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Hirokawa, Sachio
Computer Science Laboratory, Faculty of Science, Kyushu University

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Sachio Hirokawa

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Research Institute of Fundamental Information Science
Kyushu University 33
Fukuoka 812, Japan

E-mail: hirokawa@rc.kyushu-u.ac.jp Phone: 092-771-4161 ex. 468

INFINITENESS OF PROOF(α) IS POLYNOMIAL-SPACE COMPLETE

SACHIO HIROKAWA *
Computer Science Laboratory,
Faculty of Science, Kyushu University,
e-mail: hirokawa@ec.kyushu-u.ac.jp

Abstract

The infiniteness problem is investigated for the set $proof(\alpha)$ of closed λ-terms in β-normal form which has α as their types. The set is identical to the set of normal form proofs for α in the natural deduction system for implicational fragment of intuitionistic logic.

It is shown that the infiniteness is determined by checking λ-terms with the depth at most $2|\alpha|^2$. Thus the problem is solved in polynomial-space. The bound is obtained by an estimation of the length of an irredundant chain of sequents in type assignment system in sequent calculus formulation. Then the non-emptiness problem of $proof(\alpha)$, which is identical to the provability of α, is deduced to the problem of infiniteness problem by a transformation of types. Since the provability is polynomial- space complete (Statman, Theoret. Comput. Sci. 9 (1979) 97–105), the infiniteness problem is polynomial-space complete.

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1 Introduction

Our interest is in the structure of the set $proof(\alpha)$ of closed λ-terms in β-normal form which has α as their types. According to 'terms-as-proofs' correspondence [5], the set is identical to the set of normal form proofs for α in the natural deduction system NJ for implicational fragment of intuitionistic logic.

When α is in simple form, there are simple description of the set. For example, when the degree of α is at most 2 and $\alpha = \alpha_1 \to \alpha_2 \to \cdots \to \alpha_n \to a$, where $\alpha_i = a^i_1 \to a^i_2 \to \cdots \to a^i_{m_i}$ with type variables $a^i_1, \cdots, a^i_{m_i}$, we can describe the set as a context-free language [11, 13, 14]. If we extend this description to general types, we obtain a context-free-like description with infinitely many symbols and rules [11]. It seems impossible to describe $proof(\alpha)$ as a context-free language. In fact, it is known that the complexity of non-emptiness of $proof(\alpha)$, which is equivalent to the provability of α in NJ, is polynomial-space complete [10]. In this paper we show that the problem of infiniteness of $proof(\alpha)$ is polynomial-space complete. Thus the problems of the infiniteness and the problem of non-emptiness have the same complexity.

Ben-Yelles [1] showed that $proof(\alpha)$ is infinite iff there is a λ-term M in $proof(\alpha)$ whose type assignment contains a repetition. In [4], we showed a bound of the depth to test the infiniteness. Thus the problem of infiniteness of the set is decidable. However, the bound was $\mid \alpha \mid 2^{|\alpha|+1}$, where $\mid \alpha \mid$ is the size of α which is defined as the total number of occurrences of type variables and arrows in α. In the present paper, we show that the infiniteness is determined within the depth of $2 \mid \alpha \mid^2$. That depth is obtained by an analysis of type assignment figures in sequent calculus formulation. We show that the length of the chain of sequents that does not contain repetition is at most $\mid \alpha^2 \mid$. Therefore we can test the infiniteness in polynomial-space.

The polynomial-space hardness is proved in Section 3 by a deduction of the non-emptiness problem to the infiniteness problem. The transformation is defined by $F(\alpha) = (((b \to \alpha) \to \alpha) \to b) \to b$ where b is a type variable which does not occur in α. We constructed the transformation from an example $((((a \to b) \to a) \to a) \to b) \to b$ by Mints. [1] It has infinitely many

[1]Mints made this formula as a counter example to a problem by Komori [7]. This formula is essential in intuitionistic logic in the sense that it is not a non-trivial substitution

normal form proofs. Our transformation is a generalization of this example. The transformation keeps the provability, i.e., $\vdash_{NJ} \alpha$ iff $\vdash_{NJ} F(\alpha)$. Moreover, when $F(\alpha)$ is provable then $F(\alpha)$ has infinitely many normal form proofs. This infiniteness does not depend on the number of proofs for α as long as α is provable. Thus the decision problem of α is deduced to the infiniteness problem of $proof(F(\alpha))$.

2 Sequent calculus of the type assignment system to lambda-terms

We assume the familiarity to the basic notions in λ-calculus [3] and in proof theory [9].

