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THE TRANSITIVE CLOSURE OF FUZZY RELATIONS WITH A CONTRACTION PROPERTY

By

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Abstract

This paper analyses transitive closures of fuzzy relations on a compact metric space with a contraction property in [2]. We show that the transitive closure is a unique solution of a fuzzy relational equation and also has the same contraction property.

1. Introduction and notations

Let $E$ be a compact metric space and $d$ be a metric on $E$. $C(E)$ denotes the collection of all closed subsets of $E$. We put $d(x, D) := \inf_{y \in D} d(x, y)$, $x \in E$, $D \in C(E)$. Let $\rho$ be the Hausdorff metric on $C(E)$. Then it is well-known ([1]) that $(C(E), \rho)$ is a compact metric space. Let $\mathcal{F}(E)$ be the set of all fuzzy sets $\hat{s} : E \to [0, 1]$ which are upper semi-continuous and satisfy $\sup_{x \in E} \hat{s}(x) = 1$.

For a fuzzy relation $\tilde{r} : E \times E \to [0, 1]$ satisfying $\tilde{r}(x, \cdot) \in \mathcal{F}(E)$ ($x \in E$), we define a map $\tilde{r}_\alpha : C(E) \to C(E)$ ($\alpha \in [0, 1]$) by

$$
\tilde{r}_\alpha(D) := \begin{cases} 
\{y : \tilde{r}(x, y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha \neq 0, \ D \in C(E), D \neq \emptyset, \\
\text{cl}\{y : \tilde{r}(x, y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0, \ D \in C(E), D \neq \emptyset, \\
\text{cl}\{y : \tilde{r}(x, y) > 0 \text{ for some } x \in D\} & \text{for } 0 \leq \alpha \leq 1, \ D = \emptyset,
\end{cases}
$$

where cl denotes the closure of a set. Let $\mathcal{R}(E)$ be the set of all fuzzy relations $\tilde{r} : E \times E \to [0, 1]$ which satisfy $\tilde{r}(x, \cdot) \in \mathcal{F}(E)$ ($x \in E$) and

$$
\sup_{\alpha \in [0, 1]} \rho(\tilde{r}_\alpha(y), \tilde{r}_\alpha(x)) \to 0 \ (y \to x) \text{ for } x \in E. \tag{1.2}
$$

We denote the maximum operation and the minimum operation by $\vee$ and $\wedge$ respectively. Let $\tilde{q} \in \mathcal{R}(E)$ be a continuous fuzzy relation. We define sequences of fuzzy relations $\{\tilde{q}^n\}_{n=1,2,\ldots}$ and $\{\tilde{c}^m\}_{m=1,2,\ldots}$ by

$$
\tilde{q}^1 := \tilde{q} \text{ and } \tilde{q}^{n+1}(x, y) := \sup_{z \in E} \{\tilde{q}(x, z) \wedge \tilde{q}^n(z, y)\}, \ x, y \in E, \ n = 1, 2, \cdots; \tag{1.3}
$$

$$
\tilde{c}^m(x, y) := \bigvee_{n=1,2,\ldots,m} \tilde{q}^n(x, y), \ x, y \in E, \ m = 1, 2, \cdots. \tag{1.4}
$$

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If $E$ is finite, then the transitive closure of the fuzzy relation $\tilde{q}$ is given ([5]) by

$$\bigvee_{n=1,2,\ldots} \tilde{q}^n(x,y), \quad x,y \in E. \quad (1.5)$$

This paper discusses the transitive closure when $E$ is a general compact metric space. From now on we assume the following contraction property.

**Assumption** (Contraction property, [2]). There exists a real number $\beta$ ($0 < \beta < 1$) satisfying the following condition:

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \leq \beta \rho(A, B) \quad \text{for all } A, B \in \mathcal{C}(E) \text{ and all } \alpha \in [0,1]. \quad (1.6)$$

We call $\beta$ a contraction factor.

