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## Inductive Inference of Recursive Concepts

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# Inductive Inference of Recursive Concepts

December 1993

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# Abstract

# Inductive Inference of Recursive Concepts


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Inductive inference is a process of hypothesizing a general rule from examples. As a successful inference criterion for inductive inference of formal languages and models of logic programming, we have mainly used Gold's identification in the limit. An inference machine $M$ is said to infer a concept $L$ in the limit, if the sequence of guesses from $M$ which is successively fed a sequence of examples of $L$ converges to a correct expression of $L$, that is, all guesses from $M$ become a unique expression in a finite time and that the expression is a correct one. A class, or a hypothesis space, is said to be inferable in the limit, if there is an inference machine $M$ which infers every concept in the class.

In the present thesis, we mainly investigate three criteria related to the identification in the limit. As we will see, they are necessary for practical applications of machine learning or machine discovery.

The first criterion requires an inference machine to produce a unique guess. That is, we apply so-called finite identification to concept learning. As stated above, ordinary inductive inference is an infinite process. Thus we can not decide in general whether a sequence of guesses from an inference machine has converged or not at a certain time. To the contrary, in the criterion of finite identification, if an inference machine produces a guess, then it is a conclusive answer.

The second criterion requires an inference machine to refute a hypothesis space in question, if a target concept is not in the hypothesis space. In the ordinary inductive inference,

the behavior of an inference machine is not specified, when we feed examples of a target concept not belonging to the hypothesis space. That is, we implicitly assume that every target concept belongs to the hypothesis space. As far as data or facts are presented according to a concept that is unknown but guaranteed to be in the hypothesis space, the machine will eventually identify the hypothesis. However this assumption is not appropriate, if we want an inference machine to infer or to discover an unknown rule which explains examples or data obtained from scientific experiments. Thus we propose a successful inference criterion where, if there is no concept in the hypothesis space which coincides with a target concept, then an inference machine explicitly tells us this and stops in a finite time.

The third criterion requires an inference machine to infer a minimal concept within the hypothesis space concerned. In actual applications of inductive inference, there are many cases where we want an inference machine to infer an approximate concept within the hypothesis space, even when there is no concept which exactly coincides with the target concept. Here we take a minimal concept as an approximate concept within the hypothesis space, and discuss inferability of a minimal concept of the target concept which may not belong to the hypothesis space. That is, we force an inference machine to converge to an expression of a minimal concept of the target concept, if there is a minimal concept of the target concept within the hypothesis space.

In the present thesis, we discuss inferability of recursive concepts under the above three criteria, and show some necessary and sufficient conditions for inferability and some comparisons between inferable classes. Furthermore as practical and concrete hypothesis spaces, we take the classes definable by so-called length-bounded elementary formal systems (EFS's, for short) and discuss their inferability in the above three criteria. In 1990, Shinohara showed that the classes definable by length-bounded EFS's with at most $n$ axioms are inferable in the limit from positive data for any $n \geq 1$. In the present thesis, we show that the above classes are also refutably inferable from complete data, i.e. positive and negative data, as well as minimally inferable from positive data. This means that there are rich hypothesis spaces that are refutably inferable from complete data or minimally inferable from positive data.

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# Contents

# Chapter 1.

# Introduction

To research how computers assist human activities is an essential and important *aspect* of computer science. In some cases, a computer may be a tool for calculation, or in other cases, it may be a tool for retrieving a data base. Another important role for computer scientists is to make computers behave more intelligently, that is, to make computers act like human beings. The main purpose of the present thesis is to make computers assist human's scientific researches more intelligently in the framework of inductive inference of recursive concepts.

Inductive inference is a process of hypothesizing a general rule from examples. In this thesis, we call a subset of a recursively enumerable universal set $U$ a *concept*. As a successful inference criterion for inductive inference of formal languages and models of logic programming, we have mainly used Gold's identification in the limit[12]. An inference machine $M$ is said to *infer a concept $L$ in the limit*, if the sequence of guesses from $M$ which is successively fed a sequence of examples of $L$ converges to a correct expression of $L$, that is, all guesses from $M$ become a unique expression in a finite time and that the expression is a correct one. For a presentation, i.e. a sequence of examples, of a concept, we consider the following two types of presentations. A *positive presentation*, or a *text*, of a concept $L$ is an infinite sequence of all elements in $L$. A *complete presentation*, or an *informant*, $\sigma$ of a concept $L$ is an infinite sequence of elements in $U \times \{+, -\}$ such that the collection of all elements in $\sigma$ with the sign '+' coincides with $L$ and that the collection of all elements in $\sigma$ with the sign '−' coincides with the complement of $L$. We assume that every class in question is an *indexed family of recursive concepts*. This assumption is quite natural to make a grammar, i.e. a hypothesis or theory, refutable by an observation, and also to generate grammars as hypotheses automatically and successively.

Under the criterion of identification in the limit, many productive results on inferability of recursive concepts have been reported by Angluin[2], Shapiro[41], Wright[51], Shinohara[45, 46], Sato&Moriyama[39] and so on.

In the present thesis, we mainly investigate three criteria related to the identification in the limit. As we will see, they are necessary for practical applications of machine learning or machine discovery.

The first criterion requires an inference machine to produce a unique guess. That is, we apply so-called finite identification to concept learning. As stated above, ordinary inductive inference is an infinite process. Thus we can not decide in general whether a sequence of guesses from an inference machine has converged or not at a certain time. To the contrary, in the criterion of finite identification, if an inference machine produces a guess, then it is a conclusive answer.

The second criterion requires an inference machine to refute a hypothesis space in question, if a target concept is not in the hypothesis space. In the ordinary inductive inference, the behavior of an inference machine is not specified, when we feed examples of a target concept not belonging to the hypothesis space. That is, we implicitly assume that every target concept belongs to the hypothesis space. As far as data or facts are presented according to a concept that is unknown but guaranteed to be in the hypothesis space, the machine will eventually identify the hypothesis. However this assumption is not appropriate, if we want an inference machine to infer or to discover an unknown rule which explains examples or data obtained from scientific experiments. Thus we propose a successful inference criterion where, if there is no concept in the hypothesis space which coincides with a target concept, then an inference machine explicitly tells us this and stops in a finite time.

The third criterion requires an inference machine to infer a minimal concept within the hypothesis space concerned. In actual applications of inductive inference, there are many cases where we want an inference machine to infer an approximate concept within the hypothesis space, even when there is no concept which exactly coincides with the target concept. Here we take a minimal concept as an approximate concept within the hypothesis space, and discuss inferability of a minimal concept of the target concept which may not belong to the hypothesis space. That is, we force an inference machine to converge to an expression of a minimal concept of the target concept, if there is a minimal concept of the target concept within the hypothesis space.

Furthermore as practical and concrete hypothesis spaces, we take the classes definable by *elementary formal systems* (*EFS*'s, for short) and discuss their inferability in the above three criteria. The EFS's were originally introduced by Smullyan[48] to develop his recursion theory. In a word, EFS's are a kind of logic programming language which uses strings instead of terms in first order logic[52], and they are shown to be natural devices to define languages[4]. In 1990, Shinohara[45] showed that the classes definable by so-called length-bounded EFS's with at most $n$ axioms are inferable in the limit from positive data for any $n \geq 1$. In the present thesis, we show that the above classes are also refutably inferable from complete data, i.e. positive and negative data, as well as minimally inferable from positive data. This means that there are rich hypothesis spaces that are refutably inferable from complete data or minimally inferable from positive data.

This thesis is organized as follows: In Chapter 2 we prepare some definitions and notions necessary for our discussions, and review related results mainly due to Gold[12], Angluin[2], Wright[51] and Sakurai[37].

In Chapter 3 we consider an inductive inference where the number of mind changes of an inference machine is bounded by a constant number. First, we discuss inferability without any mind changes, that is, finite inferability of a class from positive data or complete data. Here we present a necessary and sufficient condition for a class to be finitely inferable from positive data or complete data. We also present examples of classes that are finitely inferable from positive data or complete data. By extending this results, when the equivalence of any two concepts in the class is recursively decidable, we show a necessary and sufficient condition for a class to be inferable within $n$ mind changes from positive data or complete data for $n \geq 0$. We also present examples of classes that are inferable within $n$ mind changes but not inferable within $n-1$ mind changes, and show that the inferability strictly increases, when the allowed number of mind changes increases. We review further results obtained by Lange&Zeugmann[21]. They discussed class-preserving learning from positive data or complete data. A class is said to be *class-preservingly learnable*, if the class is inferable with well-chosen indexing of the class. They showed the superiority of class-preserving learning and uniform characterizations of inferability with a bounded number of mind changes, and also showed that there are various hierarchies.

In Chapter 4 we discuss both refutability and inferability of a hypothesis space from examples. First we discuss some conditions on refutable inferability from positive data or complete data. Concerning refutable inferability from *positive data*, we present some necessary and sufficient conditions, and reveal that the power is very small. Then we show the differences between the inferable classes under the criteria of refutable identification, reliable identification, finite identification and identification in the limit. Among them the reliable identification is only the inference that deals with sequences from concepts not in a hypothesis space in question. However as we are seeing in Section 2.3, the reliable inference machine does not tell us that the target concept is not in the hypothesis space, but it just does not converge to any of concept in the hypothesis space. Then we show that a class which consists of unions of at most $n$ concepts from $n$ classes is refutably inferable from complete data, if each class satisfies a certain condition.

In Chapter 5 we discuss some sufficient conditions for a class to be minimally inferable from positive data. In 1989, Wright[51] showed that if a class has so-called finite elasticity, then the class is inferable in the limit from positive data. On the other hand, Sato&Moriyama[39] introduced the notion of M-finite thickness to show another condition for inferability from positive data. Here we show that the classes with both finite elasticity and M-finite thickness are minimally inferable from positive data. We also reveal the differences between the powers of inference machines whose behaviors differs from each other when there is no minimal concept of the target concept in the class concerned.

In Chapter 6 we adopt the classes definable by EFS's as practical and concrete hypothesis spaces, and discuss refutable inferability and minimal inferability of them. We show that the classes definable by length-bounded EFS's with at most $n$ axioms are refutably inferable from complete data, and reveal that there are sufficiently large classes that are refutably inferable from complete data. Furthermore we show that the above classes are also minimally inferable from positive data.

# Chapter 2.

# Preliminaries

This chapter gives definitions of ordinary inductive inference of recursive concepts and summarizes related results mainly due to Gold[12], Angluin[2], Wright[51] and Sakurai[37].

In Section 2.1 we review the definition of the criterion of identification in the limit and basic results of inferability from positive data or complete data. Here we show that every class is inferable in the limit from complete data, but the so-called super-finite classes are not inferable in the limit from positive data. In Section 2.2 we review characterization theorems and some sufficient conditions for inferability from positive data due to Angluin[2] and Wright[51]. In Section 2.3 we review definitions and characterization theorems on (semi-) reliable identification due to Sakurai[37], which has relations with definitions in Chapter 4 and 5.

In what follows, for a set $S$, $S^+$ denotes the set of all nonnull finite strings over $S$, and $\sharp S$ denotes the cardinality of $S$. For a finite sequence $\psi$, $\widetilde{\psi}$ denotes the set of all components in $\psi$, and $\sharp\psi$ denotes the length of $\psi$.

## 2.1. Inductive Inference of Recursive Concepts

We start with basic definitions and notions on inductive inference of indexed families of recursive concepts.

Let $U$ be a recursively enumerable set to which we refer as a *universal set*. Then we call $L \subseteq U$ a *concept*. In case the universal set $U$ is the set $\Sigma^+$ of all nonnull finite strings over a finite alphabet $\Sigma$, we also call $L \subseteq U$ a *language*.

**Definition 2.1.** Let $N = \{1, 2, \cdots\}$ be the set of all natural numbers. A class $\mathcal{C} = \{L_i\}_{i \in N}$ of concepts is said to be an *indexed family of recursive concepts*, if there is a recursive function $f : N \times U \to \{0, 1\}$ such that

$$f(i, w) = \begin{cases} 1, & \text{if } w \in L_i, \\ 0, & \text{otherwise.} \end{cases}$$

In what follows, we assume that a class of concepts is an indexed family of recursive concepts without any notice, and identify a class with a hypothesis space.

**Definition 2.2.** A *positive presentation*, or a *text*, of a *nonempty* concept $L$ is an infinite sequence $w_1, w_2, \cdots$ of elements in the universal set $U$ such that $\{w_1, w_2, \cdots\} = L$. A *complete presentation*, or an *informant*, of a concept $L$ is an infinite sequence $(w_1, t_1), (w_2, t_2), \cdots$ of elements in $U \times \{+, -\}$ such that $\{w_i \mid t_i = +, i \geq 1\} = L$ and $\{w_i \mid t_i = -, i \geq 1\} = L^c (= U \setminus L)$. In what follows, $\sigma$ or $\delta$ denotes a positive or complete presentation, and $\sigma[n]$ denotes the $\sigma$'s initial segment of length $n \geq 0$. For a positive or complete presentation $\sigma$, each element in $\sigma$ is called a *fact*. For a positive presentation $\sigma$, $\sigma[n]^+$ denotes the set of all facts in $\sigma[n]$. For a complete presentation $\sigma$, $\sigma[n]^+$ (resp., $\sigma[n]^-$) denotes the set of all elements in the universal set $U$ that appear in $\sigma[n]$ with the sign '+' (resp., the sign '−'), that is, $\sigma[n]^+ = \{w_i \mid (w_i, +) \in \sigma[n]\}$ and $\sigma[n]^- = \{w_i \mid (w_i, -) \in \sigma[n]\}$.

A set $T$ is said to be *consistent* with a concept $L$, if $T \subseteq L$. A pair $(T, F)$ of sets is said to be *consistent* with a concept $L$, if $T \subseteq L$ and $F \subseteq L^c$. For a positive presentation $\sigma$ and for $n \geq 0$, the finite sequence $\sigma[n]$ is said to be *consistent* with a concept $L$, if $\sigma[n]^+ \subseteq L$. For a complete presentation $\sigma$ and for $n \geq 0$, the finite sequence $\sigma[n]$ is said to be *consistent* with a concept $L$, if $\sigma[n]^+ \subseteq L$ and $\sigma[n]^- \subseteq L^c$.

For two sequences $\psi_1$ and $\psi_2$, the sequence which is obtained by concatenating $\psi_1$ with $\psi_2$ is denoted by $\psi_1 \cdot \psi_2$.

Here we note that for a class $\mathcal{C} = \{L_i\}_{i \in N}$ and explicitly given finite sets $T, F \subseteq U$, whether $T \subseteq L_i$ or not and whether $(T, F)$ is consistent with $L_i$ or not are recursively decidable for any index $i$, because for any $w \in U$, whether $w \in L_i$ or not is recursively decidable.

The following Definition 2.3 shows a method of obtaining a positive or complete presentation of a concept (cf. Lange&Zeugmann[21]).

**Definition 2.3.** Let $w_1, w_2, \cdots$ be an effective enumeration of the universal set $U$. For a nonempty concept $L$, let $i_0 = \min\{i \mid w_i \in L\}$, and for $i \geq 1$, let $v_i = w_i$ if $w_i \in L$, otherwise let $v_i = w_{i_0}$. Then the infinite sequence $v_1, v_2, \cdots$ is called the *canonical positive presentation* of $L$.

For a concept $L$ and for $i \geq 1$, let $t_i =$ '$+$' if $w_i \in L$, otherwise let $t_i =$ '$-$'. Then the infinite sequence $(w_1, t_1), (w_2, t_2) \cdots$ is called the *canonical complete presentation* of $L$.

An *inductive inference machine* (*IIM*, for short) is an effective procedure, or a certain type of Turing machine, which requests inputs from time to time and produces positive integers from time to time.

Angluin[2] defines an IIM like this: An inductive inference machine is a deterministic Turing machine with input alphabet $\Sigma$, a finite tape alphabet $\Delta$, and several one-way infinite tapes, i.e. a read-only *sample tape*, a write-only *guess tape*, and a finite number of read-write *scratch tapes.* Each tape is equipped with one head, which is initially positioned at the first square of the tape. The machine has a finite number of *states*, of which there are four distinct distinguished states: the *initial state*, the *request state*, the *answer state*, and the *guess state.* The *finite control* of the machine consists of a finite function that specifies for each tuple consisting of a state of the machine (other than the request state) and the symbols currently scanned by the heads on the sample and scratch tapes, a *move*, consisting of a state of the machine, symbols from $\Delta$ to write at the currently scanned squares on the scratch and guess tapes, and a specification for each of the tapes whether its head should be shifted one square to the right, or left or not at all. We stipulate that no move of the machine may specify that the guess tape head be shifted left, and any move of the machine that writes a nonblank symbol on the guess tape must also shift the guess tape head right.

Hereafter we will construct inductive inference machines in forms of ALGOL-like programs, but we can translate them into the above form of Turing machines. Simply we consider as follows: If it is in the request state, then it is requesting and reading the next fact (or the next coded string over $\Sigma$), and if it goes through the guess state, then it produces an integer (or a coded string over $\Delta$).

The outputs produced by an inductive inference machine are called *guess*es.

For an IIM $M$ and a nonempty initial segment $\sigma[n] = w_1, w_2, \cdots, w_n$ of a positive or complete presentation, we define $M(\sigma[n])$ as follows: Initialize $M$ and start $M$ in the initial state. If it requests a fact for the $i$-th time with $1 \le i \le n$, then feed $w_i$ and continue the execution. If it produces more than one positive integers between any two input requests before requesting the $(n+1)$-st fact, then leave $M(\sigma[n])$ undefined.

(I) In case $M$ requests the $(n+1)$-st fact, or it stops after it requested the $n$-th fact. If it produces a positive integer after it requested the $n$-th fact, then let $M(\sigma[n])$ be the last integer produced by $M$, otherwise let $M(\sigma[n]) = 0$.

(II) In case $M$ stops before requesting the $n$-th fact. Let $M(\sigma[n]) = 0$.

(III) Otherwise. Leave $M(\sigma[n])$ undefined.

By definition, it is easy to see that for any $n \ge 1$, if $M(\sigma[n])$ is defined, then $M(\sigma[1])$, $M(\sigma[2]), \cdots, M(\sigma[n-1])$ are also defined. The intended interpretation is as follows: (i) In case $M(\sigma[n])$ is defined as a positive integer $i$, the IIM $M$ guesses the $i$-th concept in the class concerned. (ii) In case $M(\sigma[n])$ is defined as the integer 0, the IIM $M$ makes no guess. (iii) In case $M(\sigma[n])$ is undefined, the IIM $M$ is out of control.

Then we define $\overline{M}(\sigma[n])$ as follows:

(I) In case $M(\sigma[n])$ is defined. If there is a positive integer in the sequence $M(\sigma[1])$, $M(\sigma[2]), \cdots, M(\sigma[n])$, then let $\overline{M}(\sigma[n])$ be the last positive integer in the sequence, otherwise let $\overline{M}(\sigma[n]) = 0$.

(II) Otherwise. Leave $\overline{M}(\sigma[n])$ undefined.

For two nonempty finite sequences $\psi_1$ and $\psi_2$, we write $M(\psi_1) = M(\psi_2)$, if (i) both $M(\psi_1)$ and $M(\psi_2)$ are undefined, or (ii) both $M(\psi_1)$ and $M(\psi_2)$ are defined and their values are identical. For a nonempty finite sequence $\psi$ and an integer $i$, we write $M(\psi) = i$ (resp., $M(\psi) > i$), if $M(\psi)$ is defined and the value of $M(\psi)$ is equal to $i$ (resp., greater than $i$). In a similar way, we also define the relations $\overline{M}(\psi_1) = \overline{M}(\psi_2)$, $\overline{M}(\psi) = i$ and $\overline{M}(\psi) > i$.

**Definition 2.4** (Gold[12]). An IIM $M$ is said to *converge to an index $i$ for a positive or complete presentation $\sigma$*, if there is an $n \ge 1$ such that for any $m \ge n$, $\overline{M}(\sigma[m]) = i$.

Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. For a concept $L_i \in \mathcal{C}$ and a positive or complete presentation $\sigma$ of $L_i$, an IIM $M$ is said to *infer the concept $L_i$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$*, if $M$ converges to an index $j$ with $L_j = L_i$ for $\sigma$.

An IIM $M$ is said to *infer a class $\mathcal{C}$ in the limit from positive data* (resp., *complete data*), if for any $L_i \in \mathcal{C}$, $M$ infers $L_i$ w.r.t. $\mathcal{C}$ in the limit from any positive presentation $\sigma$ (resp., any complete presentation $\sigma$) of $L_i$. A class $\mathcal{C}$ is said to be *inferable in the limit from positive data* (resp., *complete data*), if there is an IIM $M$ which infers the class $\mathcal{C}$ in the limit from positive data (resp., complete data).

When we consider inductive inference from positive data, we restrict every concept to a *nonempty* concept, because we can not make any positive presentation of the empty concept.

Here we note that if an inference machine $M$ does not converge to any index for a positive or complete presentation $\sigma$, then $\overline{M}(\sigma[n])$ may be undefined for some $n \geq 1$. On the other hand, we implicitly use the following Proposition 2.1 in showing some properties on various inferability.

**Proposition 2.1.** *(a) Assume that an IIM $M$ infers a class $\mathcal{C}$ in the limit from positive data. Then for any nonempty finite sequence $\psi$ consisting of elements in $U$, if there exists an $L_i \in \mathcal{C}$ such that $\widetilde{\psi} \subseteq L_i$, then $\overline{M}(\psi)$ is always defined.*

*(b) Assume that an IIM $M$ infers a class $\mathcal{C}$ in the limit from complete data. Then for any nonempty finite sequence $\psi$ consisting of elements in $U \times \{+, -\}$, if there exists an $L_i \in \mathcal{C}$ such that $\{w \mid (w, +) \in \psi\} \subseteq L_i$ and $\{w \mid (w, -) \in \psi\} \subseteq L_i^c$, then $\overline{M}(\psi)$ is always defined.*

*Proof.* We only give the proof of (a). The proof of (b) can be given in a similar way.

Assume that there are a nonempty finite sequence $\psi$ and an $L_i \in \mathcal{C}$ such that $\widetilde{\psi} \subseteq L_i$. Let $\sigma$ be an arbitrary positive presentation of $L_i$, and put $\delta = \psi \cdot \sigma$. Then $\delta$ is a positive presentation of $L_i$. Therefore $M$ converges to an index $j$ with $L_j = L_i$ for $\delta$, and it follows by definition that for any $n \geq 1$, $M(\delta[n])$ is defined. Thus $\overline{M}(\psi)$ is also defined. ∎

The above Proposition 2.1 claims that as far as we feed facts that are from a certain concept in the class, an inference machine either (i) successively requests another facts in a finite time forever or (ii) stops in a finite time after producing some positive integers.

The following Proposition 2.2 is obvious, because for a complete presentation $\sigma$ of a *nonempty* concept $L$, we can effectively obtain a positive presentation of $L$ by getting rid of all negative facts of $\sigma$ and repeating a positive fact.

**Proposition 2.2.** *If a class $\mathcal{C}$ is inferable in the limit from positive data, then $\mathcal{C}$ is also inferable in the limit from complete data.*

In this thesis, we implicitly use the following Lemma 2.3.

**Lemma 2.3.** *(a) For a concept $L$ and a set $T \subseteq U$, if $T$ is not consistent with $L$, then $T'$ is not consistent with $L$ for any set $T'$ with $T \subseteq T' \subseteq U$.*

*(b) For a concept $L$ and sets $T, F \subseteq U$, if $(T, F)$ is not consistent with $L$, then $(T', F')$ is not consistent with $L$ for any sets $T', F'$ with $T \subseteq T' \subseteq U$ and $F \subseteq F' \subseteq U$.*

By a simple enumerative method as shown below, every indexed family of recursive concepts is always inferable in the limit from *complete data*.

**Theorem 2.4** (Gold[12]). *Every indexed family $\mathcal{C} = \{L_i\}_{i \in N}$ of recursive concepts is inferable in the limit from complete data.*

*Proof.* Let us consider the procedure in Figure 2.1.

Assume that we feed a complete presentation $\sigma$ of a concept $L_h \in \mathcal{C}$ to the procedure, and let $i_0$ be the least index such that $L_{i_0} = L_h$. Then for any index $i < i_0$, $L_h \neq L_i$ holds, and it follows that there is a $w_i \in U$ such that $w_i \in L_h \setminus L_i$ or $w_i \in L_i \setminus L_h$. Since $\sigma$ is a complete presentation of $L_h$, it follows that for any index $i < i_0$, there is an $n_i \geq 1$ such that $w_i \in \sigma[n_i]^+$ or $w_i \in \sigma[n_i]^-$. Therefore after reading the $\max\{n_1, n_2, \cdots, n_{i_0-1}\}$-st fact, $(T, F)$ in the procedure is *not* consistent with $L_i$, and it follows that the index $i$ in the procedure reaches $i_0$. Furthermore for any $n \geq 1$, $\sigma[n]$ is consistent with $L_{i_0}$, because $\sigma$ is a complete presentation of $L_h = L_{i_0}$. Hence the procedure converges to $i_0$ for $\sigma$. This completes the proof. ■

**Definition 2.5.** Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. An IIM $M$ is said to be *consistently working*, if it satisfies the following condition: For any $L_i \in \mathcal{C}$, any positive or complete presentation

```
Procedure IIM M;
begin
    T = φ;    F = φ;    i = 1;
    repeat
        read_store(T, F);
        while (T, F) is not consistent with L_i do i = i + 1;
        output i;
    forever;
end;

Procedure read_store(T, F);
begin
    read the next fact (w, t);
    if t = '+' then T = T ∪ {w} else F = F ∪ {w};
end.
```

Figure 2.1: An inference machine which works on complete data

$\sigma$ of $L_i$ and any $n \geq 1$, if $M(\sigma[n]) > 0$, then $\sigma[n]$ is consistent with $L_{M(\sigma[n])}$, that is, each guess by $M$ on input $\sigma$ is consistent with all the input data read so far.

An IIM $M$ is said to be *responsively working*, if it satisfies the following condition: For any $L_i \in \mathcal{C}$, any positive or complete presentation $\sigma$ of $L_i$ and any $n \geq 1$, $M(\sigma[n]) > 0$ holds, that is, between any two input requests in the computation of $M$ on input $\sigma$, $M$ produces a guess.

An IIM $M$ is said to be *conservatively working*, if it satisfies the following condition: For any $L_i \in \mathcal{C}$, any positive or complete presentation $\sigma$ of $L_i$ and any $n, m$ with $1 \leq n < m$, if $M(\sigma[n]) > 0$, $M(\sigma[m]) > 0$ and $M(\sigma[m]) \neq M(\sigma[n])$ hold, then $\sigma[m]$ is *not* consistent with $L_{M(\sigma[n])}$, that is, $M$ never changes its guess as long as it is consistent with all the input data read so far.

A class $\mathcal{C}$ is said to be *consistently* (resp., *responsively* or *conservatively*) *inferable in the limit from positive data* or *complete data*, if there is a consistently (resp., responsively or conservatively) working IIM which infers $\mathcal{C}$ in the limit from positive data or complete data.

The procedure in Figure 2.1 is a consistently, responsively and conservatively working IIM. Thus every indexed family of recursive concepts are consistently, responsively and conservatively inferable in the limit from complete data.

On the other hand, Gold[12] showed that a class of all finite concepts and at least one infinite concept is *not* inferable in the limit from positive data. In what follows, such classes are said to be *superfinite.*

**Theorem 2.5** (Gold[12]). *None of superfinite classes is inferable in the limit from positive data.*

*Proof.* Let $\mathcal{C}$ be a superfinite class. Then suppose that an IIM $M$ infers $\mathcal{C}$ in the limit from positive data. Let $L_h \in \mathcal{C}$ be an infinite concept, and let $w_1, w_2, \cdots$ be an effective enumeration of all elements in $L_h$. Without loss of generality, we assume $w_i \neq w_j$ if $i \neq j$. Then for any $i \geq 1$, there is an index $j_i$ such that $L_{j_i} = \{w_1, w_2, \cdots, w_i\}$, because $\mathcal{C}$ contains all finite concepts.

