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AN OPTIMAL STOPPING PROBLEM IN DYNAMIC FUZZY SYSTEMS WITH FUZZY REWARDS

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Abstract : This paper deals with an optimal stopping problem in dynamic fuzzy systems with fuzzy rewards, and shows that the optimal discounted fuzzy reward is characterized by a unique solution of a fuzzy relational equation. We define a fuzzy expectation with a density given by fuzzy goals and we estimate discounted fuzzy rewards by the fuzzy expectation. This paper characterizes the optimal fuzzy expected value and gives an optimal stopping time.

Keyword : Optimal stopping; fuzzy reward; dynamic fuzzy system; fuzzy expected value.

1 Introduction

Optimal stopping problems in a fuzzy environment was studied by [4, 11, 13]. On the other hand, in the previous papers [6, 13, 14, 15], we defined a dynamic fuzzy system using a fuzzy relation and gave limit theorems for the transition of fuzzy states of the system under the contractive and nonexpansive properties of of the fuzzy relation. Recently, Kurano et al. [7] also introduced fuzzy rewards for the system and discussed fuzzy decision processes for total fuzzy rewards with respect to a partial order, which is called the fuzzy max order. In [7], we defined fuzzy rewards by maps from fuzzy states to fuzzy numbers. The definition is a natural extension of the classical rewards which are given by real valued functions on a crisp state space (see [10]). This paper deals with an optimal stopping problem in the dynamic fuzzy system with fuzzy rewards. We introduce a fuzzy expectation with a density given by fuzzy goals on the basis of the concept of decision making in Bellman and Zadeh [1]. We estimate discounted fuzzy rewards by the fuzzy expectation and we call them fuzzy expected values. In Section 3, this paper proves that the optimal fuzzy expected values equals to the fuzzy expectation of optimal fuzzy rewards regarding the fuzzy max order. In Section 2, we prove that the optimal fuzzy reward is a unique solution of a fuzzy relational equation, and, in Section 4, we give an optimal stopping time. In Section 5, a numerical example is given to illustrate our theoretical idea.

Let E be a metric space. Let $\mathcal{F}(E)$ be the set of all fuzzy sets $\tilde{s} : E \mapsto [0,1]$ which are upper semi-continuous and satisfy $\sup_{x\in E} \tilde{s}(x) = 1$. Let $\tilde{q} : E \times E \mapsto [0,1]$ be a continuous fuzzy relation satisfying $\tilde{q}(x,\cdot) \in \mathcal{F}(E)$ $(x \in E)$. Let $\tilde{s} \in \mathcal{F}(E)$. In this paper, we deal with a sequence of fuzzy states $\{\tilde{s}_n\}_{n=0}^{\infty}$ defined by the following dynamic fuzzy system (see Kurano et al. [6]) :

$$\tilde{s}_0 := \tilde{s} \quad \text{and} \quad \tilde{s}_{n+1}(y) := \sup_{x\in E} \min\{\tilde{s}_n(x), \tilde{q}(x,y)\}, \quad y \in E, \ n = 0,1,2,\cdots. \tag{1.1}$$

For simplicity, we define a map $\tilde{q} : \mathcal{F}(E) \mapsto \mathcal{F}(E)$ as follows. For any $\tilde{s} \in \mathcal{F}(E)$,

$$\tilde{q}(\tilde{s})(y) := \sup_{x\in E} \min\{\tilde{s}(x), \tilde{q}(x,y)\}, \quad y \in E. \tag{1.2}$$

Then, (1.1) is represented by

$$\tilde{q}^0(\tilde{s}) := \tilde{s} \quad \text{and} \quad \tilde{q}^n(\tilde{s}) := \tilde{q}(\tilde{q}^{n-1}(\tilde{s})), \quad n = 1,2,\cdots. \tag{1.3}$$

Firstly, we describe nonnegative fuzzy numbers in order to define fuzzy rewards. Let $\mathbf{R}_+ := [0,\infty)$. For a fuzzy set $\tilde{a}$ on $\mathbf{R}_+$ and $\alpha \in [0,1]$, the α-cut $\tilde{a}_\alpha$ is defined by

$$\tilde{a}_\alpha := \{z \in \mathbf{R}_+ \mid \tilde{a}(z) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{a}_0 := \mathrm{cl}\{z \in \mathbf{R}_+ \mid \tilde{a}(z) > 0\},$$

where cl denotes the closure of a set (for the details, refer to Novàk [8] and Zadeh [16]). Then, a fuzzy set $\tilde{a}$ on $\mathbf{R}_+$ is called a nonnegative fuzzy number if $\tilde{a}$ satisfies the following conditions (N1) — (N3) :

(N1) The α-cut $\tilde{a}_\alpha$ is a bounded closed subinterval of $\mathbf{R}_+$ for $\alpha \in [0,1]$. We represent it by $[\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$.

(N2) $\bigcap_{\alpha'<\alpha} \tilde{a}_{\alpha'} = \tilde{a}_\alpha$ for $\alpha > 0$.

(N3) $\tilde{a}$ is normal, i.e., $\sup_{z\in\mathbf{R}_+} \tilde{a}(z) = 1$.

We denote the set of all nonnegative fuzzy numbers by $\mathcal{F}_n(\mathbf{R}_+)$, and denote the set of all bounded closed subintervals of $\mathbf{R}_+$ by $\mathcal{C}(\mathbf{R}_+)$.