**Lemma 1.** The condition (1.6) is equivalent to the following condition:

$$\rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)) \leq \beta \, d(x,y) \quad \text{for all } x, y \in E \text{ and all } \alpha \in [0,1], \quad (1.7)$$

where $\tilde{q}_\alpha(x) := \tilde{q}_\alpha(\{x\})$ ($x \in E$, $\alpha \in [0,1]$).

**Proof.** We can obtain (1.7) from (1.6), taking $A = \{x\}$, $B = \{y\}$. Conversely we assume (1.7). Let $A, B \in \mathcal{C}(E)$. Since $\tilde{q}_\alpha(x) \subset \tilde{q}_\alpha(A)$ for all $x \in A$ and $\tilde{q}_\alpha(y) \subset \tilde{q}_\alpha(B)$ for all $y \in B$,

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) = \max \{ \max_{x \in \tilde{q}_\alpha(A)} d(x', \tilde{q}_\alpha(B)), \max_{y \in \tilde{q}_\alpha(B)} d(\tilde{q}_\alpha(A), y') \} \leq \max \{ \max_{x \in \tilde{q}_\alpha(A)} \min_{y \in \tilde{q}_\alpha(B)} d(x', \tilde{q}_\alpha(y)), \max_{y \in \tilde{q}_\alpha(B)} \min_{x \in \tilde{q}_\alpha(A)} d(\tilde{q}_\alpha(x), y') \}.$$

Here since $\tilde{q}_\alpha(A) = \bigcup_{x \in A} \tilde{q}_\alpha(x)$, we have

$$\max_{x \in \tilde{q}_\alpha(A)} \min_{y \in \tilde{q}_\alpha(B)} d(x', \tilde{q}_\alpha(y)) = \max_{x \in A} \max_{x' \in \tilde{q}_\alpha(x)} \min_{y \in B} d(x', \tilde{q}_\alpha(y)) \leq \max_{x \in A} \min_{y \in B} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)).$$

Similarly

$$\max_{y' \in \tilde{q}_\alpha(B)} \min_{x \in \tilde{q}_\alpha(A)} d(\tilde{q}_\alpha(x), y') \leq \max_{x \in A} \min_{y \in B} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)).$$

Therefore we obtain

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \leq \max \{ \max_{x \in A} \min_{y \in B} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)), \max_{y \in B} \min_{x \in A} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)) \}.$$

From (1.7), we get

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \leq \beta \max \{ \max_{x \in A} \min_{y \in B} d(x, y), \max_{y \in B} \min_{x \in A} d(x, y) \} = \beta \rho(A, B).$$

Therefore we obtain (1.6).
DEFINITION 1. ([3]) For $\tilde{r}_n, \tilde{r} \in \mathcal{R}(E)$,

$$\lim_{n \to \infty} \tilde{r}_n = \tilde{r}$$

means

$$\sup_{\alpha \in [0,1]} \rho(\tilde{r}_{n,\alpha}(D), \tilde{r}_{\alpha}(D)) \to 0 \quad (n \to \infty) \quad \text{for} \quad D \in \mathcal{C}(E),$$

where $\tilde{r}_{n,\alpha}, \tilde{r}_{\alpha}$ are defined by (1.1) for the fuzzy relations $\tilde{r}_n, \tilde{r}$ respectively.

LEMMA 2. ([2, Lemma 2]) Suppose a family of subsets $\{D_\alpha \mid \alpha \in [0,1]\} \subset \mathcal{C}(E)$ satisfies the following conditions:

(i) $D_\alpha \subset D_{\alpha'}$ for $\alpha' \leq \alpha$.

(ii) $\lim_{\alpha' \uparrow \alpha} D_{\alpha'} = D_\alpha$, i.e., $\lim_{\alpha' \uparrow \alpha} \rho(D_{\alpha'}, D_\alpha) = 0$ for $\alpha \in (0,1]$.