We show that $M$ changes its guess infinitely many times for a certain positive presentation of $L_h$. Put $\sigma_1 = w_1, w_1, \cdots$, and define $n_i$'s and $\sigma_{i+1}$'s $(i \geq 1)$ inductively by the following stages:

Stage $i$ $(\geq 1)$:

Since $\sigma_i$ is a positive presentation of $L_{j_i}$, there is an $n \geq 1$ such that $L_{j_i} = L_g$ and $g = \overline{M}(\sigma_i[n_1+n_2+\cdots+n_{i-1}+n])$ for some $g \geq 1$. Put $n_i = n$ and $\sigma_{i+1} = \sigma_i[n_1+n_2+\cdots+n_i], w_{i+1}, w_{i+1}, \cdots = \underbrace{w_1, w_1, \cdots, w_1}_{n_1}, \underbrace{w_2, w_2, \cdots, w_2}_{n_2}, \cdots, \underbrace{w_i, w_i, \cdots, w_i}_{n_i}, w_{i+1}, w_{i+1}, \cdots$. We note that this $\sigma_{i+1}$ becomes a positive presentation of $L_{j_{i+1}}$.

Goto Stage $i+1$.

Now we take an infinite sequence $\sigma = \underbrace{w_1, w_1, \cdots, w_1}_{n_1}, \underbrace{w_2, w_2, \cdots, w_2}_{n_2}, \cdots$. Then $\sigma$ is a positive presentation of $L_h$. However $M$ changes its guess infinitely many times for $\sigma$. Therefore $M$ does not infer $L_h$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$. This contradicts the assumption. ∎

By the above Theorem 2.5, we see that even the class of regular languages is not inferable in the limit from positive data. This result gave a negative impression to researchers in this field. In the following Section 2.2, we review the results due to Angluin[2], which gave a new life to inductive inference from positive data.

## 2.2. Inductive Inference from Positive Data

In this section we mainly review a characterization theorem for inferability from *positive data* due to Angluin[2].

**Definition 2.6** (Angluin[2]). Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. A set $T_i \subseteq U$ is said to be a *finite tell-tale of* $L_i$ *within* $\mathcal{C}$, if (i) $T_i$ is a finite subset of $L_i$ and (ii) for any $L_j \in \mathcal{C}$, $T_i \subseteq L_j$ implies $L_j \not\subsetneq L_i$.

An indexed family $\{T_i\}_{i \in N}$ of finite sets is said to be *finite tell-tales of* $\mathcal{C}$, if for any index $i$, $T_i$ is a finite tell-tale of $L_i$ within $\mathcal{C}$.

In the present thesis, an effective procedure $Q$ is said to *uniformly and recursively enumerate* an indexed family $\{T_i\}_{i \in N}$ of sets, if $Q$ on input $i$ enumerates all elements in $T_i$ for any index $i$. An indexed family $\{T_i\}_{i \in N}$ of sets is said to be *uniformly and recursively enumerable*, if there is an effective procedure which uniformly and recursively enumerates the family $\{T_i\}_{i \in N}$.

An effective procedure is said to *recursively generate* a finite set $T$, if it enumerates all elements in $T$ and *then halts*. An effective procedure $P$ is said to be *uniformly and recursively generate* a finite-set-valued function $F$ with parameters $x_1, \cdots, x_n$, if $P$ on any input $(x_1, \cdots, x_n)$ in the domain of $F$ recursively generates the finite set $F(x_1, \cdots, x_n)$. A finite-set-valued function $F$ with parameters $x_1, \cdots, x_n$ is said to be *uniformly and recursively generable*, if there is an effective procedure which uniformly and recursively generates the function $F$. An effective procedure $P$ is said to *uniformly and recursively generate* an indexed family $\{T_i\}_{i \in N}$ of *finite sets*, if $P$ on input $i$ recursively generates the finite set $T_i$ for any index $i$. An indexed family $\{T_i\}_{i \in N}$ of *finite sets* is said to be *uniformly and recursively generable*, if there is an effective procedure which uniformly and recursively generates the family $\{T_i\}_{i \in N}$.

**Theorem 2.6** (Angluin[2]). *A class* $\mathcal{C} = \{L_i\}_{i \in N}$ *is inferable in the limit from positive data, if and only if finite tell-tales of* $\mathcal{C}$ *are uniformly and recursively enumerable.*

*Proof* (Sketch). (I) The 'only if' part. Assume that an IIM $M$ infers $\mathcal{C}$ in the limit from positive data. Then finite tell-tales of $\mathcal{C}$ are uniformly and recursively enumerable by the procedure in Figure 2.2. In the procedure, we note that both $\overline{M}(\psi)$ and $\overline{M}(\psi \cdot \psi_j)$ are always defined for any $j \geq 1$ (cf. Proposition 2.1). We omit the details.

---

**Procedure** $Q(i)$;
**begin**
    let $w_1, w_2, \cdots$ be an effective enumeration of all elements in $L_i$;
    let $\psi_1, \psi_2, \cdots$ be an effective enumeration of
        all nonempty finite sequences consisting of elements in $L_i$;
    $\psi = w_1$;    $n = 1$;
    output $\psi$;
    **repeat**
        search for an index $j$ such that $\overline{M}(\psi) \neq \overline{M}(\psi \cdot \psi_j)$;
            /* It makes no matter, even if it does not terminate */
        **if** such an index $j$ is found **then begin**
            output $\psi_j$ and $w_{n+1}$;
            $\psi = \psi \cdot \psi_j \cdot w_{n+1}$;    $n = n + 1$;
        **end**;
    **forever**;
**end**.

---

Figure 2.2: An algorithm which uniformly and recursively enumerates finite tell-tales

(II) The 'if' part. Assume that a procedure $Q$ uniformly and recursively enumerates finite tell-tales of $\mathcal{C}$, and for $n \geq 1$, let $Q^{(n)}(i)$ be the set of elements produced by the procedure $Q$ on input $i$ in the first $n$ steps execution. Then let us consider the procedure in Figure 2.3.

For any $L_i \in \mathcal{C}$ and any positive presentation $\sigma$ of $L_i$, we can show that the procedure converges to $i_0$ for $\sigma$, where $i_0$ is the least index such that $L_{i_0} = L_i$. Here we note that finite tell-tales play an important role in avoiding overgeneralization. We omit the details.

■

The following Corollary 2.7 is obvious (cf. Theorem 2.13).

```
Procedure IIM M;
begin
    T = φ;    n = 0;
    repeat
        read the next fact and store it in T;
        n = n + 1;
        search for the least index i ≤ n such that Q^(n)(i) ⊆ T ⊆ L_i;
        if such an index i is found then output i else output n;
    forever;
end.
```

Figure 2.3: An inference machine which works on positive data

**Corollary 2.7.** *If a class $\mathcal{C} = \{L_i\}_{i \in N}$ contains no infinite concept, then $\mathcal{C}$ is inferable in the limit from positive data.*

*Proof.* Assume that a class $\mathcal{C}$ contains no infinite concept. It is easy to see that for any $L_i \in \mathcal{C}$, $L_i$ itself is a finite tell-tale of $L_i$ within $\mathcal{C}$. Then the procedure in Figure 2.4 uniformly and recursively enumerates finite tell-tales of $\mathcal{C}$, and thus $\mathcal{C}$ is inferable in the limit from positive data. ∎

```
Procedure Q(i);
begin
    let w_1, w_2, ··· be an effective enumeration of the universal set U;
    j = 1;
    repeat
        if w_j ∈ L_i then output w_j;
        j = j + 1;
    forever;
end.
```

Figure 2.4: An algorithm which uniformly and recursively enumerates finite tell-tales

Here we note that we can modify the procedure in Figure 2.3 so as to work consistently and responsively. Thus the requirement of both consistency and responsiveness does not restrict the power of inferability from positive data. To the contrary, it is well known that the requirement of conservativeness restricts the power (cf. Angluin[2]). In 1990, Motoki[26] obtained characterization theorems for *consistent, responsive and conservative* inferability from positive data. Furthermore Kapur[14] and Lange&Zeugmann[20] obtained characterization theorems for *consistent and conservative* inferability from positive data.

**Definition 2.7.** Let $w_1, w_2, \cdots$ be an effective enumeration of the universal set $U$, and let $L \subseteq U$ be a concept. Then the finite subset of $L$ masked by $\{w_1, w_2, \cdots, w_n\}$ is denoted by $L^{(n)}$, that is, $L^{(n)} = L \cap \{w_1, w_2, \cdots, w_n\}$.

**Lemma 2.8.** *Let $L_1, L_2 \subseteq U$ be concepts.*

*(a) If $L_1 \subsetneq L_2$, then there is a $j \geq 1$ such that for any $n \geq j$, $L_1^{(n)} \subsetneq L_2^{(n)}$.*

*(b) If $L_1 \not\subsetneq L_2$, then there is a $j \geq 1$ such that for any $n \geq j$, $L_1^{(n)} \not\subsetneq L_2^{(n)}$.*

*Proof.* (a) Assume $L_1 \subsetneq L_2$. Then (i) $L_1 \subseteq L_2$ and (ii) $L_2 \setminus L_1 \neq \phi$ hold. By (ii), there is a $j \geq 1$ such that $w_j \in L_2 \setminus L_1$. Therefore for any $n \geq j$, $w_j \in L_2^{(n)} \setminus L_1^{(n)}$ holds. On the other hand, by (i), for any $n \geq 1$, $L_1^{(n)} \subseteq L_2^{(n)}$ holds. Therefore for any $n \geq j$, $L_1^{(n)} \subsetneq L_2^{(n)}$ holds.

(b) Assume $L_1 \not\subsetneq L_2$. Then $L_1 \not\subseteq L_2$ or $L_1 = L_2$ holds.

(I) In case $L_1 \not\subseteq L_2$. There is a $j \geq 1$ such that $w_j \in L_1 \setminus L_2$. Therefore for any $n \geq j$, $w_j \in L_1^{(n)} \setminus L_2^{(n)}$, i.e. $L_1^{(n)} \not\subseteq L_2^{(n)}$, holds, and it follows that $L_1^{(n)} \not\subsetneq L_2^{(n)}$.

(II) In case $L_1 = L_2$. For any $n \geq 1$, $L_1^{(n)} = L_2^{(n)}$ holds, and it follows that $L_1^{(n)} \not\subsetneq L_2^{(n)}$.

■

We note that for any indexed family $\mathcal{C} = \{L_i\}_{i \in N}$ of recursive concepts and any $i, j, n \geq 1$, whether $L_i^{(n)} \subsetneq L_j^{(n)}$ or not is recursively decidable.

The following Theorem 2.9 is useful to show that some sufficiently large classes are inferable in the limit from positive data (cf. Example 2.2, Theorem 6.7, Chapter 5 and 6).

**Definition 2.8** (Wright[51], Motoki et al.[27])**.** A class $\mathcal{C}$ is said to *have infinite elasticity*, if there are two infinite sequences $w_0, w_1, w_2, \cdots \in U$ and $L_{j_1}, L_{j_2}, \cdots \in \mathcal{C}$ such that for any $i \geq 1$,

$$\{w_0, w_1, \cdots, w_{i-1}\} \subseteq L_{j_i} \quad \text{but} \quad w_i \notin L_{j_i}.$$

A class $\mathcal{C}$ is said to *have finite elasticity*, if $\mathcal{C}$ does not have infinite elasticity.

**Theorem 2.9** (Wright[51]). *If a class $\mathcal{C} = \{L_i\}_{i \in N}$ has finite elasticity, then finite tell-tales of $\mathcal{C}$ are uniformly and recursively enumerable, and thus it is inferable in the limit from positive data.*

*Proof* (Sketch). By Theorem 2.6, it suffices for us to show that there is an effective procedure which uniformly and recursively enumerates finite tell-tales of $\mathcal{C}$.

Let us consider the procedure in Figure 2.5 due to Angluin[2].

---

**Procedure** $Q(i)$;
**begin**
  $T = \phi$;  $n = 0$;
  **repeat**
    $n = n + 1$;
    search for an index $j \leq n$ such that $T \subseteq L_j$ and $L_j^{(n)} \subsetneq L_i^{(n)}$;
    **if** such an index $j$ is found **then begin**
      let $w$ be an element in $L_i^{(n)} \setminus L_j^{(n)}$;
      $T = T \cup \{w\}$;
      output $w$;
    **end**;
  **forever**;
**end**.

---

Figure 2.5: An algorithm which uniformly and recursively enumerates finite tell-tales

Since $\mathcal{C}$ has finite elasticity, it is guaranteed that the procedure enumerates at most finitely many elements in $L_i$. Furthermore we can show that the set of enumerated elements by the procedure on input $i$ is a finite tell-tale of $L_i$ within $\mathcal{C}$. We omit the details. ■

**Definition 2.9** (Angluin[2]). A class $\mathcal{C}$ is said to *have finite thickness*, if for any $w \in U$, $\sharp\{L_i \in \mathcal{C} \mid w \in L_i\}$ is finite.

The following Proposition 2.10 is obvious.

**Proposition 2.10** (Wright[51]). *If a class $\mathcal{C}$ has finite thickness, then $\mathcal{C}$ has finite elasticity.*

By the above Theorem 2.9 and Proposition 2.10, we have the following Corollary 2.11.

**Corollary 2.11** (Angluin[2]). *If a class $\mathcal{C}$ has finite thickness, then finite tell-tales of $\mathcal{C}$ are uniformly and recursively enumerable, and thus it is inferable in the limit from positive data.*

**Example 2.1** (Angluin[1, 2]). We consider the class $\mathcal{PAT}$ of pattern languages. Fix a finite alphabet $\Sigma$. A *pattern* is a nonnull finite string of constant and variable symbols. The pattern language $L(\pi)$ generated by a pattern $\pi$ is the set of all constant strings obtained by substituting nonnull strings of constant symbols for the variables of $\pi$.

For example, let $\Sigma = \{a, b, c\}$. Then $\pi = axbxcy$ is a pattern, and the set $\{aabac, abbbc, acbcc, aaabaac, aabbabc, \cdots\}$ is the pattern language of $\pi$.

Since two patterns that are identical except for renaming of variables generate the same pattern language, we do not distinguish one from the other. The set of all patterns is recursively enumerable and whether $w \in L(\pi)$ or not is recursively decidable for any constant string $w$ and any pattern $\pi$. Therefore we can consider the class of pattern languages as an indexed family of nonempty recursive concepts, where the pattern itself is considered to be an index.

The class $\mathcal{PAT}$ has finite thickness. In fact, fix a constant string $w$. Then if $w \in L(\pi)$, then $\pi$ is not longer than $w$. The set of all patterns shorter than a fixed length is a uniformly and recursively generable finite set, and whether $w \in L(\pi)$ or not is recursively decidable for any $w$ and $\pi$. Therefore the set $\{\pi \mid w \in L(\pi)\}$ is a uniformly and recursively generable *finite set*, and it follows that $\sharp\{L(\pi) \mid w \in L(\pi)\}$ is finite. Thus, by Corollary 2.11, $\mathcal{PAT}$ is inferable in the limit from positive data.

The advantage of finite elasticity is that if a class $\mathcal{C}$ has finite elasticity, then the union class of $\mathcal{C}$ also has finite elasticity as shown below, and thus it is inferable in the limit from positive data.

**Definition 2.10.** Let $n \geq 1$ be an integer, and let $\mathcal{C}_1, \cdots, \mathcal{C}_n$ be classes. For $i$ with $1 \leq i \leq n$ and $j \geq 1$, the $j$-th concept $L_j$ of the class $\mathcal{C}_i$ is denoted by $L_{(i,j)}$. Then let $\bigsqcup_{i=1}^{n} \mathcal{C}_i$ be the class of concepts each of which is a union of $n$ concepts from $\mathcal{C}_1, \cdots, \mathcal{C}_n$, that is,

$$\bigsqcup_{i=1}^{n} \mathcal{C}_i = \{L_{(1,j_1)} \cup \cdots \cup L_{(n,j_n)} \mid j_1, \cdots, j_n \geq 1\}.$$

By assuming a bijective coding from $N^n$ to $N$, the new class above becomes an indexed family of recursive concepts.

For a class $\mathcal{C} = \{L_i\}_{i \in N}$ and for $n \geq 1$, let

$$\mathcal{C}^{[\leq n]} = \bigsqcup_{i=1}^{n} \mathcal{C} = \{L_{j_1} \cup \cdots \cup L_{j_n} \mid j_1, \cdots, j_n \geq 1\}.$$

**Theorem 2.12** (Wright[51]). *Let $n \geq 1$ be an integer, and let $\mathcal{C}_1, \cdots, \mathcal{C}_n$ be classes. Then if each of $\mathcal{C}_1, \cdots, \mathcal{C}_n$ has finite elasticity, then the class $\bigsqcup_{i=1}^{n} \mathcal{C}_i$ also has finite elasticity, and thus it is inferable in the limit from positive data.*

**Example 2.2** (Wright[51]). We consider the class $\mathcal{PAT}$ of pattern languages. As shown in Example 2.1, $\mathcal{PAT}$ has finite thickness, and it follows by Proposition 2.10 that $\mathcal{PAT}$ also has finite elasticity. Therefore for any $n \geq 1$, $\mathcal{PAT}^{[\leq n]}$ is inferable in the limit from positive data.

We note that Sato&Umayahara[38], Sato&Moriyama[39] and Kapur[14] have obtained more general sufficient conditions for inferability from positive data than the condition of finite elasticity.

## 2.3. Reliable Identification

In Section 2.1 we have defined ordinary inductive inference. However in the definitions, the behavior of an inference machine is not specified, when we feed a positive or complete presentation of a concept not in a hypothesis space in question. In contrast, the reliable identification deals with sequences from a concept not in the hypothesis space. The notion of reliable identification was introduced by Minicozzi[24] and Blum&Blum[8] for function learning, and it was adapted to language learning by Sakurai[37].

In case we feed a positive or complete presentation of a concept not in the hypothesis space, the reliable inference machine is required not to converge to any index. However the reliable inference machine does not tell us that the target concept is not in the hypothesis space. In Chapter 4 and 5 we also define some other criteria where the behaviors of inference machines are more desirably defined, when we feed a positive or complete presentation of a concept not in the hypothesis space.

Hereafter, for a concept $L \subseteq U$, we write $L \in \mathcal{C}$, if there is an $L_i \in \mathcal{C}$ such that $L_i = L$.

Let $L \subseteq U$ be a concept, and let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. Then a concept $L_n \in \mathcal{C}$ is said to be a *minimal concept of* $L$ *within* $\mathcal{C}$, if (i) $L \subseteq L_n$ holds and (ii) for any $L_i \in \mathcal{C}$, $L \subseteq L_i$ implies $L_i \not\subsetneq L_n$.

**Definition 2.11** (Sakurai[37]). An IIM $M$ is said to *reliably infer a class* $\mathcal{C}$ *from positive data*, if it satisfies the following condition: For any nonempty concept $L$ and any positive presentation $\sigma$ of $L$, (i) if $L \in \mathcal{C}$, then $M$ infers $L$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$, (ii) otherwise $M$ does not converge to any index for $\sigma$.

An IIM $M$ is said to *semi-reliably infer a class* $\mathcal{C}$ *from positive data*, if it satisfies the following condition: For any nonempty concept $L$ and any positive presentation $\sigma$ of $L$, (i) if $L \in \mathcal{C}$, then $M$ infers $L$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$, (ii) otherwise if $M$ converges to an index $i$ for $\sigma$, then $L_i$ is a minimal concept of $L$ within $\mathcal{C}$.

A class $\mathcal{C}$ is said to be *reliably inferable* (resp., *semi-reliably inferable*) *from positive data*, if there is an IIM $M$ which reliably (resp., semi-reliably) infers $\mathcal{C}$ from positive data.

In a similar way to the case of positive data, we also define the case of complete data.

By definition, it is guaranteed that if a semi-reliable inference machine converges to an index $i$, then $L_i$ is a minimal concept of the target concept within the class. However even when there is a minimal concept of the target concept within the class, the semi-reliable inference machine may not converge to any index for its presentation. In Chapter 5 we consider an inference criterion where an inference machine converges to an index of a minimal concept of the target concept within the class, whenever there is a minimal concept of the target concept within the class.

We note that, by definition, an IIM $M$ which (semi-)reliably infers a class $\mathcal{C}$ from positive data (resp., compete data) also infers $\mathcal{C}$ in the limit from positive data (resp., complete data).

Concerning (semi-)reliable inferability from positive data, Sakurai[37] obtained the following characterizations.

**Theorem 2.13** (Sakurai[37]). *(a) A class $\mathcal{C}$ is reliably inferable from positive data, if and only if $\mathcal{C}$ contains no infinite concept.*

*(b) A class $\mathcal{C}$ is semi-reliably inferable from positive data, if and only if $\mathcal{C}$ is inferable in the limit from positive data.*

*Proof* (Sketch). (a) (I) The 'only if' part. Assume that an IIM $M$ reliably infers $\mathcal{C}$ from positive data. Then suppose that $\mathcal{C}$ contains an infinite concept.

Let $L_h \in \mathcal{C}$ be an infinite concept, and let $w_1, w_2, \cdots$ be an enumeration of $L_h$. Without loss of generality, we assume $w_i \neq w_j$ if $i \neq j$.

We show that $M$ does not converge to any index $i$ with $L_i = L_h$ for a certain positive presentation of $L_h$. Put $\sigma_1 = w_1, w_1, \cdots$, and define $n_i$'s and $\sigma_{i+1}$'s ($i \geq 1$) inductively by the following stages:

Stage $i$ ($\geq 1$):

Since $\sigma_i$ is a positive presentation of the concept $\{w_1, \cdots, w_i\}$ ($\neq L_h$), there is an $n \geq 1$ such that $L_h \neq L_g$ and $g = \overline{M}(\sigma_i[n_1 + n_2 + \cdots + n_{i-1} + n])$ for some $g \geq 1$. Put $n_i = n$ and $\sigma_{i+1} = \sigma_i[n_1 + n_2 + \cdots + n_i], w_{i+1}, w_{i+1}, \cdots = \underbrace{w_1, w_1, \cdots, w_1}_{n_1}, \underbrace{w_2, w_2, \cdots, w_2}_{n_2}, \cdots, \underbrace{w_i, w_i, \cdots, w_i}_{n_i}, w_{i+1}, w_{i+1}, \cdots$. We note that this $\sigma_{i+1}$ becomes a positive presentation of the concept $\{w_1, w_2, \cdots, w_{i+1}\}$.

Goto Stage $i + 1$.

Now we take an infinite sequence $\sigma = \underbrace{w_1, w_1, \cdots, w_1}_{n_1}, \underbrace{w_2, w_2, \cdots, w_2}_{n_2}, \cdots$. Then $\sigma$ is a positive presentation of $L_h$. However $M$ does not converge to any index $i$ with $L_i = L_h$ for $\sigma$. That is, $M$ does not infer $\mathcal{C}$ in the limit from positive data. This contradicts the assumption.

(II) The 'if' part. Assume that $\mathcal{C}$ contains no infinite concepts. It is easy to see that the procedure in Figure 2.6 reliably infers $\mathcal{C}$ from positive data. We omit the details.

(b) The 'if' part is obvious by definition. We show the 'only if' part. Assume that a class $\mathcal{C}$ is inferable in the limit from positive data. Then, by Theorem 2.6, finite tell-tales

**Procedure** IIM $M$;
**begin**
  $T = \phi$;  $n = 0$;
  **repeat**
    read the next fact and store it in $T$;
    $n = n + 1$;
    search for the least index $i \leq n$ such that $T = L_i^{(n)}$;
    **if** such an index $i$ is found **then** output $i$ **else** output $n$;
  **forever**;
**end**.

Figure 2.6: An inference machine which reliably infers a class

of $\mathcal{C}$ are uniformly and recursively enumerable. Let $Q$ be a procedure which uniformly and recursively enumerates finite tell-tales of $\mathcal{C}$, and for $n \geq 1$, let $Q^{(n)}(i)$ be the set of elements produced by the procedure $Q$ on input $i$ in the first $n$ steps execution. We reconsider the procedure in Figure 2.3 at the proof of Theorem 2.6.

Let $L \subseteq U$ be a nonempty concept, and let $\sigma$ be an arbitrary positive presentation of $L$.

**Claim:** If the procedure converges to an index $i$ for $\sigma$, then $L_i$ is a minimal concept of $L$ within $\mathcal{C}$.

*Proof of the claim.* Assume that the procedure converges to an index $i$ for $\sigma$.

(I) $Q(i) \subseteq L$ holds. In fact, suppose the converse. Then there is a $w \in U$ such that $w \in Q(i) \setminus L$, and it follows that there is an $n \geq 1$ such that $w \in Q^{(n)}(i)$. However since $w \notin L$, it follows that for any $n \geq 1$, $w \notin \sigma[n]^+$. This contradicts the fact that the procedure converges to $i$ for $\sigma$.

(II) $L \subseteq L_i$ holds. In fact, suppose the converse. Then there is a $w \in U$ such that $w \in L \setminus L_i$, and it follows that there is an $n \geq 1$ such that $w \in \sigma[n]^+$. However since $w \notin L_i$, it follows that $\sigma[n]^+ \not\subseteq L_i$. This contradicts the fact that the procedure converges to $i$ for $\sigma$.

By (I) and (II), we have $Q(i) \subseteq L \subseteq L_i$. Since $Q(i)$ is a finite tell-tale of $L_i$ within $\mathcal{C}$, it follows that there is no concept $L_j \in \mathcal{C}$ such that $Q(i) \subseteq L_j \subsetneq L_i$, and thus $L_i$ is a minimal concept of $L$ within $\mathcal{C}$. □

Furthermore, as stated in the proof of Theorem 2.6, the procedure infers $\mathcal{C}$ in the limit from positive data. Thus the procedure semi-reliably infers $\mathcal{C}$ from positive data. ■

Concerning reliable inferability from complete data, the following Proposition 2.14 holds.

**Proposition 2.14.** *Every indexed family $\mathcal{C} = \{L_i\}_{i \in N}$ of recursive concepts is reliably inferable from complete data.*

*Proof.* Let us consider the procedure in Figure 2.7, where the procedure read_store is the same one as in Figure 2.1.

---

**Procedure** IIM $M$;
**begin**
  $T = \phi$;  $F = \phi$;  $i = 1$;
  **repeat**
    read_store$(T, F)$;
    **if** $(T, F)$ is *not* consistent with $L_i$ **then** $i = i + 1$;
    output $i$;
  **forever**;
**end**.

---

Figure 2.7: An inference machine which reliably infers a class

In a similar way to the proof of Theorem 2.4, we can show that for any $L_i \in \mathcal{C}$, the procedure infers $L_i$ w.r.t. $\mathcal{C}$ in the limit from any complete presentation of $L_i$. Thus it suffices for us to show that for any concept $L \notin \mathcal{C}$, the procedure does not converge to any index for any complete presentation $\sigma$ of $L$.

Now suppose that there are a concept $L \notin \mathcal{C}$ and a complete presentation $\sigma$ of $L$ such that the procedure converges to an index $i$ for $\sigma$. Since $L \notin \mathcal{C}$, it follows that $L \neq L_i$. Therefore there is a $w \in U$ such that $w \in L \setminus L_i$ or $w \in L_i \setminus L$. Since $\sigma$ is a complete presentation of $L$, it follows that there is an $n \geq 1$ such that $w \in \sigma[n]^+$ or $w \in \sigma[n]^-$.

Therefore after reading the $n$-th fact, $(T, F)$ in the procedure is *not* consistent with $L_i$, and it follows that the procedure changes its guess. This contradicts the assumption. ■

By the above Proposition 2.14, we see that every indexed family of recursive concepts is also semi-reliably inferable from complete data.

# Chapter 3.

# Inferability with a Bounded Number of Mind Changes

In this chapter we consider inductive inference of recursive concepts with a bounded number of mind changes from positive data or complete data. We use the phrase 'mind change' to mean that an inference machine changes its guess.

The criterion of identification in the limit seems to be natural, if we consider ordinary learning process of human beings. However we can not decide in general whether a sequence of guesses from an inference machine has converged or not at a certain time, and the results of the inference necessarily involve some risks. Clearly, it is important to have a conclusive answer, when we want to use the results of machine learning.

In Section 3.1 we deal with finite identification for a class of recursive concepts. Originally, finite identification was introduced to function learning (cf. Freivald&Wiehagen[11], Klette&Wiehagen[18] and Jantke&Beick[13]). An inference machine $M$ is said to *finitely infer* a concept $L$, if $M$ which is successively fed a sequence of examples of $L$ produces a unique guess at a certain time and the guess is a correct expression of $L$. That is, the inference machine does not produce a guess until it is convinced that the guess is correct.

As stated in Section 2.2, Angluin[2] introduced the notion of a finite tell-tale of a concept to discuss inferability of an indexed family of recursive concepts from positive data, and showed that a class is inferable in the limit from positive data, if and only if finite tell-tales of the class are uniformly and recursively *enumerable*. Here we introduce a definite finite tell-tale (resp., a pair of definite finite tell-tales) of a concept, and show that a class is finitely inferable from positive data (resp., complete data), if and only if definite finite tell-tales (resp., pairs of definite finite tell-tales) of the class are uniformly and recursively *generable*.