An addition and a scalar multiplication for nonnegative fuzzy numbers are defined as follows (for example, see [3]) : For $\tilde{a}, \tilde{b} \in \mathcal{F}_n(\mathbf{R}_+)$ and $\lambda \geq 0$, the addition $\tilde{a} + \tilde{b}$ of $\tilde{a}$ and $\tilde{b}$ and the scalar multiplication $\lambda\tilde{a}$ of λ and $\tilde{a}$ are fuzzy numbers given by

$$(\tilde{a} + \tilde{b})_\alpha = [\tilde{a}_\alpha^- + \tilde{b}_\alpha^-, \tilde{a}_\alpha^+ + \tilde{b}_\alpha^+], \quad \alpha \in [0,1],$$

$$(\lambda\tilde{a})_\alpha = [\lambda\tilde{a}_\alpha^-, \lambda\tilde{a}_\alpha^+], \quad \alpha \in [0,1].$$

We define a partial order $\succeq$ on $\mathcal{F}_n(\mathbf{R}_+)$: Let $\tilde{a}, \tilde{b} \in \mathcal{F}_n(\mathbf{R}_+)$.

$$\tilde{a} \succeq \tilde{b} \quad \text{means that} \quad \tilde{a}_\alpha^- \geq \tilde{b}_\alpha^- \quad \text{and} \quad \tilde{a}_\alpha^+ \geq \tilde{b}_\alpha^+ \quad \text{for all } \alpha \in [0,1].$$

Then $(\mathcal{F}_n(\mathbf{R}_+), \succeq)$ becomes a lattice ([2]), and $\succeq$ is called the fuzzy max order. Further, for $\tilde{a}, \tilde{b} \in \mathcal{F}_n(\mathbf{R}_+)$, we define the maximum $\tilde{a} \vee \tilde{b}$ of $\tilde{a}$ and $\tilde{b}$ with respect to the order $\succeq$ by a fuzzy number such that

$$(\tilde{a} \vee \tilde{b})_\alpha = [\max\{\tilde{a}_\alpha^-, \tilde{b}_\alpha^-\}, \max\{\tilde{a}_\alpha^+, \tilde{b}_\alpha^+\}], \quad \alpha \in [0,1].$$

Next, we denote by $\mathcal{F}(E:\mathbf{R}_+)$ the family of all maps $\tilde{f}:\mathcal{F}(E)\mapsto\mathcal{F}_n(\mathbf{R}_+)$. This paper calls $\tilde{f}\in\mathcal{F}(E:\mathbf{R}_+)$ a fuzzy-number-valued function on $\mathcal{F}(E)$. We introduce an addition, a scalar multiplication and a maximum on $\mathcal{F}(E:\mathbf{R}_+)$ as follows : For $\tilde{f},\tilde{h}\in\mathcal{F}(E:\mathbf{R}_+)$ and $\lambda\geq 0$, the addition $\tilde{f}+\tilde{h}$ of $\tilde{f}$ and $\tilde{h}$, the scalar multiplication $\lambda\tilde{f}$ of λ and $\tilde{f}$, and the maximum $\tilde{f}\vee\tilde{h}$ of $\tilde{f}$ and $\tilde{h}$ are given by

$$(\tilde{f}+\tilde{h})(\tilde{s}):=\tilde{f}(\tilde{s})+\tilde{h}(\tilde{s}),\quad \tilde{s}\in\mathcal{F}(E);$$
$$(\lambda\tilde{f})(\tilde{s}):=\lambda\tilde{f}(\tilde{s}),\quad \lambda\geq 0,\ \tilde{s}\in\mathcal{F}(E);$$
$$(\tilde{f}\vee\tilde{h})(\tilde{s}):=\tilde{f}(\tilde{s})\vee\tilde{h}(\tilde{s}),\quad \tilde{s}\in\mathcal{F}(E).$$

Let $\mathbf{N}:=\{0,1,2,\cdots\}$ be a time space. Let β be a constant satisfying $0<\beta<1$ and let $\tilde{r},\tilde{c}\in\mathcal{F}(E:\mathbf{R}_+)$ be bounded in the sense that $\|\tilde{r}\|$ and $\|\tilde{c}\|$ are finite, where the norm $\|\cdot\|$ is given by (2.2) in Section 2. β means a discount rate and $\tilde{r}(\tilde{s}_n)$ and $\tilde{c}(\tilde{s}_n)$ mean a running fuzzy reward and a terminal fuzzy reward at a state $\tilde{s}_n$ ($n\in\mathbf{N}$) respectively. Then, we define discounted fuzzy rewards with stopping times m by

$$\tilde{u}(\tilde{s},m):=\sum_{n=0}^{m-1}\beta^n\tilde{r}(\tilde{s}_n)+\beta^m\tilde{c}(\tilde{s}_m)\quad\text{for } \tilde{s}\in\mathcal{F}(E),\ m\in\mathbf{N}\cup\{\infty\},\tag{1.4}$$

where $\{\tilde{s}_n\}_{n=0}^{\infty}$ is defined by (1.1) and we put the sum $\sum_{n=0}^{0-1}\beta^n\tilde{r}(\tilde{s}_n):=1_{\{0\}}\in\mathcal{F}_n(\mathbf{R}_+)$. Then, $\tilde{u}(\cdot,m)\in\mathcal{F}(E:\mathbf{R}_+)$ is trivial for $m<\infty$, and further we can check it by Yoshida et al. [14, Theorem 3.2] when $m=\infty$. This type of rewards in Markov chains are well-known and studied by Shiryayev [10]. This paper discusses the optimal stopping problem for (1.4) in dynamic fuzzy systems (1.1). Put a fuzzy goal by a fuzzy number $\tilde{g}:\mathbf{R}_+\mapsto[0,1]$ which is an upper semi-continuous and nondecreasing function with $\tilde{g}(0)=0$ and $\lim_{z\to\infty}\tilde{g}(z)=1$. Then we note that $\tilde{g}_\alpha=[\tilde{g}_\alpha^-,\infty)$ for $\alpha\in[0,1]$. In this paper, we consider the following optimal stopping problem.

Problem 1.1. Maximize

$$\tilde{E}(\tilde{u}(\tilde{s},m)):=\fint_{\mathbf{R}_+}\tilde{u}(\tilde{s},m)(z)\ \mathrm{d}\tilde{P}(z)=\sup_{z\in\mathbf{R}_+}\min\{\tilde{u}(\tilde{s},m)(z),\tilde{g}(z)\}\tag{1.5}$$

over all $m\in\mathbf{N}\cup\{\infty\}$, where $\tilde{P}$ is the possibility measure generated by the density $\tilde{g}$ and $\fint\ \mathrm{d}\tilde{P}$ denotes Sugeno integral (see [9, 12]).

The fuzzy expectation implies the degree of satisfaction of discounted fuzzy rewards, and the fuzzy goal $\tilde{g}(z)$ means a kind of utility function for fuzzy payoffs z in (1.5) (see Fig 5.1).