The set of types are constructed from type variables $a, b, \cdots$ by combining two types α and β with an arrow $\rightarrow$ obtaining a type $(\alpha \rightarrow \beta)$. A type assignment formula (TA-formula) is an expression $M : \alpha$ with an arbitrary λ-term M and a type α. M is the subject of the TA-formula and α is the predicate of the TA-formula. The type assignment system N_λ is defined by the following two inference rules.

$$\frac{M : \alpha \rightarrow \beta \quad N : \alpha}{MN : \beta} (\rightarrow E) \qquad \frac{\begin{array}{c}[x : \alpha] \\ \vdots \\ M : \beta\end{array}}{\lambda x.M : \alpha \rightarrow \beta} (\rightarrow I)$$

We assume that the subject of a TA-figure in each leaf is a variable.

A sequent is an expression $\Gamma \vdash M : \alpha$ where $M : \alpha$ is a TA-formula and Γ is a set of TA-formulas $x_1 : \alpha_1, \cdots, x_n : \alpha_n$ whose subjects are distinct variables.

The sequent calculus of the type assignment system is defined as follows.

Definition 1 $[L_\lambda]$

Axiom:

$$x : \alpha, \Gamma \vdash x : \alpha$$

---

instance of other provable formulas. Komori called such formulas minimal in intuitionistic logic. BCK-minimal formulas are defined similarly to BCK-logic. In [8], a bijection is shown between the set of $\beta\eta$-normal form BCK-proofs for BCK-formula α and the set of BCK-minimal formulas which generates α as a substitution instance.

Inference rules:

$$\frac{x:\alpha,\Gamma\vdash M:\beta}{\Gamma\vdash \lambda x.M:\alpha\to\beta}\ (\to right)$$

$$\frac{x:\alpha\to\beta,\Gamma\vdash N:\alpha \qquad y:\beta,x:\alpha\to\beta,\Gamma\vdash L:\gamma}{x:\alpha\to\beta,\Gamma\vdash L[y:=xN]:\gamma}\ (\to left)$$

The difference of our formulation to that in [2] is that the set of assumption does not decrease when we go up through an inference rule. The system L_λ is a representation of Kleene's G_3 (p. 481 of [6]) in terms of type assignment system in the following sense. If we erase all the subjects and colons from a TA-figure in L_λ, the result becomes a cut-free proof figure in Gentzen's sequent calculus $LJ'_\to$ for implicational fragment of intuitionistic logic. Conversely, given a cut-free proof figure $\mathcal{P}$ in $LJ'_\to$, we can construct λ-terms and a TA-figure $\mathcal{P}'$ whose predicate part coincides $\mathcal{P}$.

Remark 1 If $\Gamma\vdash M:\alpha$ in L_λ, then M is in β-normal form.

Remark 2 If we impose on the $(\to left)$ the restriction that y is a free variable in L, then we obtain the system L^*_λ. We can show the equivalence of L_λ and L^*_λ, i.e., $\Gamma\vdash M:\alpha\ in\ L^*_\lambda$ iff $\Gamma\vdash M:\alpha\ in\ L_\lambda$. If-part is trivial. Note that if $y:\beta,x:\alpha\to,\Gamma\vdash L:\gamma$ in L_λ and $y\notin FV(L)$ then $x:\alpha\to\beta,\Gamma\vdash L:\gamma$ in L_λ. Therefore we can prove the only-if-part by induction on the structure of the TA-figure for $\Gamma\vdash M:\alpha$ in L_λ. Thus we can assume the restriction on $(\to left)$ in the sequel of the paper. With this restriction, the size of λ-term varies according to the the size of TA-figure.

Using the translation of sequent calculus to natural deduction [15], we can see that both N_λ and L_λ are equivalent representations of Curry-Howard isomorphism.

Theorem 1 Let M be a closed λ-term in β-normal form and α be a type. Then $\vdash M:\alpha$ in N_λ iff $\vdash M:\alpha$ in L_λ.

∎

3 A bound of depth for infiniteness-test

By $proof(\alpha)$ we denote the set of closed λ-terms M such that $\vdash M : \alpha$ in L_λ. The number of λ-terms in the set is denoted by $\sharp proof(\alpha)$.

Definition 2 A **chain** in a TA-figure $\mathcal{P}$ in L_α is a sequence

$$\Gamma_1 \vdash M_1 : \alpha_1, \cdots, \Gamma_m \vdash M_m : \alpha_m$$

of occurrences of sequents such that $\Gamma_i \vdash M_i : \alpha_i$ is an upper sequent of $\Gamma_{i+1} \vdash M_{i+1} : \alpha_{i+1}$ for $i = 1, \cdots, m-1$. The length of the chain is m. A thread is a chain such that $\Gamma_1 \vdash M_1 : \alpha_1$ is an axiom and $\Gamma_m \vdash M_m : \alpha_m$ is the end-sequent. A chain is **irredundant** iff $(pred(\Gamma_i), \alpha_i) \neq (pred(\Gamma_j), \alpha_j)$ for $i \neq j$, where $pred(\Gamma) = \{\xi \mid x : \xi \in \Gamma\}$.

