The Transitive Closure of Fuzzy Relations with a Contraction Property

Yoshida, Yuji
Faculty of Economics and Business Administration, Kitakyushu University

出版情報：RIFIS Technical Report. 89, 1994-05-09. Research Institute of Fundamental Information Science, Kyushu University

RIFIS-TR-CS-89

RIFIS Technical Report

The Transitive Closure of Fuzzy Relations with a Contraction Property

Yuji Yoshida

May 9, 1994

Research Institute of Fundamental Information Science
Kyushu University 33
Fukuoka 812, Japan
E-mail: yoshida@rifis.kyushu-u.ac.jp Phone: 092-641-1101 ex. 4484

THE TRANSITIVE CLOSURE OF FUZZY RELATIONS WITH A CONTRACTION PROPERTY

By

Yuji YOSHIDA *

Abstract

This paper analyses transitive closures of fuzzy relations on a compact metric space with a contraction property in [2]. We show that the transitive closure is a unique solution of a fuzzy relational equation and also has the same contraction property.

1. Introduction and notations

Let E be a compact metric space and d be a metric on E. $\mathcal{C}(E)$ denotes the collection of all closed subsets of E. We put $d(x,D) := \inf_{y\in D} d(x,y)$, $x \in E, D \in \mathcal{C}(E)$. Let ρ be the Hausdorff metric on $\mathcal{C}(E)$. Then it is well-known ([1]) that $(\mathcal{C}(E),\rho)$ is a compact metric space. Let $\mathcal{F}(E)$ be the set of all fuzzy sets $\tilde{s} : E \to [0,1]$ which are upper semi-continuous and satisfy $\sup_{x\in E} \tilde{s}(x) = 1$.

For a fuzzy relation $\tilde{r} : E \times E \to [0,1]$ satisfying $\tilde{r}(x,\cdot) \in \mathcal{F}(E)$ $(x \in E)$, we define a map $\tilde{r}_\alpha : \mathcal{C}(E) \to \mathcal{C}(E)$ $(\alpha \in [0,1])$ by

$$\tilde{r}_\alpha(D) := \begin{cases} \{y \mid \tilde{r}(x,y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha \neq 0,\ D \in \mathcal{C}(E), D \neq \emptyset, \\ \mathrm{cl}\{y \mid \tilde{r}(x,y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0,\ D \in \mathcal{C}(E), D \neq \emptyset, \\ E & \text{for } 0 \leq \alpha \leq 1,\ D = \emptyset, \end{cases} \tag{1.1}$$

where cl denotes the closure of a set. Let $\mathcal{R}(E)$ be the set of all fuzzy relations $\tilde{r} : E \times E \to [0,1]$ which satisfy $\tilde{r}(x,\cdot) \in \mathcal{F}(E)$ $(x \in E)$ and

$$\sup_{\alpha\in[0,1]} \rho(\tilde{r}_\alpha(y), \tilde{r}_\alpha(x)) \to 0 \quad (y \to x) \quad \text{for } x \in E. \tag{1.2}$$

We denote the maximum operation and the minimum operation by $\vee$ and $\wedge$ respectively. Let $\tilde{q} \in \mathcal{R}(E)$ be a continuous fuzzy relation. We define sequences of fuzzy relations $\{\tilde{q}^n\}_{n=1,2,\cdots}$ and $\{\tilde{c}^m\}_{m=1,2,\cdots}$ by

$$\tilde{q}^1 := \tilde{q} \quad \text{and} \quad \tilde{q}^{n+1}(x,y) := \sup_{z\in E} \{\tilde{q}(x,z) \wedge \tilde{q}^n(z,y)\}, \quad x,y \in E,\ n = 1,2,\cdots; \tag{1.3}$$

$$\tilde{c}^m(x,y) := \bigvee_{n=1,2,\cdots,m} \tilde{q}^n(x,y), \quad x,y \in E,\ m = 1,2,\cdots. \tag{1.4}$$

* Department of Business Administration, Faculty of Economics and Business Administration, Kitakyushu University, Kitagata 4-2-1, Kokuraminami, Kitakyushu 802, Japan

If E is finite, then the transitive closure of the fuzzy relation $\tilde{q}$ is given ([5]) by

$$\bigvee_{n=1,2,\cdots} \tilde{q}^n(x,y), \quad x,y \in E. \tag{1.5}$$

This paper discusses the transitive closure when E is a general compact metric space. From now on we assume the following contraction property.