On the other hand, from (1.4), we can define an optimal fuzzy reward with respect to the fuzzy max order $\succeq$ as follows :

$$\tilde{v}(\tilde{s}):=\bigvee_{m\geq 0}\tilde{u}(\tilde{s},m)=\bigvee_{m\geq 0}\left\{\sum_{n=0}^{m-1}\beta^n\tilde{r}(\tilde{s}_n)+\beta^m\tilde{c}(\tilde{s}_m)\right\}\quad\text{for } \tilde{s}\in\mathcal{F}(E),\tag{1.6}$$

where $\vee$ means the supremum with respect to the fuzzy max order $\succeq$. Kurano et al. [7] discussed this type of optimal fuzzy rewards in fuzzy decision processes. Then, we can consider another problem to estimate the optimal fuzzy reward (1.6) by the fuzzy expectation $\tilde{E}(\cdot)$.

Problem 1.2. Find times m such that

$$\tilde{E}\left(\tilde{u}(\tilde{s}, m)\right) = \tilde{E}(\tilde{v}(\tilde{s})). \tag{1.7}$$

Problems 1.1 and 1.2 are different regarding the optimization orders since the fuzzy max order is used in Problem 1.2. However, in Section 3, we prove that the optimal fuzzy expected values in Problems 1.1 equals to $\tilde{E}(\tilde{v}(\tilde{s}))$.

2 Optimal fuzzy rewards

In this section, we show $\tilde{v} \in \mathcal{F}(E : \mathbf{R}_+)$ and give a fuzzy relational equation to characterize the optimal fuzzy reward (1.6).

Lemma 2.1. *Let $[a_n, b_n], [a, b] \in \mathcal{C}(\mathbf{R}_+)$ $(n = 0, 1, 2, \cdots)$. Then :*

(i)

$$\beta\left(\bigvee_{n\geq 0}[a_n, b_n]\right) = \bigvee_{n\geq 0}\beta[a_n, b_n] \qquad \textit{if } \sup_{n\geq 0} b_n < \infty;$$

(ii)

$$\beta\left(\sum_{n\geq 0}[a_n, b_n]\right) = \sum_{n\geq 0}\beta[a_n, b_n] \qquad \textit{if } \sum_{n\geq 0} b_n < \infty;$$

(iii)

$$\bigvee_{n\geq 0}\left([a_n, b_n] + [a, b]\right) = \left(\bigvee_{n\geq 0}[a_n, b_n]\right) + [a, b] \qquad \textit{if } \sup_{n\geq 0} b_n < \infty,$$

where we define $\bigvee_{n\geq 0}[a_n, b_n] := [\sup_{n\geq 0} a_n, \sup_{n\geq 0} b_n]$ *and* $\sum_{n\geq 0}[a_n, b_n] := [\sum_{n\geq 0} a_n, \sum_{n\geq 0} b_n]$.

Proof. They are trivial. □

Lemma 2.2. *Let $\tilde{f}_n, \tilde{h} \in \mathcal{F}(E : \mathbf{R}_+)$ $(n = 0, 1, 2, \cdots)$. Then, for $\tilde{s} \in \mathcal{F}(E)$, we have :*

(i)

$$\beta\left(\bigvee_{n\geq 0}\tilde{f}_n(\tilde{s})\right) = \bigvee_{n\geq 0}\beta\tilde{f}_n(\tilde{s}) \qquad \textit{if } \sup_{n\geq 0}\tilde{f}_n(\tilde{s})_0^+ < \infty;$$

(ii)

$$\beta\left(\sum_{n\geq 0}\tilde{f}_n(\tilde{s})\right)=\sum_{n\geq 0}\beta\tilde{f}_n(\tilde{s}) \qquad \textit{if } \sum_{n\geq 0}\tilde{f}_n(\tilde{s})_0^+<\infty;$$

(iii)

$$\bigvee_{n\geq 0}\left(\tilde{f}_n(\tilde{s})+\tilde{h}(\tilde{s})\right)=\left(\bigvee_{n\geq 0}\tilde{f}_n(\tilde{s})\right)+\tilde{h}(\tilde{s}) \qquad \textit{if } \sup_{n\geq 0}\tilde{f}_n(\tilde{s})_0^+<\infty.$$

Proof. They are trivial, by applying Lemma 2.1 for their α-cuts. □

Next, we introduce a distance between fuzzy-number-valued functions on $\mathcal{F}(E)$. We denote the Hausdorff metric on $\mathcal{C}(\mathbf{R}_+)$ by δ (see [5]):

$$\delta([a_1,b_1],[a_2,b_2]):=\max\{|a_1-a_2|,|b_1-b_2|\} \qquad \text{for } [a_1,b_1],[a_2,b_2]\in\mathcal{C}(\mathbf{R}).$$

Then, we define a metric on $\mathcal{F}(E:\mathbf{R}_+)$ as follows :

$$d(\tilde{f},\tilde{h}):=\sup_{\alpha\in[0,1],\tilde{s}\in\mathcal{F}(E)}\delta(\tilde{f}(\tilde{s})_\alpha,\tilde{h}(\tilde{s})_\alpha) \quad \text{for } \tilde{f},\tilde{h}\in\mathcal{F}(E:\mathbf{R}_+). \tag{2.1}$$

Further, we define a norm $\|\cdot\|$ on $\mathcal{F}(E:\mathbf{R}_+)$ by

$$\|\tilde{f}\|:=d(\tilde{f},I_{\{0\}})=\sup_{\alpha\in[0,1],\tilde{s}\in\mathcal{F}(E)}\delta(\tilde{f}(\tilde{s})_\alpha,\{0\}) \quad \text{for } \tilde{f}\in\mathcal{F}(E:\mathbf{R}_+), \tag{2.2}$$

where we put $I_{\{0\}}\in\mathcal{F}(E:\mathbf{R}_+)$ by

$$I_{\{0\}}(\tilde{s}):=\tilde{0}\ \text{(the crisp number zero)} \quad \text{for } \tilde{s}\in\mathcal{F}(E).$$

Then we have the following elementary results.