Definition 3 The size $\mid \alpha \mid$ of a type α is defined as follows. $\mid a \mid = 1$ for type variable a. $\mid \alpha \rightarrow \beta = \mid \alpha \mid + \mid \beta \mid + 1$.

Lemma 1 Let $\Gamma_1 \vdash M_1 : \alpha_1, \cdots, \Gamma_m \vdash M_m : \alpha_m$ be an irredundant chain in a TA-figure for $\vdash M : \alpha$ in L_λ. Then we have

$$m \leq \mid \alpha \mid^2.$$

Proof. By the sub-formula property, each α_i is a sub-type of α. Note that the number of sub-types is at most $\mid \alpha \mid$. Thus the number of distinct α_i's is at most $\mid \alpha \mid$. On the other hand we have $pred(\Gamma_1) \subseteq \cdots \subseteq pred(\Gamma_i) \subseteq pred(\Gamma_{i+1}) \subseteq \cdots \subseteq pred(\Gamma_m)$. By the sub-formula property, $pred(\Gamma_i)$ is a subset of all sub-types of α. Thus the number of distinct $pred(\alpha_i)$'s is at most $\mid \alpha \mid$. Thus the possibility of distinct pairs of $(pred(\Gamma_i), \alpha)$ is at most $\mid \alpha \mid^2$. Thus we have $m \leq \mid \alpha \mid^2$. ∎

Both in N_λ and L_λ, a TA-figure is a tree. A thread in the tree is a sequence of nodes from a leaf to the root.

Definition 4 Let $\mathcal{P}$ be a TA-figure in N_λ or L_λ. The depth of $\mathcal{P}$, denoted by $\mid \mathcal{P} \mid$, is the maximal length of threads in $\mathcal{P}$.

Theorem 2 For any type α, $\sharp proof(\alpha) = \infty$ iff there is a closed λ-term M in β-normal form and a TA-figure $\mathcal{P}$ for $M : \alpha$ in L_λ such that

(1) $\mid \mathcal{P} \mid \leq 2 \mid \alpha \mid^2$ and

(2) there are two distinct occurrences of sequents $\Gamma \vdash R : \xi$ and $\Delta \vdash S : \xi$ in the same thread in $\mathcal{P}$ and $pred(\Gamma) = pred(\Delta)$.

Outline of proof. (If-part) Let $\mathcal{P}$ be the TA-figure which satisfies the conditions. From this TA-figure we construct a closed λ-germ M^* such that $size(M) < size(M^*)$ and $M^* \in proof(\alpha)$. Here the size of a λ-term M is the number of occurrences of variables and λ's in M. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the sub-figures for $\Gamma \vdash R : \xi$ and $\Delta \vdash S : \xi$ respectively.

$$\mathcal{P}\left\{\begin{array}{c} \left.\begin{array}{c} \mathcal{P}_1\left\{\begin{array}{c} \vdots \\ \Gamma \vdash R : \xi \end{array}\right. \\ \vdots \\ \Delta \vdash S : \xi \end{array}\right\}\mathcal{P}_2 \\ \vdots \\ M : \alpha \end{array}\right. \quad \Rightarrow \quad \mathcal{P}^*\left\{\begin{array}{c} \mathcal{P}_2\left\{\begin{array}{c} \vdots \\ \Delta \vdash R : \xi \end{array}\right. \\ \vdots \\ \Delta^* \vdash S^* : \xi \\ \vdots \\ \vdash M^* : \alpha \end{array}\right.$$

Firstly, remove the sub-figure $\mathcal{P}_1$ from $\mathcal{P}$ and replace all the occurence of R as sub-terms which has the origin in the occurrence of R in $\Gamma \vdash R : \xi$ by S. Since $pred(\Gamma) = pred(\Delta)$, we can make a correspondence from the subjects of Γ, which are variables, to the subjects of Δ. According to this correspondence, rename each variable in Γ by the corresponding variable in Δ. At the root of this figure we obtain a closed λ-term M^* in β-normal form which is similar to M except that the sub-term R is replace by S and that in the construction in M around R the name of variables are replaced. Finally put $\mathcal{P}_2$ at the original position of $\mathcal{P}_1$. Thus we obtain a TA-figure $\mathcal{P}^*$ in L_λ for $\vdash M^* : \alpha$. Continuing this rewriting, we can construct infinitely many λ-terms with type α. Thus $\sharp proof(\alpha) = \infty$.

