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Abstract— We analyse P-superharmonic fuzzy sets on recurrent sets in dynamic fuzzy systems and we derive a simple estimation for the fuzzy sets. This paper presents a method to calculate an optimal value for Snell's optimal stopping problem.

Keywords— Recurrence; superharmonic fuzzy set; dynamic fuzzy system, optimal value.

1. INTRODUCTION

Bellman and Zadeh [1], Esogbue and Bellman [2] and some authors studied fuzzy decision processes and fuzzy dynamic programming. In the dynamic fuzzy systems, a sequence of fuzzy states are successively defined by fuzzy relations. Kurano et al. [3] discussed the limiting behavior of the sequence, and Yoshida [4] studied the recurrence for the systems. Yoshida [5] discussed Snell's optimal stopping problem for the systems, which is found in the classical probability theory (Neveu [6, Sect.VI-2]), and showed that the problem is solved by a finite-step fuzzy dynamic program under a transient condition ([5, Condition (II)]).

The optimal value for Snell's problem is given by a fuzzy set which satisfies a fuzzy relational equation ([5, (4.3)]), in which the solutions are not unique in general. To obtain the optimal value, we need to analyse it not only on transient sets but also on recurrent sets. While, the optimal value has a Q-superharmonic property of [5]. This paper derives a simple estimation for Q-superharmonic fuzzy sets on recurrent sets and gives a method to calculate the optimal value for Snell's problem.

In Section 2 we describe notations and definitions of dynamic fuzzy systems and a recurrence of the system in [4]. In Section 3 we show main theorems regarding P-superharmonic fuzzy sets on α-recurrent sets. In Section 4, we give a numerical example with a one-dimensional state space to comprehend our idea in this paper, and we calculate the optimal value for Snell's problem, using the results in Section 3.

2. DYNAMIC FUZZY SYSTEMS AND RECURRENCE

For a metric space S, we write a fuzzy set on S by its membership function $\tilde{s} : S \mapsto [0,1]$ and a crisp set $A(\subset S)$ by its indicator function $1_A : S \mapsto \{0,1\}$. The α-cut $\tilde{s}_\alpha$ is defined by

$$\tilde{s}_\alpha := \{x \in S \mid \tilde{s}(x) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{s}_0 := \mathrm{cl}\{x \in S \mid \tilde{s}(x) > 0\},$$

where cl denotes the closure of a set. $\mathcal{F}(S)$ denotes the set of all fuzzy sets $\tilde{s}$ on S satisfying the following conditions (i) and (ii) :

(i) $\tilde{s}_\alpha \in \mathcal{E}(S)$ for $\alpha \in [0,1]$;

(ii) $\bigcap_{\alpha' < \alpha} \tilde{s}_{\alpha'} = \tilde{s}_\alpha$ for $\alpha \in (0,1]$,

where we put $\mathbf{N} := \{0,1,2,3,\cdots\}$ and

$$\mathcal{E}(S) := \left\{ A \mid A = \bigcup_{n=0}^{\infty} C_n, \ C_n \text{ are closed subsets of } S \ (n \in \mathbf{N}) \right\}.$$

Then we define

$$\mathcal{G}(S) := \left\{ \text{fuzzy sets } \tilde{s} \text{ on } S \ \mid \text{there exists } \{\tilde{s}_n\}_{n \in \mathbf{N}} \subset \mathcal{F}(S) \text{ satisfying } \tilde{s} = \bigvee_{n \in \mathbf{N}} \tilde{s}_n \right\},$$

where $\bigvee_{n \in \mathbf{N}} \tilde{s}_n(x) := \sup_{n \in \mathbf{N}} \tilde{s}_n(x), \ x \in S$.

