further, (τ^1, τ^2) is a saddle point in the class of all pairs of finite stopping times :*

$\mathcal{S} := \{(m^1, m^2) \mid m^1$ *are player 1's stopping times and* m^2 *are player 2's stopping times* $\}$.

Namely, it holds that

$$\tilde{E}(\tilde{u}(\tilde{s}, m^1, \tau^2)) \leq \tilde{E}(\tilde{u}(\tilde{s}, \tau^1, \tau^2)) = \tilde{E}(\tilde{v}(\tilde{s})) \leq \tilde{E}(\tilde{u}(\tilde{s}, \tau^1, m^2)) \quad \textit{for all } (m^1, m^2) \in \mathcal{S}. \tag{4.12}$$

Proof. From Proposition 4.1 and (3.2), we have

$$\tilde{g}^-_{\alpha^*} \leq \tilde{u}(\tilde{s}, \tau^1, *)^+_{\alpha^*}.$$

By Lemma 3.1(ii), we have

$$\tilde{E}(\tilde{u}(\tilde{s},\tau^1,*)) \geq \alpha^*.$$

By Theorem 3.1 and (3.5), we get

$$\tilde{E}(\tilde{u}(\tilde{s},\tau^1,*)) = \tilde{E}(\tilde{v}(\tilde{s})) = \alpha^*.$$

By (3.8), we obtain

$$\inf_{m^2\geq 0} \tilde{E}(\tilde{u}(\tilde{s},\tau^1,m^2)) = \tilde{E}(\tilde{u}(\tilde{s},\tau^1,*)) = \tilde{E}(\tilde{v}(\tilde{s})). \tag{4.13}$$

Similarly, we obtain

$$\sup_{m^1\geq 0} \tilde{E}(\tilde{u}(\tilde{s},m^1,\tau^2)) = \tilde{E}(\tilde{u}(\tilde{s},*,\tau^2)) = \tilde{E}(\tilde{v}(\tilde{s})). \tag{4.14}$$

Thus, (4.13) and (4.14) imply (4.12). Therefore, τ^1 is player 1's optimal stopping time and τ^2 is player 2's optimal stopping time for Problem 1.1. The proof is completed. □

5 A numerical example

We consider a numerical example with a one-dimensional state space. Take the state space $E := [-2,2]$ and the discount rate $\beta := 0.5$. Give a fuzzy relation by

$$\tilde{q}(x,y) = \max\{1-3|y-0.5x|,0\}, \quad x,y\in E.$$

We take an initial fuzzy state by

$$\tilde{s}_0(x) = \tilde{s}(x) = \max\{1-1.5|x|,0\}, \quad x\in E,$$

and we give a sequence of the fuzzy states $\{\tilde{s}_n\}_{n=0}^{\infty}$ by (1.1). Further, in the same way as [15, 8], we give fuzzy relations

$$\tilde{r}(x,z) = \begin{cases} \max\left\{1-2\left|\frac{5x}{z}-1\right|,0\right\}, & x\in E,\ z>0, \\ 1_{\{x\}}, & x\in E,\ z=0, \end{cases}$$

$$\tilde{c}^1(x,z) = \max\left\{1-\frac{1}{2}|z-x-1|,0\right\}, \quad x\in E,\ z\geq 0,$$

$$\tilde{c}^0(x,z) = \max\left\{1-\frac{1}{2}|z-x-1.1|,0\right\}, \quad x\in E,\ z\geq 0,$$

$$\tilde{c}^2(x,z) = \max\left\{1-\frac{1}{2}|z-x-1.2|,0\right\}, \quad x\in E,\ z\geq 0.$$

In a way similar to (1.2), we define a running fuzzy reward and a terminal fuzzy reward by

$$\tilde{r}(\tilde{s})(z) := \sup_{x\in E}\min\{\tilde{s}(x),\tilde{r}(x,z)\}, \quad z\geq 0 \quad \text{for } \tilde{s}\in\mathcal{F}(E),$$

$$\tilde{c}^i(\tilde{s})(z) := \sup_{x\in E} \min\{\tilde{s}(x), \tilde{c}^i(x,z)\}, \quad z \geq 0 \quad \text{for } \tilde{s} \in \mathcal{F}(E) \quad i = 0,1,2,$$