In Section 3.2, by extending the above results, when the equivalence of any two concepts in the class is recursively decidable, we show a necessary and sufficient condition for a class to be inferable within $n$ mind changes from positive data or complete data for $n \geq 0$.

Case&Smith[10] discussed inductive inference of a class of recursive functions from view points of anomalies and mind changes, and showed that there are natural hierarchies. Case&Lynes[9] also showed that an anomaly hierarchy exists even in case of a class of recursive languages.

Here we also present examples of classes that are inferable within $n$ mind changes but not inferable within $n-1$ mind changes, and show that the inferability strictly increases, when the allowed number of mind changes increases.

In Section 3.3 we review further results obtained by Lange&Zeugmann[21]. They discussed class-preserving learning of a class from positive data or complete data. A class is said to be *class-preservingly learnable*, if the class is inferable with well-chosen indexing of the class. They showed the superiority of class-preserving learning and uniform characterizations for inferability with a bounded number of mind changes, and also showed that there are various hierarchies.

Section 3.1 and 3.2 are based on Mukouchi[28, 29], and Section 3.3 is based on Lange& Zeugmann[21].

## 3.1. Finite Identification

First of all, we define finite identification for a class of recursive concepts.

**Definition 3.1.** An IIM $M$ is said to *finitely converge to an index $i$ for a positive or complete presentation $\sigma$*, if there is an $n \geq 1$ such that (i) $M(\sigma[n]) = i$ and that (ii) for any $m$ with $1 \leq m \neq n$, $M(\sigma[m]) = 0$. In this case we also say that $M$ finitely converges to the index $i$ from $\sigma[n]$.

Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. For a concept $L_i \in \mathcal{C}$ and a positive or complete presentation $\sigma$ of $L_i$, an IIM $M$ is said to *finitely infer the concept $L_i$ w.r.t. $\mathcal{C}$ from $\sigma$*, if $M$ finitely converges to an index $j$ with $L_j = L_i$ for $\sigma$.

An IIM $M$ is said to *finitely infer a class $\mathcal{C}$ from positive data* (resp., *complete data*), if for any $L_i \in \mathcal{C}$, $M$ finitely infers $L_i$ w.r.t. $\mathcal{C}$ from any positive presentation $\sigma$ (resp., any complete presentation $\sigma$) of $L_i$. A class $\mathcal{C}$ is said to be *finitely inferable from positive data* (resp., *complete data*), if there is an IIM $M$ which finitely infers the class $\mathcal{C}$ from positive data (resp., complete data).

We can consider that an IIM $M$ finitely converges to an index $i$ for $\sigma$, if $M$ which is successively fed $\sigma$'s facts produces the unique guess $i$ and stops. Thus, in criterion of finite identification, an inference machine produces a unique guess when the inference process terminates.

Now we show our definition and theorem that form a remarkable contrast to Definition 2.6 and Theorem 2.6 concerning inferability from positive data.

**Definition 3.2.** Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. A set $T_i$ is said to be a *definite finite tell-tale of $L_i$ within $\mathcal{C}$*, if (i) $T_i$ is a finite subset of $L_i$ and (ii) for any $L_j \in \mathcal{C}$, $T_i \subseteq L_j$ implies $L_j = L_i$.

A pair $(T_i, F_i)$ of sets is said to be *a pair of definite finite tell-tales of $L_i$ within $\mathcal{C}$*, if it satisfies the following conditions: (i) $T_i$ is a finite subset of $L_i$, (ii) $F_i$ is a finite subset of $L_i^c$, and (iii) for any $L_j \in \mathcal{C}$, if $(T_i, F_i)$ is consistent with $L_j$, then $L_j = L_i$ holds.

An indexed family $\{T_i\}_{i \in N}$ of finite sets is said to be *definite finite tell-tales of $\mathcal{C}$*, if for any index $i$, $T_i$ is a definite finite tell-tale of $L_i$ within $\mathcal{C}$.

Two indexed families $\{T_i\}_{i \in N}$ and $\{F_i\}_{i \in N}$ of finite sets is said to be *pairs of definite finite tell-tales of $\mathcal{C}$*, if for any index $i$, $(T_i, F_i)$ is a pair of definite finite tell-tales of $L_i$ within $\mathcal{C}$.

By definition, the definite finite tell-tale has a more specific meaning than the finite tell-tale.

**Theorem 3.1** (Mukouchi[28], Lange&Zeugmann[20], Kapur[15])**.** *A class $\mathcal{C}$ is finitely inferable from positive data, if and only if definite finite tell-tales of $\mathcal{C}$ are uniformly and recursively generable.*

*Proof.* (I) The 'only if' part. Assume that an IIM $M$ finitely infers $\mathcal{C}$ from positive data. Let us consider the procedure in Figure 3.1. In the procedure we note that we can effectively take a positive presentation $\sigma$ of $L_i$ (cf. Definition 2.3).

---

**Procedure** $Q(i)$;
**begin**
  let $\sigma$ be an arbitrary positive presentation of $L_i$;
  search for an integer $n \geq 1$ such that $M(\sigma[n]) > 0$;
  **if** such an integer $n$ is found **then** output $\sigma[n]$;
**end**.

---

Figure 3.1: An algorithm which recursively generates definite finite tell-tales

Let $\sigma$ be the positive presentation of $L_i$ which we used in the procedure. Since $M$ finitely infers $\mathcal{C}$ from positive data, there is an $n \geq 1$ such that $M(\sigma[n]) > 0$, and thus the procedure on input $i$ recursively generates the finite set $\sigma[n]^+$. Now we show by contradiction that $Q(i)$ $(= \sigma[n]^+)$ is a definite finite tell-tale of $L_i$. Suppose that $Q(i)$ is not a definite finite tell-tale of $L_i$. It is easy to see that $Q(i)$ is a finite subset of $L_i$. Therefore there is an $L_j \in \mathcal{C}$ such that $Q(i) \subseteq L_j$ and $L_j \neq L_i$. Let $\delta$ be an arbitrary positive presentation of $L_j$. Then $\sigma[n] \cdot \delta$ is a positive presentation of $L_j$. Since $M$ finitely converges to an index $k$ with $L_k = L_i$ for $\sigma[n]$, it follows that $M$ does not finitely infer $L_j$ w.r.t. $\mathcal{C}$ from $\sigma[n] \cdot \delta$. This contradicts the assumption.

(II) The 'if' part. Assume that a procedure $Q$ uniformly and recursively generates definite finite tell-tales of $\mathcal{C}$. Then let us consider the procedure in Figure 3.2. In the procedure we note that whether $Q(i) \subseteq T$ or not is recursively decidable, because $Q(i)$ and $T$ are explicitly given finite sets.

Assume that we feed a positive presentation $\sigma$ of a concept $L_h \in \mathcal{C}$ to the procedure.

(1) If the procedure finitely converges to an index $k$, then $L_k = L_h$ holds. In fact, when the procedure terminates, $Q(k) \subseteq T \subseteq L_h$ holds, and it follows that $L_k = L_h$ by Definition 3.2.

**Procedure** IIM $M$;
**begin**
$T = \phi$; $n = 1$;
**repeat**
read the next fact and store it in $T$;
search for an index $i \leq n$ such that $Q(i) \subseteq T$;
**if** such an index $i$ is found **then** output $i$ and stop;
$n = n + 1$;
**forever**;
**end**.

Figure 3.2: An inference machine which finitely infers a class

(2) The procedure terminates in a finite time. In fact, let

$$k = \min\{j \mid Q(h) \subseteq \sigma[j]^+\} \qquad \text{and} \qquad m = \max\{k, h\}.$$

We note that $h \leq m$. Suppose that the procedure does not terminate. Then it reaches the case of $n = m$ and $i = h$. In this case, $Q(i) \subseteq T\,(= \sigma[m]^+)$ holds, and it follows that the procedure outputs a guess and terminates. This contradicts the assumption. ■

In a similar way, we can also show the following Theorem 3.2.

**Theorem 3.2.** *A class $\mathcal{C}$ is finitely inferable from complete data, if and only if pairs of definite finite tell-tales of $\mathcal{C}$ are uniformly and recursively generable.*

The following Corollary 3.3 presents an interesting relation between presentations and mind changes.

**Corollary 3.3** (Lange&Zeugmann[19])**.** *If a class $\mathcal{C}$ is finitely inferable from complete data, then $\mathcal{C}$ is also inferable in the limit from positive data.*

*Proof.* Assume that a class $\mathcal{C}$ is finitely inferable from complete data. Then, by Theorem 3.2, pairs of definite finite tell-tales of $\mathcal{C}$ are uniformly and recursively generable.

**Claim:** For any index $i$, if $(T_i, F_i)$ is a pair of definite finite tell-tales of $L_i$ within $\mathcal{C}$, then $T_i$ is a finite tell-tale of $L_i$ within $\mathcal{C}$.

*Proof of the claim.* Let $(T_i, F_i)$ be a pair of definite finite tell-tales of $L_i$ within $\mathcal{C}$. Then (i) $T_i \subseteq L_i$ holds, (ii) $F_i \subseteq L_i^c$ holds, and (iii) $L_i \neq L_j$ implies $T_i \not\subseteq L_j$ or $F_i \not\subseteq L_j^c$. Suppose that $T_i$ is not a finite tell-tale of $L_i$ within $\mathcal{C}$. Therefore, by (i), there is an $L_j \in \mathcal{C}$ such that $T_i \subseteq L_j \subsetneq L_i$. Since $L_j \subsetneq L_i$, it follows that $L_i^c \subsetneq L_j^c$. Hence by (ii), we have $F_i \subsetneq L_j^c$. This contradicts (iii). □

We see by this claim that finite tell-tales of $\mathcal{C}$ are uniformly and recursively *generable*, and it follows that they are uniformly and recursively *enumerable*. Hence $\mathcal{C}$ is inferable in the limit from positive data. ■

The following Corollary 3.4 is obvious from Definition 3.2 and Theorem 3.1.

**Corollary 3.4.** *Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. Then if there are two concepts $L_i, L_j \in \mathcal{C}$ such that $L_i \subsetneq L_j$, then $\mathcal{C}$ is not finitely inferable from positive data.*

Here we present examples of classes that are finitely inferable from positive data.

**Example 3.1.** For $i \geq 1$, let $p_i$ be the $i$-th prime number, and let $L_i$ be the set of all multiples of $p_i$. Then let $\mathcal{PR}_1 = \{L_i\}_{i \in N}$ be the class of interest.

Since $p_i$ is a primitive recursive function of $i \geq 1$, $\mathcal{PR}_1$ is an indexed family of nonempty recursive concepts on $N$. This class is finitely inferable from positive data. In fact, we can take the set $\{p_i\}$ as a definite finite tell-tale of $L_i$.

Furthermore we have the following Example 3.2.

**Example 3.2.** For $n \geq 1$, let $\mathcal{FC}_n$ be the class of all nonempty finite concepts on the universal set $U$ each of which cardinality is *just n*. By assuming a bijective coding from $\mathcal{FC}_n$ to $N$, this class becomes an indexed family of nonempty recursive concepts. This class is finitely inferable from positive data. In fact, each concept itself is its definite finite tell-tale within $\mathcal{FC}_n$, and they are uniformly and recursively generable.

For $n \geq 1$, let $\mathcal{PR}_n$ be the class of all concepts each of which consists of all multiples of $n$ distinct prime numbers. By assuming a bijective coding from the set of all $n$ distinct prime numbers to $N$, this class becomes an indexed family of nonempty recursive concepts on $N$. For example, the concept $\{2, 4, 6, \cdots, 7, 14, 21, \cdots\}$ is in $\mathcal{PR}_2$ but not in $\mathcal{PR}_i$ with

$i = 1$ or $i \geq 3$. This class is finitely inferable from positive data. In fact, a set of $n$ distinct prime numbers is a definite finite tell-tale of the corresponding concept within $\mathcal{PR}_n$, and they are uniformly and recursively generable.

We note that if $n \geq 2$, the above two classes $\mathcal{FC}_n$ and $\mathcal{PR}_n$ do not have finite thickness (cf. Definition 2.9).

We present a sufficient condition for a class to be finitely inferable from complete data. This condition has more specific meaning than the condition of finite thickness (cf. Definition 2.9).

**Definition 3.3.** Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class, and let $\mathcal{S}$ be a subclass of $\mathcal{C}$. A set $I$ of indices is said to be a *cover-index set* of $S$, if the collection of all concepts each of which has an index in $I$ is equal to $\mathcal{S}$, that is, $\mathcal{S} = \{L_i \in \mathcal{C} \mid i \in I\}$.

**Theorem 3.5.** *A class $\mathcal{C} = \{L_i\}_{i \in N}$ is finitely inferable from complete data, if*

*(i) $\mathcal{C}$ does not contain the empty concept as its member,*

*(ii) for any $w \in U$, there is a uniformly and recursively generable finite cover-index set of the subclass $\{L_i \in \mathcal{C} \mid w \in L_i\}$ of $\mathcal{C}$, and*

*(iii) whether $L_i = L_j$ or not is recursively decidable for any indices $i$ and $j$.*

*Proof.* Assume that a class $\mathcal{C}$ satisfies the above three conditions. Then the procedure in Figure 3.3 uniformly and recursively generates definite finite tell-tales of $\mathcal{C}$.

In fact, it is easy to see that for any index $i$, the procedure on input $i$ terminates in a finite time, and that the output of the procedure on input $i$ is a pair of definite finite tell-tales of $L_i$ within $\mathcal{C}$. ∎

Here we present an example of a class of languages which is finitely inferable from complete data.

**Example 3.3** (Mukouchi[28], Lange&Zeugmann[19]). We consider the class $\mathcal{PAT}$ of pattern languages (cf. Example 2.1).

(I) By definition, it is easy to see that $\mathcal{PAT}$ satisfies the condition (i) of Theorem 3.5.

**Procedure** $Q(i)$;
**begin**
  let $w$ be an arbitrary element in $L_i$;
  $T = \{w\}$;    $F = \phi$;
  recursively generate a cover-index set of $\{L_j \in \mathcal{C} \mid w \in L_j\}$, and set it to $I$;
  **for each** $j \in I$ **do**
    **if** $L_i \neq L_j$ **then begin**
      search for an element $w$ such that $w \in L_i \setminus L_j$ or $w \in L_j \setminus L_i$;
      **if** $w \in L_i \setminus L_j$ **then** $T = T \cup \{w\}$ **else** $F = F \cup \{w\}$;
    **end**;
  output the pair $(T, F)$;
**end**.

Figure 3.3: An algorithm which recursively generates a pair of definite finite tell-tales

(II) $\mathcal{PAT}$ also satisfies the condition (ii) of Theorem 3.5. This is already shown in Example 2.1.

(III) Angluin[1] showed that $L(\pi) = L(\tau)$ if and only if $\pi = \tau$, and it follows that whether $L(\pi) = L(\tau)$ or not is recursively decidable for any patterns $\pi$ and $\tau$.

Therefore we see by Theorem 3.5 that $\mathcal{PAT}$ is finitely inferable from complete data.

By theorems in Angluin[1], we can also show that $(T, F)$ is a pair of definite finite tell-tales of $L(\pi)$ within $\mathcal{PAT}$, where $T$ is the set of all elements in $L(\pi)$ of the same length as $\pi$, and $F$ is the set of all constant strings each of which is not longer than $\pi$ and is not in $T$. Furthermore we see by Corollary 3.4 that $\mathcal{PAT}$ is *not* finitely inferable from *positive data.*

**Example 3.4** (Tanimizu&Sato[49]). Let $n \geq 2$ be an integer. We consider the class $\mathcal{PAT}^{[\leq n]}$ of unions of at most $n$ pattern languages (cf. Example 2.2). This class $\mathcal{PAT}^{[\leq n]}$ is *not* finitely inferable from *complete data.*

In fact, let $w_1 \in \Sigma^+$ be an arbitrary constant string. We show that there is no pair of definite finite tell-tales of $L(w_1)$ within $\mathcal{PAT}^{[\leq n]}$. Suppose that there is a pair $(T, F)$ of definite finite tell-tales of $L(w_1)$ within $\mathcal{PAT}^{[\leq n]}$. Since $F$ is a finite set, it follows that there is a $w_2 \in \Sigma^+$ such that $w_1 \neq w_2$ and $w_2 \notin F$. Then $(T, F)$ is consistent with $L(w_1) \cup L(w_2) \in \mathcal{PAT}^{[\leq n]}$, which contradicts the assumption.

The following Corollary 3.6 shows a necessary condition for a class to be finitely inferable from positive data or complete data.

**Corollary 3.6.** *If a class $\mathcal{C}$ is finitely inferable from positive data or complete data, then whether $L_i = L_j$ or not is recursively decidable for any indices $i$ and $j$.*

*Proof.* Clearly, if $\mathcal{C}$ is finitely inferable from positive data, then $\mathcal{C}$ is also finitely inferable from complete data (cf. Proposition 2.2). Therefore it suffices for us to show the case of complete data.

Assume that $\mathcal{C}$ is finitely inferable from complete data. For any indices $i$ and $j$, we can recursively decide whether $L_i = L_j$ or not as follows: To begin with, recursively generate a pair of definite finite tell-tales of $L_i$ within $\mathcal{C}$, and set it to $(T_i, F_i)$. Then check whether $(T_i, F_i)$ is consistent with $L_j$. If $(T_i, F_i)$ is *not* consistent with $L_j$, then we conclude $L_i \neq L_j$, because $(T_i, F_i)$ is consistent with $L_i$. Otherwise, we conclude $L_i = L_j$ by Definition 3.2. ∎

## 3.2. Bounded Mind Changes

In this section we extend the results obtained in the previous section to the case of inductive inference where the number of mind changes is bounded by a constant number.

**Definition 3.4.** Let $M$ be an IIM. For a positive or complete presentation $\sigma$ and for $n \geq 1$, we define $M[\sigma[n]]$ and $\overline{M}[\sigma[n]]$ as follows: (i) In case $M(\sigma[n])$ is defined. Let $M[\sigma[n]]$ be the finite sequence $M(\sigma[1]), M(\sigma[2]), \cdots, M(\sigma[n])$, and let $\overline{M}[\sigma[n]]$ be the finite sequence obtained by getting rid of every 0 from $M[\sigma[n]]$. (ii) Otherwise. Leave $M[\sigma[n]]$ and $\overline{M}[\sigma[n]]$ undefined.

Let $n \geq 0$ be an integer. An IIM $M$ is said to *converge to an index $i$ within $n$ mind changes for a positive or complete presentation $\sigma$*, if (i) there is an $m \geq 1$ such that for any $k \geq m$, $\overline{M}(\sigma[k]) = i$ and (ii) for any $k \geq 1$, $\sharp\overline{M}[\sigma[k]] \leq n + 1$.

Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. For a concept $L_i \in \mathcal{C}$ and a positive or complete presentation $\sigma$ of $L_i$, an IIM $M$ is said to *infer the concept* $L_i$ *w.r.t.* $\mathcal{C}$ *within* $n$ *mind changes from* $\sigma$, if $M$ converges to an index $j$ with $L_j = L_i$ within $n$ mind changes for $\sigma$.

An IIM $M$ is said to *infer a class* $\mathcal{C}$ *within* $n$ *mind changes from positive data* (resp., *complete data*), if for any $L_i \in \mathcal{C}$, $M$ infers $L_i$ w.r.t. $\mathcal{C}$ within $n$ mind changes from any positive presentation $\sigma$ (resp., any complete presentation $\sigma$) of $L_i$. A class $\mathcal{C}$ is said to be *inferable within* $n$ *mind changes from positive data* (resp., *complete data*), if there is an IIM $M$ which infers the class $\mathcal{C}$ within $n$ mind changes from positive data (resp., complete data).

In the above definition, $\overline{M}[\sigma[n]]$ represents the sequence of guesses produced by $M$ which is successively fed $\sigma[n]$'s facts on its input requests, and thus $\sharp\overline{M}[\sigma[n]]$ represents the number of guesses produced by $M$. We can simply consider that an IIM $M$ converges to an index $i$ within $n$ mind changes for $\sigma$, if $M$ which is successively fed $\sigma$'s facts produces at most $n+1$ guesses and its last guess is $i$.

For $n \geq 0$, if a class $\mathcal{C}$ is inferable within $n$ mind changes from positive data (resp., complete data), we also say that $\mathcal{C}$ is *$EX_n$-TXT inferable* (resp., *$EX_n$-INF inferable*). Furthermore, if a class $\mathcal{C}$ is inferable in the limit from positive data (resp., complete data), we also say that $\mathcal{C}$ is *$EX_*$-TXT inferable* (resp., *$EX_*$-INF inferable*).

For $n \geq 0$ or $n = *$, by the same notation $\mathrm{EX}_n$-TXT (resp., $\mathrm{EX}_n$-INF), we also denote the collection of all classes that are $\mathrm{EX}_n$-TXT inferable (resp., $\mathrm{EX}_n$-INF inferable).

We note that, by definition, finite identification is equivalent to inferability within 0 mind change, that is, inferability without any mind changes.

First of all, we present a necessary condition for a class to be $\mathrm{EX}_n$-TXT inferable, which is a natural extension of Corollary 3.4.

**Proposition 3.7.** *Let $n \geq 0$ be an integer, and let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. Then if there are concepts $L_{j_1}, L_{j_2}, \cdots, L_{j_{n+2}} \in \mathcal{C}$ such that $L_{j_1} \subsetneq L_{j_2} \subsetneq \cdots \subsetneq L_{j_{n+2}}$, then $\mathcal{C}$ is not $EX_n$-TXT inferable.*

*Proof.* Assume that there are concepts $L_{j_1}, L_{j_2}, \cdots, L_{j_{n+2}} \in \mathcal{C}$ such that $L_{j_1} \subsetneq L_{j_2} \subsetneq \cdots \subsetneq L_{j_{n+2}}$. Then suppose that an IIM $M$ infers $\mathcal{C}$ in the limit from positive data. We show that $M$ changes its guesses at least $n+1$ times to infer $L_{j_{n+2}}$ w.r.t. $\mathcal{C}$ in the limit from a certain positive presentation of $L_{j_{n+2}}$. For $i$ with $1 \leq i \leq n+2$, let $\sigma_i$ be an arbitrary positive presentation of $L_{j_i}$. Put $\delta_1 = \sigma_1$, and define $n_i$'s and $\delta_{i+1}$'s $(1 \leq i \leq n+1)$ inductively by the following stages:

Stage $i$ $(1 \leq i \leq n+1)$:

Since $\delta_i$ is a positive presentation of $L_{j_i}$, there is an $n \geq 1$ such that $L_{j_i} = L_g$ and $g = M(\delta_i[n_1 + n_2 + \cdots + n_{i-1} + n])$ for some $g \geq 1$. Put $n_i = n$ and $\delta_{i+1} = \delta_i[n_1 + n_2 + \cdots + n_i] \cdot \sigma_{i+1}$. We note that this $\delta_{i+1}$ becomes a positive presentation of $L_{j_{i+1}}$, because $L_{j_i} \subsetneq L_{j_{i+1}}$.

If $i < n+1$ then goto Stage $i+1$ else stop.

Then $\delta_{n+2}$ is a positive presentation of $L_{j_{n+2}}$. However $M$ changes its guesses at least $n+1$ times for $\delta_{n+2}$. This completes the proof. ∎

**Corollary 3.8.** *For any $n \geq 0$, the class $\mathcal{PAT}$ of pattern languages is not EX$_n$-TXT inferable.*

*Proof.* By Proposition 3.7, it suffices for us to show that for any $n \geq 0$, there are patterns $\pi_1, \pi_2, \cdots, \pi_{n+2}$ such that $L(\pi_1) \subsetneq L(\pi_2) \subsetneq \cdots \subsetneq L(\pi_{n+2})$. In fact, let $\pi_1 = x_1 x_2 \cdots x_{n+2}, \pi_2 = x_1 x_2 \cdots x_{n+1}, \cdots, \pi_{n+2} = x_1$. Then $L(\pi_i)$ is the set of all constant strings of length more than $n+2-i$, and thus

$$L(\pi_1) \subsetneq L(\pi_2) \subsetneq \cdots \subsetneq L(\pi_{n+2})$$

holds. ∎

The above Corollary 3.8 means $\mathcal{PAT} \notin \bigcup_{i=1}^{\infty}$ EX$_i$-TXT. To the contrary, as shown in Example 2.1, $\mathcal{PAT} \in$ EX$_*$-TXT holds.

**Definition 3.5.** Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. A set $T_i$ is said to be a 0-*bounded finite tell-tale* ($FT_0$, for short) of $L_i$ within $\mathcal{C}$, if $T_i$ is a definite finite tell-tale of $L_i$ within $\mathcal{C}$, that is, (i) $T_i$ is a finite subset of $L_i$ and (ii) for any $L_j \in \mathcal{C}$, $T_i \subseteq L_j$ implies $L_j = L_i$.

A set $T_i$ is said to be an *$n$-bounded finite tell-tale* ($FT_n$, for short; $n \geq 1$) of $L_i$ within $\mathcal{C}$, if it satisfies the following conditions: (i) $T_i$ is a finite subset of $L_i$ and (ii) for any $L_j \in \mathcal{C}$, if $L_j \neq L_i$ and $T_i \subseteq L_j$ hold, then there exists an $FT_{n-1}$ of $L_j$ within $\mathcal{C}$.

A pair $(T_i, F_i)$ is said to be a *pair of 0-bounded finite tell-tales* ($PFT_0$, for short) of $L_i$ within $\mathcal{C}$, if it satisfies the following conditions: $(T_i, F_i)$ is a pair of definite finite tell-tales of $L_i$ within $\mathcal{C}$, that is, (i) $T_i$ is a finite subset of $L_i$, (ii) $F_i$ is a finite subset of $L_i^c$, and (iii) for any $L_j \in \mathcal{C}$, if $(T_i, F_i)$ is consistent with $L_j$, then $L_j = L_i$ holds.

A pair $(T_i, F_i)$ is said to be a *pair of $n$-bounded finite tell-tales* ($PFT_n$, for short; $n \geq 1$) of $L_i$ within $\mathcal{C}$, if it satisfies the following conditions: (i) $T_i$ is a finite subset of $L_i$, (ii) $F_i$ is a finite subset of $L_i^c$, and (iii) for any $L_j \in \mathcal{C}$, if $L_j \neq L_i$ holds and $(T_i, F_i)$ is consistent with $L_j$, then there exists a $PFT_{n-1}$ of $L_j$ within $\mathcal{C}$.

Intuitively, an $FT_n$ or a $PFT_n$ of $L_i$ within $\mathcal{C}$ is a tell-tale which validates producing the guess $i$, when the inference machine is allowed to produce another $n+1$ guesses.

For $n \geq 0$, we can easily show by mathematical induction on $n$ that if a finite set $T$ is an $FT_n$ of $L_i$ within $\mathcal{C}$, then it is also an $FT_{n+1}$ of $L_i$ within $\mathcal{C}$. In a similar way, we can show that if a pair $(T, F)$ of finite sets is a $PFT_n$ of $L_i$ within $\mathcal{C}$, then it is also a $PFT_{n+1}$ of $L_i$ within $\mathcal{C}$.

**Definition 3.6.** Let $n \geq 0$ be an integer. An effective procedure *Ft* is said to *uniformly and recurrently construct $FT_n$'s of $\mathcal{C}$*, if it satisfies the following conditions:

(i) For any index $i$, the procedure *Ft* on inputs $n$ and $i$ recursively generates a finite set.

(ii) For any $m$ with $0 \leq m \leq n$ and any index $i$, if the procedure *Ft* on inputs $m$ and $i$ recursively generates a finite set $S$, then

(a) $S$ is an $FT_m$ of $L_i$ within $\mathcal{C}$, and

(b) for any index $j$, if $L_j \neq L_i$ and $S \subseteq L_j$ hold, then the procedure *Ft* on inputs $m-1$ and $j$ also recursively generates a finite set.

For $n \geq 0$, $FT_n$'s of $\mathcal{C}$ are said to be *uniformly and recurrently constructible*, if there is an effective procedure which uniformly and recurrently construct $FT_n$'s of $\mathcal{C}$.

In a similar way, we also define the uniform and recurrent constructibility of $PFT_n$'s of a class.