We describe dynamic fuzzy systems in [3, 5]. $\mathbf{N}$ denotes a time space. Let a state space E be a complete metric space. Let $\tilde{q}$ be an upper semi-continuous fuzzy relation on $E \times E$ satisfying the following normality condition :

$$\sup_{x \in E} \tilde{q}(x,y) = 1 \ (y \in E) \quad \text{and} \quad \sup_{y \in E} \tilde{q}(x,y) = 1 \ (x \in E).$$

Let $\tilde{c}$ be a continuous fuzzy set on E, which denotes a fuzzy constraint on E. Define maps P and $Q : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$P\tilde{s}(x) := \sup_{y \in E} \{\tilde{q}(x,y) \wedge \tilde{s}(y)\} \quad x \in E \qquad \text{for } \tilde{s} \in \mathcal{G}(E), \tag{2.1}$$

$$Q\tilde{s}(x) := \tilde{c}(x) \wedge \sup_{y \in E} \{\tilde{q}(x,y) \wedge \tilde{s}(y)\} \quad x \in E \qquad \text{for } \tilde{s} \in \mathcal{G}(E), \tag{2.2}$$

where we write binary operations $a \wedge b := \min\{a,b\}$ for real numbers $a, b \in [0,1]$. We call P (Q) a fuzzy transition defined by the fuzzy relation $\tilde{q}$ (with a fuzzy constraint $\tilde{c}$ resp.). We define a partial order $\geq$ on $\mathcal{G}(E)$: For $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$

$$\tilde{s} \geq \tilde{r} \iff \tilde{s}(x) \geq \tilde{r}(x) \quad x \in E.$$

Definition ([5, Section 4]). A fuzzy set $\tilde{s}$ $(\in \mathcal{G}(E))$ is called P-superharmonic (Q-superharmonic) if

$$\tilde{s} \geq P\tilde{s} \quad (\tilde{s} \geq Q\tilde{s} \text{ resp.}).$$

The optimal value for Snell's problem is a Q-superharmonic fuzzy set ([5, Lemma 4.1]). For simplicity of the proofs in this paper, we deal with the case of $c = 1$ and we analyse P-superharmonic fuzzy sets. The results in Sections 2 and 3 still hold for Q-superharmonic fuzzy sets. We also define n-steps fuzzy transitions $P_n : \mathcal{G}(E) \mapsto \mathcal{G}(E),\ n \in \mathbf{N}$, by

$$P_n \tilde{s}(x) := \sup_{y \in E} \{\tilde{q}^n(x, y) \wedge \tilde{s}(y)\} \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E), \tag{2.3}$$

where, for $n \in \mathbf{N}$, we put

$$\tilde{q}^1(x, y) := \tilde{q}(x, y) \ \text{ and } \ \tilde{q}^{n+1}(x, y) := \sup_{z \in E} \{\tilde{q}^n(x, z) \wedge \tilde{q}(z, y)\} \quad x, y \in E.$$

We put a path space by $\Omega := \prod_{k=0}^{\infty} E$ and we write a path by $\omega = (\omega(0), \omega(1), \omega(2), \cdots) \in \Omega$. Define a map $X_n(\omega) := \omega(n)$ and a shift $\theta_n(\omega) := (\omega(n), \omega(n+1), \omega(n+2), \cdots)$ for $n \in \mathbf{N}$ and $\omega = (\omega(0), \omega(1), \omega(2), \cdots) \in \Omega$. Put σ-fields by $\mathcal{M}_n := \sigma(X_0, X_1, \cdots, X_n)$ [1] for $n \in \mathbf{N}$ and $\mathcal{M} := \sigma(\bigcup_{n \in \mathbf{N}} \mathcal{M}_n)$ [2]. We define a fuzzy expectation : For an initial state $x \in E$ and an $\mathcal{M}$-measurable fuzzy set $h \in \mathcal{F}(\Omega)$,

$$E_x(h) := \int\!\!\!\!-_{\{\omega \in \Omega : \omega(0) = x\}} h(\omega) \, \mathrm{d}\tilde{P}(\omega)$$

where $\tilde{P}$ is the following possibility measure :

$$\tilde{P}(\Lambda) := \sup_{\omega \in \Lambda} \bigwedge_{n \in \mathbf{N}} \tilde{q}(X_n \omega, X_{n+1} \omega) \qquad \Lambda \in \mathcal{M}$$

and $\int\!\!\!\!- \, \mathrm{d}\tilde{P}$ denotes Sugeno integral (Sugeno [7]).