Lemma 2.3. *Let* $[a_1,b_1],[c_1,d_1],[a_2,b_2],[c_2,d_2]\in\mathcal{C}(\mathbf{R}_+)$. *Then*

(i) $\delta([a_1,b_1]\vee[c_1,d_1],[a_2,b_2]\vee[c_2,d_2])\leq\max(\delta([a_1,b_1],[a_2,b_2]),\delta([c_1,d_1],[c_2,d_2]))$;

(ii) $\delta([a_1,b_1]+[c_1,d_1],[a_2,b_2]+[c_2,d_2])\leq\delta([a_1,b_1],[a_2,b_2])+\delta([c_1,d_1],[c_2,d_2])$;

(iii) $\delta(\beta[a_1,b_1],\beta[c_1,d_1])=\beta\delta([a_1,b_1],[c_1,d_1])$.

Proof. We can easily check them. □

Lemma 2.4. *Let* $\tilde{f}_1,\tilde{f}_2,\tilde{h}_1,\tilde{h}_2\in\mathcal{F}(E:\mathbf{R}_+)$. *Then*

(i) $d(\tilde{f}_1\vee\tilde{h}_1,\tilde{f}_2\vee\tilde{h}_2)\leq\max(d(\tilde{f}_1,\tilde{f}_2),d(\tilde{h}_1,\tilde{h}_2))$;

(ii) $d(\tilde{f}_1+\tilde{h}_1,\tilde{f}_2+\tilde{h}_2)\leq d(\tilde{f}_1,\tilde{f}_2)+d(\tilde{h}_1,\tilde{h}_2)$;

(iii) $d(\beta \tilde{f}_1, \beta \tilde{f}_2) = \beta d(\tilde{f}_1, \tilde{f}_2)$;

(iv) $d(\tilde{f}_1(\tilde{q}), \tilde{f}_2(\tilde{q})) \leq d(\tilde{f}_1, \tilde{f}_2)$.

Proof. (i) — (iii) are trivial from the definition (2.1) and Lemma 2.3. (iv) Let $\tilde{s} \in \mathcal{F}(E)$. Since $\tilde{q}(\tilde{s}) \in \mathcal{F}(E)$, we have

$$\begin{aligned} \delta(\tilde{f}_1(\tilde{q}(\tilde{s}))_\alpha, \tilde{f}_2(\tilde{q}(\tilde{s}))_\alpha) &\leq \sup_{\alpha \in [0,1], \tilde{s}' \in \mathcal{F}(E)} \delta(\tilde{f}_1(\tilde{s}')_\alpha, \tilde{f}_2(\tilde{s}')_\alpha) \\ &= d(\tilde{f}_1, \tilde{f}_2) \quad \text{for all } \alpha \in [0,1], \ \tilde{s} \in \mathcal{F}(E). \end{aligned}$$

This yields (iv). □

We obtain the following theorems for the optimal fuzzy reward $\tilde{v}$ in (1.6).

Theorem 2.1. *It holds that* $\tilde{v} \in \mathcal{F}(E : \mathbf{R}_+)$.

Proof. Let $\tilde{s} \in \mathcal{F}(E)$. From Lemmas 2.1 and 2.3, for $m' < m''$ and $\alpha \in [0,1]$, we have

$$\begin{aligned} &\delta\left(\bigvee_{0 \leq m \leq m''} \tilde{u}(\tilde{s}, m)_\alpha, \bigvee_{0 \leq m \leq m'} \tilde{u}(\tilde{s}, m)_\alpha\right) \\ &\leq \delta\left(\bigvee_{m' \leq m \leq m''} \tilde{u}(\tilde{s}, m)_\alpha, \tilde{u}(\tilde{s}, m')_\alpha\right) \\ &\leq \max_{m' \leq m \leq m''} \delta\left(\left\{\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_\alpha + \beta^m \tilde{c}(\tilde{q}^m(\tilde{s}))_\alpha\right\}, \left\{\sum_{n=0}^{m'-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_\alpha + \beta^{m'} \tilde{c}(\tilde{q}^{m'}(\tilde{s}))_\alpha\right\}\right) \\ &\leq \max_{m' \leq m \leq m''} \left\{\delta\left(\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_\alpha, \sum_{n=0}^{m'-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_\alpha\right) + \delta(\beta^m \tilde{c}(\tilde{q}^m(\tilde{s}))_\alpha, \beta^{m'} \tilde{c}(\tilde{q}^{m'}(\tilde{s}))_\alpha)\right\} \\ &\leq \max_{m' \leq m \leq m''} \left\{\delta\left(\sum_{n=m'}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_\alpha, \{0\}\right) + \delta(\beta^m \tilde{c}(\tilde{q}^m(\tilde{s}))_\alpha, \beta^{m'} \tilde{c}(\tilde{q}^{m'}(\tilde{s}))_\alpha)\right\} \\ &\leq \sum_{n=m'}^{\infty} \delta(\beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))_\alpha, \{0\}) + 2 \sup_{m \geq m'} \delta(\beta^m \tilde{c}(\tilde{q}^m(\tilde{s}))_\alpha, \{0\}) \\ &\leq \sum_{n=m'}^{\infty} \beta^n \|\tilde{r}\| + 2\beta^{m'} \|\tilde{c}\|. \end{aligned}$$

By letting $m'' \to \infty$, we obtain

$$\delta\left(\tilde{v}(\tilde{s})_\alpha, \bigvee_{0 \leq m \leq m'} \tilde{u}(\tilde{s}, m)_\alpha\right) \leq \epsilon(m') \quad \text{for all } m' \geq 1, \ \alpha \in [0,1],$$

where

$$\epsilon(m') := \sum_{n=m'}^{\infty} \beta^n \|\tilde{r}\| + 2\beta^{m'} \|\tilde{c}\| \quad \text{for } m' \geq 1.$$

By Lemma 2.3(i), we obtain

$$\begin{aligned}
&\delta\left(\tilde{v}(\tilde{s})_\alpha, \tilde{v}(\tilde{s})_{\alpha'}\right)\\
\leq\ &\delta\left(\tilde{v}(\tilde{s})_\alpha, \bigvee_{0\leq m\leq m'} \tilde{u}(\tilde{s},m)_\alpha\right) + \delta\left(\bigvee_{0\leq m\leq m'} \tilde{u}(\tilde{s},m)_\alpha, \bigvee_{0\leq m\leq m'} \tilde{u}(\tilde{s},m)_{\alpha'}\right)\\
&+\delta\left(\bigvee_{0\leq m\leq m'} \tilde{u}(\tilde{s},m)_{\alpha'}, \tilde{v}(\tilde{s})_{\alpha'}\right)\\
\leq\ &2\epsilon(m') + \max_{0\leq m\leq m'} \delta\left(\tilde{u}(\tilde{s},m)_\alpha, \tilde{u}(\tilde{s},m)_{\alpha'}\right) \quad \text{for } m'\geq 1,\ \alpha'<\alpha.
\end{aligned}$$