ASSUMPTION (Contraction property, [2]). There exists a real number β $(0 < \beta < 1)$ satisfying the following condition :

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \le \beta\ \rho(A,B) \quad \text{for all } A,B \in \mathcal{C}(E) \text{ and all } \alpha \in [0,1]. \tag{1.6}$$

We call β a contraction factor.

LEMMA 1. *The condition (1.6) is equivalent to the following condition :*

$$\rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)) \le \beta\ d(x,y) \quad \textit{for all } x,y \in E \textit{ and all } \alpha \in [0,1], \tag{1.7}$$

where $\tilde{q}_\alpha(x) := \tilde{q}_\alpha(\{x\})$ $(x \in E, \alpha \in [0,1])$.

PROOF. We can obtain (1.7) from (1.6), taking $A = \{x\}$, $B = \{y\}$. Conversely we assume (1.7). Let $A, B \in \mathcal{C}(E)$. Since $\tilde{q}_\alpha(x) \subset \tilde{q}_\alpha(A)$ for all $x \in A$ and $\tilde{q}_\alpha(y) \subset \tilde{q}_\alpha(B)$ for all $y \in B$,

$$\begin{aligned} \rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) &= \max\{\max_{x' \in \tilde{q}_\alpha(A)} d(x', \tilde{q}_\alpha(B)), \max_{y' \in \tilde{q}_\alpha(B)} d(\tilde{q}_\alpha(A), y')\} \\ &\le \max\{\max_{x' \in \tilde{q}_\alpha(A)} \min_{y \in B} d(x', \tilde{q}_\alpha(y)), \max_{y' \in \tilde{q}_\alpha(B)} \min_{x \in A} d(\tilde{q}_\alpha(x), y')\}. \end{aligned}$$

Here since $\tilde{q}_\alpha(A) = \bigcup_{x \in A} \tilde{q}_\alpha(x)$, we have

$$\max_{x' \in \tilde{q}_\alpha(A)} \min_{y \in B} d(x', \tilde{q}_\alpha(y)) = \max_{x \in A} \max_{x' \in \tilde{q}_\alpha(x)} \min_{y \in B} d(x', \tilde{q}_\alpha(y)) \le \max_{x \in A} \min_{y \in B} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)).$$

Similarly

$$\max_{y' \in \tilde{q}_\alpha(B)} \min_{x \in A} d(\tilde{q}_\alpha(x), y') \le \max_{y \in B} \min_{x \in A} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)).$$

Therefore we obtain

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \le \max\{\max_{x \in A} \min_{y \in B} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)), \max_{y \in B} \min_{x \in A} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y))\}.$$

From (1.7), we get

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \le \beta\ \max\{\max_{x \in A} \min_{y \in B} d(x,y), \max_{y \in B} \min_{x \in A} d(x,y)\} = \beta\ \rho(A,B).$$

Therefore we obtain (1.6). ■

DEFINITION 1. ([3]) For $\tilde{r}_n, \tilde{r} \in \mathcal{R}(E)$,

$$\lim_{n\to\infty} \tilde{r}_n = \tilde{r}$$

means

$$\sup_{\alpha\in[0,1]} \rho(\tilde{r}_{n,\alpha}(D), \tilde{r}_\alpha(D)) \to 0 \quad (n \to \infty) \quad \text{for } D \in \mathcal{C}(E),$$

where $\tilde{r}_{n,\alpha}, \tilde{r}_\alpha$ are defined by (1.1) for the fuzzy relations $\tilde{r}_n, \tilde{r}$ respectively.

LEMMA 2. ([2,Lemma 2]) *Suppose a family of subsets $\{D_\alpha \mid \alpha \in [0,1]\} \subset \mathcal{C}(E)$ satisfies the following conditions:*

(i) $D_\alpha \subset D_{\alpha'}$ *for* $\alpha' \leq \alpha$.

(ii) $\lim_{\alpha'\uparrow\alpha} D_{\alpha'} = D_\alpha$, *i.e.*, $\lim_{\alpha'\uparrow\alpha} \rho(D_{\alpha'}, D_\alpha) = 0$ *for* $\alpha \in (0,1]$.