We put

$$\mathcal{E} := \{A \mid A \in \mathcal{E}(E) \text{ and } E \setminus A \in \mathcal{E}(E)\}$$

and we call a map $\tau : \Omega \mapsto \mathbf{N} \cup \{\infty\}$ an $\mathcal{E}$-stopping time if

$$\{\tau = n\} \in \mathcal{M}_n \cap \mathcal{E}(\Omega) \qquad n \in \mathbf{N}.$$

For $A \in \mathcal{E}$, we put the first hitting time of A by

$$\sigma_A(\omega) := \inf\{n \in \mathbf{N} \mid n \geq 1, X_n(\omega) \in A\} \quad \omega \in \Omega,$$

where the infimum of the empty set is understood to be $+\infty$ ([5, Lemma 1.5]). Further a fuzzy transition $P_\tau : \mathcal{G}(E) \mapsto \mathcal{G}(E)$, τ is an $\mathcal{E}$-stopping time, is defined by

$$P_\tau \tilde{s} := E_.(\tilde{s}(X_\tau)) \quad \text{for } \tilde{s} \in \mathcal{G}(E), \tag{2.4}$$

[1] It denotes the smallest σ-field on Ω relative to which $X_0, X_1, \cdots, X_n$ are measurable.

[2] It denotes the smallest σ-field generated by $\bigcup_{n \in \mathbf{N}} \mathcal{M}_n$.

where $X_\tau := X_n$ on $\{\tau = n\}$, $n \in \mathbf{N} \cup \{\infty\}$. We note that (2.4) is an extension of (2.3) (see [5, (1.1)]) since

$$P_n \tilde{s}(x) := E_x(\tilde{s}(X_n)) \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E), n \in \mathbf{N}. \tag{2.5}$$

We define an operator $G := \bigvee_{n \in \mathbf{N}} P_n$ on $\mathcal{G}(E)$. Then we note that

$$PG1_{\{y\}}(x) = \bigvee_{n \geq 1} P_n 1_{\{y\}}(x) = \sup_{n \geq 1} \tilde{q}^n(x, y) \qquad x, y \in E.$$

Now, Snell's optimal stopping problem when $c = 1$ is described as follows : Let $x \in E$ be a initial state and let $\tilde{s} \in \mathcal{G}(E)$ denote a fuzzy goal. The problem is to find a finite $\mathcal{E}$-stopping time τ which maximizes τ-step fuzzy transitions to the fuzzy goal $\tilde{s}$:

$$P_\tau \tilde{s}(x), \quad x \in E.$$

From [5, Lemma 4.1(ii) and Theorem 4.1], the optimal value, $\tilde{v}(x) := \sup_\tau P_\tau \tilde{s}(x)$, satisfies

$$\tilde{v} = \tilde{s} \vee P(\tilde{v}). \tag{2.6}$$

However is characterization for $\tilde{v}$ is not sufficient because the solutions in (2.6) is not unique in general. Then we also have $\tilde{v} = G\tilde{s}$. In Section 4, we estimate the optimal value $\tilde{v}$, using this fact.

In [4], α-recurrent sets are defined as follows.

Definition ([4]). Let $\alpha \in (0, 1]$. A non-empty set $A \in \mathcal{E}(E)$ is called α-recurrent if $P_{\sigma_B^n} 1 \geq \alpha$ on A for all $n \in \mathbf{N}$ and all non-empty $B \in \mathcal{E}$ satisfying $B \subset A$, where σ_B^n means the nth hitting time of B :

$$\sigma_B^n := \begin{cases} 0 & \text{if } n = 0 \\ \sigma_B^{n-1} + \sigma_B \circ \theta_{\sigma_B^{n-1}} & \text{if } n \geq 1. \end{cases}$$

For $\beta \in [0, 1]$, we represent a constant fuzzy set $\beta 1_E$ by β simply. Then we have the following results regarding α-recurrent sets.

