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In [2] M. Hanazawa showed a characterization of axiom schema by each of which a system of classical logic is obtained from any system of intuitionistic logic. He defined a three valued logic and he showed that *if and only if a formula α is a tautology in usual sense (or in the usual two-valued logic) and is not incidentally true in the three-valued logic, the classical propositional calculus is obtained from the intuitionistic propositional calculus by adjoining α as an axiom schema.* In [3] V. A. Jankov gave a good perspective to this fact by associating a characteristic formula to each finite algebra.

Hanazawa proved the fact concerning propositional calculus with logical connectives conjunction, disjunction, implication and negation. We cannot apply his proof to implicational fragment, since negation is essential in his proof. In this note, we prove the same characterization for implication fragment.

We consider implicational formulas. In [2] Hanazawa defined a three valued logic with the following truth-table.

			α	
$\alpha \to \beta$		t	u	f
	t	t	u	f
β	u	t	t	f
	f	t	t	t

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where t,f and u stand for 'true', 'false' and 'unknown', respectively. A formula is a *t-formula* iff the value of the formula is t for all valuation with respect to the above truth-table. For example, let τ be a valuation such that $\tau(p) = u$ and $\tau(q) = f$. Then we have $\tau(p \to q) = f$ and $\tau((p \to q) \to p) = t$ and $\tau(((p \to q) \to p) \to p) = u$. Therefore Peirce's formula $((p \to q) \to p) \to p$ is not a t-formula. By I and C we denote the set of implicational formulas provable in intuitionistic logic and in classical logic respectively. We denote by $I + \alpha$ the set of implicational formulas that is provable in intuitionistic logic adjoined α as an axiom scheme. Two formulas α and β are equivalent iff both $\alpha \to \beta$ and $\beta \to \alpha$ is provable in intuitionistic logic.

In [2] Hanazawa proved the characterization by reducing to the case where only one propositional variable occurs. We cannot apply his proof, since no implicational formula with only one propositional variable yield classical logic. We need at least two variables to obtain an axiom scheme that yields classical logic. It is known that the number of nonequivalent formulas with n variables is finite in implicational intuitionistic logic [1]. In [4] A. Urquhart showed an estimation of the number of nonequivalent formulas. Katsumi Sasaki wrote a computer program that computes such nonequivalent formulas. He obtained the list of nonequivalent formulas for the case $n = 2$ and showed the following lemma.

Lemma 1 *Every implicational formulas with variables p and q is equivalent to one of the following 14 formulas:*

$$\begin{gathered}
p, \quad q, \\
p \to p, \\
p \to q, q \to p, \\
(p \to q) \to p, \quad (q \to p) \to q, \\
(p \to q) \to q, \quad (q \to p) \to p, \\
((p \to q) \to p) \to p, \quad ((q \to p) \to q) \to q, \\
((p \to q) \to q) \to p, \quad ((q \to p) \to p) \to q, \\
(p \to q) \to (q \to p) \to p.
\end{gathered}$$

Proof. Let $\mathcal{F}$ be the set of above 14 formulas. By direct calculation, we see that the formulas in $\mathcal{F}$ are not equivalent. Similarly, by checking the equivalence of $\delta_i \to \delta_j$ and δ_k for δ_i, δ_j and δ_k in $\mathcal{F}$, we see that $\delta_i \to \delta_j$ is equivalent to some δ_k in $\mathcal{F}$. Let γ be a formula with two variables p and

q. By induction on the number m of implication in γ, we prove that γ is equivalent to some δ in $\mathcal{F}$. For the case $m = 0$, we have $\gamma = p$ or $\gamma = q$. Then we can put $\delta = \gamma$. Hence Lemma holds trivially. Let $\gamma = \gamma_1 \to \gamma_2$. By induction hypothesis, each γ_1 and γ_2 is equivalent to some formulas δ_1 and δ_2 in $\mathcal{F}$ respectively. Therefore γ is equivalent to $\delta_1 \to \delta_2$. Since $\delta_1 \to \delta_2$ is equivalent to some δ in $\mathcal{F}$, γ is equivalent to δ. ∎

Lemma 2 *Let β be a tautology with two variables. If β is not a t-formula, then β is equivalent to Peirce's formula.*

Proof. By Lemma 1, β is equivalent to some formula δ in $\mathcal{F}$. Since β is a tautology and is not a t-formula, δ is a tautology and is not a t-formula. By direct calculation of valuation of each formula in $\mathcal{F}$, we see that $((p \to q) \to p) \to p$ and $((q \to p) \to q) \to q$ are the unique two formulas in $\mathcal{F}$ that is a tautology and is not t-formula. Therefore $\delta = ((p \to q) \to p) \to p$ or $\delta = ((q \to p) \to q) \to q$. Hence β is equivalent to Peirce's formula. ∎

Theorem 1 *If a tautology α is not a t-formula, then there is a substitution instance β of α such that β contains only two variables and that β is equivalent to Peirce's formula.*

Proof. Let $p_1, \cdots, p_n$ be the propositional variables in α and ρ be a valuation such that $\rho(\alpha) \neq t$. We construct β by $\beta = \alpha[p_1 := \beta_1, \cdots, p_n := \beta_n]$ where

$$\beta_i = \begin{cases} (p \to q) \to p, & if \quad \rho(p_i) = t \\ p, & if \quad \rho(p_i) = u \\ q, & if \quad \rho(p_i) = f \end{cases}$$

and p and q are new variables. Since α is a tautology, β is a tautology. Consider a valuation τ such that $\tau(p) = u$ and $\tau(q) = f$. Then we have $\rho(p_i) = \tau(\beta_i)$ $(i = 1, \cdots, n)$. Thus we have $\tau(\beta) = \tau(\alpha[p_1 := \beta_1, p_2 := \beta_2, \cdots, p_n := \beta_n]) = \rho(\alpha) \neq t$. Therefore β is not a t-formula. By Lemma 2 β is equivalent to Peirce's formula. ∎

Theorem 2 *$I + \alpha = C$ iff α is a tautology and α is not a t-formula.*

Proof. (Only-if-part) Assume that $I + \alpha = C$. First we see that α is a tautology, since α is provable in C. Since $C = I + \alpha$, every tautology is provable in intuitionistic logic using α as axiom scheme. Therefore so is Peirce's formula. Note that every intuitionistic formula is a t-formula and being a t-formula is preserved by substitution and modus ponens. Therefore, if α is a t-formula, then Peirce's formula is a t-formula. A contradiction. Thus α is not a t-formula.

(If-part) Let α be a tautology which is not a t-formula. By Theorem 1 α is equivalent to Peirce's formula. Therefore $I + \alpha = C$. ∎

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