Then it holds that

$$\lim_{\alpha'\uparrow\alpha} \tilde{q}_{\alpha'}(D_{\alpha'}) = \tilde{q}_\alpha(D_\alpha) \quad \textit{for } \alpha \in (0,1]. \tag{1.8}$$

LEMMA 3. *We suppose that a family of subsets $\{D_\alpha(x) \mid x \in E, \alpha \in [0,1]\} (\subset \mathcal{C}(E))$ satisfies the following conditions (i) – (iii) :*

(i) $D_\alpha(x) \subset D_{\alpha'}(x)$ *for* $x \in E,\ 0 \leq \alpha' < \alpha \leq 1$.

(ii) $\lim_{\alpha'\uparrow\alpha} D_{\alpha'}(x) = D_\alpha(x)$ *for* $x \in E,\ \alpha \in (0,1]$.

(iii) $\sup_{\alpha\in[0,1]} \rho(D_\alpha(y), D_\alpha(x)) \to 0 \quad (y \to x)$ *for* $x \in E$.

Then

$$\tilde{r}(x,y) := \sup_{\alpha\in[0,1]} \{\alpha \wedge I_{D_\alpha(x)}(y)\}, \quad x, y \in E,$$

satisfies $\tilde{r} \in \mathcal{R}(E)$ *and* $\tilde{r}_\alpha(x) = D_\alpha(x)$ *for all* $x \in E,\ \alpha \in [0,1]$.

PROOF. Fix any $x \in E$. By [4], from the conditions (i) and (ii), we have $\tilde{r}(x,\cdot) \in \mathcal{F}(E)$ and $\tilde{r}_\alpha(x) = \{y \in E \mid \tilde{r}(x,y) \geq \alpha\} = D_\alpha(x)$ for $\alpha \in (0,1]$. Therefore (1.2) holds from (iii). Thus we get $\tilde{r} \in \mathcal{R}(E)$. ■

We define maps $\tilde{q}^n_\alpha : C(E) \mapsto C(E)$ $(n = 1,2,\cdots, \alpha \in [0,1])$ by $\tilde{q}^1_\alpha := \tilde{q}_\alpha$ and $\tilde{q}^{n+1}_\alpha := \tilde{q}_\alpha(\tilde{q}^n_\alpha)$ $(n = 1,2,\cdots)$.

LEMMA 4. *Let $\alpha \in [0,1]$. Then :*

(i) $(\tilde{q}^n)_\alpha(D) = \tilde{q}^n_\alpha(D)$, $\quad D \in \mathcal{C}(E)$ *for* $n = 1,2,\cdots$;

(ii) $(\tilde{c}^m)_\alpha(D) = \bigcup_{n=1,2,\cdots,m} \tilde{q}^n_\alpha(D)$, $\quad D \in \mathcal{C}(E)$ *for* $m = 1,2,\cdots$.

PROOF. We have (i) from [2,Lemma 1] and (1.3). Further (ii) is trivial from (1.4). ■

LEMMA 5. It holds that

$$\rho(A \cup C, B \cup D) \leq \max\{\rho(A,B), \rho(C,D)\} \quad \text{for } A,B,C,D \in \mathcal{C}(E).$$

PROOF. Let $A,B,C,D \in \mathcal{C}(E)$. Then

$$\begin{aligned}
&\rho(A \cup C, B \cup D) \\
&= \max\{\max_{x \in A \cup C} d(x, B \cup D), \max_{y \in B \cup D} d(A \cup C, y)\} \\
&= \max\{\max_{x \in A} d(x, B \cup D), \max_{y \in B} d(A \cup C, y), \max_{x \in C} d(x, B \cup D), \max_{y \in D} d(A \cup C, y)\} \\
&\leq \max\{\max_{x \in A} d(x, B), \max_{y \in B} d(A, y), \max_{x \in C} d(x, D), \max_{y \in D} d(C, y)\} \\
&= \max\{\rho(A,B), \rho(C,D)\}.
\end{aligned}$$

Therefore we obtain this lemma. ■

2. Main results

We discuss the convergence of the sequence of fuzzy relations $\{\tilde{c}^m\}_{m=1}^{\infty}$.

THEOREM 1.