Lemma 2.1 ([4, Lemma 3.1]). *Let $\beta (\in [0, 1])$ be a constant fuzzy set. It holds that*

$$G(\tilde{s} \wedge \beta) = G\tilde{s} \wedge \beta \quad \text{and} \quad PG(\tilde{s} \wedge \beta) = PG\tilde{s} \wedge \beta \quad \text{for } \tilde{s} \in \mathcal{G}(E).$$

Especially,

$$GI_A(\beta) = G1_A \wedge \beta \quad \text{and} \quad PGI_A(\beta) = PG1_A \wedge \beta \quad \text{for } A \in \mathcal{E}(E),$$

where, for $A \in \mathcal{E}(E)$, an operator $I_A : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ is defined by

$$I_A \tilde{s} := \tilde{s} \wedge 1_A \qquad \tilde{s} \in \mathcal{G}(E).$$

LEMMA 2.2 ([4, Proposition 3.1]). *Let $\alpha \in (0,1]$ and let non-empty $A \in \mathcal{E}(E)$. Then the following statements are equivalent :*

(i) *A is α-recurrent;*

(ii) $PG1_B \geq \alpha \wedge 1_A$ *for non-empty $B \in \mathcal{E}(E)$ satisfying $B \subset A$;*

(iii) $PG1_{\{y\}} \geq \alpha \wedge 1_A$ *for $y \in A$.*

LEMMA 2.3 ([4, Theorem 3.1]). *It holds that*

$$\bigcup_{A \in \mathcal{E}(E)\,:\,\alpha-\text{recurrent sets}} A = \left\{ x \in E \mid \sup_{n \geq 1} \tilde{q}^n(x,x) \geq \alpha \right\} \qquad \text{for } \alpha \in (0,1].$$

3. P-SUPERHARMONIC FUZZY SETS

We show main theorems, which characterize P-superharmonic fuzzy sets on α-recurrent sets. The classical Harris chain in probability theory has similar properties (Revuz [8, Proposition III-2.10]).

THEOREM 3.1. *Let $\alpha \in (0,1]$ and let $A(\in \mathcal{E}(E))$ be α-recurrent. Then, for any P-superharmonic fuzzy set $\tilde{s}$, we have*

$$\tilde{s} \begin{cases} \geq \alpha \quad \text{on } A & \text{if } \{\tilde{s} > \alpha\} \cap A \neq \emptyset, \\ \text{is constant on } A & \text{if } \{\tilde{s} > \alpha\} \cap A = \emptyset. \end{cases} \tag{3.1}$$

PROOF. Let $\tilde{s}$ be P-superharmonic and define a positive constant $\beta := \sup_{x \in A} \tilde{s}(x)$. Given a positive number ϵ, we put a non-empty set $B := \{\tilde{s} > \beta - \epsilon\} \cap A$. We check $B \in \mathcal{E}(E)$. Since $\tilde{s} \in \mathcal{G}(E)$, there exists a sequence $\{\tilde{s}_n\}_{n \in \mathbf{N}} \subset \mathcal{F}(E)$ such that $\tilde{s}_n \uparrow \tilde{s}$ $(n \to \infty)$. By [5, Lemma1.1(i)], we have

$$B = \bigcup_{n \in \mathbf{N}} \{\tilde{s}_n > \beta - \epsilon\} \cap A = \bigcup_{n \in \mathbf{N}} \bigcup_{m=1,2,\cdots} \left\{ \tilde{s}_n \geq \beta - \epsilon + \frac{1}{m} \right\} \cap A \in \mathcal{E}(E).$$

So $B \in \mathcal{E}(E)$. Using Lemmas 2.1 and 2.2, we obtain

$$\tilde{s} \geq PG\tilde{s} \geq PGI_B\tilde{s} \geq PGI_B(\beta - \epsilon) \geq PG1_B \wedge (\beta - \epsilon) \geq \alpha \wedge (\beta - \epsilon) \quad \text{on } A.$$

Therefore, we get $\tilde{s} \geq \alpha \wedge (\beta - \epsilon)$ on A for all $\epsilon > 0$. So, $\tilde{s} \geq \alpha \wedge \beta$ on A. This implies (3.1) and we establish this theorem. ∎

We give a kind of sufficient conditions for Theorem 3.1.