**Theorem 3.9.** *Let $n \geq 0$ be an integer, and let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. Assume that whether $L_i = L_j$ or not is recursively decidable for any indices $i$ and $j$. Then $\mathcal{C}$ is $EX_n$-TXT inferable, if and only if $FT_n$'s of $\mathcal{C}$ are uniformly and recurrently constructible.*

*Proof.* (I) The 'only if' part. Assume that an IIM $M$ infers $\mathcal{C}$ within $n$ mind changes from positive data. Let us consider the procedure in Figure 3.4.

---

**Procedure** $Ft(m, i)$;
**begin**
let $\psi_1, \psi_2, \cdots$ be an effective enumeration of
all nonempty finite sequences consisting of elements in $L_i$;
search for an integer $j \geq 1$ such that $\sharp\overline{M}[\psi_j] > n - m$ and $L_{\overline{M}(\psi_j)} = L_i$;
**if** such an integer $j$ is found **then** output $\psi_j$ and stop;
**end**.

---

Figure 3.4: An algorithm which uniformly and recurrently constructs $FT_n$'s of a class

(1) For any index $i$, the procedure on inputs $n$ and $i$ recursively generates a finite set. This is because $M$ infers $\mathcal{C}$ in the limit from positive data.

(2) If the procedure on inputs $m$ and $i$ recursively generates a finite set, then it is a finite subset of $L_i$.

(3) If the procedure on inputs 0 and $i$ recursively generates a finite set $S$, then $S$ is an $FT_0$ of $L_i$ within $\mathcal{C}$. In fact, suppose that $S$ is not an $FT_0$ of $L_i$ within $\mathcal{C}$. Then there is an $L_j \in \mathcal{C}$ such that $L_j \neq L_i$ and $S \subseteq L_j$. Let $\sigma$ be an arbitrary positive presentation of $L_j$, and let $\psi$ be the finite sequence enumerated by the procedure on inputs 0 and $i$. Since $\psi \cdot \sigma$ is a positive presentation of $L_j$, it follows that $M$ does not infer $L_j$ w.r.t. $\mathcal{C}$ within $n$ mind changes from $\psi \cdot \sigma$, which contradicts the assumption.

(4) For any $m$ with $0 < m \leq n$ and any index $i$, if the procedure on inputs $m$ and $i$ recursively generates a finite set $S$ and there is an index $j$ such that $L_j \neq L_i$ and $S \subseteq L_j$, then the procedure on inputs $m - 1$ and $j$ also recursively generates a finite set. In fact, suppose that the procedure on inputs $m - 1$ and $j$ does not recursively generate a finite set. Let $\sigma$ be an arbitrary positive presentation of $L_j$, let $\psi$ be the finite sequence enumerated

by the procedure on inputs $m$ and $i$, and put $\delta = \psi \cdot \sigma$. Then there is a $k > \sharp\psi$ such that $\overline{M}(\delta[k]) > 0$ and $L_j = L_{\overline{M}(\delta[k])}$, because $M$ infers $L_j$ w.r.t. $\mathcal{C}$ in the limit from $\delta$. Since $\delta[k]$ is a finite sequence consisting of elements in $L_i$, there is a case where the search statement in the procedure is executed with this finite sequence. Furthermore, by its construction, $\sharp\overline{M}[\delta[k]] > n - m + 1$ holds. Hence the procedure will produce a guess. This is a contradiction.

By (1), (2), (3) and (4), we see that the procedure uniformly and recurrently constructs $FT_n$'s of $\mathcal{C}$. Therefore $FT_n$'s of $\mathcal{C}$ are uniformly and recurrently constructible.

(II) The 'if' part. Assume that a procedure $Ft$ uniformly and recurrently constructs $FT_n$'s of $\mathcal{C}$. Then let us consider the procedure in Figure 3.5.

---

```
Procedure IIM M;
begin
    m = n;    k = 0;    S = φ;    T = φ;
    j = 1;
    repeat
        read the next fact and store it in T;
        for i = 1 to j do
            if k = 0 or (L_k ≠ L_i and S ⊆ L_i) then
                if Ft(m, i) ⊆ T then begin
                    output i;
                    if m = 0 then stop;
                    m = m − 1;    k = i;    S = Ft(m, i);
                end;
        j = j + 1;
    forever;
end.
```

---

Figure 3.5: An inference machine which infers a class within $n$ mind changes

It is easy to see that the procedure produces at most $n + 1$ guesses. Assume that we feed a positive presentation $\sigma$ of a concept $L_h \in \mathcal{C}$ to the procedure.

(1) The procedure produces at least one guess. In fact, suppose that the procedure never produces a guess. When it reaches the case

$$j = \max\{h, \min\{l \mid Ft(n,h) \subseteq \sigma[l]^+\}\} \quad \text{and} \quad i = h,$$

the procedure should produce the guess $h$, which contradicts the assumption.

(2) Suppose the last guess $g$ produced by the procedure is not correct.

(a) In case the value of $m$ in the procedure is equal to 0, when the procedure produced the last guess. It contradicts the definition of an $FT_0$.

(b) Otherwise. In the procedure, we note that $S \subseteq T$ and $L_g \neq L_h$. When it reaches the case

$$j \geq \max\{h, \min\{l \mid Ft(n,h) \subseteq \sigma[l]^+\}\} \quad \text{and} \quad i = h,$$

the procedure should produce the next guess $h$, which contradicts the assumption. ∎

In a similar way to the proof of Theorem 3.9, we can show the following Theorem 3.10.

**Theorem 3.10.** *Let $n \geq 0$ be an integer, and let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. Assume that whether $L_i = L_j$ or not is recursively decidable for any indices $i$ and $j$. Then $\mathcal{C}$ is EX$_n$-INF inferable, if and only if PFT$_n$'s of $\mathcal{C}$ are uniformly and recurrently constructible.*

We note that in case $n = 0$, the above Theorem 3.9 or Theorem 3.10 is equivalent to Theorem 3.1 or Theorem 3.2 by Corollary 3.6. In the following Section 3.3, we show another complete characterizations on EX$_n$-TXT or EX$_n$-INF inferability without the assumption of recursive decidability of equivalence of two concepts.

The following Example 3.5 contrasts well with Example 3.2.

**Example 3.5.** For $n \geq 1$, let $\mathcal{FC}_{\leq n}$ be the class of all nonempty finite concepts on the universal set $U$ each of which cardinality is *at most* $n$. By assuming a bijective coding from $\mathcal{FC}_{\leq n}$ to $N$, this class becomes an indexed family of nonempty recursive concepts. Then it is clear that whether $L_i = L_j$ or not is recursively decidable for any indices $i$ and $j$.

This class is $\mathrm{EX}_{n-1}$-TXT inferable. In fact, for a concept $\{w_1, \cdots, w_k\}$ $(1 \leq k \leq n)$, there is an $FT_m$ $(n-k \leq m \leq n-1)$ of it within $\mathcal{FC}_{\leq n}$ such as

$$FT_{n-k} = FT_{n-k+1} = \cdots = FT_{n-1} = \{w_1, \cdots, w_k\},$$

and it follows that $FT_{n-1}$'s of $\mathcal{FC}_{\leq n}$ are uniformly and recurrently constructible.

On the other hand, we see by Proposition 3.7 that this class is not $\mathrm{EX}_{n-2}$-TXT inferable, if $n \geq 2$. Furthermore this class is shown not to be $\mathrm{EX}_{n-2}$-INF inferable. This is because there is no $PFT_{n-2}$ of the concept $\{w_1\}$ within $\mathcal{FC}_{\leq n}$, where $w_1$ is an arbitrary element in the universal set $U$. In fact, suppose that there is a $PFT_{n-2}$ $(T_1, F_1)$ of the concept $\{w_1\}$ within $\mathcal{FC}_{\leq n}$. We define $w_i$'s and $(T_i, F_i)$'s $(2 \leq i \leq n-2)$ inductively by the following stages:

Stage $i$ $(2 \leq i \leq n-1)$:

Let $w_i \in U$ be an arbitrary element such that $w_i \notin \{w_1, \cdots, w_{i-1}\}$ and $w_i \notin \bigcup_{j=1}^{i-1} F_j$. Since $(T_{i-1}, F_{i-1})$ is a $PFT_{n-i}$ of the concept $\{w_1, \cdots, w_{i-1}\}$ within $\mathcal{FC}_{\leq n}$ and it is consistent with the concept $\{w_1, \cdots, w_i\}$, there is a $PFT_{n-i-1}$ $(T, F)$ of the concept $\{w_1, \cdots, w_i\}$ within $\mathcal{FC}_{\leq n}$. Put $(T_i, F_i) = (T, F)$.

If $i < n-1$ then goto Stage $i+1$ else stop.

Thus we have a $PFT_0$ $(T_{n-1}, F_{n-1})$ of the concept $\{w_1, \cdots, w_{n-1}\}$ within $\mathcal{FC}_{\leq n}$. Let $w_n \in U$ be an arbitrary element such that $w_n \notin \{w_1, \cdots, w_{n-1}\}$ and $w_n \notin \bigcup_{i=1}^{n-1} F_i$. Then the concept $\{w_1, \cdots, w_n\}$ is in $\mathcal{FC}_{\leq n}$ and that $(T_{n-1}, F_{n-1})$ is consistent with this concept. This is a contradiction, because $(T_{n-1}, F_{n-1})$ is a $PFT_0$ of the concept $\{w_1, \cdots, w_{n-1}\}$ within $\mathcal{FC}_{\leq n}$. Hence $\mathcal{FC}_{\leq n}$ is *not* $\mathrm{EX}_{n-2}$-INF inferable.

Let $\mathcal{FC}_*$ be the class of all nonempty finite concepts on the universal set $U$. By assuming a bijective coding from $\mathcal{FC}_*$ to $N$, this class becomes an indexed family of nonempty recursive concepts. This class is $\mathrm{EX}_*$-TXT inferable (cf. Corollary 2.7), and thus it is $\mathrm{EX}_*$-INF inferable. Furthermore in a similar way to the case of $\mathcal{FC}_{\leq n}$, this class is shown to be neither $\mathrm{EX}_n$-TXT nor $\mathrm{EX}_n$-INF inferable for any $n \geq 0$.

By Corollary 3.8 and Example 3.5, we see that there are hierarchies such as

$$\mathrm{EX}_0\text{-TXT} \subsetneq \mathrm{EX}_1\text{-TXT} \subsetneq \cdots \subsetneq \mathrm{EX}_n\text{-TXT} \subsetneq \cdots \subsetneq \bigcup_{i=1}^{\infty} \mathrm{EX}_i\text{-TXT} \subsetneq \mathrm{EX}_*\text{-TXT},$$

$$\mathrm{EX}_0\text{-INF} \subsetneq \mathrm{EX}_1\text{-INF} \subsetneq \cdots \subsetneq \mathrm{EX}_n\text{-INF} \subsetneq \cdots \subsetneq \bigcup_{i=1}^{\infty} \mathrm{EX}_i\text{-INF} \subsetneq \mathrm{EX}_*\text{-INF}.$$

## 3.3. Further Results

In this section we review further results obtained by Lange&Zeugmann[21]. Here we mainly consider class-preserving learning defined below.

**Definition 3.7.** A class $\mathcal{C} = \{L_i\}_{i\in N}$ is said to be *class-preservingly inferable in the limit from positive data* (resp., *complete data*), if there is a class $\mathcal{C}' = \{L'_i\}_{i\in N}$ such that (i) $\mathcal{C}'$ is inferable in the limit from positive data (resp., positive data) and that (ii) $\{L_i \in \mathcal{C} \mid i \geq 1\} = \{L'_i \in \mathcal{C}' \mid i \geq 1\}$.

That is, class-preserving learnability means inferability with well-chosen indexing of the class.

In a similar way, we also define class-preserving inferability with a bounded number of mind changes. For $n \geq 0$, if a class $\mathcal{C}$ is class-preservingly inferable within $n$ mind changes from positive data (resp., complete data), we also say that $\mathcal{C}$ is *CPEX$_n$-TXT inferable* (resp., *CPEX$_n$-INF inferable*). Furthermore, if a class $\mathcal{C}$ is class-preservingly inferable in the limit from positive data (resp., complete data), we also say that $\mathcal{C}$ is *CPEX$_*$-TXT inferable* (resp., *CPEX$_*$-INF inferable*).

For $n \geq 0$ or $n = *$, by the same notation CPEX$_n$-TXT (resp., CPEX$_n$-INF), we also denote the collection of all classes that are CPEX$_n$-TXT inferable (resp., CPEX$_n$-INF inferable).

By definition, it is clear that for any $n \geq 0$ or $n = *$, EX$_n$-TXT $\subseteq$ CPEX$_n$-TXT and EX$_n$-INF $\subseteq$ CPEX$_n$-INF. The following Theorem 3.11 shows superiority of class-preserving learning, if the number of mind changes is bounded by a constant number greater than 0.

**Theorem 3.11** (Lange&Zeugmann[21]). *(a) The following four equations are valid:*

$$CPEX_0\text{-}TXT = EX_0\text{-}TXT, \quad CPEX_0\text{-}INF = EX_0\text{-}INF,$$

$$CPEX_*\text{-}TXT = EX_*\text{-}TXT, \quad CPEX_*\text{-}INF = EX_*\text{-}INF.$$

*(b) For any $n \geq 1$, $EX_n\text{-}TXT \subsetneq CPEX_n\text{-}TXT$ holds.*

It is unknown at present whether $\mathrm{EX}_n\text{-INF} \subsetneq \mathrm{CPEX}_n\text{-INF}$ holds or not for any $n \geq 1$.

Lange&Zeugmann[21] also obtained very interesting and surprising results as follows:

**Theorem 3.12** (Lange&Zeugmann[21]). *(a) For any $n \geq 0$, $EX_{n+1}\text{-}TXT \not\subseteq CPEX_n\text{-}INF$ holds. Therefore for any $n \geq 0$, $CPEX_{n+1}\text{-}TXT \not\subseteq CPEX_n\text{-}INF$ holds.*

*(b) $EX_1\text{-}INF \not\subseteq CPEX_*\text{-}TXT$ holds. Therefore for any $n \geq 1$, $EX_n\text{-}INF \not\subseteq CPEX_*\text{-}TXT$ and $CPEX_n\text{-}INF \not\subseteq CPEX_*\text{-}TXT$ hold.*

*(c) $\bigcup_{i=1}^{\infty} EX_i\text{-}INF \# CPEX_*\text{-}TXT$ and $\bigcup_{i=1}^{\infty} EX_i\text{-}INF \# CPEX_*\text{-}TXT$ hold, where for two sets $S$ and $T$, the relation $T \# S$ means $T \not\subseteq S$ and $S \not\subseteq T$.*

Furthermore Lange&Zeugmann[21] showed that there are hierarchies such as

$$\begin{aligned}\mathrm{CPEX}_0\text{-TXT} \subsetneq \mathrm{CPEX}_1\text{-TXT} \subsetneq \cdots \subsetneq \mathrm{CPEX}_n\text{-TXT} \subsetneq \cdots \\ \subsetneq \bigcup_{i=1}^{\infty} \mathrm{CPEX}_i\text{-TXT} \subsetneq \mathrm{CPEX}_*\text{-TXT},\end{aligned}$$

$$\begin{aligned}\mathrm{CPEX}_0\text{-INF} \subsetneq \mathrm{CPEX}_1\text{-INF} \subsetneq \cdots \subsetneq \mathrm{CPEX}_n\text{-INF} \subsetneq \cdots \\ \subsetneq \bigcup_{i=1}^{\infty} \mathrm{CPEX}_i\text{-INF} \subsetneq \mathrm{CPEX}_*\text{-INF}.\end{aligned}$$

Concerning characterizations for inferability with a bounded number of mind changes, the following Theorem 3.13 and 3.14 hold:

**Theorem 3.13** (Lange&Zeugmann[21]). *Let $n \geq 0$ be an integer, and let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. Then $\mathcal{C}$ is $CPEX_n\text{-}TXT$ inferable, if and only if there are a class $\mathcal{C}' = \{L'_i\}_{i \in N}$, a computable relation $\prec$ over $N$, and a uniformly and recursively generable family $\{T_i\}_{i \in N}$ of finite sets such that*

*(i) $\{L_i \in \mathcal{C} \mid i \geq 1\} = \{L'_i \in \mathcal{C}' \mid i \geq 1\}$,*

*(ii) for any index $i$, $T_i \subseteq L_i$ holds,*

*(iii) for any indices $i$ and $j$, if $T_i \subseteq L_j$ and $L_j \neq L_i$ hold, then there is an index $k$ such that $i \prec k$, $T_i \subseteq T_k$ and $L_k = L_i$,*

*(iv) for any index $i$, there is no sequence $i_1, i_2, \cdots, i_{n+2}$ of indices such that $i_1 \prec i_2 \prec \cdots \prec i_{n+2}$ and $T_{i_1} \subseteq T_{i_2} \subseteq \cdots \subseteq T_{i_{n+2}} \subseteq L_i$.*

**Theorem 3.14** (Lange&Zeugmann[21]). *Let $n \geq 0$ be an integer, and let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. Then $\mathcal{C}$ is CPEX$_n$-INF inferable, if and only if there are a class $\mathcal{C}' = \{L'_i\}_{i \in N}$, and uniformly and recursively generable families $\{T_i\}_{i \in N}$ and $\{F_i\}_{i \in N}$ of finite sets such that*

*(i) $\{L_i \in \mathcal{C} \mid i \geq 1\} = \{L'_i \in \mathcal{C}' \mid i \geq 1\}$,*

*(ii) for any index $i$, $(T_i, F_i)$ is consistent with $L_i$,*

*(iii) for any indices $i$ and $j$, if $(T_i, F_i)$ is consistent with $L_j$ and $L_j \neq L_i$ holds, then there is an index $k$ such that $(T_i, F_i) \sqsubset (T_k, F_k)$ and $L_k = L_i$,*

*(iv) for any index $i$, there is no sequence $i_1, i_2, \cdots, i_{n+2}$ of indices such that $(T_{i_1}, F_{i_1}) \sqsubset (T_{i_2}, F_{i_2}) \sqsubset \cdots \sqsubset (T_{i_{n+2}}, F_{i_{n+2}}) \sqsubset (L_i, L_i^c)$,*

*where for sets $T, F, T'$ and $F'$, the relation $(T, F) \sqsubset (T', F')$ means $T \subseteq T'$, $F \subseteq F'$ and $(T, F) \neq (T', F')$.*

Furthermore we have the following characterizations for EX$_n$-TXT or EX$_n$-INF inferability for $n \geq 0$.

**Corollary 3.15** (Sato[40]). *Let $n \geq 0$ be an integer, and let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. Then $\mathcal{C}$ is EX$_n$-TXT inferable, if and only if there are a uniformly and recursively generable family $\{T_i\}_{i \in N}$ of finite sets, a computable relation $\prec$ over $N$, and a recursive function $h : N \to N$ such that*

*(i) for any index $i$, there exists a $j \geq 1$ such that $L_i = L_{h(j)}$,*

*(ii) for any index $i$, $T_i \subseteq L_{h(i)}$,*

*(iii) for any indices $i$ and $j$, if $T_i \subseteq L_j$ and $L_j \neq L_{h(i)}$ hold, then there is an index $k$ such that $i \prec k$ and $L_j = L_{h(k)}$,*

*(iv) for any index $i$, there is no sequence $i_1, i_2, \cdots, i_{n+2}$ of indices such that $i_1 \prec i_2 \prec \cdots \prec i_{n+2}$ and $T_{i_1} \subseteq T_{i_2} \subseteq \cdots \subseteq T_{i_{n+2}} \subseteq L_i$.*

*Proof* (Based on the proof of Theorem 3.13). (I) The 'only if' part. Assume that an IIM $M$ infers $\mathcal{C}$ within $n$ mind changes from positive data.

Let $\psi_1, \psi_2, \cdots$ be an enumeration of all nonempty finite sequences consisting of elements in $U$ such that if $\psi_i$ is an initial segment of $\psi_j$ then $i < j$ holds. Let $c : N \times N \to N$ be Cantor's pairing function. For $i, j \geq 1$, put $L'_{c(i,j)} = L_i$, and define $T'_{c(i,j)}$ as follows:

(1) In case the length of $\psi_i$ is 1. We put

$$T'_{c(i,j)} = \begin{cases} \widetilde{\psi}_j, & \text{if } \widetilde{\psi}_j \subseteq L_i \, (= L'_{c(i,j)}) \text{ and } \overline{M}(\psi_j) = i, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

(2) Otherwise. Then there are a finite sequence $\psi_k$ and an element $w$ in $U$ such that $\psi_i = \psi_k w$. We note that $k < i$ by assumption. We put

$$T'_{c(i,j)} = \begin{cases} T'_{c(i,k)}, & \text{if } \widetilde{\psi}_j \subseteq L_i \, (= L'_{c(i,j)}) \text{ and } \overline{M}(\psi_k) = \overline{M}(\psi_j) = i, \\ \widetilde{\psi}_j, & \text{if } \widetilde{\psi}_j \subseteq L_i \, (= L'_{c(i,j)}) \text{ and } \overline{M}(\psi_k) \neq \overline{M}(\psi_j) = i, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Next, we define a function $g : N \to N$ as follows:

$$g(0) = 0 \quad \text{and} \quad g(i+1) = \min\{j > g(i) \mid T'_j \text{ is defined}\} \quad \text{for } i \geq 0.$$

Then for $i \geq 1$, put $T_i = T'_{g(i)}$, and let $h(i)$ be the integer $j$ such that $g(i) = c(j, m)$ for some $m \geq 1$.

We note that the above $\{T_i\}_{i \in N}$ is a uniformly and recursively generable family of finite sets and that two functions $g$ and $h$ are recursive.

Finally we define the relation $\prec$ as follows: For $i, j \geq 1$, $i \prec j$ if and only if $g(i) = c(i', k)$, $g(j) = c(j', l)$, $i' \neq j'$ hold and $\psi_k$ is an initial segment of $\psi_l$ for some $i', j', k, l \geq 1$.

Now we verify that the family $\{T_i\}_{i \in N}$, the relation $\prec$ and the function $h$ satisfy the conditions (i)-(iv). Since $M$ infers $\mathcal{C}$ in the limit from positive data, it is easy to see that (i) holds. It is also easy to see that (ii) holds.

Assume $T_i \subseteq L_j$ and $L_j \neq L_{h(i)}$. Let $i', k'$ be the integers such that $h(i) = i'$ and $g(i) = c(i', k')$, and let $\delta$ be an arbitrary positive presentation of $L_j$. Since $T'_{c(i',k')} = T_i \subseteq L_j$,

$\overline{M}(\psi_{k'}) = i'$ and $L_j \neq L_{h(i)} (= L_{i'})$, it follows that there are an $l \geq 1$ and an index $j'$ such that $\overline{M}(\psi_{k'} \cdot \delta[l-1]) \neq \overline{M}(\psi_{k'} \cdot \delta[l]) = j'$ and $L_{j'} = L_j$. Let $l'$ be an integer such that $\psi_{l'} = \psi_{k'} \cdot \delta[l]$, and let $k$ be an index such that $h(k) = j'$ and $g(k) = c(j', l')$. Then $i \prec k$ and $L_j = L_{h(k)}$ hold. Hence (iii) holds.

Finally suppose that there are an index $i$ and a sequence $i_1, i_2, \cdots, i_{n+2}$ of indices such that $i_1 \prec i_2 \prec \cdots \prec i_{n+2}$ and $T_{i_1} \subseteq T_{i_2} \subseteq \cdots \subseteq T_{i_{n+2}} \subseteq L_i$. Then it is easy to see that $M$ makes at least $n+1$ mind changes for a certain positive presentation of $L_i$, which contradicts the assumption. Hence (iv) holds.

(II) The 'if' part. Assume that there are a uniformly and recursively generable family $\{T_i\}_{i \in N}$ of finite sets, a computable relation $\prec$ over $N$, and a recursive function $h : N \to N$ that satisfy (i)-(iv). Let us consider the procedure in Figure 3.6, where we put $T_0 = \phi$ and $0 \prec i$ for any $i \geq 1$.

---

**Procedure** IIM $M$;
**begin**
    $T = \phi$;
    $m = 0$; $j = 0$;
    **repeat**
        read the next fact and store it in $T$;
        $m = m + 1$;
        search for the least index $i \leq m$ such that $T_j \subseteq T_i \subseteq T$ and $j \prec i$;
        **if** such an index $i$ is found **then** let $j = i$, and output $h(i)$;
    **forever**;
**end**.

---

Figure 3.6: An inference machine which infers a class within $n$ mind changes

Since $\{T_i\}_{i \in N}$ is a uniformly and recursively generable family of finite sets, the relation $\prec$ over $N$ is computable, and the function $h : N \to N$ is recursive, it follows that the procedure is effective.

By the conditions (i), (ii) and (iii), it is easy to see that the procedure infers $\mathcal{C}$ in the limit from positive data, and by the condition (iv), it is also easy to see that the procedure makes at most $n$ mind changes. ∎

**Corollary 3.16** (Sato[40]). *Let $n \geq 0$ be an integer, and let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. Then $\mathcal{C}$ is $EX_n$-INF inferable, if and only if there are uniformly and recursively generable families $\{T_i\}_{i \in N}$ and $\{F_i\}_{i \in N}$ of finite sets and a recursive function $h : N \to N$ such that*

*(i) for any index $i$, there exists a $j \geq 1$ such that $L_i = L_{h(j)}$,*

*(ii) for any index $i$, $(T_i, F_i)$ is consistent with $L_{h(i)}$,*

*(iii) for any indices $i$ and $j$, if $(T_i, F_i)$ is consistent with $L_j$ and $L_j \neq L_{h(i)}$ holds, then there is an index $k$ such that $(T_i, F_i) \sqsubset (T_k, F_k)$ and $L_j = L_{h(k)}$,*

*(iv) for any index $i$, there is no sequence $i_1, i_2, \cdots, i_{n+2}$ of indices such that $(T_{i_1}, F_{i_1}) \sqsubset (T_{i_2}, F_{i_2}) \sqsubset \cdots \sqsubset (T_{i_{n+2}}, F_{i_{n+2}}) \sqsubset (L_i, L_i^c)$,*

*where for sets $T, F, T'$ and $F'$, the relation $(T, F) \sqsubset (T', F')$ means $T \subseteq T'$, $F \subseteq F'$ and $(T, F) \neq (T', F')$.*

*Proof* (Based on the proof of Theorem 3.14). (I) The 'only if' part. Assume that an IIM $M$ infers $\mathcal{C}$ within $n$ mind changes from complete data. Without loss of generality, we assume that $M$ works conservatively.

Let $c : N \times N \to N$ be Cantor's pairing function. For index $i$, let $\sigma_i$ be the canonical complete presentation of $L_i$ (cf. Definition 2.3). For $i, j \geq 1$, put $L'_{c(i,j)} = L_i$ and

$$T'_{c(i,j)} = \begin{cases} \sigma_i[\min\{k \leq j \mid \overline{M}(\sigma_i[k]) = i\}]^+, & \text{if } \exists k \leq j \text{ s.t. } \overline{M}(\sigma_i[k]) = i, \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

$$F'_{c(i,j)} = \begin{cases} \sigma_i[\min\{k \leq j \mid \overline{M}(\sigma_i[k]) = i\}]^-, & \text{if } \exists k \leq j \text{ s.t. } \overline{M}(\sigma_i[k]) = i, \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

Next, we define a function $g : N \to N$ as follows:

$$g(0) = 0 \quad \text{and} \quad g(i+1) = \min\{j > g(i) \mid T'_j \text{ is defined}\} \quad \text{for } i \geq 0.$$

Then for $i \geq 1$, put $T_i = T'_{g(i)}$ and $F_i = F'_{g(i)}$, and let $h(i)$ be the integer $j$ such that $g(i) = c(j, m)$ for some $m \geq 1$.

We note that the above $\{T_i\}_{i \in N}$ and $\{F_i\}_{i \in N}$ are uniformly and recursively generable families of finite sets and that two functions $g$ and $h$ are recursive.

Now we verify that two families $\{T_i\}_{i\in N}$ and $\{F_i\}_{i\in N}$ and the function $h$ satisfy the conditions (i)-(iv). Since $M$ infers $\mathcal{C}$ in the limit from complete data, it is easy to see that (i) holds. It is also easy to see that (ii) holds.