Since $\epsilon(m')$ is independent of α and α' and we have $\tilde{u}(\tilde{s},m)\in\mathcal{F}_n(\mathbf{R}_+)$, this yields

$$\lim_{\alpha'\uparrow\alpha} \delta(\tilde{v}(\tilde{s})_\alpha, \tilde{v}(\tilde{s})_{\alpha'}) \leq 2\epsilon(m') \quad \text{for all } m'\geq 1, \alpha>0.$$

Therefore

$$\lim_{\alpha'\uparrow\alpha} \delta(\tilde{v}(\tilde{s})_\alpha, \tilde{v}(\tilde{s})_{\alpha'}) = 0.$$

Since $\tilde{v}(\tilde{s})_\alpha \subset \tilde{v}(\tilde{s})_{\alpha'}$ holds trivially for $\alpha'<\alpha$, we obtain $\tilde{v}(\tilde{s})\in\mathcal{F}_n(\mathbf{R}_+)$ for all $\tilde{s}\in\mathcal{F}(E)$, using [6, Lemma 3]. Thus we get $\tilde{v}\in\mathcal{F}(E:\mathbf{R}_+)$. □

Theorem 2.2. *The optimal fuzzy reward $\tilde{v}\in\mathcal{F}(E:\mathbf{R}_+)$ is a unique solution of the following fuzzy relational equation :*

$$\tilde{v}(\tilde{s}) = \tilde{c}(\tilde{s}) \vee \{\tilde{r}(\tilde{s}) + \beta\tilde{v}(\tilde{q}(\tilde{s}))\} \quad \textit{for } \tilde{s}\in\mathcal{F}(E). \tag{2.3}$$

Proof. Let $\tilde{s}\in\mathcal{F}(E)$. From (1.1) and Lemma 2.2, we have

$$\begin{aligned}
\tilde{v}(\tilde{s}) &= \bigvee_{m\geq 0}\left\{\sum_{n=0}^{m-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{s})) + \beta^m\tilde{c}(\tilde{q}^m(\tilde{s}))\right\}\\
&= \tilde{c}(\tilde{s}) \vee \left\{\tilde{r}(\tilde{s}) + \beta\bigvee_{m\geq 1}\left\{\sum_{n=1}^{m-1}\beta^{n-1}\tilde{r}(\tilde{q}^{n-1}(\tilde{q}(\tilde{s}))) + \beta^{m-1}\tilde{c}(\tilde{q}^{m-1}(\tilde{q}(\tilde{s})))\right\}\right\}\\
&= \tilde{c}(\tilde{s}) \vee \{\tilde{r}(\tilde{s}) + \beta\tilde{v}(\tilde{q}(\tilde{s}))\} \quad \text{for } \tilde{s}\in\mathcal{F}(E).
\end{aligned}$$

Therefore $\tilde{v}$ satisfies (2.2). If $\tilde{w}\in\mathcal{F}(E:\mathbf{R}_+)$ is another solution of (2.2), then by Lemma 2.4 we have

$$\begin{aligned}
d(\tilde{v},\tilde{w}) &= d(\tilde{c}\vee\{\tilde{r}+\beta\tilde{v}(\tilde{q})\}, \tilde{c}\vee\{\tilde{r}+\beta\tilde{w}(\tilde{q})\})\\
&\leq d(\tilde{r}+\beta\tilde{v}(\tilde{q}), \tilde{r}+\beta\tilde{w}(\tilde{q}))\\
&\leq d(\beta\tilde{v}(\tilde{q}), \beta\tilde{w}(\tilde{q}))\\
&= \beta d(\tilde{v}(\tilde{q}), \tilde{w}(\tilde{q}))\\
&\leq \beta d(\tilde{v},\tilde{w}).
\end{aligned}$$

So we obtain $\tilde{v}=\tilde{w}$ since $0<\beta<1$. Therefore $\tilde{v}$ is a unique solution of (2.2). □

3 Optimal fuzzy expected values

In this section, we discuss the fuzzy expectation of the optimal fuzzy reward. From now on, we fix an initial fuzzy state $\tilde{s} \in \mathcal{F}(E)$. Define a level α^* by

$$\alpha^* := \sup\{\alpha \in [0,1] |\ \tilde{g}\,_\alpha^- \leq \tilde{v}(\tilde{s})_\alpha^+\}. \tag{3.1}$$

Then, since $\tilde{g}$ and $\tilde{v}(\tilde{s})$ are upper semi-continuous, we have

$$\tilde{g}\,_{\alpha^*}^- = \tilde{v}(\tilde{s})_{\alpha^*}^+. \tag{3.2}$$

Theorem 3.1. *It holds that*

$$\alpha^* = \tilde{E}(\tilde{v}(\tilde{s})). \tag{3.3}$$

Proof. Let α satisfy $0 \leq \alpha < \tilde{E}(\tilde{v}(\tilde{s}))$. By the definition of the fuzzy expectation $\tilde{E}(\tilde{v}(\tilde{s}))$, we have

$$\tilde{g}\,_\alpha \cap \tilde{v}(\tilde{s})_\alpha \neq \emptyset.$$

Since $\tilde{g}$ is nondecreasing and $\tilde{v}(\tilde{s}) \in \mathcal{F}_n(\mathbf{R}_+)$, it is equivalent to

$$\tilde{g}\,_\alpha^- \leq \tilde{v}(\tilde{s})_\alpha^+.$$

Therefore $\alpha \leq \alpha^*$. Thus, we get $\alpha^* \geq \tilde{E}(\tilde{v}(\tilde{s}))$.