Corollary 3.1. *Let $\alpha \in (0,1]$. If (3.1) holds for $A = E$ and any P-superharmonic $\tilde{s}$, then E is α-recurrent.*

Proof. Let $y \in E$. We take $\tilde{s} = G1_{\{y\}}$ in (3.1). Then $G1_{\{y\}} \geq \alpha \wedge \sup_{x\in E} G1_{\{y\}}(x) = \alpha$. Therefore we obtain $PG1_{\{y\}} \geq P(\alpha) = \alpha$. By Lemma 2.2, E is α-recurrent. ∎

Next we extend Theorem 3.1 to be applicable for calculation of optimal value for Snell's problem. We introduce the following notations. Let $x \in E$, $A \in \mathcal{E}(E)$ and $\alpha \in (0,1]$. We write $x \rightarrow_\alpha A$ if $G1_A(x) \geq \alpha$. This means that the possibility to transit in some time from x to A is greater than α. Especially if $A = \{y\}$, we write $x \rightarrow_\alpha y$ simply. We define a set

$$T_\alpha(A) := \{x \in E \mid x \rightarrow_\alpha A\} = \{G1_A \geq \alpha\} \quad \text{for } \alpha \in (0,1] \text{ and } A \in \mathcal{E}(E).$$

Then $T_\alpha(A) \supset A$. The following theorems are useful to calculate P-superharmonic fuzzy sets.

Theorem 3.2. *Let $\alpha \in (0,1]$ and let $A(\in \mathcal{E}(E))$ be α-recurrent. Then, for P-superharmonic fuzzy set $\tilde{s}$, we have*

$$\tilde{s} \geq \alpha \wedge \sup_{x\in A} \tilde{s}(x) \quad \text{on } T_\alpha(A). \tag{3.2}$$

Proof. From Theorem 3.1

$$\tilde{s} \geq \alpha \wedge \sup_{x\in A} \tilde{s}(x) \quad \text{on } A. \tag{3.3}$$

Put $B := \{PG1_A \geq \alpha\}$. From Lemma 2.1 and (3.3) we have

$$\tilde{s} \geq PGI_A\tilde{s} \geq PGI_A(\alpha \wedge \sup_{x\in A} \tilde{s}(x)) = PG1_A \wedge (\alpha \wedge \sup_{x\in A} \tilde{s}(x)) = \alpha \wedge \sup_{x\in A} \tilde{s}(x) \quad \text{on } B. \tag{3.4}$$

Since $T_\alpha(A) = A \cup B$, (3.3) and (3.4) complete the proof of this theorem. ∎

Theorem 3.3. *Let $\alpha \in (0,1]$ and let $A(\in \mathcal{E}(E))$ be α-recurrent. Then :*

(i) *for $\tilde{s} \in \mathcal{G}(E)$, we have*

$$GI_A\tilde{s} \begin{cases} = \sup_{x\in A} \tilde{s}(x) & \text{on } A \quad \text{if } \sup_{x\in A} \tilde{s}(x) < \alpha, \\ \geq \alpha \quad \text{on } A & \qquad\quad \text{if } \sup_{x\in A} \tilde{s}(x) \geq \alpha; \end{cases} \tag{3.5}$$

(ii) *for $\tilde{s} \in \mathcal{G}(E)$, we have*

$$GI_A\tilde{s} \geq \alpha \wedge \sup_{x\in A} \tilde{s}(x) \quad \text{on } T_\alpha(A). \tag{3.6}$$

Proof. (i) By Lemma 2.1, $GI_A\tilde{s}\wedge\alpha = G(I_A\tilde{s}\wedge\alpha)$. It is P-superharmonic from [4, Lemma 2.4]. From Theorem 3.1, $GI_A\tilde{s} \wedge \alpha$ is constant on A. Therefore

$$GI_A\tilde{s} \wedge \alpha = \sup_{x\in A} GI_A\tilde{s}(x) \wedge \alpha \quad \text{on } A.$$

Then we have

$$\sup_{x \in A} GI_A \tilde{s}(x) = \sup_{x \in A} \tilde{s}(x)$$

since

$$\sup_{x \in A} \tilde{s}(x) \leq \sup_{x \in A} GI_A \tilde{s}(x) \leq \sup_{x \in A} G(\sup_{y \in A} \tilde{s}(y))(x) \leq \sup_{y \in A} \tilde{s}(y).$$

Thus we obtain

$$GI_A \tilde{s} \wedge \alpha = \sup_{x \in A} \tilde{s}(x) \wedge \alpha \quad \text{on } A.$$

and so (i) holds. (ii) is trivial from (i) and Theorem 3.2. ∎

4. A NUMERICAL EXAMPLE

In this section, we consider a numerical example with a one-dimensional state space $E = \mathbf{R}$, where $\mathbf{R}$ is the set of all real numbers. The purpose is to calculate the optimal value $\tilde{v} = G\tilde{s}$ for Snell's problem, using the results in Section 3.