Then it holds that

$$\lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}(D_{\alpha'}) = \tilde{q}_\alpha(D_\alpha) \quad \text{for} \quad \alpha \in (0,1]. \quad (1.8)$$

LEMMA 3. We suppose that a family of subsets $\{D_\alpha(x) \mid x \in E, \alpha \in [0,1]\}(\subset \mathcal{C}(E))$ satisfies the following conditions (i) - (iii):

(i) $D_\alpha(x) \subset D_{\alpha'}(x)$ for $x \in E$, $0 \leq \alpha' < \alpha \leq 1$.

(ii) $\lim_{\alpha' \uparrow \alpha} D_{\alpha'}(x) = D_\alpha(x)$ for $x \in E$, $\alpha \in (0,1]$.

(iii) $\sup_{\alpha \in [0,1]} \rho(D_\alpha(y), D_\alpha(x)) \to 0 \quad (y \to x) \quad \text{for} \quad x \in E$.

Then

$$\bar{r}(x,y) := \sup_{\alpha \in [0,1]} \{\alpha \land I_{D_\alpha(x)}(y)\}, \quad x,y \in E,$$

satisfies $\bar{r} \in \mathcal{R}(E)$ and $\bar{r}_\alpha(x) = D_\alpha(x)$ for all $x \in E$, $\alpha \in [0,1]$.

PROOF. Fix any $x \in E$. By [4], from the conditions (i) and (ii), we have $\bar{r}(x,\cdot) \in \mathcal{F}(E)$ and $\bar{r}_\alpha(x) = \{y \in E \mid \bar{r}(x,y) \geq \alpha\} = D_\alpha(x)$ for $\alpha \in (0,1]$. Therefore (1.2) holds from (iii). Thus we get $\bar{r} \in \mathcal{R}(E)$.

We define maps $\tilde{q}_\alpha^n : \mathcal{C}(E) \mapsto \mathcal{C}(E)$ ($n = 1,2,\cdots, \alpha \in [0,1]$) by $\tilde{q}_\alpha^1 := \tilde{q}_\alpha$ and $\tilde{q}_\alpha^{n+1} := \tilde{q}_\alpha(\tilde{q}_\alpha^n)$ ($n = 1,2,\cdots$).

LEMMA 4. Let $\alpha \in [0,1]$. Then:

(i) $(\tilde{q}^n)_\alpha(D) = \tilde{q}_\alpha^n(D), \quad D \in \mathcal{C}(E) \quad \text{for} \quad n = 1,2,\cdots$;

(ii) $(\tilde{e}^m)_\alpha(D) = \bigcup_{n=1,2,\cdots,m} \tilde{q}_\alpha^n(D), \quad D \in \mathcal{C}(E) \quad \text{for} \quad m = 1,2,\cdots$. 

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PROOF. We have (i) from [2,Lemma 1] and (1.3). Further (ii) is trivial from (1.4).

**Lemma 5.** It holds that
\[
\rho(A \cup C, B \cup D) \leq \max\{\rho(A, B), \rho(C, D)\} \quad \text{for } A, B, C, D \in C(E).
\]

**Proof.** Let \( A, B, C, D \in C(E) \). Then
\[
\rho(A \cup C, B \cup D) = \max\{\max_{x \in A \cup C} d(x, B \cup D), \max_{y \in B \cup D} d(A \cup C, y)\}
\]
\[
= \max\{\max_{x \in A} d(x, B \cup D), \max_{y \in B} d(A \cup C, y)\} \leq \max\{\max_{x \in A} d(x, B), \max_{y \in B} d(A, y), \max_{x \in C} d(x, B), \max_{y \in D} d(C, y)\}
\]
\[
= \max\{\rho(A, B), \rho(C, D)\}.
\]
Therefore we obtain this lemma.

2. **Main results**

We discuss the convergence of the sequence of fuzzy relations \( \{\tilde{c}^m\}_{m=1}^\infty \).