(i) *There exists a unique solution $\tilde{c} \in \mathcal{R}(E)$ of the following fuzzy relational equation :*

$$\tilde{c}(x,y) = \tilde{q}(x,y) \vee \max_{z \in E}\{\tilde{c}(x,z) \wedge \tilde{q}(z,y)\} \quad x, y \in E. \tag{2.1}$$

(ii) *The fuzzy relation $\tilde{c}$ also has the contraction property with the same contraction factor β.*

(iii) *The fuzzy relation $\tilde{c}$ equals to the limit of $\{\tilde{c}^m\}_{m=1}^{\infty}$:*

$$\tilde{c} = \lim_{m \to \infty} \tilde{c}^m. \tag{2.2}$$

PROOF. Define a map $T_{x,\alpha} : \mathcal{C}(E) \to \mathcal{C}(E)$ $(x \in E,\ \alpha \in [0,1])$ by

$$T_{x,\alpha}(D) := \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D), \quad D \in \mathcal{C}(E).$$

From Lemma 5,

$$\begin{aligned}
\rho(T_{x,\alpha}(D), T_{x,\alpha}(D')) &= \rho(\tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D), \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D')) \\
&\leq \rho(\tilde{q}_\alpha(D), \tilde{q}_\alpha(D')) \\
&\leq \beta\rho(D, D'), \quad D, D' \in \mathcal{C}(E),\ x \in E,\ \alpha \in [0,1].
\end{aligned}$$

Since the metric space $(\mathcal{C}(E), \rho)$ is compact, from the Banach's fixed point theorem, there exists a family $\{A_\alpha(x) \mid x \in E,\ \alpha \in [0,1]\} \subset \mathcal{C}(E)$ such that

$$\tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(A_\alpha(x)) = T_{x,\alpha}(A_\alpha(x)) = A_\alpha(x), \quad x \in E,\ \alpha \in [0,1],$$

and $\lim_{n\to\infty} T^n_{x,\alpha}(D) = A_\alpha(x)$ for any $D \in \mathcal{C}(E)$. From the definition of $\tilde{q}_\alpha$, $T_{x,\alpha'}(D) = \tilde{q}_{\alpha'}(x) \cup \tilde{q}_{\alpha'}(D) \supset \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D) = T_{x,\alpha}(D)$ for $\alpha' \le \alpha$. Inductively we have $T^n_{x,\alpha'}(D) \supset T^n_{x,\alpha}(D)$ for $n = 1, 2, \cdots$. By letting $n \to \infty$, we obtain

$$A_{\alpha'}(x) \supset A_\alpha(x) \quad \text{for } \alpha' \le \alpha. \tag{2.3}$$

Let $\alpha' \le \alpha$. Inductively we have

$$\begin{aligned}
\rho(A_\alpha(x), A_{\alpha'}(x)) &= \rho(T^n_{x,\alpha}(A_\alpha(x)), T^n_{x,\alpha'}(A_{\alpha'}(x))) \\
&\le \rho(T^n_{x,\alpha'}(A_\alpha(x)), T^n_{x,\alpha}(A_\alpha(x))) + \rho(T^n_{x,\alpha'}(A_{\alpha'}(x)), T^n_{x,\alpha'}(A_\alpha(x))) \\
&\le \rho(T^n_{x,\alpha'}(A_\alpha(x)), T^n_{x,\alpha}(A_\alpha(x))) + \beta^n \rho(A_{\alpha'}(x), A_\alpha(x)), \quad n = 1, 2, \cdots.
\end{aligned}$$

Then $\rho(A_{\alpha'}(x), A_\alpha(x))$ is uniformly bounded since E is compact. We put $\rho(A_{\alpha'}(x), A_\alpha(x)) \le M$ for some $M > 0$. Therefore

$$\rho(A_\alpha(x), A_{\alpha'}(x)) \le \rho(T^n_{x,\alpha'}(A_\alpha(x)), T^n_{x,\alpha}(A_\alpha(x))) + \beta^n M, \quad n = 1, 2, \cdots. \tag{2.4}$$

By Lemma 2, we have $\lim_{\alpha'\uparrow\alpha} T_{x,\alpha'}(A_\alpha(x)) = \lim_{\alpha'\uparrow\alpha}\{\tilde{q}_{\alpha'}(x) \cup \tilde{q}_{\alpha'}(A_\alpha(x))\} = \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(A_\alpha(x)) = T_{x,\alpha}(A_\alpha(x))$. Repeating these arguments inductively,

$$\lim_{\alpha'\uparrow\alpha} T^n_{x,\alpha'}(A_\alpha(x)) = T^n_{x,\alpha}(A_\alpha(x)), \quad n = 1, 2, \cdots.$$