Next, let $\alpha < \alpha^*$. Then, we have

$$\tilde{g}\,_\alpha^- \leq \tilde{v}(\tilde{s})_\alpha^+.$$

This is equivalent to

$$\tilde{g}\,_\alpha \cap \tilde{v}(\tilde{s})_\alpha \neq \emptyset.$$

Therefore $\alpha \leq \tilde{E}(\tilde{v}(\tilde{s}))$. Thus, we get $\alpha^* \leq \tilde{E}(\tilde{v}(\tilde{s}))$. Therefore, we obtain (3.3). □

Theorem 3.2. *It holds that*

$$\tilde{E}(\tilde{v}(\tilde{s})) = \sup_{m \geq 0} \tilde{E}(\tilde{u}(\tilde{s}, m)). \tag{3.4}$$

Therefore, the optimal fuzzy expected value in Problem 1.1 coincides with the optimal fuzzy expected value in Problem 1.2.

Proof. We have

$$\tilde{u}(\tilde{s}, m)_\alpha^+ \leq \tilde{v}(\tilde{s})_\alpha^+. \quad \text{for all } \alpha \in [0,1] \text{ and } m \geq 0.$$

Since $\tilde{g}$ is nondecreasing, by the definition of the fuzzy expectation $\tilde{E}(\tilde{v}(\tilde{s}))$ and $\tilde{E}(\tilde{u}(\tilde{s}, m))$, we get

$$\tilde{E}(\tilde{v}(\tilde{s})) \geq \tilde{E}(\tilde{u}(\tilde{s}, m)). \quad \text{for all } m \geq 0. \tag{3.5}$$

If $\alpha^* = 0$, then (3.4) is trivial from (3.5) and Theorem 3.1. We assume $\alpha^* > 0$. Let ϵ be an arbitrary real number such that $0 < \epsilon < \alpha^*$. Let α satisfy $\alpha^* - \epsilon < \alpha < \alpha^*$. We consider the following case :

$$\tilde{u}(\tilde{s}, m)^+_{\alpha^*} < \tilde{v}(\tilde{s})^+_{\alpha^*} \quad \text{for all } m \geq 0. \tag{3.6}$$

From the definition of $\tilde{v}(\tilde{s})$, there exists a sequence $\{m'\}$ such that

$$\tilde{u}(\tilde{s}, m')^+_\alpha \uparrow \tilde{v}(\tilde{s})^+_\alpha \quad \text{as } m' \to \infty.$$

Since

$$\tilde{g}^-_\alpha < \tilde{g}^-_{\alpha^*} \leq \tilde{v}(\tilde{s})^+_{\alpha^*} \leq \tilde{v}(\tilde{s})^+_\alpha,$$

there exists m'_0 such that

$$\tilde{g}^-_\alpha < \tilde{u}(\tilde{s}, m'_0)^+_\alpha. \tag{3.7}$$

If the condition (3.6) does not hold, then there exists m'_0 such that

$$\tilde{u}(\tilde{s}, m'_0)^+_{\alpha^*} = \tilde{v}(\tilde{s})^+_{\alpha^*}. \tag{3.8}$$

Then

$$\tilde{g}^-_\alpha < \tilde{g}^-_{\alpha^*} \leq \tilde{v}(\tilde{s})^+_{\alpha^*} = \tilde{u}(\tilde{s}, m'_0)^+_{\alpha^*} \leq \tilde{u}(\tilde{s}, m'_0)^+_\alpha.$$

So (3.7) holds clearly. Therefore, in the both case (3.6) and (3.8), we obtain (3.7). Since the $\tilde{g}$ is nondecreasing and upper semi-continuous, from (3.7), we obtain

$$\tilde{g}_\alpha \cap \tilde{u}(\tilde{s}, m'_0)_\alpha \neq \emptyset.$$

By the definition of the fuzzy expectation $\tilde{E}(\tilde{u}(\tilde{s}, m'_0))$, we get

$$\tilde{E}(\tilde{u}(\tilde{s}, m'_0)) \geq \alpha > \alpha^* - \epsilon.$$

Since ϵ is arbitrary, we get

$$\sup_{m \geq 0} \tilde{E}(\tilde{u}(\tilde{s}, m)) \geq \alpha^*.$$

Together with (3.5) and Theorem 3.1, we obtain this theorem. □

4 Optimal stopping times

In this section, we give an optimal stopping time for Problem 1.1. The following lemma is trivial from the definitions.

Lemma 4.1. *Let $\tilde{a}_n, \tilde{a} \in \mathcal{F}_n(\mathbf{R}_+)$ $(n = 0, 1, 2, \cdots)$. Then, for $\alpha \in [0, 1]$,*

(i)

$$\left(\sum_{n \geq 0} \tilde{a}_n \right)^+_\alpha = \sum_{n \geq 0} \tilde{a}^+_{n,\alpha};$$

(ii)

$$\left(\bigvee_{n\geq 0}\tilde{a}_n\right)_\alpha^+ = \sup_{n\geq 0}\tilde{a}_{n,\alpha}^+;$$

(iii)

$$(\beta\tilde{a})_\alpha^+ = \beta\tilde{a}_\alpha^+,$$

where $\tilde{a}_{n,\alpha}$ denotes the α-cut of $\tilde{a}_n$.

We define a time

$$m^* := \inf\{m \in \mathbf{N} \mid \tilde{v}(\tilde{s}_m)_{\alpha^*}^+ = \tilde{c}(\tilde{s}_m)_{\alpha^*}^+\}, \tag{4.1}$$

where the infimum of the empty set is understood to be $+\infty$.

We have the following proposition by the standard methods in the classical optimal stopping problems (see [10]).