**Theorem 1.**

(i) There exists a unique solution \( \tilde{c} \in R(E) \) of the following fuzzy relational equation:
\[
\tilde{c}(x, y) = \tilde{q}(x, y) \lor \max_{z \in E} \{\tilde{c}(x, z) \land \tilde{q}(z, y)\} \quad x, y \in E.
\]

(ii) The fuzzy relation \( \tilde{c} \) also has the contraction property with the same contraction factor \( \beta \).

(iii) The fuzzy relation \( \tilde{c} \) equals to the limit of \( \{\tilde{c}^m\}_{m=1}^\infty \) :
\[
\tilde{c} = \lim_{m \to \infty} \tilde{c}^m.
\]

**Proof.** Define a map \( T_{x, \alpha} : C(E) \to C(E) \) \((x \in E, \alpha \in [0,1])\) by
\[
T_{x, \alpha}(D) := \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D), \quad D \in C(E).
\]

From Lemma 5,
\[
\rho(T_{x, \alpha}(D), T_{x, \alpha}(D')) = \rho(\tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D), \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D'))
\]
\[
\leq \rho(\tilde{q}_\alpha(D), \tilde{q}_\alpha(D'))
\]
\[
\leq \beta \rho(D, D'), \quad D, D' \in C(E), \ x \in E, \ \alpha \in [0,1].
\]
Since the metric space \((C(E), \rho)\) is compact, from the Banach’s fixed point theorem, there exists a family \(\{A_\alpha(x) \mid x \in E, \ \alpha \in [0, 1]\} \subset C(E)\) such that

\[
\tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(A_\alpha(x)) = T_{x, \alpha}(A_\alpha(x)) = A_\alpha(x), \quad x \in E, \ \alpha \in [0, 1],
\]

and \(\lim_{n \to \infty} T^n_{x, \alpha}(D) = A_\alpha(x)\) for any \(D \in C(E)\). From the definition of \(\tilde{q}_\alpha\), \(T^n_{x, \alpha'}(D) = \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D) \supset \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D) = T^n_{x, \alpha}(D)\) for \(\alpha' \leq \alpha\). Inductively we have \(T^n_{x, \alpha'}(D) \supset T^n_{x, \alpha}(D)\) for \(n = 1, 2, \ldots\). By letting \(n \to \infty\), we obtain

\[
A_{\alpha'}(x) \supset A_\alpha(x) \quad \text{for} \quad \alpha' \leq \alpha. \tag{2.3}
\]

Let \(\alpha' \leq \alpha\). Inductively we have

\[
\rho(A_\alpha(x), A_{\alpha'}(x)) = \rho(T^n_{x, \alpha}(A_\alpha(x)), T^n_{x, \alpha'}(A_{\alpha'}(x)))
\]

\[
\leq \rho(T^n_{x, \alpha'}(A_\alpha(x)), T^n_{x, \alpha'}(A_{\alpha'}(x))) + \rho(T^n_{x, \alpha'}(A_{\alpha'}(x)), T^n_{x, \alpha'}(A_{\alpha'}(x)))
\]

\[
\leq \rho(T^n_{x, \alpha'}(A_\alpha(x)), T^n_{x, \alpha'}(A_{\alpha'}(x))) + \beta^n \rho(A_{\alpha'}(x), A_{\alpha'}(x)), \quad n = 1, 2, \ldots.
\]

Then \(\rho(A_{\alpha'}(x), A_\alpha(x))\) is uniformly bounded since \(E\) is compact. We put \(\rho(A_{\alpha'}(x), A_\alpha(x)) \leq M\) for some \(M > 0\). Therefore

\[
\rho(A_\alpha(x), A_{\alpha'}(x)) \leq \rho(T^n_{x, \alpha'}(A_\alpha(x)), T^n_{x, \alpha'}(A_{\alpha'}(x))) + \beta^n M, \quad n = 1, 2, \ldots. \tag{2.4}
\]