(Only-if-part) If $\sharp proof(\alpha) = \infty$ then for any integer d there is a λ-term M in $proof(\alpha)$ and a TA-figure $\mathcal{P}$ in L_α for $\vdash M : \alpha$ such that $\mid \mathcal{P} \mid > d$. Let $\mathcal{P}$ be such a TA-figure for $\vdash M : \alpha$ for $d = 2\mid \alpha \mid^2$. By Lemma 1, the length of an irredundant chain is at most $\mid \alpha \mid^2$. Since $\mid \mathcal{P} \mid > 2\mid \alpha \mid^2$, there is a thread which contains three distinct occurrences of sequents $\Gamma \vdash Q : \xi$, $\Delta \vdash R : \xi$ and $\Sigma \vdash S : \xi$ such that $pred(\Gamma) = pred(\Delta) = pred(\Sigma)$. Select one of the longest thread among such threads.

$$
\left.\begin{array}{c}
\left.\begin{array}{c} \vdots \\ \Gamma \vdash Q : \xi \end{array}\right\} \mathcal{P}_1 \\
\vdots \\
\Delta \vdash R : \xi
\end{array}\right\} \mathcal{P}_2
$$

$$
\begin{array}{c}
\vdots \\
\Sigma \vdash S : \xi \\
\vdots \\
\vdash M : \alpha
\end{array}
$$

Replace $\mathcal{P}_2$ by $\mathcal{P}_1$ and rename the variables in Δ. Then we obtain a λ-term M' and a TA-figure $\mathcal{P}'$ for $\vdash M' : \alpha$ such that $\mid \mathcal{P}' \mid < \mid \mathcal{P} \mid$. We can continue this shrinking while $\mid \mathcal{P}' \mid > 2 \mid \alpha \mid^2$. Finally we obtain a closed λ-term M^* and a TA-figure P^* for $\vdash M^* : \alpha$ such that $\mid \mathcal{P}' \mid \leq 2 \mid \alpha \mid^2$. ■

Theorem 3 Given a type α, we can decide the infiniteness of $proof(\alpha)$ in polynomial-space with respect to $\mid \alpha \mid$.

Proof. By Theorem 2, we can decide the infiniteness of $proof(\alpha)$ by searching a λ-term in $proof(\alpha)$ and a TA-figure $\mathcal{P}$ with the depth $\leq 2 \mid \alpha \mid^2$ which contains the repetition in the thread. Therefore, it can be determined in polynomial-space. ■

4 A transformation and polynomial-space completeness

According to Theorem 3 the infiniteness of $proof(\alpha)$ is decidable in polynomial-space. Statman proved that the decidability of α, which is equivalent to the non-emptiness of $proof(\alpha)$, is polynomial-space complete [10]. In this section we prove that the problem of infiniteness is polynomial-space complete. The proof is by a deduction of the non-emptiness problem into the infiniteness problem. We use N_λ for this analysis.

Definition 5 Let α be a type and b We define $F(\alpha)$ by

Theorem 4 For a type α, we put $F(\alpha) = (((b \to \alpha) \to \alpha) \to b) \to b$ where b is a type variable which does not occur in α. Then the following (a),(b) and (c) are equivalent.

(a) $\sharp proof(\alpha) > 0$

(b) $\sharp proof(F(\alpha)) > 0$

(c) $\sharp proof(F(\alpha)) = \infty$

Proof. **(a) $\Rightarrow$ (b).** Consider the following TA-figure for $z : \alpha \vdash \lambda x.x(\lambda y.y\ (x(\lambda u.z))) : F(\alpha)$.

$$\dfrac{x : ((b \to \alpha) \to \alpha) \to b^3 \qquad \dfrac{\dfrac{y : b \to \alpha^2 \qquad \dfrac{x : ((b \to \alpha) \to \alpha) \to b^3 \qquad \dfrac{z : \alpha}{\lambda u.z : (b \to \alpha) \to \alpha}\,1}{x(\lambda u.z) : b}}{y(x(\lambda u.z)) : \alpha}}{\lambda y.y(x(\lambda u.z)) : (b \to \alpha) \to \alpha}\,2}{\dfrac{x(\lambda y.y(x(\lambda u.z))) : b}{\lambda x.x(\lambda y.y(x(\lambda u.z))) : (((b \to \alpha) \to \alpha) \to b) \to b}\,3}$$