Assume that $(T_i, F_i)$ is consistent with $L_j$ and that $L_j \neq L_{h(i)}$. Let $l_0 = \sharp(T_i \cup F_i)$. Since $(T_i, F_i)$ is consistent with $L_j$ and $\sigma_j$ is the canonical complete presentation of $L_j$, it follows that $T_i = \sigma_j[l_0]^+$ and $F_i = \sigma_j[l_0]^-$. Since $M$ conservatively infers $L_j$ w.r.t. $\mathcal{C}$ in the limit from $\sigma_j$, there are an $l > l_0$ and an index $k'$ such that $L_j = L_{k'}$ and $k' = \overline{M}(\sigma_j[l])$. Therefore $T'_{c(k',l)}$ is defined, and thus there is an index $k$ such that $g(k) = c(k', l)$ and $h(k) = k'$. Then we have $(T_i, F_i) \sqsubset (T'_{c(k',l)}, F'_{c(k',l)}) = (T_k, F_k)$ and $L_j = L_{k'} = L_{h(k)}$. Hence (iii) holds.

Finally suppose that there are an index $i$ and a sequence $i_1, i_2, \cdots, i_{n+2}$ of indices such that $(T_{i_1}, F_{i_1}) \sqsubset (T_{i_2}, F_{i_2}) \sqsubset \cdots \sqsubset (T_{i_{n+2}}, F_{i_{n+2}}) \sqsubset (L_i, L_i^c)$. Then it is easy to see that $M$ makes at least $n+1$ mind changes for the canonical complete presentation of $L_i$, which contradicts the assumption. Hence (iv) holds.

(II) The 'if' part. Assume that there are uniformly and recursively generable families $\{T_i\}_{i\in N}$ and $\{F_i\}_{i\in N}$ of finite sets and a recursive function $h : N \to N$ that satisfy (i)-(iv). Let us consider the procedure in Figure 3.7, where the procedure read_store is the same one as in Figure 2.1, and we put $T_0 = \phi$ and $F_0 = \phi$.

Since $\{T_i\}_{i\in N}$ and $\{F_i\}_{i\in N}$ are uniformly and recursively generable families of finite sets, and the function $h : N \to N$ is recursive, it follows that the procedure is effective.

By the conditions (i), (ii) and (iii), it is easy to see that the procedure infers $\mathcal{C}$ in the limit from complete data, and by the condition (iv), it is also easy to see that the procedure makes at most $n$ mind changes. ∎

## 3.4. Discussion

In this chapter we have discussed conditions for a class to be inferable with a bounded number of mind changes from positive data or complete data. We also presented some classes that are inferable within $n$ mind changes but not inferable within $n-1$ mind changes from positive data or complete data for $n \geq 1$.

```
Procedure IIM M;
begin
    T = φ;    F = φ;
    m = 0;    j = 0;
    repeat
        read_store(T, F);
        m = m + 1;
        search for the least index i ≤ m such that
                T_i ⊆ T, F_i ⊆ F and (T_j, F_j) ⊏ (T_i, F_i) hold;
        if such an index i is found then let j = i, and output h(i);
    forever;
end.
```

Figure 3.7: An inference machine which infers a class within $n$ mind changes

Among them, finitely inferable classes are much smaller than those that are inferable in the limit, but the finite identification seems to be much more significant than it is thought of. For example, it is not only a base case of inductive inference but also a base case of PAC and MAT learning[50, 3], and its characterization theorems (cf. Theorem 3.1 and 3.2) seem to be depicting everything that we can finitely identify.

Concerning conditions for inferability with a bounded number of mind changes, Corollary 3.4, 3.6 and Proposition 3.7 are very interesting, when we consider the necessity of mind changes. Furthermore Corollary 3.3 and Theorem 3.12 by Lange&Zeugmann[21] present interesting relations between presentations and mind changes.

The class of pattern languages, which was introduced by Angluin[1, 2] as a concrete class which is inferable in the limit from positive data (cf. Example 2.1), is also very interesting from view points of presentations and mind changes. That is, this class is *finitely* inferable from *complete data* (cf. Example 3.3) but *not* inferable *within n mind changes* from *positive data* for $n \geq 0$ (cf. Corollary 3.8). Moreover the class of unions of at most $n$ pattern languages is shown to be inferable in the limit from positive data (cf. Example 2.2), while it is shown not to be finitely inferable from complete data if $n \geq 2$ (cf. Example 3.4). However it is unknown at present whether or not the class of unions of at most $n$ pattern languages is inferable within $m$ mind changes from complete data for $n, m \geq 2$.

In Chapter 6 we discuss inferability of the class of languages definable by EFS's. We can regard this class as a natural extension of a class of unions of pattern languages, and it also has many interesting properties.

# Chapter 4.

# Refutability and Inferability from Examples

In the middle of this century the logic of scientific discovery was deeply discussed by philosophers[32, 33]. Recently in Artificial Intelligence, especially in Cognitive Science, researchers are extensively discussing frameworks for scientific discovery from various viewpoints[47].

Before going into such detailed discussions we need to set up a computational logic of scientific discovery in a mathematical way so that we can precisely discuss what kinds of machine discovery can work. One of the best ways to this should be to reexamine the philosophical results from computational viewpoints. In the present chapter we start with making the Popperian logic of scientific discovery[32, 33] computational.

The Popperian logic of scientific discovery concentrated on the testability, falsifiability or refutability of hypotheses or scientific theories. Popper also asserted that scientific theory should have been refuted by observed facts and any such theory could by no means be verified[32]. Thus we tentatively believe the current theory until we face with an observation which is inconsistent with the theory.

Hence the consistent and conservative inductive inference can be viewed as a computational realization of the Popperian notion of refutability. In the inductive inference, the inference machine requests data or facts from time to time and produces hypotheses from time to time. The hypotheses produced by the machine are to be consistent with the facts read so far, and each of them is to be refuted when the machine faces with inconsistent data or facts.

Thus the Popperian logic of scientific discovery can be viewed as a basis of the modern inductive inference studies. The inductive inference is thus a mathematical basis of machine learning. *Then what should be a logic of machine discovery or a computational logic of scientific discovery?*

The machine discovery we are concerned with in this thesis is to make computers discover some scientific theories from given data or facts. Hence machine learning should be a key technology for machine discovery. In machine learning first we must select a hypothesis space from which the learning machine proposes theories or hypotheses. The space is naturally required to be large, but to make the learning efficient it is required to be small. As far as data or facts are presented according to a hypothesis that is unknown but guaranteed to be in the space as in the ordinal inductive inference, the machine will eventually identify the hypothesis, and hence no problem may arise. In machine discovery, however, we can not assume this. God knows whether or not a hypothesis behinds the data or facts belongs to the space.

If the hypothesis is not in the space, most learning machines will continue for ever to search the space for a new hypothesis. Usually we can not know the time when to stop such an ineffective searching. This is the most crucial problem we must solve in realizing machine discovery systems. In machine discovery the sequences of data or facts are given at first independently of the space. We can not give in advance the space that includes the desired theory. If the learning machine can explicitly tell us that there are no theories in the space which explain the given sequence, the machine will work for machine discovery.

Hence the essence of a computational logic of scientific discovery should be that the entire hypothesis space is refutable by a sequence of observed data or facts. If there exist rich hypothesis spaces that can be refuted, we can give a space and a sequence to the machine, and then we can just wait for an output from it. The machine will discover a hypothesis which is producing the sequence if it is in the space, otherwise it will refute the whole of the space and stop. When the space is refuted, we may give another space to the machine and try to make such a discovery in the new space.

In the present thesis, we choose the inductive inference as the framework for machine learning. Then the machine discovery system is an inductive inference machine that can refute hypothesis spaces.

If the class is a finite set of recursive concepts, it is trivially refutable from complete data. Also if the class contains all finite concepts, it is easily shown not to be refutable. *Then are there any meaningful classes, i.e. hypothesis spaces, that are identifiable and refutable?* We give a positive answer to this question. We will say such classes to be *refutably inferable.*

In Section 4.2 we discuss some conditions on refutable inferability from positive data or complete data. Concerning refutable inferability from *positive data*, we present some necessary and sufficient conditions, and reveal that the power is very small. Then in Section 4.3 we show the differences between the inferable classes under the criteria of refutable identification, reliable identification, finite identification and identification in the limit. In Section 4.4 we show that a class which consists of unions of at most $n$ concepts from $n$ classes is refutably inferable from complete data, if each class satisfies a certain condition. Furthermore in Section 6.2 we show that for any $n \geq 0$, the classes definable by length-bounded EFS's with at most $n$ axioms are refutably inferable from complete data, and reveal that there are sufficiently large classes that are refutably inferable from complete data.

This chapter is based on Mukouchi&Arikawa[30].

## 4.1. Definitions of Refutability and Inferability

First of all, we give the ability to refute a hypothesis space to an inference machine.

**Definition 4.1.** An *inductive inference machine that can refute hypothesis spaces* (*RIIM*, for short) is an effective procedure, or a certain type of Turing machine, which requests inputs from time to time and either (i) produces positive integers from time to time or (ii) refutes the class and stops after producing some positive integers.

For an RIIM $M$ and a nonempty initial segment $\sigma[n] = w_1, w_2, \cdots, w_n$ of a positive or complete presentation, we define $M(\sigma[n])$ as follows: Initialize $M$ and start $M$ in the initial state. If it requests a fact for the $i$-th time with $1 \leq i \leq n$, then feed $w_i$ and continue the execution. If it produces more than one positive integers or 'refutation' signs between any two input requests before requesting the $(n+1)$-st fact, then leave $M(\sigma[n])$ undefined.

(I) In case $M$ requests the $(n+1)$-st fact, or it stops after it requested the $n$-th fact. If it produces a positive integer or the 'refutation' sign after it requested the $n$-th fact, then let $M(\sigma[n])$ be the last integer or the 'refutation' sign produced by $M$, otherwise let $M(\sigma[n]) = 0$.

(II) In case $M$ stops before requesting the $n$-th fact. Let $M(\sigma[n]) = 0$.

(III) Otherwise. Leave $M(\sigma[n])$ undefined.

The intended interpretation is as follows: (i) In case $M(\sigma[n])$ is defined as the 'refutation' sign, the RIIM $M$ refutes the class concerned. (ii) In case $M(\sigma[n])$ is defined as a positive integer $i$, the RIIM $M$ guesses the $i$-th concept in the class. (iii) In case $M(\sigma[n])$ is defined as the integer 0, the RIIM $M$ makes no guess. (iv) In case $M(\sigma[n])$ is undefined, the RIIM $M$ is out of control.

Furthermore we define $\overline{M}(\sigma[n])$ as follows:

(I) In case $M(\sigma[n])$ is defined. (i) If there is the 'refutation' sign in the sequence $M(\sigma[1]), M(\sigma[2]), \cdots, M(\sigma[n])$, then let $\overline{M}(\sigma[n])$ be the 'refutation' sign. (ii) Otherwise. If there is a positive integer in the sequence $M(\sigma[1]), M(\sigma[2]), \cdots, M(\sigma[n])$, then let $\overline{M}(\sigma[n])$ be the last positive integer in the sequence, otherwise let $\overline{M}(\sigma[n]) = 0$.

(II) Otherwise. Leave $\overline{M}(\sigma[n])$ undefined.

An RIIM $M$ is said to *converge to an index $i$ for a positive or complete presentation $\sigma$*, if there is an $n \geq 1$ such that for any $m \geq n$, $\overline{M}(\sigma[m]) = i$.

An RIIM $M$ is said to *refute a class $\mathcal{C}$ from a positive or complete presentation $\sigma$*, if there is an $n \geq 1$ such that $\overline{M}(\sigma[n])$ is the 'refutation' sign. In this case we also say that $M$ refutes the class $\mathcal{C}$ from $\sigma[n]$.

Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. For a concept $L_i \in \mathcal{C}$ and a positive or complete presentation $\sigma$ of $L_i$, an RIIM $M$ is said to *infer the concept $L_i$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$*, if $M$ converges to an index $j$ with $L_j = L_i$ for $\sigma$.

An RIIM $M$ is said to *refutably infer a class $\mathcal{C}$ from positive data* (resp., *complete data*), if it satisfies the following condition: For any nonempty concept $L$ (resp., any concept $L$)

and any positive presentation $\sigma$ (resp., any complete presentation $\sigma$) of $L$, (i) if $L \in \mathcal{C}$, then $M$ infers $L$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$, (ii) otherwise $M$ refutes the class $\mathcal{C}$ from $\sigma$. A class $\mathcal{C}$ is said to be *refutably inferable from positive data* (resp., *complete data*), if there is an RIIM $M$ which refutably infers the class $\mathcal{C}$ from positive data (resp., complete data).

Corresponding to Proposition 2.1, we have the following Proposition 4.1, which we implicitly use in showing some properties.

**Proposition 4.1.** *(a) Assume that an RIIM $M$ refutably infers a class $\mathcal{C}$ from positive data. Then for any nonempty finite sequence $\psi$ consisting of elements in $U$, $\overline{M}(\psi)$ is always defined.*

*(b) Assume that an RIIM $M$ refutably infers a class $\mathcal{C}$ from complete data. Then for any nonempty finite sequence $\psi$ consisting of elements in $U \times \{+, -\}$, $\overline{M}(\psi)$ is always defined.*

*Proof.* We only give the proof of (a). The proof of (b) can be given in a similar way.

Let $\psi = w_1, w_2, \cdots, w_n$ be a nonempty finite sequence, and put $\sigma = \psi, w_1, w_1, w_1, \cdots$. Then $\sigma$ is a positive presentation of the concept $\{w_1, w_2, \cdots, w_n\}$. Therefore $M$ either (i) converges to an index $j$ with $L_j = \{w_1, w_2, \cdots, w_n\}$ for $\sigma$ or (ii) refutes the class $\mathcal{C}$ from $\sigma$, and it follows by definition that for any $n \geq 1$, $M(\sigma[n])$ is defined. Thus $\overline{M}(\psi)$ is also defined. ∎

The above Proposition 4.1 claims that even when we feed *any* facts that may not be from any concept in the class, an RIIM either (i) successively requests another facts in a finite time forever or (ii) stops in a finite time after producing some positive integers.

Since we are considering an indexed family of recursive concepts, every class can be inferred from complete data by a simple enumerative method (cf. Theorem 2.4). However we can not take the class of all recursive concepts as a hypothesis space, because the following Proposition 4.2 holds.

**Proposition 4.2.** *The class $\mathcal{C}$ of all recursive concepts is not an indexed family of recursive concepts.*

*Proof.* Suppose that the class $\mathcal{C} = \{L_i\}_{i \in N}$ is an indexed family of *all* recursive concepts. Let $w_1, w_2, \cdots$ be an effective enumeration of the universal set $U$. Without loss of generality, we assume $w_i \neq w_j$ if $i \neq j$. Then we take a concept $L$ as follows:

$$L = \{w_i \mid w_i \notin L_i,\ i \geq 1\}.$$

As easily seen, this $L$ is recursive, and it differs from any concept in $\mathcal{C}$. This is a contradiction. ∎

In case an RIIM $M$ is fed a positive or complete presentation of a non-recursive concept, $M$ should refute the class. Therefore even if we could take the class of all recursive concepts, it would be still significant to consider refutable inferability.

## 4.2. Characterizations

In order to characterize the refutable inferability, we need the following Lemma 4.3.

**Lemma 4.3.** *Let $M$ be an RIIM which refutably infers a class $\mathcal{C}$ from positive data (resp., complete data). Then for a nonempty concept $L$ (resp., a concept $L$), for a positive presentation $\sigma$ (resp., a complete presentation $\sigma$) of $L$ and for $n \geq 1$, if $M$ refutes the class $\mathcal{C}$ from $\sigma[n]$, then $\sigma[n]$ is not consistent with any $L_i \in \mathcal{C}$.*

*Proof.* Assume that an RIIM $M$ refutes a class $\mathcal{C}$ from $\sigma[n]$. Then suppose that there is an $L_i \in \mathcal{C}$ such that $\sigma[n]$ is consistent with $L_i$. Let $\delta$ be a positive presentation (resp., a complete presentation) of $L_i$. Then the infinite sequence $\sigma[n] \cdot \delta$ becomes a positive presentation (resp., a complete presentation) of $L_i$. Therefore $M$ can not infer $L_i$ w.r.t. $\mathcal{C}$ in the limit from $\sigma[n] \cdot \delta$, which contradicts the assumption. ∎

By the above Lemma 4.3, we obtain the following Proposition 4.4.

**Proposition 4.4.** *(a) If a class $\mathcal{C}$ is refutably inferable from positive data, then*

*(4.1) for any nonempty concept $L \notin \mathcal{C}$, there is a finite set $T \subseteq L$ such that $T$ is not consistent with any $L_i \in \mathcal{C}$.*

*(b) If a class $\mathcal{C}$ is refutably inferable from complete data, then*

*(4.2) for any concept $L \notin \mathcal{C}$, there are finite sets $T \subseteq L$ and $F \subseteq L^c$ such that $(T, F)$ is not consistent with any $L_i \in \mathcal{C}$.*

*Proof.* We only give the proof of (a). The proof of (b) can be given in a similar way.

Assume that an RIIM $M$ refutably infers a class $\mathcal{C}$ from positive data. Let $L \notin \mathcal{C}$ be a nonempty concept, and let $\sigma$ be an arbitrary positive presentation of $L$. By definition, there is an $n \geq 1$ such that $M$ refutes the class $\mathcal{C}$ from $\sigma[n]$. Put $T = \sigma[n]^+$. Then, by Lemma 4.3, $T$ is *not* consistent with any $L_i \in \mathcal{C}$. ■

**Corollary 4.5.** *If a class $\mathcal{C}$ contains all nonempty finite concepts, then $\mathcal{C}$ is not refutably inferable from positive data or complete data.*

*Proof.* We only give the proof of the case of complete data. The proof for positive data can be given in a similar way.

Assume that a class $\mathcal{C}$ contains all nonempty finite concepts. Then let $L \notin \mathcal{C}$ be a concept. Let $T \subseteq L$ and $F \subseteq L^c$ be finite sets. (i) In case $T$ is not empty. There is an $L_i \in \mathcal{C}$ with $T = L_i$, and it follows that $(T, F)$ is consistent with $L_i$. (ii) In case $T$ is empty. Since $F$ is a finite set, there is an $L_i \in \mathcal{C}$ such that $F \subseteq L_i^c$, which means $(T, F)$ is consistent with $L_i$. Therefore by Proposition 4.4, we see that $\mathcal{C}$ is not refutably inferable from complete data. ■

In characterizing the refutable inferability, the notion of consistency plays an important role (cf. Definition 2.5).

**Definition 4.2.** Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. An RIIM $M$ which refutably infers a class $\mathcal{C}$ from positive data (resp., complete data) is said to be *consistently working*, if it satisfies the following condition: For any nonempty concept $L$ (resp., any concept $L$), any positive presentation $\sigma$ (resp., any complete presentation $\sigma$) of $L$ and any $n \geq 1$, (i) if $M(\sigma[n])$ is the 'refutation' sign, then $\sigma[n]$ is *not* consistent with any $L_i \in \mathcal{C}$, (ii) if $M(\sigma[n]) > 0$, then $\sigma[n]$ is consistent with $L_{M(\sigma[n])}$.

An RIIM $M$ which refutably infers a class $\mathcal{C}$ from positive data (resp., complete data) is said to be *responsively working*, if it satisfies the following condition: For any nonempty concept $L$ (resp., any concept $L$), any positive presentation $\sigma$ (resp., any complete presentation $\sigma$) of $L$ and any $n \geq 1$, if $M$ does not refute the class $\mathcal{C}$ from $\sigma[n]$, then $M(\sigma[n]) > 0$ holds, that is, while $M$ does not refute the class, $M$ produces a guess between any two input requests in the computation of $M$ on input $\sigma$.

A class $\mathcal{C}$ is said to be *refutably, consistently and responsively inferable* from positive data (resp., complete data), if there is a consistently and responsively working RIIM which refutably infers the class $\mathcal{C}$ from positive data (resp., complete data).

Here we note that a consistently and responsively working RIIM refutes a class immediately after the observed data become *not* consistent with any concept in the class.

Since we are considering an indexed family of recursive concepts, we can easily show that if a class $\mathcal{C}$ is inferable in the limit from positive data or complete data, then it can be achieved by a *consistently and responsively* working IIM (cf. Section 2.1 and 2.2). Furthermore, as shown later, if a class $\mathcal{C}$ is refutably inferable from positive data or complete data, then it can be achieved by a *consistently and responsively* working RIIM.

**Definition 4.3.** For a finite set $T \subseteq U$, let

$$\mathrm{econs}_p(T) = \begin{cases} 1, & \text{if there exists an } L_i \in \mathcal{C} \text{ such that} \\ & \qquad\qquad T \text{ is consistent with } L_i, \\ 0, & \text{otherwise.} \end{cases}$$

For finite sets $T, F \subseteq U$, let

$$\mathrm{econs}_c(T, F) = \begin{cases} 1, & \text{if there exists an } L_i \in \mathcal{C} \text{ such that} \\ & \qquad\qquad (T, F) \text{ is consistent with } L_i, \\ 0, & \text{otherwise.} \end{cases}$$

For any $L_i \in \mathcal{C}$, whether $T \subseteq L_i$ and $F \subseteq L_i^c$ or not is recursively decidable, because $L_i$ is recursive, and $T$ and $F$ are explicitly given finite sets. Therefore in general, the above functions are regarded as recursively enumerable predicates.

**Proposition 4.6.** *If a class $\mathcal{C}$ is refutably inferable from positive data, then*

*(4.3) the function* $\mathrm{econs}_p$ *for $\mathcal{C}$ is recursive.*

*Proof.* Assume that an RIIM $M$ refutably infers $\mathcal{C}$ from positive data. Let $T = \{w_1, \cdots, w_n\} \subseteq U$ be a nonempty finite set, and put $\sigma = w_1, w_2, \cdots, w_n, w_1, w_1, w_1, \cdots$. Clearly, the infinite sequence $\sigma$ is a positive presentation of the concept $T$. Thus when we successively feed $\sigma$, $M$ either refutes the class $\mathcal{C}$ or produces an index $i$ with $T = L_i$ after producing some positive integers. Therefore we can recursively compute the function $\mathrm{econs}_p(T)$ for $\mathcal{C}$ as follows: Simulate $M$ with presenting $\sigma$. During the simulation, (i) if $M$ refutes the class $\mathcal{C}$, then output 0 and stop, (ii) if $M$ produces an index $i$ with $T \subseteq L_i$, then output 1 and stop, (iii) otherwise continue the simulation. We note that whether $T \subseteq L_i$ or not is recursively decidable. By Lemma 4.3, it is clear that the above output agrees with the $\mathrm{econs}_p(T)$. ■

**Theorem 4.7** (Based on Kinber[17]). *If a class $\mathcal{C}$ is refutably inferable from complete data, then*

*(4.4) the function* $\mathrm{econs}_c$ *for $\mathcal{C}$ is recursive.*

*Proof.* Assume that an RIIM $M$ refutably infers $\mathcal{C}$ from complete data. Let $T = \{w_1, \cdots, w_n\} \subseteq U$ and $F = \{w_{n+1}, \cdots, w_m\} \subseteq U$ be finite sets.

It is easy to see that if $T \cap F \neq \phi$, then $\mathrm{econs}_c(T, F) = 0$. Thus in what follows, we assume $T \cap F = \phi$.

Let $\psi_0 = (w_1, +), \cdots, (w_n, +), (w_{n+1}, -), \cdots, (w_m, -)$, and let $u_1, u_2, \cdots$ be an effective enumeration of $U \setminus (T \cup F)$. Then let $\mathcal{T}$ be the set of all initial segments of $u_1, u_2, \cdots$ coupled with $+$ and $-$, that is, $\mathcal{T} = \{\phi;\ (u_1, +);\ (u_1, -);\ (u_1, +), (u_2, +);\ (u_1, +), (u_2, -);\ (u_1, -), (u_2, +);\ (u_1, -), (u_2, -);\ \cdots\}$. We define the binary relation $\sqsubseteq$ over $\mathcal{T}$ as follows: $\psi_1 \sqsubseteq \psi_2$ if and only if $\psi_1$ is an initial segment of $\psi_2$. This gives a partial ordering of $\mathcal{T}$, and it becomes a binary tree, which can be diagrammed in Figure 4.1.

Then we define a subtree $\mathcal{S}$ of $\mathcal{T}$ as follows:

$$\mathcal{S} = \{\psi \in \mathcal{T} \mid \overline{M}(\psi_0 \cdot \psi) \neq \text{‘refutation’}\}.$$

Here we note that if $\psi_1 \sqsubseteq \psi_2$ and $\overline{M}(\psi_0 \cdot \psi_1) = \text{‘refutation’}$, then $\overline{M}(\psi_0 \cdot \psi_2) = \text{‘refutation’}$.

**Claim A:** The subtree $\mathcal{S}$ is finite, if and only if there is no concept $L_i \in \mathcal{C}$ such that $(T, F)$ is consistent with $L_i$.

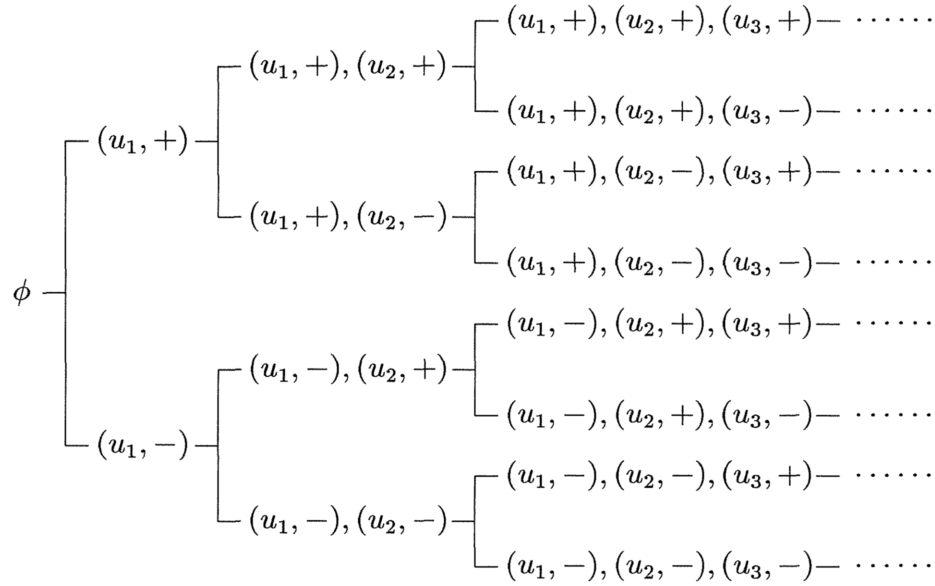


Figure 4.1: The binary tree $\mathcal{T}$

*Proof of the claim.* (I) The 'if' part. Assume that there is no concept $L_i \in \mathcal{C}$ such that $(T, F)$ is consistent with $L_i$. Then suppose that the subtree $\mathcal{S}$ has an infinite branch, say $\psi_1, \psi_2, \cdots$. By the construction of the subtree $\mathcal{S}$, there is an infinite sequence $t_1, t_2, \cdots \in \{+, -\}$ such that for any $i \geq 1$, $\psi_i = (u_1, t_1), (u_2, t_2), \cdots, (u_i, t_i)$. Put $\sigma = \psi_0, (u_1, t_1), (u_2, t_2), \cdots$, and let $L = \{u_i \in U \mid (u_i, +) \in \sigma,\ i \geq 1\}$ be a concept. Then $(T, F)$ is consistent with $L$, and $\sigma$ is a complete presentation of $L$. By assumption, $L$ is not in $\mathcal{C}$, and it follows that $M$ refutes $\mathcal{C}$ from $\sigma[n]$ for some $n \geq 1$. However, by the construction, there is a $j \geq 1$ such that $\sigma[n] = \psi_0 \cdot \psi_j$. This contradicts the assumption of $\psi_j \in \mathcal{S}$.

Thus we see that the subtree $\mathcal{S}$ has no infinite branch, and it follows by Endlichkeitslemma for trees with finite branching (cf. e.g. Rogers[36], Exercise 9.40) that the subtree $\mathcal{S}$ is finite.

(II) The 'only if' part. Assume that the subtree $\mathcal{S}$ is finite. Therefore the subtree $\mathcal{S}$ has no infinite branch. Then suppose that there is an $L_i \in \mathcal{C}$ such that $(T, F)$ is consistent with $L_i$. For $j \geq 1$, let $t_j$ = '+' if $u_j \in L_i$, otherwise let $t_j$ = '−'. Then put $\sigma = \psi_0, (u_1, t_1), (u_2, t_2), \cdots$. By the construction, $\sigma$ is a complete presentation of $L_i$. Thus $M$ does not refute $\mathcal{C}$ from $\sigma$. It is easy to see that this contradicts the assumption. $\square$

**Claim B:** There is an $L_i \in \mathcal{C}$ such that $(T, F)$ is consistent with $L_i$, if and only if there is a $\psi \in \mathcal{S}$ such that $\overline{M}(\psi_0 \cdot \psi) > 0$ and $(T, F)$ is consistent with $L_{\overline{M}(\psi_0 \cdot \psi)}$.