By Lemma 2, we have \(\lim_{\alpha' \searrow \alpha} T^n_{x, \alpha'}(A_\alpha(x)) = \lim_{\alpha' \searrow \alpha} \{\tilde{q}_{\alpha'}(x) \cup \tilde{q}_{\alpha'}(A_\alpha(x))\} = \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(A_\alpha(x)) = T_{x, \alpha}(A_\alpha(x))\). Repeating these arguments inductively,

\[
\lim_{\alpha' \searrow \alpha} T^n_{x, \alpha'}(A_\alpha(x)) = T^n_{x, \alpha}(A_\alpha(x)), \quad n = 1, 2, \ldots.
\]

Therefore (2.4) follows

\[
\lim_{\alpha' \searrow \alpha} \rho(A_\alpha(x), A_{\alpha'}(x)) \leq \beta^n M, \quad n = 1, 2, \ldots.
\]

By letting \(n \to \infty\), we obtain

\[
\lim_{\alpha' \searrow \alpha} A_{\alpha'}(x) = A_\alpha(x). \tag{2.5}
\]

Let \(\alpha \in [0, 1]\) and \(x, y \in E\). From Lemma 5 and the contraction property of \(\tilde{q}\), we have

\[
\rho(T_{y, \alpha}(D), T_{x, \alpha}(D')) = \rho(\tilde{q}_\alpha(y) \cup \tilde{q}_\alpha(D), \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D'))
\]

\[
\leq \max\{\rho(\tilde{q}_\alpha(y), \tilde{q}_\alpha(x)), \rho(\tilde{q}_\alpha(D), \tilde{q}_\alpha(D'))\}
\]

\[
\leq \beta \max\{d(y, x), \rho(D, D')\}, \quad D, D' \in C(E).
\]

Repeating these arguments inductively,

\[
\rho(T^n_{x, \alpha}(D), T^n_{x, \alpha}(D')) \leq \max\{\beta \rho(D, D'), \beta^n \rho(D, D')\}, \quad D, D' \in C(E), \quad n = 1, 2, \ldots.
\]