Therefore (2.4) follows

$$\lim_{\alpha'\uparrow\alpha} \rho(A_\alpha(x), A_{\alpha'}(x)) \le \beta^n M, \quad n = 1, 2, \cdots.$$

By letting $n \to \infty$, we obtain

$$\lim_{\alpha'\uparrow\alpha} A_{\alpha'}(x) = A_\alpha(x). \tag{2.5}$$

Let $\alpha \in [0,1]$ and $x, y \in E$. From Lemma 5 and the contraction property of $\tilde{q}$, we have

$$\begin{aligned}
\rho(T_{y,\alpha}(D), T_{x,\alpha}(D')) &= \rho(\tilde{q}_\alpha(y) \cup \tilde{q}_\alpha(D), \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D')) \\
&\le \max\{\rho(\tilde{q}_\alpha(y), \tilde{q}_\alpha(x)), \rho(\tilde{q}_\alpha(D), \tilde{q}_\alpha(D'))\} \\
&\le \beta \ \max\{d(y,x), \rho(D, D')\}, \quad D, D' \in \mathcal{C}(E).
\end{aligned}$$

Repeating these arguments inductively,

$$\rho(T^n_{x,\alpha}(D), T^n_{x,\alpha}(D')) \le \max\{\beta\ d(y,x), \beta^n \rho(D, D')\}, \quad D, D' \in \mathcal{C}(E),\ n = 1, 2, \cdots.$$

Since $\rho(D, D')$ is uniformly bounded, letting $n \to \infty$, we obtain

$$\rho(A_\alpha(y), A_\alpha(x)) \le \beta\ d(y,x) \quad \text{for } x, y \in E. \tag{2.6}$$

Therefore

$$\sup_{\alpha\in[0,1]} \rho(A_\alpha(y), A_\alpha(x)) \to 0 \quad (y \to x) \quad \text{for } x \in E.$$

Thus the family $\{A_\alpha(x) \mid x \in E,\ \alpha \in [0,1]\}$ satisfies the conditions (i) – (iii) of Lemma 3. By Lemma 3, we can define a fuzzy relation $\tilde{c} \in \mathcal{R}(E)$ by

$$\tilde{c}(x,y) := \sup_{\alpha\in[0,1]} \{\alpha \wedge I_{A_\alpha(x)}(y)\}, \quad x, y \in E.$$

Then $\tilde{c}_\alpha(x) = A_\alpha(x)$ $(x \in E, \alpha \in [0,1])$. Since $A_\alpha(x)$ is a unique fixed point of $T_{x,\alpha}$,

$$\lim_{n\to\infty} (\tilde{c}^n)_\alpha(x) = \lim_{n\to\infty} T^n_{x,\alpha}(\{x\}) = A_\alpha(x) = \tilde{c}_\alpha(x), \quad \alpha \in [0,1].$$

We get (iii) since the convergence is uniform in $\alpha \in [0,1]$.

Next we show that $\tilde{c}$ is a solution of (2.1). Since $\tilde{c}_\alpha(x) = A_\alpha(x)$, we note that

$$\tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(\tilde{c}_\alpha(x)) = T_\alpha(\tilde{c}_\alpha(x)) = \tilde{c}_\alpha(x), \quad \alpha \in [0,1]. \tag{2.7}$$

If $\alpha > 0$, then we have

$$\left\{y \in E \mid \tilde{q}(x,y) \vee \max_{z\in E}\{\tilde{c}(x,z) \wedge \tilde{q}(z,y)\} \geq \alpha\right\} = \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(\tilde{c}_\alpha(x)).$$

If $\alpha = 0$, then in a similar way to the proof of [2,Lemma 1] we have

$$\mathrm{cl}\left\{y \in E \mid \max_{x\in E}\{\tilde{q}(x,y) \vee \max_{z\in E}\{\tilde{c}(x,z) \wedge \tilde{q}(z,y)\} > 0\right\} = \tilde{q}_0(x) \cup \tilde{q}_0(\tilde{c}_0(x)).$$

Therefore

$$\left\{y \in E \mid \tilde{q}(x,y) \vee \max_{z\in E}\{\tilde{c}(x,z) \wedge \tilde{q}(z,y)\} \geq \alpha\right\} = \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(\tilde{c}_\alpha(x)) \quad \text{for } \alpha \in [0,1].$$