*Proof of the claim.* The 'if' part is obvious. Thus we only give the proof of the 'only if' part. Assume that there is an $L_i \in \mathcal{C}$ such that $(T, F)$ is consistent with $L_i$. For $j \geq 1$, let $t_j =$ '+' if $u_j \in L_i$, otherwise let $t_j =$ '−'. Then put $\sigma = \psi_0, (u_1, t_1), (u_2, t_2), \cdots$. By the construction, $\sigma$ is a complete presentation of $L_i$. Since $M$ infers $L_i$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$, it follows that there is an $n \geq 1$ such that $\overline{M}(\sigma[n]) > 0$ and $L_i = L_{\overline{M}(\sigma[n])}$. It is easy to see that there is a $\psi \in \mathcal{S}$ such that $\psi_0 \cdot \psi = \sigma[n]$. □

Therefore we can compute $\mathrm{econs}_c(T, F)$ as follows: Search for a node $\psi \in \mathcal{S}$ such that $\overline{M}(\psi_0 \cdot \psi) > 0$ and $(T, F)$ is consistent with $L_{\overline{M}(\psi_0 \cdot \psi)}$. By Claim A and B, we see that there must happen one of two cases:

(1) Such a node is found.

(2) The subtree $\mathcal{S}$ is confirmed to be finite.

In case (1), put $\mathrm{econs}_c(T, F) = 1$ and in case (2), put $\mathrm{econs}_c(T, F) = 0$. It is easy to see that the obtained results agrees with the definition of $\mathrm{econs}_c(T, F)$. ■

**Theorem 4.8.** *(a) If a class $\mathcal{C}$ satisfies the following three conditions, then $\mathcal{C}$ is refutably, consistently and responsively inferable from positive data.*

*(4.1) For any nonempty concept $L \notin \mathcal{C}$, there is a finite set $T \subseteq L$ such that $T$ is not consistent with any $L_i \in \mathcal{C}$.*

*(4.3) The function $\mathrm{econs}_p$ for $\mathcal{C}$ is recursive.*

*(4.5) The class $\mathcal{C}$ is inferable in the limit from positive data.*

*(b) If a class $\mathcal{C}$ satisfies the following two conditions, then $\mathcal{C}$ is refutably, consistently and responsively inferable from complete data.*

*(4.2) For any concept $L \notin \mathcal{C}$, there are finite sets $T \subseteq L$ and $F \subseteq L^c$ such that $(T, F)$ is not consistent with any $L_i \in \mathcal{C}$.*

*(4.4) The function $\mathrm{econs}_c$ for $\mathcal{C}$ is recursive.*

*Proof.* We only give the proof of (a). The proof of (b) can be given in a similar way, where we note that every class is consistently and responsively inferable in the limit from complete data (cf. Theorem 2.4 and Definition 2.5).

Assume that a class $\mathcal{C}$ satisfies the above three conditions (4.1), (4.3) and (4.5). Let $M$ be an IIM which infers $\mathcal{C}$ in the limit from positive data. Without loss of generality, we can assume that it works consistently and responsively. Then let us consider the procedure in Figure 4.2.

---

**Procedure** RIIM $M$;
**begin**
  $T = \phi$;
  **repeat**
    read the next fact $w$ and store it in $T$;
    **if** $\mathrm{econs}_p(T) = 0$ **then**
      refute the class $\mathcal{C}$ and stop;
    **else begin**
      simulate $M$ with presenting the fact $w$ until requesting the next fact;
      **if** $M$ produces a guess **then** output it;
    **end**;
  **forever**;
**end**.

---

Figure 4.2: An inference machine that can refute a hypothesis space

Assume that we feed a positive presentation $\sigma$ of a nonempty concept $L$ to the procedure.

(I) In case $L \notin \mathcal{C}$. By the condition (4.1), there is a finite set $T \subseteq L$ such that $\mathrm{econs}_p(T) = 0$, and by the definition of a positive presentation, we see that there is an $n \geq 1$ such that $T \subseteq \sigma[n]^+$. Therefore the procedure refutes the class $\mathcal{C}$ from $\sigma[n]$ and stops.

(II) In case $L \in \mathcal{C}$. As easily seen, for any finite set $T \subseteq L$, the value of $\mathrm{econs}_p(T)$ never becomes 0. Since the IIM $M$ infers $L$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$, it follows that the procedure infers $L$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$.

Furthermore it is easy to see that the procedure works consistently and responsively. ∎

By Proposition 4.4, 4.6, Theorem 4.7 and 4.8, we have the following Corollary 4.9.

**Corollary 4.9.** *(a) For a class $\mathcal{C}$, the following three statements are equivalent:*

*(i) $\mathcal{C}$ is refutably inferable from positive data.*

*(ii) $\mathcal{C}$ is refutably, consistently and responsively inferable from positive data.*

*(iii) $\mathcal{C}$ satisfies the conditions (4.1), (4.3) and (4.5).*

*(b) For a class $\mathcal{C}$, the following three statements are equivalent:*

*(i) $\mathcal{C}$ is refutably inferable from complete data.*

*(ii) $\mathcal{C}$ is refutably, consistently and responsively inferable from complete data.*

*(iii) $\mathcal{C}$ satisfies the conditions (4.2) and (4.4).*

The above conditions heavily depend on each concept rather than the properties of the class concerned. Thus we need to investigate another conditions concerned with the properties of the class itself.

**Definition 4.4.** A class $\mathcal{C}$ is said to be *closed under the subset operation*, if for any $L_i \in \mathcal{C}$, all nonempty subsets of $L_i$ are also in the class $\mathcal{C}$.

A class $\mathcal{C}$ is said to be *of finite hierarchy*, if there is no infinite sequence of concepts $L_{i_1}, L_{i_2}, \cdots \in \mathcal{C}$ such that $L_{i_1} \subsetneq L_{i_2} \subsetneq \cdots$.

As easily seen, if a class $\mathcal{C}$ has finite elasticity (cf. Definition 2.8), then $\mathcal{C}$ is of finite hierarchy. Moreover, if a class $\mathcal{C}$ is closed under the subset operation and it is of finite hierarchy, then $\mathcal{C}$ contains no infinite concept, as shown in the proof of Lemma 4.11.

**Lemma 4.10.** *If a class $\mathcal{C}$ is refutably inferable from positive data, then $\mathcal{C}$ satisfies the following two conditions:*

*(4.6) $\mathcal{C}$ is closed under the subset operation.*

*(4.7) $\mathcal{C}$ is of finite hierarchy.*

*Proof.* (I) Suppose that the condition (4.6) does not hold, that is, there is a nonempty concept $L$ such that $L \notin \mathcal{C}$ and $L \subseteq L_i$ for some $L_i \in \mathcal{C}$. Then every subset of $L$ is consistent with $L_i$. Therefore by Proposition 4.4, $\mathcal{C}$ is not refutably inferable from positive data.

(II) Assume that a class $\mathcal{C}$ is refutably inferable from positive data. Then, by the above (I), the condition (4.6) holds.

**Claim:** The class $\mathcal{C}$ contains no infinite concept.

*Proof of the claim.* Suppose that $\mathcal{C}$ contains an infinite concept $L_i$. Then, by the condition (4.6), we see that $\mathcal{C}$ contains all nonempty finite subset of $L_i$. By modifying the proof of Theorem 2.5, we can show that any RIIM does not infer $L_i$ w.r.t. $\mathcal{C}$ in the limit from a certain positive presentation of $L_i$. This is a contradiction. □

Now suppose that the condition (4.7) does not hold, that is, there is an infinite sequence of concepts $L_{i_1}, L_{i_2}, \cdots \in \mathcal{C}$ such that $L_{i_1} \subsetneq L_{i_2} \subsetneq \cdots$. Then we consider the concept $L = \bigcup_{j=0}^{\infty} L_{i_j}$. Since $L$ is an infinite concept, it follows by the above claim that $L \notin \mathcal{C}$. By the definition of the concept $L$, we see that for any finite set $T \subseteq L$, there is an $L_{i_j} \in \mathcal{C}$ such that $T$ is consistent with $L_{i_j}$. This contradicts the assumption by Proposition 4.4. ■

**Lemma 4.11.** *If a class $\mathcal{C}$ satisfies the following three conditions, then $\mathcal{C}$ is refutably, consistently and responsively inferable from positive data.*

*(4.6) $\mathcal{C}$ is closed under the subset operation.*

*(4.7) $\mathcal{C}$ is of finite hierarchy.*

*(4.3) The function* $\mathrm{econs}_p$ *for $\mathcal{C}$ is recursive.*

*Proof.* Assume that a class $\mathcal{C}$ satisfies the above three conditions. Then, by Theorem 4.8, it suffices for us to show that $\mathcal{C}$ satisfies the conditions (4.1) and (4.5).

**Claim:** The class $\mathcal{C}$ contains no infinite concept.

*Proof of the claim.* Suppose that $\mathcal{C}$ contains an infinite concept $L_i$. Then, by the condition (4.6), all subsets of $L_i$ are also in $\mathcal{C}$, and it follows that there is an infinite sequence of concepts $L_{i_1}, L_{i_2}, \cdots \in \mathcal{C}$ such that $L_{i_1} \subsetneq L_{i_2} \subsetneq \cdots$, which contradicts the condition (4.7). □

(I) By this claim, we see that $\mathcal{C}$ is inferable in the limit from positive data, that is, the condition (4.5) is satisfied (cf. Corollary 2.7).

(II) Let $L \notin \mathcal{C}$ be a nonempty concept. (i) In case $L$ is a finite concept. By the condition (4.6), we see that $L$ is *not* consistent with any $L_i \in \mathcal{C}$, because the existence of $L_i \in \mathcal{C}$ with

$L \subseteq L_i$ means $L \in \mathcal{C}$. Therefore it suffices for us to take $L$ itself as $T$ in the condition (4.1). (ii) In case $L$ is an infinite concept. Suppose that the condition (4.1) does not hold. Then for any finite set $T \subseteq L$, there is an $L_i \in \mathcal{C}$ such that $T$ is consistent with $L_i$. However by the condition (4.6), we see that the above $T$'s themselves are in $\mathcal{C}$. To sum up, every finite set $T \subseteq L$ is in $\mathcal{C}$, and it follows that there is an infinite sequence $L_{i_1}, L_{i_2}, \cdots$ such that $L_{i_1} \subsetneq L_{i_2} \subsetneq \cdots$, because $L$ is an infinite concept. This contradicts the condition (4.7). ∎

By Corollary 4.9, Lemma 4.10 and 4.11, we have the following Theorem 4.12.

**Theorem 4.12.** *For a class $\mathcal{C}$, the following four statements are equivalent:*

*(i) $\mathcal{C}$ is refutably inferable from positive data.*

*(ii) $\mathcal{C}$ is refutably, consistently and responsively inferable from positive data.*

*(iii) $\mathcal{C}$ satisfies the conditions (4.1), (4.3) and (4.5).*

*(iv) $\mathcal{C}$ satisfies the conditions (4.3), (4.6) and (4.7).*

**Example 4.1.** Let $\mathcal{FC}_n$ be the class of all nonempty finite concepts each of which cardinality is *just* $n$ (cf. Example 3.2). Then this class is *not* refutably inferable from positive data for any $n \geq 2$, because it is not closed under the subset operation.

In contrast with the above class, let $\mathcal{FC}_{\leq n}$ be the class of all nonempty finite concepts each of which cardinality is *at most* $n$ (cf. Example 3.5). We note that $\mathcal{FC}_{\leq 1} = \mathcal{FC}_1$. As easily seen, the function $\mathrm{econs}_p$ for $\mathcal{FC}_{\leq n}$ is recursive, because $\mathrm{econs}_p(T) = 1$ if and only if $\sharp T$ is not greater than $n$. Furthermore this class is closed under the subset operation and of finite hierarchy. Therefore $\mathcal{FC}_{\leq n}$ is refutably inferable from positive data.

Lastly, let $\mathcal{FC}_*$ be the class of all nonempty finite concepts (cf. Corollary 4.5 and Example 3.5). The function $\mathrm{econs}_p$ for $\mathcal{FC}_*$ is recursive, because $\mathrm{econs}_p(T) = 1$ for any finite set $T \subseteq U$. This class is closed under the subset operation but is not of finite hierarchy. Therefore $\mathcal{FC}_*$ is *not* refutably inferable from positive data.

Here we present a sufficient condition for a class to be refutably inferable from complete data, which is very strict but widely applicable as shown in Section 4.4 and 6.2.

The following Lemma 4.13 is basic.

**Lemma 4.13.** *Let $n \geq 1$ be an integer, let $L_1, \cdots, L_n \subseteq U$ be concepts, and let $L \subseteq U$ be a concept which differs from $L_1, \cdots, L_n$. Then for any complete presentation $\sigma$ of $L$, there is an $m \geq 1$ such that $\sigma[m]$ is not consistent with any concept $L_i$ with $1 \leq i \leq n$.*

**Theorem 4.14.** *If a class $\mathcal{C}$ satisfies the following two conditions, then $\mathcal{C}$ is refutably inferable from complete data.*

*(4.8) For any $w \in U$, there is a uniformly and recursively generable finite cover-index set of the subclass $\{L_i \in \mathcal{C} \mid w \in L_i\}$ of $\mathcal{C}$.*

*(4.9) The class $\mathcal{C}$ contains the empty concept as its member.*

*Proof.* Assume that a class $\mathcal{C}$ satisfies the conditions (4.8) and (4.9). Then let us consider the procedure in Figure 4.3, where the procedure read_store is the same one as in Figure 2.1.

---

**Procedure** RIIM $M$;
**begin**
  $T = \phi$;  $F = \phi$;  $i = 1$;
  read_store($T$, $F$);
  **while** $T = \phi$ **do begin** ........................................ (1)
    **while** $F \not\subseteq L_i^c$ **do** $i = i + 1$;
    output $i$;
    read_store($T$, $F$);
  **end**;
  let $\{w\} = T$;
  recursively generate a cover-index set of $\{L_i \in \mathcal{C} \mid w \in L_i\}$, and set it to $I$;
  **for each** $j \in I$ **do** ........................................ (2)
    **while** $(T, F)$ is consistent with $L_j$ **do begin** .......................... (3)
      output $j$;
      read_store($T$, $F$);
    **end**;
  refute the class $\mathcal{C}$ and stop;
**end**.

---

Figure 4.3: An inference machine that can refute a hypothesis space

Assume that we feed a complete presentation $\sigma$ of a concept $L$ to the procedure.

(I) In case $L = \phi$. It is easy to see that the while-loop (1) never terminates and that the procedure infers the empty concept w.r.t. $\mathcal{C}$ in the limit from $\sigma$.

(II) In case $L \neq \phi$. The procedure terminates the while-loop (1) in a finite time.

(i) In case $L \in \mathcal{C}$. It is easy to see that $L$ is in the subclass $\{L_i \in \mathcal{C} \mid w \in L_i\}$, and it follows that there is an index $j \in I$ such that $L_j = L$. Since $I$ is a finite set, we see by Lemma 4.13 that the for-loop (2) is eventually executed with $j \in I$ such that $L_j = L$, and the while-loop (3) never terminates. That is, the procedure infers $L$ w.r.t. $\mathcal{C}$ in the limit from $\sigma$.

(ii) In case $L \notin \mathcal{C}$. Since $I$ is an explicitly given finite set, we see by Lemma 4.13 that the procedure refutes the class $\mathcal{C}$ from $\sigma$. ∎

**Example 4.2.** We consider the class $\mathcal{PAT}$ of pattern languages (cf. Example 2.1). As easily seen, the empty concept $L = \phi$ is not in $\mathcal{PAT}$. Furthermore, for any finite set $F \subseteq U$, there is an $L_i \in \mathcal{PAT}$ such that $(\phi, F)$ is consistent with $L_i$. In fact, let $l$ be the length of the longest string in $F$. Then $(\phi, F)$ is consistent with the language of the pattern $x_1 x_2 \cdots x_{l+1}$. Therefore by Proposition 4.4, we see that $\mathcal{PAT}$ is *not* refutably inferable from complete data.

However $\mathcal{PAT}$ satisfies the condition (4.8) as shown in Example 2.1. Thus, by Theorem 4.14, we see that if we add the empty concept to the class of pattern languages, then the obtained class is refutably inferable from complete data.

## 4.3. Comparisons with Other Identifications

In this section by some distinctive examples of classes, we compare the criterion of refutable identification with some other criteria. This is motivated by the following question: What should we do if we face with facts that are *not* consistent with a finitely inferred hypothesis?

For the purpose of comparing the inferability from positive data with the inferability from complete data, we assume that every concept is *nonempty* throughout this section. The following Proposition 4.15 is obvious (cf. Proposition 2.2).

**Proposition 4.15.** *If a class $\mathcal{C}$ is inferable in the limit from positive data, then $\mathcal{C}$ is also inferable in the limit from complete data.*

*Furthermore the above assertion is still valid, if we replace the phrase 'inferable in the limit' with the phrase 'finitely inferable', 'reliably inferable' or 'refutably inferable'.*

In the following examples, we assume appropriate universal sets and indexings of the classes.

**Example 4.3.** We consider the classes $\mathcal{FC}_n$, $\mathcal{FC}_{\leq n}$ and $\mathcal{FC}_*$ (cf. Theorem 2.13, Example 3.2 and 4.1).

(A.1) For any $n \geq 2$, the class $\mathcal{FC}_n$ is *not* refutably inferable from *complete data.*

In fact, let $L \subseteq U$ be a concept with cardinality 1. It is easy to see that there are no finite sets $T \subseteq L$ and $F \subseteq L^c$ that satisfy the condition (4.2) in Proposition 4.4.

(A.2) For any $n \geq 1$, the class $\mathcal{FC}_n$ is reliably inferable from *positive data.*

(A.3) For any $n \geq 1$, the class $\mathcal{FC}_n$ is finitely inferable from *positive data.*

(B.1) For any $n \geq 1$, the class $\mathcal{FC}_{\leq n}$ is refutably inferable from *positive data.*

(B.2) For any $n \geq 1$, the class $\mathcal{FC}_{\leq n}$ is reliably inferable from *positive data.*

(B.3) For any $n \geq 2$, the class $\mathcal{FC}_{\leq n}$ is *not* finitely inferable from *complete data.*

(C.1) $\mathcal{FC}_*$ is *not* refutably inferable from *complete data* (cf. Corollary 4.5).

(C.2) $\mathcal{FC}_*$ is reliably inferable from *positive data.*

(C.3) $\mathcal{FC}_*$ is *not* finitely inferable from *complete data.*

**Example 4.4.** Let $\Sigma = \{a\}$, $L_1 = \{a^j \mid j \geq 1\}$, and $L_i = \{a^j \mid 1 \leq j \leq i-1\}$ for $i \geq 2$. Then let $\mathcal{CLC} = \{L_i\}_{i \in N}$ be the class of interest.

(D.1) The class $\mathcal{CLC}$ is refutably inferable from *complete data.*

In fact, let $L \notin \mathcal{CLC}$ be a nonempty concept. Then as easily seen, there is a $j \geq 1$ such that $a^j \notin L$ but $a^{j+1} \in L$. Therefore $T = \{a^{j+1}\}$ and $F = \{a^j\}$ satisfy the condition (4.2). It is easy to see that the condition (4.4) is also satisfied (cf. Theorem 4.8).

(D.2) The class $\mathcal{CLC}$ is *not* inferable from *positive data.*

We note that, in Lange&Zeugmann[21], this class was shown to be inferable within one mind change from complete data but not inferable in the limit from positive data.

**Example 4.5.** Let $\mathcal{PR}_{\le n}$ be the class of concepts each of which consists of all multiples of at most $n$ prime numbers (cf. Example 3.2).

(E.1) For any $n \ge 1$, the class $\mathcal{PR}_{\le n}$ is refutably inferable from *complete data.*

In fact, let $L \in \mathcal{PR}_{\le n}$ be a nonempty concept. (i) In case $L$ contains more than $n$ prime numbers. Let $T$ be a set of some $n+1$ prime numbers in $L$, and let $F = \phi$. (ii) In case $L$ contains no prime number. Let $m$ be the least integer in $L$. Then let $T = \{m\}$ and $F = \{1, \cdots, m-1\}$. (iii) Otherwise. Let $p_1, \cdots, p_k$ be all prime numbers in $L$. As easily seen, the following (1) or (2) holds. (1) There is an $m \in L$ which is not a multiple of any $p_i$ with $1 \le i \le k$. Then let $T = \{m\}$, and let $F$ be the finite set of all prime numbers less than $m$ each of which differs from $p_1, \cdots, p_k$. (2) There is an $m \notin L$ which is a multiple of some $p_i$. Then let $T = \{p_1, \cdots, p_k\}$ and $F = \{m\}$. It is easy to see that the above defined $T$ and $F$ satisfy the condition (4.2). Furthermore it is also easy to see that the condition (4.4) is satisfied.

(E.2) For any $n \ge 1$, the class $\mathcal{PR}_{\le n}$ is *not* reliably inferable from *positive data.*

(E.3) For any $n \ge 2$, the class $\mathcal{PR}_{\le n}$ is *not* finitely inferable from *complete data.*

(E.4) For any $n \ge 1$, the class $\mathcal{PR}_{\le n}$ is inferable in the limit from *positive data.*

Let $\mathcal{PR}_n$ be the class of concepts each of which consists of all multiples of $n$ distinct prime numbers (cf. Example 3.2). We note that $\mathcal{PR}_1 = \mathcal{PR}_{\le 1}$.

(F.1) For any $n \ge 2$, the class $\mathcal{PR}_n$ is *not* refutably inferable from *complete data.*

In fact, let $L = \{2\}$. Then for any finite set $F \subseteq L^c$, there is a prime number which is greater than any integer in $F$. Therefore there are no finite sets $T \subseteq L$ and $F \subseteq L^c$ that satisfy the condition (4.2).

(F.2) For any $n \ge 1$, the class $\mathcal{PR}_n$ is *not* reliably inferable from *positive data.*

(F.3) For any $n \ge 1$, the class $\mathcal{PR}_n$ is finitely inferable from *positive data.*

**Example 4.6.** We consider the class $\mathcal{PAT}$ of pattern languages (cf. Example 4.2). As easily seen from Theorem 4.14 and Example 4.2, the empty concept is only the concept that does not satisfy the condition (4.2). Since all concepts are assumed to be nonempty in this section, the class of pattern languages is shown to be refutably inferable from complete data.

(G.1) $\mathcal{PAT}$ is refutably inferable from *complete data.*

(G.2) $\mathcal{PAT}$ is *not* reliably inferable from *positive data.*

(G.3) $\mathcal{PAT}$ is finitely inferable from *complete data* but *not* finitely inferable from *positive data* (cf. Example 3.3).

(G.4) $\mathcal{PAT}$ is inferable in the limit from *positive data* (cf. Example 2.1).

We can summarize the above comparisons in Figure 4.4.

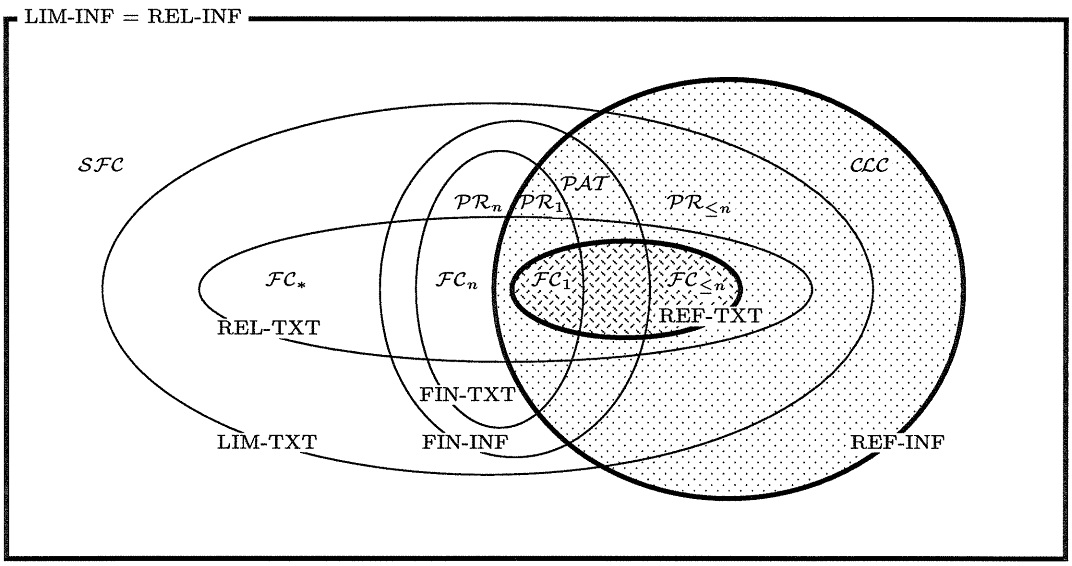


Figure 4.4: Comparisons with other identifications

In the figure, the prefix 'LIM', 'FIN', 'REL' or 'REF' means the collection of all classes that are 'inferable in the limit', 'finitely inferable', 'reliably inferable' or 'refutably inferable', respectively, and the postfix 'TXT' or 'INF' means 'from positive data' or 'from complete data', respectively. For example, REF-TXT is the collection of all classes that are refutably inferable from positive data. By definition, FIN-TXT = $\mathrm{EX}_0$-TXT, FIN-INF = $\mathrm{EX}_0$-INF, LIM-TXT = $\mathrm{EX}_*$-TXT, and LIM-INF = $\mathrm{EX}_*$-INF hold.

The classes $\mathcal{FC}_1$, $\mathcal{FC}_n$, $\mathcal{FC}_{\leq n}$ and $\mathcal{FC}_*$ consist of all finite concepts each of which cardinality is just 1, $n \geq 2$, at most $n$ and unrestricted finite, respectively (cf. Example 4.3).

The class $\mathcal{CLC}$ has been shown in Example 4.4. The classes $\mathcal{PR}_1$, $\mathcal{PR}_n$ and $\mathcal{PR}_{\leq n}$ consist of all multiples of a prime number, $n \geq 2$ distinct prime numbers and at most $n$ prime numbers, respectively (cf. Example 4.5). The class $\mathcal{PAT}$ is the class of pattern languages (cf. Example 4.6). The class $\mathcal{SFC}$ is the so-called superfinite class (cf. Theorem 2.5), that is, a class contains all finite concepts and at least one infinite concept.

In Figure 4.4, we see that a subclass of a refutably inferable class is not always refutably inferable.

## 4.4. Unions of Some Classes

In this section we consider two types of union classes. First we take a class as the collection of all concepts from $n$ classes.

**Definition 4.5.** Let $n \geq 1$ be an integer, and let $\mathcal{C}_1, \cdots, \mathcal{C}_n$ be classes. For $i$ with $1 \leq i \leq n$ and $j \geq 1$, the $j$-th concept $L_j$ of the class $\mathcal{C}_i$ is denoted by $L_{(i,j)}$. Then the union class of $\mathcal{C}_1, \cdots, \mathcal{C}_n$ is represented as:

$$\bigcup_{i=1}^{n} \mathcal{C}_i = \{L_{(i,j)}\}_{1 \leq i \leq n,\, j \in N}.$$

By assuming a bijective coding from $\{1, \cdots, n\} \times N$ to $N$, the new class above becomes an indexed family of recursive concepts.

**Theorem 4.16.** *Let $n \geq 1$ be an integer, and let $\mathcal{C}_1, \cdots, \mathcal{C}_n$ be classes each of which is refutably inferable from positive data (resp., complete data). Then the class $\bigcup_{i=1}^{n} \mathcal{C}_i$ is refutably inferable from positive data (resp., complete data).*

*Proof.* We only give the proof of the case of positive data. The proof for complete data can be given in a similar way.

For any $i$ with $1 \leq i \leq n$, let $M_i$ be an RIIM which refutably infers $\mathcal{C}_i$ from positive data. Then let us consider the procedure in Figure 4.5.

Assume that we feed a positive presentation $\sigma$ of a nonempty concept $L$ to the procedure.