and a fuzzy goal by

$$\tilde{g}\,(z) = \frac{1}{1+e^{-z}}, \qquad z \geq 0.$$

Then we have

$$\|\tilde{r}\| \leq \sup_{\tilde{s}\in\mathcal{F}(E)} \tilde{r}(\tilde{s})_0^+ = \tilde{r}(1_{[-2,2]})_0^+ \leq 20 < \infty,$$

where $1_{[-2,2]}$ is the classical indicator function of $[-2,2]$. Similarly $\|\tilde{c}\| \leq 3.2 < \infty$. Therefore, we can calculate fuzzy rewards $\tilde{u}(\tilde{s}, m^1, m^2)$ $(m^1, m^2 \geq 0)$. From Fig. 5.1, in this example we find that the optimal fuzzy expected value is given by

$$\alpha^* = \tilde{E}\{\tilde{v}(\tilde{s})\} \approx 0.81613.$$

at an optimal payoff

$$z^* \approx 1.49033.$$

Then, players' optimal stopping time are $\tau^1 = \tau^2 = 0$.

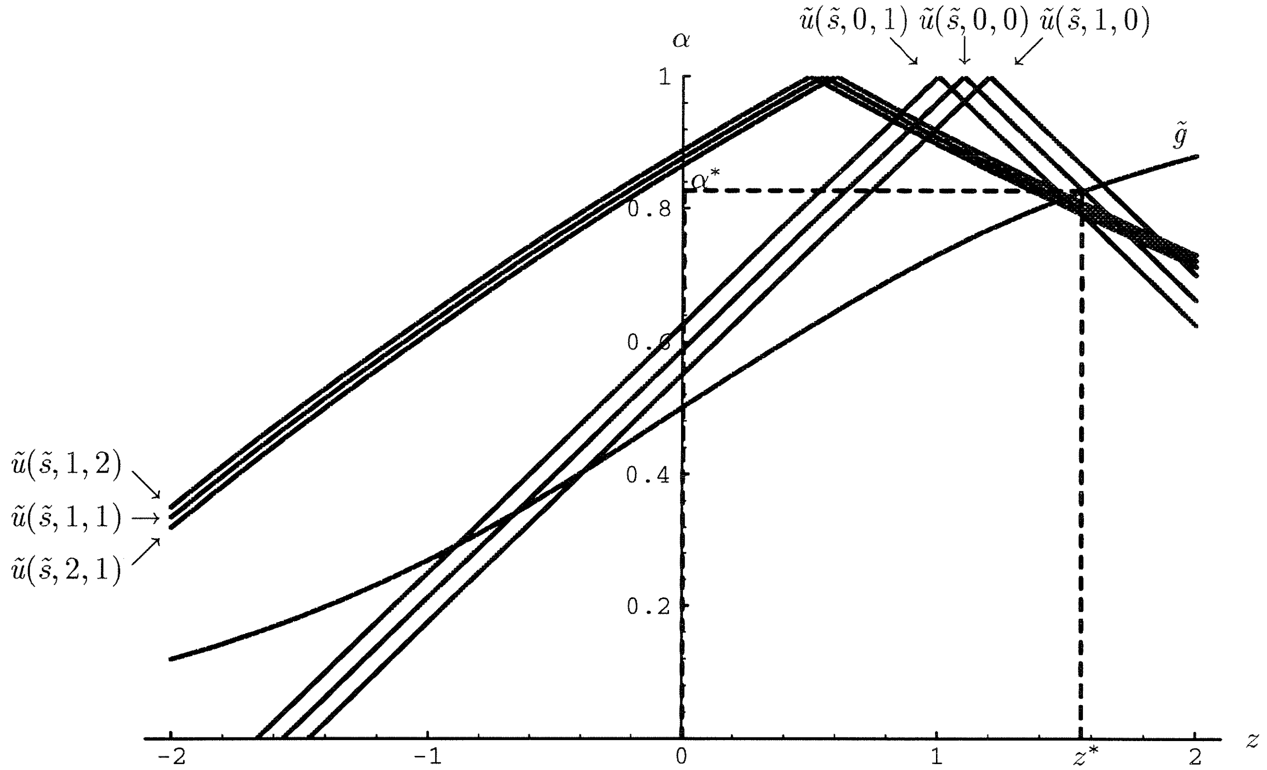

Fig. 5.1. The fuzzy rewards $\tilde{u}(\tilde{s}, m^1, m^2)$ $(\min\{m^1, m^2\} \leq 1)$ and the fuzzy goal $\tilde{g}$.

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