Since \(\rho(D, D')\) is uniformly bounded, letting \(n \to \infty\), we obtain

\[
\rho(A_\alpha(y), A_\alpha(x)) \leq \beta \rho(D, D') \quad \text{for} \quad x, y \in E. \tag{2.6}
\]
Therefore
\[ \sup_{\alpha \in [0,1]} \rho(A_\alpha(y), A_\alpha(x)) \to 0 \quad (y \to x) \quad \text{for } x \in E. \]
Thus the family \( \{ A_\alpha(x) \mid x \in E, \alpha \in [0,1] \} \) satisfies the conditions (i) – (iii) of Lemma 3. By Lemma 3, we can define a fuzzy relation \( \bar{c} \in \mathcal{R}(E) \) by
\[ \bar{c}(x, y) := \sup_{\alpha \in [0,1]} \{ \alpha \land I_{A_\alpha}(y) \}; \quad x, y \in E. \]
Then \( \bar{c}_\alpha(x) = A_\alpha(x) \quad (x \in E, \alpha \in [0,1]) \). Since \( A_\alpha(x) \) is a unique fixed point of \( T_{x,\alpha} \),
\[ \lim_{n \to \infty} (\bar{c}^n)_\alpha(x) = \lim_{n \to \infty} T_{x,\alpha}^n(\{ x \}) = A_\alpha(x) = \bar{c}_\alpha(x), \quad \alpha \in [0,1]. \]
We get (iii) since the convergence is uniform in \( \alpha \in [0,1] \).
Next we show that \( \bar{c} \) is a solution of (2.1). Since \( \bar{c}_\alpha(x) = A_\alpha(x) \), we note that
\[ \bar{q}_\alpha(x) \cup \bar{q}_\alpha(\bar{c}_\alpha(x)) = T_\alpha(\bar{c}_\alpha(x)) = \bar{c}_\alpha(x), \quad \alpha \in [0,1]. \quad (2.7) \]
If \( \alpha > 0 \), then we have
\[ \left\{ y \in E \mid \bar{q}(x, y) \lor \max_{z \in E} \{ \bar{c}(x, z) \land \bar{q}(z, y) \} \geq \alpha \right\} = \bar{q}_\alpha(x) \cup \bar{q}_\alpha(\bar{c}_\alpha(x)). \]
If \( \alpha = 0 \), then in a similar way to the proof of [2, Lemma 1] we have
\[ \text{cl} \left\{ y \in E \mid \max_{x \in E} \{ \bar{q}(x, y) \lor \max_{z \in E} \{ \bar{c}(x, z) \land \bar{q}(z, y) \} \geq 0 \right\} = \bar{q}_0(x) \cup \bar{q}_0(\bar{c}_0(x)). \]
Therefore
\[ \left\{ y \in E \mid \bar{q}(x, y) \lor \max_{z \in E} \{ \bar{c}(x, z) \land \bar{q}(z, y) \} \geq \alpha \right\} = \bar{c}_\alpha(x) \quad \text{for } \alpha \in [0,1]. \]
Together with (2.7), we get
\[ \left\{ y \in E \mid \bar{q}(x, y) \lor \max_{z \in E} \{ \bar{c}(x, z) \land \bar{q}(z, y) \} \geq \alpha \right\} = \bar{c}_\alpha(x) \quad \text{for } \alpha \in [0,1]. \]
Therefore \( \bar{c} \) satisfies (2.1). We prove the uniqueness of solution of (2.1). Let us denote by \( \bar{c}' \in \mathcal{R}(E) \) another solution of (2.1). For \( x \in E, \alpha \in [0,1] \), it is shown similarly that \( \bar{c}'_\alpha(x) = \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(\bar{c}'_\alpha(x)) \). That is, \( \bar{c}'_\alpha(x) \) is a fixed point of \( T_{x,\alpha} : \mathcal{C}(E) \to \mathcal{C}(E) \). From the uniqueness of the fixed point, we get \( \bar{c}'_\alpha(x) = \bar{c}_\alpha(x) \) for \( x \in E, \alpha \in [0,1] \). By Lemma 3, \( \bar{c}' = \bar{c} \). Thus we get (i).
Finally (ii) is trivial from (2.6), using Lemma 1 and \( \bar{c}_\alpha(x) = A_\alpha(x) \) for \( x \in E, \alpha \in [0,1] \). Thus the proof is completed.

**Theorem 2.** It holds that
\[ \bar{c}(x, y) = \bigvee_{n=1,2,...} \bar{q}^n(x, y), \quad x, y \in E. \quad (2.8) \]
Further \( \bar{c} \) is the transitive closure of the fuzzy relation \( \bar{q} \), namely \( \bar{c} \) satisfies (i) – (iii) :
(i) \( \hat{c} \geq \hat{q} \).

(ii) \( \hat{c} \) has the transitive property:

\[
\hat{c}(x, y) \geq \sup_{z \in E} \{ \hat{c}(x, z) \land \hat{c}(z, y) \}, \quad x, y \in E. \tag{2.9}
\]

(iii) If \( \hat{\tau} \in \mathcal{R}(E) \) satisfies \( \hat{\tau} \geq \hat{q} \) and has the transitive property, then \( \hat{\tau} \geq \hat{c} \).