**Procedure** RIIM $M$;
**begin**
  **for** $i = 1$ to $n$ **do**
    **while** $M_i$ does not refute the class $\mathcal{C}_i$ **do begin**
      simulate $M_i$ with presenting facts read so far;
      during the simulation,
        **if** $M_i$ requests another fact **then**
          read the next fact and present it to $M_i$;
        **if** $M_i$ produces a guess $j$ **then**
          output the coding of $(i, j)$;
    **end**;
  refute the class $\bigcup_{i=1}^{n} \mathcal{C}_i$ and stop;
**end**.

Figure 4.5: An inference machine that can refute a hypothesis space

(I) In case $L \notin \bigcup_{i=1}^{n} \mathcal{C}_i$. Then for any $i$ with $1 \leq i \leq n$, $L \notin \mathcal{C}_i$ holds, and it follows that $M_i$ refutes the class $\mathcal{C}_i$ from $\sigma$. Thus the procedure refutes the class $\bigcup_{i=1}^{n} \mathcal{C}_i$ from $\sigma$.

(II) In case $L \in \bigcup_{i=1}^{n} \mathcal{C}_i$. Let $i_0$ be the least integer such that $L \in \mathcal{C}_{i_0}$. Then for any $i$ with $1 \leq i < i_0$, $L \notin \mathcal{C}_i$ holds, and it follows that $M_i$ refutes the class $\mathcal{C}_i$ from $\sigma$. Therefore the for-loop in the procedure reaches the case of $i = i_0$. Since $M_{i_0}$ infers $L$ w.r.t. $\mathcal{C}_{i_0}$ in the limit from $\sigma$, it follows that $M_{i_0}$ converges to an index $j$ with $L_{(i_0,j)} = L$ for $\sigma$. Thus the procedure converges to the coding of $(i_0, j)$ for $\sigma$. That is, the procedure infers $L$ w.r.t. $\bigcup_{i=1}^{n} \mathcal{C}_i$ in the limit from $\sigma$.

Thus the procedure refutably infers the class $\bigcup_{i=1}^{n} \mathcal{C}_i$ from positive data. ∎

Now we consider a class of concepts each of which is a union of at most $n$ concepts from $n$ classes (cf. Definition 2.10).

**Definition 4.6.** Let $n \geq 1$ be an integer, and let $\mathcal{C}_1, \cdots, \mathcal{C}_n$ be classes. For $i$ with $1 \leq i \leq n$ and $j \geq 0$, $L_{\langle i,j \rangle}$ denotes the empty concept if $j = 0$, otherwise the $j$-th concept $L_j$ of the class $\mathcal{C}_i$. Then we define a class generated by $\mathcal{C}_1, \cdots, \mathcal{C}_n$ as follows:

$$\widetilde{\bigsqcup_{i=1}^{n}} \mathcal{C}_i = \left\{ \bigcup_{i=1}^{n} L_{\langle i,j_i \rangle} \;\middle|\; (j_1, \cdots, j_n) \in \mathcal{N}^n \right\},$$

where $\mathcal{N}^n$ is the set of all $n$-tuples of nonnegative integers, that is, $\mathcal{N}^n = \{0, 1, 2, \cdots\}^n$.

By assuming a bijective coding from $\mathcal{N}^n$ to $N$, the new class above becomes an indexed family of recursive concepts.

If each class satisfies the condition (4.8), then the above class is shown to be refutably inferable from complete data.

**Theorem 4.17.** *Let $n \geq 1$ be an integer, and let $\mathcal{C}_1, \cdots, \mathcal{C}_n$ be classes each of which satisfies the condition (4.8). Then the class $\widetilde{\bigsqcup}_{i=1}^{n} \mathcal{C}_i$ is refutably inferable from complete data.*

*Proof.* Let us consider the procedure in Figure 4.6, where the procedure read_store is the same one as in Figure 2.1 and $\lambda$ is a special element not in the universal set $U$.

Assume that we feed a complete presentation $\sigma$ of a concept $L_{\text{base}}$ to the procedure.

(A) In case $L_{\text{base}} \in \widetilde{\bigsqcup}_{i=1}^{n} \mathcal{C}_i$. Let $NE(m) = \{(j_1, \cdots, j_n) \in \mathcal{N}^n \mid$ there are just $m$ nonempty concepts among $L_{\langle 1,j_1 \rangle}, \cdots, L_{\langle n,j_n \rangle}$ $\}$.

**Claim:** In the procedure, for any $m$ with $0 \leq m \leq n$, if $T_m$ and $F_m$ are defined, then $(T_m, F_m)$ is *not* consistent with $(L_{\langle 1,j_1 \rangle} \cup \cdots \cup L_{\langle n,j_n \rangle})$ for any $(j_1, \cdots, j_n) \in NE(m)$.

*Proof of the claim.* This proof is given by mathematical induction on $m$.

(I) In case $m = 0$. It is clear because $T_0$ is nonempty and the union of $n$ empty concepts is empty.

(II) In case $m \geq 1$. We assume the claim for $m-1$, and assume that $T_m$ and $F_m$ are defined. Then suppose that there is an $n$-tuple $(k_1, \cdots, k_n) \in NE(m)$ such that $(T_m, F_m)$ is consistent with $(L_{\langle 1,k_1 \rangle} \cup \cdots \cup L_{\langle n,k_n \rangle})$. Then $(T_{m-1}, F_{m-1})$ is consistent with $(L_{\langle 1,k_1 \rangle} \cup \cdots \cup L_{\langle n,k_n \rangle})$, because $T_{m-1} \subseteq T_m$ and $F_{m-1} \subseteq F_m$ hold.

Here suppose that there is an $i \geq 1$ such that $L_{\langle i,k_i \rangle} \neq \phi$ and $(L_{\langle i,k_i \rangle} \cap T_{m-1}) = \phi$. Then $T_{m-1} \subseteq (L_{\langle 1,k_1 \rangle} \cup \cdots \cup L_{\langle i-1,k_{i-1} \rangle} \cup L_{\langle i+1,k_{i+1} \rangle} \cup \cdots \cup L_{\langle n,k_n \rangle})$ and $F_{m-1} \subseteq (L_{\langle 1,k_1 \rangle} \cup \cdots \cup L_{\langle n,k_n \rangle})^c \subseteq (L_{\langle 1,k_1 \rangle} \cup \cdots \cup L_{\langle i-1,k_{i-1} \rangle} \cup L_{\langle i+1,k_{i+1} \rangle} \cup \cdots \cup L_{\langle n,k_n \rangle})^c$ hold. This means that $(T_{m-1}, F_{m-1})$ is consistent with $(L_{\langle 1,k_1 \rangle} \cup \cdots \cup L_{\langle i-1,k_{i-1} \rangle} \cup L_{\langle i+1,k_{i+1} \rangle} \cup \cdots \cup L_{\langle n,k_n \rangle})$. This contradicts the induction hypothesis, because $(k_1, \cdots, k_{i-1}, 0, k_{i+1}, \cdots, k_n) \in NE(m-1)$.

```
Procedure RIIM M;
begin
  T = φ;    F = φ;
  read_store(T, F);
  while T = φ do begin  ........................................ (1)
    output the coding of (0, ···, 0);
    read_store(T, F);
  end;
  T_0 = T;    F_0 = F;
  for m = 1 to n do begin
    let 𝒯_m = { (w_1, ···, w_n) ∈ (𝒯_{m-1} ∪ {λ})^n | the number of w_i's other than λ is just m };
    for each (w_1, ···, w_n) ∈ 𝒯_m do begin  ..................... (2)
      for i = 1 to n do
        if w_i = λ then let S_i = {0}
        else recursively generate a cover-index set w.r.t. 𝒞_i
                 of {L_⟨i,j⟩ ∈ 𝒞_i | w_i ∈ L_⟨i,j⟩}, and set it to S_i;
      if all S_i's are nonempty then
        for each (j_1, ···, j_n) ∈ S_1 × ··· × S_n do  .............. (3)
          while (T, F) is consistent with (L_⟨1,j_1⟩ ∪ ··· ∪ L_⟨n,j_n⟩) do begin . (4)
            output the coding of (j_1, ···, j_n);
            read_store(T, F);
          end;
    end;
    T_m = T;    F_m = F;
  end;
  refute the class ⊔~_{i=1}^{n} 𝒞_i and stop;
end.
```

Figure 4.6: An inference machine that can refute a hypothesis space

Thus we have $(L_{\langle i,k_i\rangle} \cap T_{m-1}) \neq \phi$ for any $i$ with $L_{\langle i,k_i\rangle} \neq \phi$. Therefore we can take $u_i$'s as follows: If $L_{\langle i,k_i\rangle} \neq \phi$, then $u_i \in (L_{\langle i,k_i\rangle} \cap T_{m-1})$, otherwise $u_i = \lambda$. Since all $u_i$'s are in $(T_{m-1} \cup \{\lambda\})$ and the number of $u_i$'s other than $\lambda$ is just $m$, the $n$-tuple $(u_1, \cdots, u_n)$ is in $\mathcal{T}_m$. Thus there is a case where the for-loop (2) is executed with $(u_1, \cdots, u_n)$. In this case there is an $n$-tuple $(k_1', \cdots, k_n') \in S_1 \times \cdots \times S_n$ such that for any $i$ with $1 \leq i \leq n$, $L_{k_i'} = L_{k_i}$. Thus there is a case where the for-loop (3) is executed with $(k_1', \cdots, k_n')$. Since $(T_m, F_m)$ is consistent with $(L_{\langle 1,k_1'\rangle} \cup \cdots \cup L_{\langle n,k_n'\rangle})$, it follows that for any finite sets $T \subseteq T_m$ and $F \subseteq F_m$, $(T, F)$ is consistent with $(L_{\langle 1,k_1'\rangle} \cup \cdots \cup L_{\langle n,k_n'\rangle})$. However the while-loop (4) should terminate with $T \subseteq T_m$ and $F \subseteq F_m$, because $T_m$ and $F_m$ are assumed to be defined. This is a contradiction. □

Since $L_{\text{base}} \in \widetilde{\bigsqcup}_{i=1}^{n} \mathcal{C}_i$, there are an $m_0$ with $0 \leq m_0 \leq n$ and an $n$-tuple $(j_1, \cdots, j_n)$ such that $(j_1, \cdots, j_n) \in NE(m_0)$ and $L_{\text{base}} = (L_{\langle 1,j_1\rangle} \cup \cdots \cup L_{\langle n,j_n\rangle})$. By this claim, we see that $T_{m_0}$ and $F_{m_0}$ are never defined. By Lemma 4.13, this means that the procedure outputs the coding of an $n$-tuple $(j_1, \cdots, j_n)$ with $L_{\text{base}} = (L_{\langle 1,j_1\rangle} \cup \cdots \cup L_{\langle n,j_n\rangle})$ and never terminates the while-loop (1) or (4).

(B) In case $L_{\text{base}} \notin \widetilde{\bigsqcup}_{i=1}^{n} \mathcal{C}_i$. By using Lemma 4.13 $n$ times, we see that the procedure refutes the class $\widetilde{\bigsqcup}_{i=1}^{n} \mathcal{C}_i$ from $\sigma$. ■

**Example 4.7.** We consider the class $\mathcal{PAT}$ of pattern languages. As shown in Example 4.2, the class satisfies the condition (4.8). Therefore by Theorem 4.17, for any $n \geq 1$, the class of unions of at most $n$ pattern languages is refutably inferable from complete data.

By Corollary 4.5, we see that if the number of patterns is not bounded by a constant number, then the class is not refutably inferable from complete data, because it contains all nonempty finite languages.

We note that as shown in Example 2.2, for any $n \geq 1$, the class of unions of at most $n$ pattern languages is inferable in the limit from positive data.

## 4.5. Discussion

We have pointed out that the essence of the computational logic of scientific discovery or the logic of machine discovery should be the refutability of the whole space of hypotheses by observed data or given facts.

The refutability we proposed here forms an interesting contrast to the original one: In the logic of scientific discovery for scientists *each theory* in a hypothesis space is to be *refutable*, while in the computational version of the logic for machines or the logic of machine discovery the *space* itself is to be *refutable* by an observation. Popper contributed to the modern theory of inductive inference. In his books[32, 33], however, Popper strongly denied the induction so far developed by, for example, J. Stuart Mill[23], which had two stages of mechanical creation of a hypothesis from observations and proof of its validity. In fact, he said that the induction, i.e. inference based on many observations, was a myth, and it was neither a psychological fact, nor a fact of ordinary life, nor one of scientific procedure[32]. What he wanted to assert in those books was that scientific theory should have been refuted by observed data or facts and any such theory could by no means be verified. He could not agree with the assertion that the induction should have proved the validity of theory. But we think there were no reasons to deny the stage of mechanical creation of hypotheses.

The inductive inference machine that can refute the hypothesis space itself works as an automatic system for scientific discovery. If the machine for scientific discovery can not refute the whole space of hypotheses, it can just work for *computer aided* scientific discovery. That is, we need to check from time to time whether the machine is still searching for a possible hypothesis.

As a future work we have left the refutability of hypothesis spaces in another two major frameworks of PAC and MAT learning[50, 3]. Searching for classes which can be refutably inferred by *efficient* algorithms will be another important and interesting work.

# Chapter 5.

# Inferability of Approximate Concepts from Positive Data

In the previous chapter, we have discussed both refutability and inferability of a hypothesis space from examples. If a target concept is in the hypothesis space, then an inference machine should identify the target concept in the limit, otherwise it should refute the hypothesis space itself in a finite time. Unfortunately the refutably inferable classes from only positive data were shown to be very small.

In practical applications of inductive inference, there are many cases where we want an inference machine to infer an approximate concept within the hypothesis space concerned, even when there is no concept which exactly coincides with the target concept. In this chapter we take a minimal concept as an approximate concept within the hypothesis space, and discuss inferability of a minimal concept of the target concept which may not belong to the hypothesis space. That is, we force an inference machine to converge to a minimal concept of the target concept, if there is a minimal concept of the target concept within the hypothesis space. We introduce some criteria which specify behaviors of inference machines in case there is no minimal concept of the target concept within the hypothesis space.

In 1989, Wright[51] showed that if a class has finite elasticity, then the class is inferable in the limit from positive data. Using this result, Shinohara[45, 46] showed that the classes definable by length-bounded EFS's with at most $n$ axioms are inferable in the limit from positive data. Furthermore Moriyama&Sato[25] discussed closure properties of the classes with finite elasticity and inferability of the classes definable by so-called max-length bounded EFS's from positive data. On the other hand, Sato&Moriyama[39] introduced the notion of M-finite thickness to show another condition for inferability from positive data. In this chapter we show that the classes with both finite elasticity and M-finite thickness are minimally inferable from positive data. In Section 6.3, by using the result, we show that the classes that were introduced by Shinohara[45, 46] are also minimally inferable from positive data. This means that there are rich hypothesis spaces that are minimally inferable from positive data.

In Section 5.1 we prepare some necessary concepts for our discussions and introduce our definitions of inferability. In Section 5.2 we discuss some sufficient conditions for a class to be minimally inferable from positive data. In Section 5.3 we also show the differences between the powers of inference machines whose behaviors differs from each other when there is no minimal concept of the target concept in the class concerned.

This chapter is based on Mukouchi[31].

## 5.1. Definitions of Minimal Inferability

First of all, we introduce our definitions of minimal inferability.

**Definition 5.1.** An IIM $M$ is said to *minimally infer a class $\mathcal{C}$ from positive data*, if it satisfies the following condition: For any nonempty concept $L$ and any positive presentation $\sigma$ of $L$, if there exists a minimal concept of $L$ within $\mathcal{C}$, then $M$ converges to an index of a minimal concept of $L$ within $\mathcal{C}$ for $\sigma$.

An RIIM $M$ is said to *refutably minimally infer a class $\mathcal{C}$ from positive data*, if it satisfies the following condition: For any nonempty concept $L$ and any positive presentation $\sigma$ of $L$, (i) if there exists a minimal concept of $L$ within $\mathcal{C}$, then $M$ converges to an index of a minimal concept of $L$ within $\mathcal{C}$ for $\sigma$, (ii) otherwise $M$ refutes the class $\mathcal{C}$ from $\sigma$.

An IIM $M$ is said to *reliably minimally infer a class $\mathcal{C}$ from positive data*, if it satisfies the following condition: For any nonempty concept $L$ and any positive presentation $\sigma$ of $L$, (i) if there exists a minimal concept of $L$ within $\mathcal{C}$, then $M$ converges to an index of a minimal concept of $L$ within $\mathcal{C}$ for $\sigma$, (ii) otherwise $M$ does not converge to any index for $\sigma$.

An IIM $M$ is said to *strong-minimally infer a class $\mathcal{C}$ from positive data*, if for any nonempty concept $L$ and any positive presentation $\sigma$ of $L$, $M$ converges to an index of a minimal concept of $L$ within $\mathcal{C}$ for $\sigma$.

A class $\mathcal{C}$ is said to be *minimally* (resp., *strong-minimally*, *refutably minimally* or *reliably minimally*) *inferable from positive data*, if there is an IIM $M$ (resp., an IIM $M$, an RIIM $M$ or an IIM $M$) which minimally (resp., strong-minimally, refutably minimally or reliably minimally) infers $\mathcal{C}$ from positive data.

We note that a strong-minimally inferable class $\mathcal{C}$ has the strong property that for any nonempty concept $L$, there always exists a minimal concept of $L$ within $\mathcal{C}$. To the contrary, by definition, for a class with this property, (refutably or reliably) minimal inferability is equivalent to strong-minimal inferability. In Section 6.3 we show that there are some rich hypothesis spaces that are strong-minimally inferable from positive data.

Let $M$ be an IIM or an RIIM which (refutably, reliably or strong-) minimally infers a class $\mathcal{C}$ from positive data. If we feed a positive presentation $\sigma$ of a concept $L_i \in \mathcal{C}$ to $M$, then $M$ converges to an index $j$ with $L_j = L_i$ for $\sigma$, because $L_i$ itself is the unique minimal concept, i.e. the least concept, of $L_i$ within $\mathcal{C}$. Therefore $M$ also infers $\mathcal{C}$ in the limit from positive data.

By definitions and Theorem 2.13, it is easy to see that the following implications hold:

$\mathcal{C}$ is strong-minimally inferable from positive data,

$\downarrow$

$\mathcal{C}$ is refutably minimally inferable from positive data,

$\downarrow$

$\mathcal{C}$ is reliably minimally inferable from positive data,

$\downarrow$

$\mathcal{C}$ is minimally inferable from positive data,

$\downarrow$

$\mathcal{C}$ is inferable in the limit from positive data,

$\downarrow\uparrow$

$\mathcal{C}$ is semi-reliably inferable from positive data.

In Section 5.3 we will sharpen the above separations.

As stated in Section 2.2, a class $\mathcal{C}$ is inferable in the limit from positive data, if and only if finite tell-tales of $\mathcal{C}$ are uniformly and recursively enumerable (cf. Theorem 2.6).

Hence many studies on inferability from positive data concentrate on uniform and recursive enumerability of finite tell-tales (cf. Theorem 2.6, 2.9, Proposition 2.10, Corollary 2.11, Sato&Umayahara[38] and Kapur[14]).

In the ordinary inductive inference of an indexed family of nonempty recursive concepts from positive data, an inference machine takes the following strategy (cf. Theorem 2.6):

- Search for an index $i$ such that $L_i$ contains the obtained data and that a currently enumerated finite tell-tale of $L_i$ within $\mathcal{C}$ is contained in the set of obtained data.
- If such an index is found, output it.

However if an inference machine takes the same strategy on a positive presentation of a concept not in $\mathcal{C}$, then it may not converge to any index for the presentation.

By the following Proposition 5.1, we see that if we force an inference machine to converge to an index of a minimal concept of a target concept within the class concerned, finite tell-tales turn to be of no use.

**Proposition 5.1.** *Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class of concepts each of which has a finite tell-tale within $\mathcal{C}$, let $L \notin \mathcal{C}$ be a concept, and let $L_{j_1}, L_{j_2}, \cdots \in \mathcal{C}$ be minimal concepts of $L$ within $\mathcal{C}$. Then for any $i \geq 1$, there is a finite tell-tale $T_i$ of $L_{j_i}$ within $\mathcal{C}$ such that $T_i \not\subseteq L$.*

*Proof.* For $i \geq 1$, let $T_i'$ be a finite tell-tale of $L_{j_i}$ within $\mathcal{C}$. Since $L_{j_i}$ is a minimal concept of $L$ within $\mathcal{C}$ and $L \neq L_{j_i}$ holds, it follows that $L \subsetneq L_{j_i}$. Thus there is a $w_i \in L_{j_i} \setminus L$. Put $T_i = T_i' \cup \{w_i\}$. Then it is easy to see that $T_i$ is a finite tell-tale of $L_{j_i}$ within $\mathcal{C}$ which satisfies the proposition. ■

## 5.2. Some Sufficient Conditions

We start with some basic definitions and lemmas necessary for showing some sufficient conditions for minimal inferability.

**Definition 5.2** (Sato&Moriyama[39]). A class $\mathcal{C}$ is said to *satisfy MEF-condition*, if for any nonempty finite set $T \subseteq U$ and any $L_i \in \mathcal{C}$ with $T \subseteq L_i$, there is a minimal concept $L_j$ of $T$ within $\mathcal{C}$ such that $L_j \subseteq L_i$.

A class $\mathcal{C}$ is said to *satisfy MFF-condition*, if for any nonempty finite set $T \subseteq U$, $\sharp\{L_i \in \mathcal{C} \mid L_i$ is a minimal concept of $T$ within $\mathcal{C}\}$ is finite.

A class $\mathcal{C}$ is said to *have M-finite thickness*, if $\mathcal{C}$ satisfies both MEF-condition and MFF-condition.

We note that if a class $\mathcal{C}$ contains all nonempty finite concepts, then $\mathcal{C}$ has M-finite thickness.

For a class with M-finite thickness, the existence of a finite tell-tale of each concept in the class leads to the enumerability of it.

**Theorem 5.2** (Sato&Moriyama[39]). *If a class $\mathcal{C}$ has M-finite thickness and each concept in $\mathcal{C}$ has a finite tell-tale within $\mathcal{C}$, then finite tell-tales of $\mathcal{C}$ are uniformly and recursively enumerable, and thus it is inferable in the limit from positive data.*

Here we note that the condition of M-finite thickness alone is not sufficient for inferability from positive data.

**Example 5.1.** Let $\mathcal{SFC}$ be the so-called superfinite class, that is, a class contains all finite concepts and at least one infinite concept.

It is easy to see that this class has M-finite thickness. However as shown in Theorem 2.5, this class is not inferable in the limit from positive data.

Therefore this class is also not (refutably, reliably or strong-) minimally inferable from positive data.

**Lemma 5.3.** *Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class. Then if there are two infinite sequences $T_1, T_2, \cdots \subseteq U$ of nonempty finite sets and $L_{j_1}, L_{j_2}, \cdots \in \mathcal{C}$ of concepts such that*

*(i) $T_1 \subseteq T_2 \subseteq \cdots$, and*

*(ii) for any $i \geq 1$, $L_{j_i}$ is a minimal concept of $T_i$ within $\mathcal{C}$ but is not that of $T_{i+1}$,*

*then $\mathcal{C}$ has infinite elasticity.*

*Proof.* Assume that there are two infinite sequences $T_1, T_2, \cdots \subseteq U$ of nonempty finite sets and $L_{j_1}, L_{j_2}, \cdots \in \mathcal{C}$ of concepts that satisfy both (i) and (ii).

Since $L_{j_i}$ is a minimal concept of $T_i$ within $\mathcal{C}$, (a) $T_i \subseteq L_{j_i}$ holds and (b) for any $L_j \in \mathcal{C}$, $T_i \subseteq L_j$ implies $L_j \not\subsetneq L_{j_i}$. On the other hand, since $L_{j_i}$ is not a minimal concept of $T_{i+1}$ within $\mathcal{C}$, either (c) $T_{i+1} \not\subseteq L_{j_i}$ holds or (d) there is an $L_j \in \mathcal{C}$ such that $T_{i+1} \subseteq L_j$ and $L_j \subsetneq L_{j_i}$. Since $T_i \subseteq T_{i+1}$, (d) contradicts (b). Therefore (a), (b) and (c) hold.

Thus we see by (c) that there is an infinite sequence $w_0, w_1, w_2, \cdots$ such that

$$w_0 \in T_1 \quad \text{and} \quad w_i \in T_{i+1} \setminus L_{j_i} \quad \text{for } i \geq 1.$$

Furthermore we see by (a) that two infinite sequences $w_0, w_1, w_2, \cdots$ and $L_{j_1}, L_{j_2}, \cdots$ satisfy the following condition: For $i \geq 1$,

$$\{w_0, w_1, \cdots, w_{i-1}\} \subseteq L_{j_i} \quad \text{but} \quad w_i \notin L_{j_i}.$$

That is, $\mathcal{C}$ has infinite elasticity. ■

**Lemma 5.4.** *Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class, let $L \subseteq U$ be a nonempty concept, and let $L_n \in \mathcal{C}$ be a concept. Then if there is a finite subset $T$ of $L$ such that for any finite set $T'$ with $T \subseteq T' \subseteq L$, $L_n$ is a minimal concept of $T'$ within $\mathcal{C}$, then $L_n$ is also a minimal concept of $L$ within $\mathcal{C}$.*

*Proof.* Assume that there is a finite subset $T$ of $L$ such that for any finite set $T'$ with $T \subseteq T' \subseteq L$, $L_n$ is a minimal concept of $T'$ within $\mathcal{C}$.

(I) $L \subseteq L_n$ holds. In fact, suppose the converse. Then there is a $w \in L \setminus L_n$. Let $T' = T \cup \{w\}$. Then $T \subseteq T' \subseteq L$ holds, and it follows by assumption that $L_n$ is a minimal concept of $T'$ within $\mathcal{C}$. Thus we have $w \in L_n$, which contradicts the assumption of $w \in L \setminus L_n$.

(II) For any $L_i \in \mathcal{C}$, $L \subseteq L_i$ implies $L_i \not\subsetneq L_n$. In fact, suppose the converse. Then there is an $L_i \in \mathcal{C}$ such that $L \subseteq L_i$ and $L_i \subsetneq L_n$. Since $T \subseteq L$, it follows that $T \subseteq L_i \subsetneq L_n$, which contradicts the assumption that $L_n$ is a minimal concept of $T$ within $\mathcal{C}$. ■

**Lemma 5.5.** *Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class which satisfies MEF-condition and has finite elasticity, let $L \subseteq U$ be a nonempty concept, and let $L_n \in \mathcal{C}$ be a concept.*

*(a) If $L \subseteq L_n$, then there is a minimal concept $L_j$ of $L$ within $\mathcal{C}$ such that $L_j \subseteq L_n$.*

*(b) If $L_n$ is a minimal concept of $L$ within $\mathcal{C}$, then there is a finite subset $T$ of $L$ such that $L_n$ is a minimal concept of $T$ within $\mathcal{C}$.*

*Proof.* (a) Assume $L \subseteq L_n$.

**Claim:** There are a finite subset $T$ of $L$ and an $L_j \in \mathcal{C}$ with $L_j \subseteq L_n$ such that for any finite set $T'$ with $T \subseteq T' \subseteq L$, $L_j$ is a minimal concept of $T'$ within $\mathcal{C}$.

*Proof of the claim.* Suppose the converse. Then for any finite subset $T$ of $L$ and any $L_j \in \mathcal{C}$ with $L_j \subseteq L_n$, there is a finite set $T'$ with $T \subseteq T' \subseteq L$ such that $L_j$ is not a minimal concept of $T'$ within $\mathcal{C}$.

We define $T_i$'s and $j_i$'s ($i \geq 1$) inductively by the following stages:

Stage 1:

Let $T_1$ be any nonempty finite subset of $L$. Since $T_1 \subseteq L \subseteq L_n$, it follows by MEF-condition that there is a minimal concept $L_j$ of $T$ within $\mathcal{C}$ such that $L_j \subseteq L_n$. Put $j_1 = j$, and goto Stage 2.

Stage $i$ ($\geq 2$):

Since $T_{i-1}$ is a finite subset of $L$ and $L_{j_{i-1}} \subseteq L_n$ holds, it follows by assumption that there is a finite set $T'$ with $T_{i-1} \subseteq T' \subseteq L$ such that $L_{j_{i-1}}$ is not a minimal concept of $T'$ within $\mathcal{C}$. Put $T_i = T'$. Since $T_i \subseteq L \subseteq L_n$, it follows by MEF-condition that there is a minimal concept $L_j$ of $T_i$ within $\mathcal{C}$ such that $L_j \subseteq L_n$. Put $j_i = j$, and goto Stage $i+1$.