Together with (2.7), we get

$$\left\{y \in E \mid \tilde{q}(x,y) \vee \max_{z\in E}\{\tilde{c}(x,z) \wedge \tilde{q}(z,y)\} \geq \alpha\right\} = \tilde{c}_\alpha(x) \quad \text{for } \alpha \in [0,1].$$

Therefore $\tilde{c}$ satisfies (2.1). We prove the uniqueness of solution of (2.1). Let us denote by $\tilde{c}' \in \mathcal{R}(E)$ another solution of (2.1). For $x \in E,\ \alpha \in [0,1]$, it is shown similarly that $\tilde{c}'_\alpha(x) = \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(\tilde{c}'_\alpha(x))$. That is, $\tilde{c}'_\alpha(x)$ is a fixed point of $T_{x,\alpha} : \mathcal{C}(E) \to \mathcal{C}(E)$. From the uniqueness of the fixed point, we get $\tilde{c}'_\alpha(x) = \tilde{c}_\alpha(x)$ for $x \in E,\ \alpha \in [0,1]$. By Lemma 3, $\tilde{c}' = \tilde{c}$. Thus we get (i).

Finally (ii) is trivial from (2.6), using Lemma 1 and $\tilde{c}_\alpha(x) = A_\alpha(x)$ for $x \in E,\ \alpha \in [0,1]$. Thus the proof is completed. ∎

THEOREM 2. *It holds that*

$$\tilde{c}(x,y) = \bigvee_{n=1,2,\cdots} \tilde{q}^n(x,y), \quad x, y \in E. \tag{2.8}$$

Further $\tilde{c}$ is the transitive closure of the fuzzy relation $\tilde{q}$, namely $\tilde{c}$ satisfies (i) – (iii) :

(i) $\tilde{c} \geq \tilde{q}$.

(ii) *$\tilde{c}$ has the transitive property :*

$$\tilde{c}(x,y) \geq \sup_{z \in E} \{\tilde{c}(x,z) \wedge \tilde{c}(z,y)\}, \quad x,y \in E. \tag{2.9}$$

(iii) *If $\tilde{r} \in \mathcal{R}(E)$ satisfies $\tilde{r} \geq \tilde{q}$ and has the transitive property, then $\tilde{r} \geq \tilde{c}$.*

PROOF. Let $\tilde{r}(x,y) := \bigvee_{n=1,2,\cdots} \tilde{q}^n(x,y)$, $x,y \in E$. Then we have $\tilde{r} \geq \bigvee_{n=1,2,\cdots,m} \tilde{q}^n = \tilde{c}^m$ for $m = 1,2,\cdots$. Therefore $\tilde{r}_\alpha(x) \supset (\tilde{c}^m)_\alpha(x)$ for $x \in E, \alpha \in [0,1], m = 1,2,\cdots$. From (2.2) we obtain $\tilde{r}_\alpha(x) \supset \tilde{c}_\alpha(x)$ for $x \in E, \alpha \in [0,1]$. Thus we get $\tilde{r} \geq \tilde{c}$.

On the other hand, from (2.1), we obtain $\tilde{c} \geq \tilde{q}$ and

$$\tilde{c}(x,y) \geq \sup_{z \in E}\{\tilde{c}(x,z) \wedge \tilde{q}(z,y)\} \geq \sup_{z \in E}\{\tilde{q}(x,z) \wedge \tilde{q}(z,y)\} = \tilde{q}^2(x,y), \quad x,y \in E.$$

Repeating this argument inductively, we obtain $\tilde{c} \geq \tilde{q}^n$ for $n = 1,2,\cdots$. Therefore $\tilde{c}(x,y) \geq \tilde{c}^n(x,y)$ for $x,y \in E, n = 1,2,\cdots$. Thus we get $\tilde{c} \geq \tilde{r}$. Therefore we obtain (2.8).