Proposition 4.1. *If m^* is finite, then*

$$\tilde{v}(\tilde{s})_{\alpha^*}^+ = \tilde{u}(\tilde{s}, m^*)_{\alpha^*}^+ \tag{4.2}$$

Proof. From (4.2) and Lemmas 2.2 and 4.1, we have

$$\begin{aligned}
&\sum_{n=0}^{m^*-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \beta^{m^*}\tilde{c}(\tilde{q}^{m^*}(\tilde{s}))_{\alpha^*}^+ \\
&= \sum_{n=0}^{m^*-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \beta^{m^*}\tilde{v}(\tilde{q}^{m^*}(\tilde{s}))_{\alpha^*}^+ \\
&= \sum_{n=0}^{m^*-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \beta^{m^*}\sup_{m\geq 0}\left\{\sum_{n=0}^{m-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{q}^{m^*}(\tilde{s})))_{\alpha^*}^+ + \beta^m\tilde{c}(\tilde{q}^m(\tilde{q}^{m^*}(\tilde{s})))_{\alpha^*}^+\right\} \\
&= \sum_{n=0}^{m^*-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \sup_{m\geq 0}\left\{\sum_{n=m^*}^{m^*+m-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \beta^{m^*+m}\tilde{c}(\tilde{q}^{m^*+m}(\tilde{s}))_{\alpha^*}^+\right\} \\
&= \sum_{n=0}^{m^*-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \sup_{m\geq m^*}\left\{\sum_{n=m^*}^{m-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \beta^m\tilde{c}(\tilde{q}^m(\tilde{s}))_{\alpha^*}^+\right\} \\
&= \sup_{m\geq m^*}\left\{\sum_{n=0}^{m-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \beta^m\tilde{c}(\tilde{q}^m(\tilde{s}))_{\alpha^*}^+\right\}.
\end{aligned}$$

Therefore, we obtain

$$\sum_{n=0}^{m^*-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \beta^{m^*}\tilde{c}(\tilde{q}^{m^*}(\tilde{s}))_{\alpha^*}^+ = \sup_{m\geq m^*}\left\{\sum_{n=0}^{m-1}\beta^n\tilde{r}(\tilde{q}^n(\tilde{s}))_{\alpha^*}^+ + \beta^m\tilde{c}(\tilde{q}^m(\tilde{s}))_{\alpha^*}^+\right\}. \tag{4.3}$$

On the other hand, from Theorem 2.2 and Lemmas 2.2 and 4.1, we have

$$\begin{aligned}
&\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{v}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*} \\
&= \sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \{\tilde{c}(\tilde{q}^m(\tilde{s})) \vee \{\tilde{r}(\tilde{q}^m(\tilde{s})) + \beta \tilde{v}(\tilde{q}^{m+1}(\tilde{s}))\}\}^+_{\alpha^*} \\
&= \sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \max\{\beta^m \tilde{c}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*}, \{\beta^m \tilde{r}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*} + \beta^{m+1} \tilde{v}(\tilde{q}^{m+1}(\tilde{s}))^+_{\alpha^*}\}\} \\
&= \max\left\{\left\{\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{c}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*}\right\}, \left\{\sum_{n=0}^{m} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{m+1} \tilde{v}(\tilde{q}^{m+1}(\tilde{s}))^+_{\alpha^*}\right\}\right\}.
\end{aligned}$$

Therefore, for $m < m^*$, we obtain

$$\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{v}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*} = \sum_{n=0}^{m} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{m+1} \tilde{v}(\tilde{q}^{m+1}(\tilde{s}))^+_{\alpha^*}.$$

This yields

$$\begin{aligned}
&\sum_{n=0}^{m^*-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{m^*} \tilde{c}(\tilde{q}^{m^*}(\tilde{s}))^+_{\alpha^*} \\
&= \sum_{n=0}^{m^*-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{m^*} \tilde{v}(\tilde{q}^{m^*}(\tilde{s}))^+_{\alpha^*} \\
&= \max_{m \le m^*} \left\{\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{v}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*}\right\} \\
&= \max_{m \le m^*} \left\{\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{c}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*}\right\}. \qquad (4.4)
\end{aligned}$$

Together with (4.3), we get

$$\sum_{n=0}^{m^*-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{m^*} \tilde{c}(\tilde{q}^{m^*}(\tilde{s}))^+_{\alpha^*} = \sup_{m \ge 0} \left\{\sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{c}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*}\right\}.$$

From Lemma 4.1, this yields $\tilde{u}(\tilde{s}, m^*)^+_{\alpha^*} = \tilde{v}(\tilde{s})^+_{\alpha^*}$. Thus we obtain (4.2). □

Theorem 4.1. *If m^* is finite, then m^* is an optimal stopping time for Problem 1.1. Further, m^* is the shortest in the class of optimal stopping times.*

Proof. From Proposition 4.1 and (3.2), we have

$$\tilde{g}^-_{\alpha^*} \le \tilde{u}(\tilde{s}, m^*)^+_{\alpha^*}. \qquad (4.5)$$

Since the $\tilde{g}$ is nondecreasing and $\tilde{u}(\tilde{s}, m^*) \in \mathcal{F}_n(\mathbf{R}_+)$, it is equivalent to

$$\tilde{g}_{\alpha^*} \cap \tilde{u}(\tilde{s}, m^*)_{\alpha^*} \ne \emptyset.$$

By the definition of the fuzzy expectation $\tilde{E}(\tilde{u}(\tilde{s}, m^*))$, we obtain

$$\tilde{E}(\tilde{u}(\tilde{s}, m^*)) \geq \alpha^*.$$

By Theorems 3.1 and 3.2, we get

$$\tilde{E}(\tilde{u}(\tilde{s}, m^*)) = \tilde{E}(\tilde{v}(\tilde{s})) = \alpha^*.$$

Therefore, m^* is an optimal stopping time for Problem 1.1.

Further, from (4.4), we have

$$\sum_{n=0}^{m^*-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^{m^*} \tilde{c}(\tilde{q}^{m^*}(\tilde{s}))^+_{\alpha^*} > \max_{m<m^*} \left\{ \sum_{n=0}^{m-1} \beta^n \tilde{r}(\tilde{q}^n(\tilde{s}))^+_{\alpha^*} + \beta^m \tilde{c}(\tilde{q}^m(\tilde{s}))^+_{\alpha^*} \right\}. \quad (4.6)$$

From (3.2), (4.6) and Proposition 4.1, we have

$$\tilde{u}(\tilde{s}, m)^+_{\alpha^*} < \tilde{u}(\tilde{s}, m^*)^+_{\alpha^*} = \tilde{v}(\tilde{s})^+_{\alpha^*} = \tilde{g}^-_{\alpha^*} \quad \text{for all } m < m^*.$$

This yields

$$\tilde{E}(\tilde{u}(\tilde{s}, m)) < \alpha^* \quad \text{for all } m < m^*.$$

Therefore, m^* is the shortest in the class of optimal stopping times. This completes the proof. □