EXAMPLE 4.1. We give a fuzzy goal $\tilde{s}$ and a fuzzy relation $\tilde{q}$ by

$$\tilde{s}(x) = (0.8 - 0.1|1 - x|) \vee 0, \quad x \in \mathbf{R}, \tag{4.1}$$

$$\tilde{q}(x, y) = (1 - |y - x^3|) \vee 0, \quad x, y \in \mathbf{R}. \tag{4.2}$$

(4.2) is shown in Yoshida [4, Example 6.1]. By [4, Example 6.1], there exist three maximal α-recurrent sets $F_{\alpha,0}, F_{\alpha,1}, F_{\alpha,2}$, which are closed intervals such that

$$\{x \in \mathbf{R} \mid \tilde{q}(x, x) \geq \alpha\} = F_{\alpha,0} \cup F_{\alpha,1} \cup F_{\alpha,2} \quad \text{if } \alpha > \alpha_1,$$

where $\alpha_1 := 1 - 1/\sqrt{3} + 3^{3/2} \approx 0.6151$. Now we calculate the optimal value $\tilde{v} = G\tilde{s}$. First, we note

$$\sup_{x \in \mathbf{R}} \tilde{s}(x) = 0.8.$$

Solving $\tilde{q}(x, x) = \tilde{s}(x)$, we have the solutions $x = x_1, x_2, x_3, x_4, x_5, x_6$ such that

$$x_1 \approx -1.1651, \quad x_2 \approx -0.6749, \quad x_3 \approx -0.4098,$$

$$x_4 \approx 0.2964, \quad x_5 \approx 0.8687, \quad x_6 \approx 1.0916.$$

We put

$$\alpha_2 := \tilde{s}(x_4) \approx 0.7296.$$

Figure 4.1 shows them.

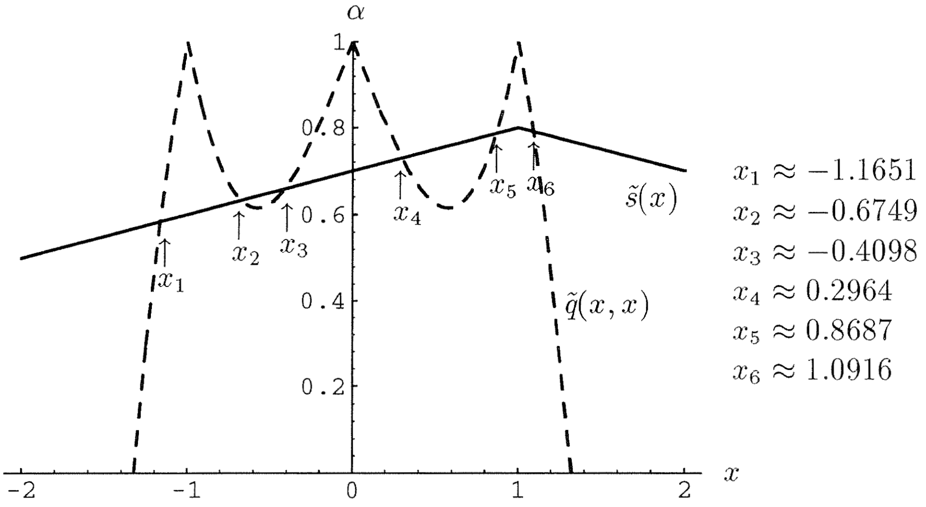

Fig. 4.1. The fuzzy set $\tilde{q}(x,x)$ and the fuzzy goal $\tilde{s}(x)$.