Next we prove (i) – (iii). (i) is trivial from (2.1). From (2.8), we have

$$\begin{aligned}
\tilde{c}(x,y) &\geq \bigvee_{n=1,2,\cdots,m;\ n'=1,2,\cdots,m'} \tilde{q}^{n+n'}(x,y) \\
&= \sup_{z \in E} \left\{ \bigvee_{n=1,2,\cdots,m} \tilde{q}^n(x,z) \wedge \bigvee_{n'=1,2,\cdots,m'} \tilde{q}^{n'}(z,y) \right\} \\
&= \sup_{z \in E}\{\tilde{c}^m(x,z) \wedge \tilde{c}^{m'}(z,y)\}, \quad x,y \in E.
\end{aligned}$$

Taking the supremum over $m = 1,2,\cdots$ and $m' = 1,2,\cdots$, we obtain (ii). Finally let $\tilde{r} \in \mathcal{R}(E)$ satisfy $\tilde{r} \geq \tilde{q}$ and have the transitive property. Then

$$\tilde{r}(x,y) \geq \sup_{z \in E}\{\tilde{r}(x,z) \wedge \tilde{r}(z,y)\} \geq \sup_{z \in E}\{\tilde{q}(x,z) \wedge \tilde{q}(z,y)\} = \tilde{q}^2(x,y), \quad x,y \in E.$$

Repeating this argument inductively, we obtain $\tilde{r} \geq \tilde{q}^n$ for $n = 1,2,\cdots$. Therefore $\tilde{r}(x,y) \geq \tilde{c}^n(x,y)$ for $x,y \in E, n = 1,2,\cdots$. Thus we get $\tilde{r} \geq \tilde{c}$. Therefore (iii) holds. The proof is completed. ■

3. Numerical example

Let $E = [-2,2]$ be a space of states. We consider a fuzzy relation (see [2,Figure 1])

$$\tilde{q}(x,y) = 1 - \left| y - \left(\frac{x}{2} + \frac{1}{4} \right) \right|, \quad x,y \in E. \tag{3.1}$$

Then we have

$$\tilde{q}^n(x,y) = 1 - \left| y - \left(\frac{1}{2^n}x + \frac{1}{2} - \frac{1}{2^{n+1}} \right) \right| \Big/ \left(2 - \frac{1}{2^{n-1}} \right), \quad x,y \in E, n = 1,2,\cdots. \tag{3.2}$$

From Theorem 2, we obtain

$$\tilde{c}(x,y) = \bigvee_{n=1,2,\cdots} \left\{1 - \left|y - \left(\frac{1}{2^n}x + \frac{1}{2} - \frac{1}{2^{n+1}}\right)\right| \Big/ \left(2 - \frac{1}{2^{n-1}}\right)\right\}, \quad x, y \in E. \qquad (3.3)$$

Then (3.3) is the unique solution of Theorem 1 (see Figure 1).

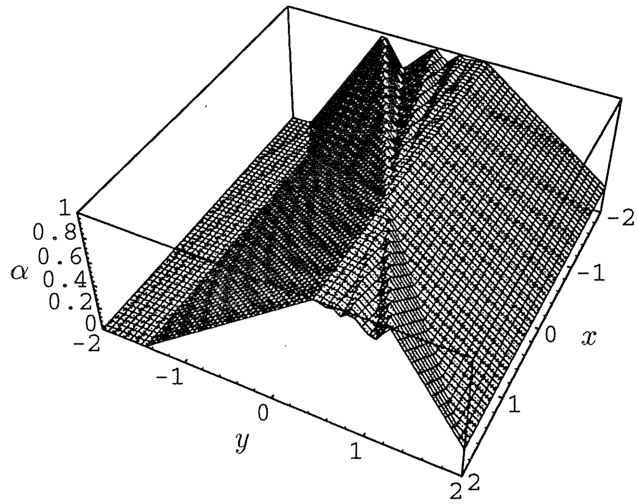

Fig. 1 : The transitive closure $\tilde{c}(x,y)$.

References

[1] KURATOWSKI, K.: *Topology I*, Academic Press, New York, (1966).

[2] KURANO, M., YASUDA, M., NAKAGAMI, J. and YOSHIDA, Y.: *A limit theorem in some dynamic fuzzy systems*, Fuzzy Sets and Systems, **51** (1992), 83-88.

[3] NANDA, S.: *On sequences of fuzzy numbers*, Fuzzy Sets and Systems, **33** (1989), 123-126.

[4] NOVÀK, U.: *Fuzzy Sets and Their Applications*, Adam Hilder, Bristol-Boston, (1989).

[5] ZIMMERMANN, H.-J.: *Fuzzy Set Theory and its Applications*, Kluwer-Nijhoff, Boston, (1985).