5 A numerical example

We consider a numerical example with a one-dimensional state space. Take the state space $E := [-2, 2]$ and the discount rate $\beta := 0.5$. Give a fuzzy relation by

$$\tilde{q}(x, y) = \max\{1 - 3|y - 0.5x|, 0\}, \quad x, y \in E.$$

We take an initial fuzzy state by

$$\tilde{s}_0(x) = \tilde{s}(x) = \max\{1 - 4|x - 1|, 0\}, \quad x \in E,$$

and we give a sequence of the fuzzy states $\{\tilde{s}_n\}_{n=0}^{\infty}$ by (1.1). Further, in the same way as [14, 6], we give fuzzy relations

$$\tilde{r}(x, z) = \begin{cases} \max\left\{1 - 2\left|\frac{5x}{z} - 1\right|, 0\right\}, & x \in E, \ z > 0, \\ 1_{\{x\}}, & x \in E, \ z = 0, \end{cases}$$

$$\tilde{c}(x, z) = \max\left\{1 - \frac{1}{2}|z - x - 10|, 0\right\}, \quad x \in E, \ z \geq 0.$$

In a way to similar (1.2), we define a running fuzzy reward and a terminal fuzzy reward by

$$\tilde{r}(\tilde{s})(z) := \sup_{x \in E} \min\{\tilde{s}(x), \tilde{r}(x, z)\}, \quad z \geq 0 \quad \text{for } \tilde{s} \in \mathcal{F}(E),$$

$$\tilde{c}(\tilde{s})(z) := \sup_{x \in E} \min\{\tilde{s}(x) \wedge \tilde{c}(x, z)\}, \quad z \geq 0 \quad \text{for } \tilde{s} \in \mathcal{F}(E),$$

and a fuzzy goal by

$$\tilde{g}\,(z) = 1 - e^{-0.08z}, \quad z \geq 0.$$

Then we have

$$\|\tilde{r}\| \leq \sup_{\tilde{s} \in \mathcal{F}(E)} \tilde{r}(\tilde{s})_0^+ = \tilde{r}(1_{[-2,2]})_0^+ \leq 20 < \infty,$$

where $1_{[-2,2]}$ is the classical indicator function of $[-2, 2]$. Similarly $\|\tilde{c}\| \leq 14 < \infty$. Therefore, we can calculate fuzzy rewards $\{\tilde{u}(\tilde{s}, m)\}_{m=0}^{\infty}$. They are shown by Fig. 5.1. The optimal fuzzy expected value is given by

$$\alpha^* = \tilde{E}\{\tilde{v}(\tilde{s})\} \approx 0.62949.$$

at an optimal payoff

$$z^* \approx 12.41078.$$

Then, the optimal stopping time is $m^* = 1$.

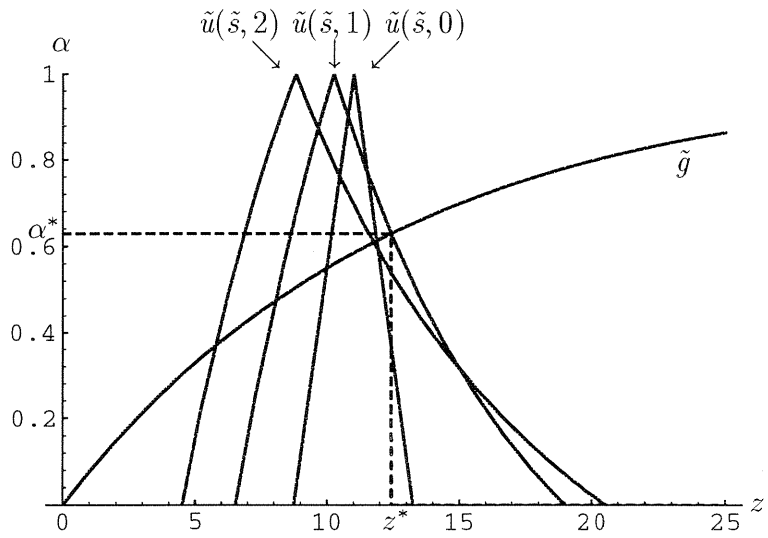

Fig. 5.1. The fuzzy rewards $\{\tilde{u}(\tilde{s}, m)\}_{m=0}^{2}$ and the fuzzy goal $\tilde{g}$.

References

[1] R.E.Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Management Sci. Ser B.* **17** (1970) 141-164.

[2] G.Birkhoff, Lattice Theory, *Amer. Math. Soc., Coll. Pub.*, **25** (1940).

[3] S.Gähler and W.Gähler, Fuzzy real numbers, *Fuzzy Sets and Systems* **66** (1994) 137-158.

[4] J.Kacprzyk, Decision making in a fuzzy environment with fuzzy termination time, *Fuzzy Sets and Systems* **1** (1978) 169-179.

[5] K.Kuratowski, *Topology I* (Academic Press, New York, 1966).

[6] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[7] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, Markov-type fuzzy decision processes with a discounted reward on a closed interval, *Technical Reports of Mathematical Sciences, Chiba University,* No.10, Vol.4 (1992).

[8] V.Novàk, *Fuzzy Sets and Their Applications* (Adam Hilder, Bristol-Boston, 1989).

[9] D.Ralescu and Adams, The fuzzy integral, *J. Math. Anal. Appl.* **75** (1980) 562-570.

[10] A.N.Shiryayev, *Optimal Stopping Rules* (Springer, New York, 1979).

[11] W.E.Stein, Optimal stopping in a fuzzy environment, *Fuzzy Sets and Systems* **3** (1980) 253-259.

[12] M.Sugeno, Fuzzy measures and fuzzy integral : a survey in M.M.Gupta, G.N.Saridis and B.R.Gaines, Eds., *Fuzzy Automata and Decision Processes* (North-Holland, Amsterdam, 1977) 89-102.

[13] Y.Yoshida, Markov chains with a transition possibility measure and fuzzy dynamic programming, *Fuzzy Sets and Systems* **66** (1994) 39-57.

[14] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure : The contractive case, *Fuzzy Sets and Systems* **60** (1993) 283-294.

[15] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure : The unbounded case, *Fuzzy Sets and Systems* **66** (1994) 83-95.

[16] L.A.Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965) 338-353.