Next, solving $\tilde{q}(x,x) = \alpha_2$, we have the solutions $x = -x_7, -x_4, x_4, x_7$ such that

$$x_7 \approx 1.1147.$$

Also solving $\tilde{q}(x,x) = 0.8$, we have the solutions $x = -x_{10}, -x_9, -x_8, x_8, x_9, x_{10}$ such that

$$x_8 \approx 0.2091, \quad x_9 \approx 0.8789, \quad x_{10} \approx 1.0880.$$

Figure 4.2 shows them. By Theorem 3.3,

$$\tilde{v} = \sup_{x \in [x_9, x_{10}]} \tilde{s}(x) = 0.8 \quad \text{on } F_{0.8,2} = [x_9, x_{10}].$$

From [4, Figure 6.2], we have

$$T_{\alpha_2}(F_{\alpha_2,0}) \subset F_{\alpha_2,0}, \quad T_{\alpha_2}(F_{\alpha_2,1}) = [-x_7, x_7], \quad T_{\alpha_2}(F_{\alpha_2,2}) \subset F_{\alpha_2,2}.$$

Therefore, by Theorem 3.3,

$$\tilde{v} = \sup_{x \in [-x_4, x_4]} \tilde{s}(x) = \sup_{x \in F_{\alpha_2,2}} \tilde{s}(x) = \alpha_2 \approx 0.7296 \quad \text{on } [-x_7, x_4].$$

Every starting point in (x_4, x_9) transits to $F_{\alpha_2,1} = [-x_4, x_4]$ monotonically and every point in $(-\infty, -x_7)$ transits to $-\infty$ and every point in $(x_{10}, +\infty)$ transits to $+\infty$ monotonically. Therefore we obtain the optimal value :

$$\tilde{v}(x) = \begin{cases} \tilde{s}(x) & \text{for } x \in (-\infty, x_1) \approx (-\infty, -1.1651) \\ \tilde{q}(x,x) & \text{for } x \in [x_1, -x_7) \approx [-1.1651, -1.1147) \\ \alpha_2 \approx 0.7296 & \text{for } x \in [-x_7, x_4) \approx [-1.1147, 0.2964) \\ \tilde{s}(x) & \text{for } x \in [x_4, x_5) \approx [0.2964, 0.8687) \\ \tilde{q}(x,x) & \text{for } x \in [x_5, x_9) \approx [0.8687, 0.8789) \\ 0.8 & \text{for } x \in [x_9, x_{10}) \approx [0.8789, 1.0880) \\ \tilde{q}(x,x) & \text{for } x \in [x_{10}, x_6) \approx [1.0880, 1.0916) \\ \tilde{s}(x) & \text{for } x \in [x_6, +\infty) \approx [1.0916, +\infty). \end{cases}$$

The optimal value $\tilde{v}(x)$ is shown in Figure 4.2.

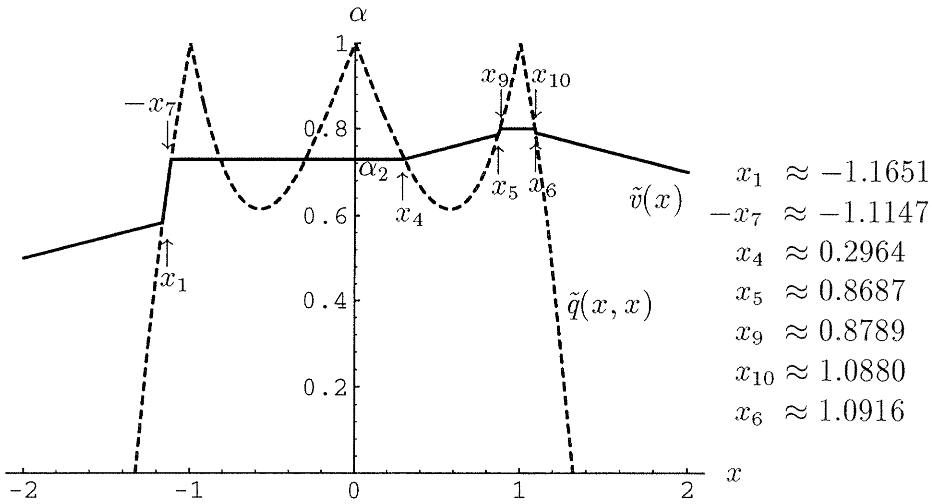

Fig. 4.2. The fuzzy set $\tilde{q}(x,x)$ and the optimal value $\tilde{v}(x)$.

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