If $\sharp proof(\alpha) > 0$ then there is a closed λ-term M such that $\vdash M : \alpha$. Replace the assumption $z : \alpha$ by the TA-figure for $\vdash M : \alpha$. Then we have a TA-figure for $\vdash \lambda x.x(\lambda y.y(x(\lambda u.M))) : F(\alpha)$. Thus $\sharp proof(F(\alpha)) > 0$.
(b) $\Rightarrow$ (a). Assume that $\sharp proof(F(\alpha)) > 0$. Consider a substitution $b := \alpha$. Then we have $\vdash_{NJ} (((\alpha \to \alpha) \to \alpha) \to \alpha) \to \alpha$. Thus we $\vdash_{NJ} \alpha$ by the following proof figure. Therefore $\sharp proof(\alpha) > 0$.

$$\dfrac{(((\alpha \to \alpha) \to \alpha) \to \alpha) \to \alpha \qquad \dfrac{\dfrac{(\alpha \to \alpha) \to \alpha^2 \qquad \dfrac{\alpha^1}{\alpha \to \alpha}\,1}{\alpha}}{((\alpha \to \alpha) \to \alpha) \to \alpha}\,2}{\alpha}$$

(c) $\Rightarrow$ (b) is trivial. Thus it suffices to show **(b) $\Rightarrow$ (c)**. Assume that $\sharp proof(F(\alpha)) > 0$ then we have $\sharp proof(\alpha) > 0$ by (b) $\Rightarrow$ (a). Therefore we have a closed λ-term M in β-normal form such that $\vdash M : \alpha$. Similarly to the proof of (a) $\Rightarrow$ (b), we construct a TA-figure in N_λ for $\lambda x.x(\lambda y.y(x(\lambda u.M))) : F(\alpha)$. Replace the sub-figure above $x(\lambda u.M) : b$ by the TA-figure above $x(\lambda y.y(x(\lambda u.M))) : b$. Then we have a TA-figure for

$\lambda x.x(\lambda y.y(x(\lambda y'.y'(x(\lambda u.M))))) : F(\alpha)$. We can repeat this transformation infinitely many times. Thus $\sharp proof(F(\alpha)) = \infty$. ■

By Theorem 3 and Theorem 4 we have the main theorem.

Theorem 5 The problem of infiniteness of $proof(\alpha)$ for given type α is polynomial-space complete.

■

Remark 3 The transformation $F(\alpha) = (((b \to \alpha) \to \alpha) \to b) \to b$ is not the unique transformation which satisfies the equivalences in Theorem 4. We can construct another transformation $G(\alpha) = (\alpha \to b) \to (b \to \alpha) \to \alpha$ from a closed λ-term $\lambda xy.x(y(\lambda z.xz))$ whose principal type-scheme is $(a \to b) \to ((a \to b) \to a) \to b$ by the following TA-figure.

$$\dfrac{\dfrac{\dfrac{x : a \to b \qquad \dfrac{y : (a \to b) \to a \qquad \dfrac{\dfrac{x : a \to b \quad z : a}{xz : b}}{\lambda z.xz : a \to b}}{y(\lambda z.xz) : a}}{x(y(\lambda z.xz)) : b}}{\lambda y.x(y(\lambda z.xz)) : ((a \to b) \to a) \to b}}{\lambda xy.x(y(\lambda z.xz)) : (a \to b) \to ((a \to b) \to a) \to b}$$

Note that the local assumption $z : a$ is discharged at $\lambda z.xz : a \to b$. If we does not discharge $z : a$ and leave it until we reach the conclusion, we have the following TA-figure for $\lambda xy.x(y(xz)) : (a \to b) \to (b \to a) \to b$. The term is not closed and $(a \to b) \to (b \to a) \to b$ is not NJ provable unless we use an assumption $z : a$.

$$\dfrac{\dfrac{\dfrac{x : a \to b \qquad \dfrac{y : b \to a \qquad \dfrac{x : a \to b \quad z : a}{xz : b}}{y(xz) : a}}{x(y(xz)) : b}}{\lambda y.x(y(xz)) : (b \to a) \to b}}{z : a \vdash \lambda xy.x(y(xz)) : (a \to b) \to (b \to a) \to b}$$

Thus we obtain a transformation $G(\alpha) = (\alpha \to b) \to (b \to \alpha) \to \alpha$.

Remark 4 In [14], Zaionc constructed a system of polynomial equation given a type α and showed that $\sharp proof(\alpha)$ is obtained as a fixpoint solution of the equation. To reach the fixpoint, we iterate the system of polynomial calculation. It is worth while investigating the relation between this number of iteration and the depth of proof in the present paper.

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