**Proof.** Let \( \hat{r}(x, y) := \vee_{n=1,2,\ldots} \hat{q}^n(x, y), \ x, y \in E.\) Then we have \( \hat{r} \geq \vee_{n=1,2,\ldots,m} \hat{q}^n = \hat{c}^m \) for \( m = 1, 2, \ldots \). Therefore \( \hat{r}_\alpha(x) \supseteq (\hat{c}^m)_\alpha(x) \) for \( x \in E, \alpha \in [0,1], m = 1, 2, \ldots. \) From (2.2) we obtain \( \hat{r}_\alpha(x) \supseteq \hat{c}_\alpha(x) \) for \( x \in E, \alpha \in [0,1]. \) Thus we get \( \hat{r} \geq \hat{c}. \)

On the other hand, from (2.1), we obtain \( \hat{c} \geq \hat{q} \) and

\[
\hat{c}(x, y) \geq \sup_{z \in E} \{ \hat{c}(x, z) \land \hat{q}(z, y) \} \geq \sup_{z \in E} \{ \hat{q}(x, z) \land \hat{q}(z, y) \} = \hat{q}^2(x, y), \quad x, y \in E.
\]

Repeating this argument inductively, we obtain \( \hat{c} \geq \hat{q}^n \) for \( n = 1, 2, \ldots \). Therefore \( \hat{c}(x, y) \geq \hat{c}^n(x, y) \) for \( x, y \in E, n = 1, 2, \ldots \). Thus we get \( \hat{c} \geq \hat{r}. \) Therefore we obtain (2.8).

Next we prove (i) – (iii). (i) is trivial from (2.1). From (2.8), we have

\[
\hat{c}(x, y) \geq \left( \bigvee_{n=1,2,\ldots,m; n'=1,2,\ldots,m'} \hat{q}^{m+n'}(x, y) \right) = \sup_{z \in E} \left\{ \bigvee_{n=1,2,\ldots,m} \hat{q}^n(x, z) \land \bigvee_{n'=1,2,\ldots,m'} \hat{q}^{n'}(z, y) \right\} = \sup_{z \in E} \{ \hat{c}^m(x, z) \land \hat{c}^{n'}(z, y) \}, \quad x, y \in E.
\]

Taking the supremum over \( m = 1, 2, \ldots \) and \( m' = 1, 2, \ldots \), we obtain (ii). Finally let \( \hat{\tau} \in \mathcal{R}(E) \) satisfy \( \hat{\tau} \geq \hat{q} \) and have the transitive property. Then

\[
\hat{\tau}(x, y) \geq \sup_{z \in E} \{ \hat{\tau}(x, z) \land \hat{\tau}(z, y) \} \geq \sup_{z \in E} \{ \hat{q}(x, z) \land \hat{q}(z, y) \} = \hat{q}^2(x, y), \quad x, y \in E.
\]

Repeating this argument inductively, we obtain \( \hat{\tau} \geq \hat{q}^n \) for \( n = 1, 2, \ldots \). Therefore \( \hat{\tau}(x, y) \geq \hat{c}^n(x, y) \) for \( x, y \in E, n = 1, 2, \ldots \). Thus we get \( \hat{\tau} \geq \hat{c}. \) Therefore (iii) holds. The proof is completed.

**3. Numerical example**

Let \( E = [-2, 2] \) be a space of states. We consider a fuzzy relation (see [2, Figure 1])

\[
\hat{q}(x, y) = 1 - \left| y - \left( \frac{x}{2} + \frac{1}{4} \right) \right|, \quad x, y \in E. \tag{3.1}
\]

Then we have

\[
\hat{q}^n(x, y) = 1 - \left| y - \left( \frac{1}{2^n} x + \frac{1}{2} - \frac{1}{2^n+1} \right) \right| \left/ \left( 2 - \frac{1}{2^n-1} \right) \right., \quad x, y \in E, n = 1, 2, \ldots. \tag{3.2}
\]
From Theorem 2, we obtain

\[
\bar{c}(x, y) = \bigvee_{n=1,2,\ldots} \left\{ 1 - \left| y - \left( \frac{1}{2^n} x + \frac{1}{2} - \frac{1}{2^{n+1}} \right) \right| \left( 2 - \frac{1}{2^n-1} \right) \right\}, \quad x, y \in E. \tag{3.3}
\]

Then (3.3) is the unique solution of Theorem 1 (see Figure 1).

Fig. 1: The transitive closure \( \bar{c}(x, y) \).

References