Then two infinite sequences $T_1, T_2, \cdots$ and $L_{j_1}, L_{j_2}, \cdots$ satisfy the condition of Lemma 5.3, and it follows that $\mathcal{C}$ has infinite elasticity. This contradicts the assumption. □

By this claim and Lemma 5.4, we have the lemma.

(b) Assume that $L_n$ is a minimal concept of $L$ within $\mathcal{C}$. Then suppose that for any finite subset $T$ of $L$, $L_n$ is not a minimal concept of $T$ within $\mathcal{C}$.

We define $T_i$'s and $j_i$'s ($i \geq 1$) inductively by the following stages:

Stage 1:

Let $T_1$ be any nonempty finite subset of $L$. Since $T_1 \subseteq L \subseteq L_n$, it follows by MEF-condition that there is a minimal concept $L_j$ of $T$ within $\mathcal{C}$ such that $L_j \subseteq L_n$. Put $j_1 = j$, and goto Stage 2.

Stage $i$ ($\geq 2$):

Since $T_{i-1}$ is a finite subset of $L$, it follows by assumption that $L_n$ is not a minimal concept of $T_{i-1}$ within $\mathcal{C}$.

Here suppose that for any finite set $T$ with $T_{i-1} \subseteq T \subseteq L$, $L_{j_{i-1}}$ is a minimal concept of $T$ within $\mathcal{C}$. Then, by Lemma 5.4, $L_{j_{i-1}}$ is a minimal concept of $L$ within $\mathcal{C}$. Since $L_n$ is not a minimal concept of $T_{i-1}$ within $\mathcal{C}$, it follows that $L_{j_{i-1}} \neq L_n$. Thus we have $L_{j_{i-1}} \subsetneq L_n$, which is impossible because both $L_{j_{i-1}}$ and $L_n$ are minimal concepts of $L$ within $\mathcal{C}$.

Therefore there is a finite set $T$ with $T_{i-1} \subseteq T \subseteq L$ such that $L_{j_{i-1}}$ is not a minimal concept of $T$ within $\mathcal{C}$. Put $T_i = T$. Since $T_i \subseteq L \subseteq L_n$, it follows by MEF-condition that there is a minimal concept $L_j$ of $T_i$ within $\mathcal{C}$ such that $L_j \subseteq L_n$. Put $j_i = j$, and goto Stage $i+1$.

Then two infinite sequences $T_1, T_2, \cdots$ and $L_{j_1}, L_{j_2}, \cdots$ satisfy the condition of Lemma 5.3, and it follows that $\mathcal{C}$ has infinite elasticity. This contradicts the assumption. ■

In the above Lemma 5.5 (b), the condition that $\mathcal{C}$ has finite elasticity is necessary. In fact, we consider the class $\mathcal{SFC}$ (cf. Theorem 2.5 and Example 5.1). It is easy to see that this class has infinite elasticity. Put $L = U$. Then the unique minimal concept, i.e. the least concept, of $L$ within $\mathcal{SFC}$ is $L$ itself and that for any nonempty finite set $T \subseteq U$, the unique minimal concept of $T$ within $\mathcal{SFC}$ is $T$ itself. Thus for any finite set $T \subseteq U$, $L$ is not a minimal concept of $T$ within $\mathcal{SFC}$.

**Theorem 5.6.** *If a class $\mathcal{C}$ has both finite elasticity and M-finite thickness, then $\mathcal{C}$ is reliably minimally inferable from positive data.*

*Proof.* Let us consider the procedure in Figure 5.1, where the notation $L_i^{(n)}$ is defined in Definition 2.7.

**Procedure** IIM $M$;
**begin**
  $T = \phi$;  $n = 0$;
  **repeat**
    read the next fact and store it in $T$;
    $n = n + 1$;
    search for the least index $i \leq n$ such that
      (1) $T \subseteq L_i$, and
      (2) $\forall j \leq n$, $[T \subseteq L_j \Rightarrow L_j^{(n)} \not\subsetneq L_i^{(n)}]$;
    **if** such an index $i$ is found **then** output $i$ **else** output $n$;
  **forever**;
**end**.

Figure 5.1: An IIM which reliably minimally infers a class

Assume that we feed a positive presentation $\sigma$ of a nonempty concept $L$ to the procedure.

(I) In case there is no minimal concept of $L$ within $\mathcal{C}$. Suppose that the procedure converges to an index $i$ for $\sigma$. Then we see by Lemma 5.5 (a) that $L \not\subseteq L_i$. Therefore there is an $n \geq 1$ such that $\sigma[n]^+ \not\subseteq L_i$, and it follows that the index $i$ does not satisfy the condition (1) in the procedure after reading the $n$-th fact. This is a contradiction.

(II) In case there is a minimal concept of $L$ within $\mathcal{C}$. Let $i_0$ be the least index $i$ such that $L_i$ is a minimal concept of $L$ within $\mathcal{C}$, that is,

$$i_0 = \min\{i \mid L \subseteq L_i \in \mathcal{C} \text{ and } \forall j,\ [L \subseteq L_j \Rightarrow L_j \not\subsetneq L_i]\}.$$

**Claim A:** There is an $n \geq 1$ such that any index $i < i_0$ does not satisfy the condition (1) or (2), after reading the $n$-th fact.

*Proof of the claim.* We define $m_i$'s ($1 \leq i < i_0$) as follows:

(i) In case $L \not\subseteq L_i$. It is easy to see that there is an $m \geq 1$ such that for any $j \geq m$, $\sigma[j]^+ \not\subseteq L_i$. Put $m_i = m$.

(ii) Otherwise. By the definition of $i_0$ and the fact $i < i_0$, there is a $j \geq 1$ such that $L \subseteq L_j$ and $L_j \subsetneq L_i$. Since $L_j \subsetneq L_i$, we see by Lemma 2.8 that there is an $m \geq 1$ such that for any $n \geq m$, $L_j^{(n)} \subsetneq L_i^{(n)}$. Put $m_i = m$.

Then any index $i < i_0$ does not satisfy the condition (1) or (2), after reading the $\max\{m_i \mid 1 \leq i < i_0\}$-th fact. □

It is clear that $i_0$ satisfies the condition (1) at any point.

**Claim B:** There is an $n \geq 1$ such that $i_0$ always satisfies the condition (2), after reading the $n$-th fact.

*Proof of the claim.* For $i \geq 1$, put $T_i = \sigma[i]^+$. Since $L_{i_0}$ is a minimal concept of $L$ within $\mathcal{C}$, it follows by Lemma 5.5 (b) that there is a finite subset $T$ of $L$ such that $L_{i_0}$ is a minimal concept of $T$ within $\mathcal{C}$. Since $\sigma$ is a positive presentation of $L$, it follows that there is an $m \geq 1$ such that $T \subseteq T_m$. Since $T \subseteq T_m \subseteq L \subseteq L_{i_0}$, it follows that $L_{i_0}$ is also a minimal concept of $T_m$ within $\mathcal{C}$.

Let $\{L_{j_1}, \cdots, L_{j_k}\}$ be the collection of all minimal concepts of $T_m$ within $\mathcal{C}$, which is of finite cardinality by MFF-condition. Since $L_{i_0}$ is a minimal concept of $T_m$ within $\mathcal{C}$ and $T_m \subseteq L_{j_i}$ holds, it follows that $L_{j_i} \not\subsetneq L_{i_0}$ for any $i$ with $1 \leq i \leq k$. Therefore, by Lemma 2.8, we can take $n_i$'s $(1 \leq i \leq k)$ such that for any $n \geq n_i$, $L_{j_i}^{(n)} \not\subsetneq L_{i_0}^{(n)}$. Let $n_{\max} = \max\{n_i \mid 1 \leq i \leq k\}$. Then for any $i$ with $1 \leq i \leq k$ and any $n \geq n_{\max}$, $L_{j_i}^{(n)} \not\subsetneq L_{i_0}^{(n)}$ holds.

On the other hand, for any $n \geq 1$, if $L_{j_i}^{(n)} \not\subsetneq L_{i_0}^{(n)}$, then for any $L_j \supseteq L_{j_i}$, $L_j^{(n)} \not\subsetneq L_{i_0}^{(n)}$ holds. By MEF-condition and the definition of $\{L_{j_1}, \cdots, L_{j_k}\}$, for any index $j$, if $T_m \subseteq L_j$, then there is an $i$ with $1 \leq i \leq k$ such that $T_m \subseteq L_{j_i} \subseteq L_j$. Therefore for any $n \geq n_{\max}$ and any $j \geq 1$, $T_m \subseteq L_j$ implies $L_j^{(n)} \not\subsetneq L_{i_0}^{(n)}$. Since $T_m \subseteq T_{m+1} \subseteq \cdots$, it follows that for any $n \geq \max\{m, n_{\max}\}$ and any $j \geq 1$, $T_n \subseteq L_j$ implies $L_j^{(n)} \not\subsetneq L_{i_0}^{(n)}$. Therefore $i_0$ always satisfies the condition (2), after reading the $\max\{m, n_{\max}\}$-th fact. □

By Claim A and B, the procedure converges to $i_0$ for $\sigma$. ■

Here we note that the procedure in Figure 5.1 is a sufficiently general one in the following sense: By directly using the procedure, we can show that (i) the classes with finite elasticity or (ii) the classes with M-finite thickness, each of which concept has a finite tell-tale within the class, are inferable in the limit from positive data (cf. Theorem 2.9 and 5.2).

For a class $\mathcal{C}$ which has finite elasticity but does not have M-finite thickness, the procedure in Figure 5.1 may not minimally infer $\mathcal{C}$ from positive data, even when $\mathcal{C}$ is reliably minimally inferable from positive data.

**Example 5.2** (Kapur[16]). Let $w_1, w_2, \cdots$ be an effective enumeration of the universal set $U$, which we used in defining $L_i^{(n)}$ (cf. Definition 2.7). Without loss of generality, we assume $w_i \neq w_j$ if $i \neq j$.

We put

$$L = \{w_1\} \quad \text{and} \quad L_i = \{w_1, w_{i+1}\} \quad \text{for } i \geq 1.$$

Then let $\mathcal{C} = \{L_i\}_{i \in N}$ be the class of interest. It is easy to see that this class has finite elasticity but does not have M-finite thickness.

On the other hand, any concept in $\mathcal{C}$ is a minimal concept of $L$ within $\mathcal{C}$. Let $\sigma = w_1, w_1, w_1, \cdots$ be the positive presentation of $L$. Since for any $n \geq 1$ and any $i$ with $1 \leq i < n$, $L_n^{(n)} = \{w_1\}$ and $L_i^{(n)} = \{w_1, w_{i+1}\}$ hold, it follows that the procedure does not converge to any index for $\sigma$.

That is, the procedure does not minimally infer $\mathcal{C}$ from positive data.

However it is easy to see that this class is refutably minimally inferable from positive data. We omit the details.

In Example 5.1, we have seen that the condition of M-finite thickness is not sufficient for inferability from positive data. Furthermore, by the following Example 5.3, the condition for a class with M-finite thickness to have finite elasticity is not necessary for minimal inferability.

**Example 5.3.** Let $\mathcal{FC}_*$ be the class of all nonempty finite concepts on the universal set $U$. It is easy to see that this class has M-finite thickness, but does not have finite elasticity.

Furthermore it is also easy to see that for any nonempty concept $L$, there is a minimal concept of $L$ within $\mathcal{C}$ if and only if $L$ is a finite concept. Since this class is reliably inferable from positive data (cf. Theorem 2.13), it follows that this class is reliably minimally inferable from positive data.

**Corollary 5.7.** *Let $\mathcal{C} = \{L_i\}_{i\in N}$ be a class with both finite elasticity and M-finite thickness. Then if $\mathcal{C}$ contains the universal set $U$ as its member, then $\mathcal{C}$ is strong-minimally inferable from positive data.*

*Proof.* Assume that $\mathcal{C}$ contains the universal set $U$ as its member. Then, by Lemma 5.5 (a), for any concept $L$, there is a minimal concept of $L$ within $\mathcal{C}$. Thus, by Theorem 5.6, we have the corollary. ∎

**Lemma 5.8.** *If a class $\mathcal{C}$ does not satisfy MEF-condition, then there are a nonempty finite set $T \subseteq U$ and an infinite sequence $L_{j_1}, L_{j_2}, \cdots \in \mathcal{C}$ such that $L_{j_1} \supsetneq L_{j_2} \supsetneq \cdots \supsetneq T$.*

*Proof.* Assume that $\mathcal{C}$ does not satisfy MEF-condition. Then, by Definition 5.2, there are a nonempty finite set $T \subseteq U$ and an $L_i \in \mathcal{C}$ such that (i) $T \subseteq L_i$ and that (ii) there is no minimal concept $L_j$ of $T$ within $\mathcal{C}$ such that $L_j \subseteq L_i$.

Put $j_1 = i$, and define $j_i$'s ($i \geq 2$) inductively by the following stages:

Stage $i$ ($\geq 2$):

Since $L_{j_{i-1}} \subseteq L_i$, we see by (ii) that $L_{j_{i-1}}$ is not a minimal concept of $T$ within $\mathcal{C}$. Therefore there is a $j \geq 1$ such that $T \subseteq L_j \subsetneq L_{j_{i-1}}$. Put $j_i = j$, and goto Stage $i+1$.

Then it is clear that $L_{j_1} \supsetneq L_{j_2} \supsetneq \cdots \supsetneq T$. ∎

The following Corollary 5.9 is a weak form of Theorem 5.6 and Corollary 5.7.

**Corollary 5.9.** *(a) If a class $\mathcal{C}$ has finite thickness, then $\mathcal{C}$ is reliably minimally inferable from positive data.*

*(b) If a class $\mathcal{C}$ with finite thickness contains the universal set $U$ as its member, then $\mathcal{C}$ is strong-minimally inferable from positive data.*

*Proof.* (a) Assume that a class $\mathcal{C}$ has finite thickness. By Theorem 5.6, it suffices for us to show that $\mathcal{C}$ has finite elasticity and satisfies both MEF-condition and MFF-condition. By Proposition 2.10, $\mathcal{C}$ has finite elasticity. It is easy to see that $\mathcal{C}$ satisfies MFF-condition.

Suppose that $\mathcal{C}$ does not satisfy MEF-condition. Then, by Lemma 5.8, there are a nonempty finite set $T \subseteq U$ and an infinite sequence $L_{j_1}, L_{j_2}, \cdots \in \mathcal{C}$ of concepts such that

$L_{j_1} \supsetneq L_{j_2} \supsetneq \cdots \supsetneq T$. This means that $\mathcal{C}$ has infinitely many distinct concepts that include $T$. This contradicts the assumption that $\mathcal{C}$ has finite thickness.

(b) is clear by the above (a) and Corollary 5.7. ∎

**Example 5.4.** We consider the class $\mathcal{PAT}$ of pattern languages. As shown in Example 2.1, $\mathcal{PAT}$ has finite thickness. Furthermore $\mathcal{PAT}$ contains the universal set $U\,(=\Sigma^+)$ as its member. Thus, by Corollary 5.9 (b), $\mathcal{PAT}$ is strong-minimally inferable from positive data.

**Lemma 5.10** (Kapur[16]). *Let $\mathcal{C} = \{L_i\}_{i \in N}$ be a class with finite elasticity, and let $L \subseteq U$ be a concept. Then if $L$ is not a subset of any $L_i \in \mathcal{C}$, then there is a finite subset $T$ of $L$ such that $T$ is not a subset of any $L_i \in \mathcal{C}$.*

*Proof.* Assume that $L$ is not a subset of any $L_i \in \mathcal{C}$. Thus wee see that $L$ is nonempty.

Here suppose that for any finite subset $T$ of $L$, there is an $L_i \in \mathcal{C}$ such that $T \subseteq L_i$. Let $w_0$ be an arbitrary element in $L$, and define $w_i$'s and $L_{j_i}$'s ($i \geq 1$) inductively by the following stages:

Stage $i$ ($\geq 1$):

Since $\{w_0, w_1, \cdots, w_{i-1}\} \subseteq L$, it follows by assumption that there is an $L_j \in \mathcal{C}$ such that $\{w_0, w_1, \cdots, w_{i-1}\} \subseteq L_j$. Put $j_i = j$. Then, by assumption, $L$ is not a subset of $L_{j_i}$, it follows that there is a $w \in L \setminus L_{j_i}$. Put $w_i = w$, and goto Stage $i+1$.

Then, by the construction, two infinite sequences $w_0, w_1, w_2, \cdots$ and $L_{j_1}, L_{j_2}, \cdots$ satisfy the following condition: For $i \geq 1$,

$$\{w_0, w_1, \cdots, w_{i-1}\} \subseteq L_{j_i} \quad \text{but} \quad w_i \notin L_{j_i}.$$

That is, $\mathcal{C}$ has infinite elasticity. This contradicts the assumption. ∎

If a class with both finite elasticity and M-finite thickness has a computable function $\mathrm{econs}_p$ (cf. Definition 4.3), then an inference machine can refute the class when there is no minimal concept of the target concept within the class.

**Corollary 5.11.** *Let $\mathcal{C}$ be a class with both finite elasticity and M-finite thickness. Then if the function* $\mathrm{econs}_p$ *for $\mathcal{C}$ is recursive, then $\mathcal{C}$ is refutably minimally inferable from positive data.*

*Proof.* Assume that the function $\mathrm{econs}_p$ for $\mathcal{C}$ is recursive. Then let us consider the procedure in Figure 5.2.

---

**Procedure** RIIM $M$;
**begin**
  $T = \phi$;  $n = 0$;
  **repeat**
    read the next fact and store it in $T$;
    $n = n + 1$;
    **if** $\mathrm{econs}_p(T) = 0$ **then** refute the class and stop;
    search for the least index $i \leq n$ such that
      (1) $T \subseteq L_i$, and
      (2) $\forall j \leq n$, $[T \subseteq L_j \Rightarrow L_j^{(n)} \not\subsetneq L_i^{(n)}]$;
    **if** such an index $i$ is found **then** output $i$ **else** output $n$;
  **forever**;
**end**.

---

Figure 5.2: An RIIM which refutably minimally infers a class

Assume that we feed a positive presentation $\sigma$ of a nonempty concept $L$ to the procedure.

**Claim:** There is an $n \geq 1$ such that $\mathrm{econs}_p(\sigma[n]^+) = 0$, if and only if there is no minimal concept of $L$ within $\mathcal{C}$.

*Proof of the claim.* (I) The 'if' part. Assume that there is no minimal concept of $L$ within $\mathcal{C}$. Then, by Lemma 5.5 (a), for any $L_i \in \mathcal{C}$, $L \not\subseteq L_i$ holds. Therefore, by Lemma 5.10, there is a finite set $T \subseteq L$ such that for any $L_i \in \mathcal{C}$, $T \not\subseteq L_i$. Hence for any positive presentation $\sigma$ of $L$, there is an $n \geq 1$ such that $T \subseteq \sigma[n]^+$, and it follows that $\mathrm{econs}_p(\sigma[n]^+) = 0$.

(II) The 'only if' part. Assume that there is a minimal concept of $L$ within $\mathcal{C}$. It is easy to see that for any positive presentation $\sigma$ of $L$ and any $n \geq 1$, $\mathrm{econs}_p(\sigma[n]^+) = 1$. □

By this claim and the proof of Theorem 5.6, it is easy to see that the procedure is an RIIM which refutably minimally infers $\mathcal{C}$ from positive data. ■

By a similar discussion to that in Corollary 5.9, we have the following Corollary 5.12.

**Corollary 5.12.** *Let $\mathcal{C}$ be a class with finite thickness. Then if the function* $\mathrm{econs}_p$ *for $\mathcal{C}$ is recursive, then $\mathcal{C}$ is refutably minimally inferable from positive data.*

## 5.3. Separations

In this section, we show that there are differences between the powers of strong-minimal inferability, refutably minimal inferability, reliably minimal inferability and inferability in the limit from positive data.

First we present a class which is refutably minimally inferable but not strong-minimally inferable from positive data.

**Example 5.5.** Let $\Sigma = \{a\}$ be a finite alphabet, and let $\mathcal{PAT}'$ be the class of pattern languages over $\Sigma$ each of which does not contain the string '$a$', that is, $\mathcal{PAT}' = \mathcal{PAT} \setminus \{L(x), L(a)\}$ (cf. Example 5.4).

Then it is easy to see that this class has finite thickness. Furthermore we define the function $\mathrm{econs}_p$ for $\mathcal{C}$ as follows:

$$\mathrm{econs}_p(T) = \begin{cases} 1, & \text{if } T \text{ does not contain the string '}a\text{'}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that this function $\mathrm{econs}_p$ agrees with Definition 4.3 and it is recursively computable. Thus, by Corollary 5.12, this class is refutably minimally inferable from positive data.

However there is no minimal concept of the concept $\Sigma^+$ within $\mathcal{PAT}'$, it follows that this class is not strong-minimally inferable from positive data.

Next, we present a class which is reliably minimally inferable but not refutably minimally inferable from positive data.

**Example 5.6.** We consider the class $\mathcal{FC}_*$ of all nonempty finite concepts on the universal set $U$. As shown in Example 5.3, this class is reliably minimally inferable from positive data.

On the other hand, this class is not refutably minimally inferable from positive data. In fact, suppose that there is an RIIM $M$ which refutably minimally infers $\mathcal{FC}_*$ from positive data. Then let $L \subseteq U$ be an infinite concept, and let $\sigma$ be an arbitrary positive presentation of $L$. Since there is no minimal concept of $L$ within $\mathcal{FC}_*$, $M$ refutes the class $\mathcal{FC}_*$ from $\sigma[n]$ for some $n$. Let $w$ be the last element in $\sigma[n]$, let $T = \sigma[n]^+$ be a finite set, and put $\delta = \sigma[n], w, w, \cdots$. Since $T$ is in $\mathcal{FC}_*$ and $\delta$ is a positive presentation of $T$, $M$ should infer $T$ w.r.t. $\mathcal{FC}_*$ in the limit from $\delta$. This is a contradiction.

Finally, the following Theorem 5.13 shows that there is a class which is inferable in the limit but not minimally inferable from positive data.

**Theorem 5.13** (Kapur[16]). *There is an indexed family $\mathcal{C}$ of recursive concepts such that $\mathcal{C}$ is inferable in the limit but not minimally inferable from positive data.*

*Proof.* Let $M_1, M_2, \cdots$ be an enumeration of all inference machines, and let $c : N \times N \to N$ be Cantor's pairing function. For $j \geq 1$, let $p_j$ be the $j$-th prime number, and put $\sigma_j = p_j, p_j, p_j, \cdots$. For $j, n \geq 1$, let $\overline{M}_j^{(n)}(\sigma_j)$ be the last guess of $M_j$ executed in $n$ steps on input $\sigma_j$.

For $j, k \geq 1$, let

$$L_{c(j,k)} = \begin{cases} \{p_j, p_j^{n+1}\}, & \text{if there is an } m \geq 1 \text{ such that } \overline{M}_j^{(m)}(\sigma_j) = c(j,k), \\ & \qquad \text{where } n = \min\{m \mid \overline{M}_j^{(m)}(\sigma_j) = c(j,k)\}, \\ \{p_j\}, & \text{otherwise.} \end{cases}$$

Then let $\mathcal{C} = \{L_i\}_{i \in N}$. This class is an indexed family of recursive concepts. In fact, for any $i, q \geq 1$, we can decide whether $q \in L_i$ or not as follows: Let $j, k$ be integers such that $i = c(j,k)$.

(i) In case $q = p_j$. Then $q$ is in $L_i$.

(ii) In case $q = p_j^{n+1}$ for some $n \geq 1$. Then we execute $M_j$ in $n$ steps on input $\sigma_j$. If it outputs $i\,(= c(j,k))$ for the first time at just $n$-th step, then $q\,(= p_j^{n+1})$ is in $L_i$. Otherwise $q$ is not in $L_i$.

(iii) Otherwise $q$ is not in $L_i$.

Thus this class is an indexed family of recursive concepts.

(I) This class $\mathcal{C}$ is inferable in the limit from positive data. This is because this class consists of finite concepts (cf. Corollary 2.7).

(II) This class is not minimally inferable from positive data. In fact, suppose that there is an IIM $M_j$ which minimally infers $\mathcal{C}$ from positive data. Therefore $M_j$ should converge to an index $i$ for $\sigma_j$ such that $L_i$ is a minimal concept of $\{p_j\}$ within $\mathcal{C}$.

Since $p_j \in L_i$, it is easy to see that there is a $k \geq 1$ such that $i = c(j,k)$. Furthermore since $M_j$ outputs $i$, there is an $m \geq 1$ such that $\overline{M}_j^{(m)}(\sigma_j) = i\,(= c(j,k))$. Let $n = \min\{m \mid \overline{M}_j^{(m)}(\sigma_j) = c(j,k)\}$. Then, by definition, $L_i = L_{c(j,k)} = \{p_j, p_j^{n+1}\}$ holds.

However there is a $k' \geq 1$ such that $L_{c(j,k')} = \{p_j\}$. In fact, suppose the converse. Then for any $k' \geq 1$, $L_{c(j,k')} \neq \{p_j\}$. By the construction, this means that $M_j$ changes its mind infinitely many times for $\sigma_j$, which contradicts the fact that $M_j$ converges to the index $i$ for $\sigma_j$.

This means that $L_i$ is not a minimal concept of $\{p_j\}$ within $\mathcal{C}$, which contradicts the assumption. ∎

The separations so far known are summarized as follows:

$\mathcal{C}$ is strong-minimally inferable from positive data,

$\downarrow \not\uparrow$

$\mathcal{C}$ is refutably minimally inferable from positive data,

$\downarrow \not\uparrow$

$\mathcal{C}$ is reliably minimally inferable from positive data,

$\downarrow$ ($\not\uparrow$?)

$\mathcal{C}$ is minimally inferable from positive data,

$\downarrow \not\uparrow$

$\mathcal{C}$ is inferable in the limit from positive data,

$\downarrow \uparrow$

$\mathcal{C}$ is semi-reliably inferable from positive data.

It is unknown at present whether the classes that are minimally inferable from positive data are reliably minimally inferable from positive data or not.

## 5.4. Discussion

We have introduced the notion of minimal inferability for a class of recursive concepts and showed some sufficient conditions and separations.

Using only positive data, it is natural to consider a minimal concept of a target concept within the class concerned, because a minimal concept explains all obtained facts and, in a sense, it is one of the best concept within the class. Moreover we can regard it as a natural extension of ordinary inferability from positive data, because minimal inferability directly leads to inferability of the class in the ordinary sense.

As stated in Section 5.1, if we consider an inference of a minimal concept, finite tell-tales turn to be of no use. However various conditions, properties and notions introduced to show uniform and recursive enumerability of finite tell-tales seem to be valid to some degree. I think this is because these notions, including finite tell-tales, more or less lead to how to avoid overgeneralization, that is, how to identify a minimal concept. In order to work finite tell-tales intendedly, the target concept should be in the class (cf. Proposition 5.1).

In Section 5.2 we showed some sufficient conditions for minimal inferability. These conditions are so general and practical that by using them we can show that the classes definable by length-bounded EFS's with at most $n$ axioms are (refutably or strong-) minimally inferable from positive data (cf. Section 6.3). As far as I know, these classes are one of the largest classes that are inferable in the limit from positive data.

# Chapter 6.

# Inferability of EFS Definable Classes

In the previous chapters, we have discussed characterizations, comparisons and some other properties of indexed families of recursive concepts under various inference criteria. In this chapter we fix our attention to the classes of models or languages definable by *elementary formal systems* (*EFS*'s, for short) and discuss their inferability under the criteria of finite identification, refutable identification and minimal identification.

The EFS's were originally introduced by Smullyan[48] to develop his recursion theory. In a word, EFS's are a kind of logic programming language which uses patterns instead of terms in first order logic[52], and they are shown to be natural devices to define languages[4].

The classes of models or languages definable by EFS's are regarded as indexed families of recursive concepts, if we put a syntactical restriction of so-called length-bounded on EFS's, which we will define later.

In Section 6.1 we recall definitions and properties of EFS's, and review results on inferability of the classes definable by length-bounded EFS's from positive data due to Shinohara[45].

In Section 6.2 we discuss refutable inferability of the classes definable by length-bounded EFS's from *complete data*. In Chapter 4 we have discussed refutable inferability of a class from positive data or complete data, and showed that the refutably inferable classes from *positive data* are very small. Here we show that there are rich hypothesis spaces that are refutably inferable from *complete data*.

In Section 6.3 we also discuss minimal inferability of the classes definable by length-bounded EFS's from positive data. In Chapter 5 we have shown that the classes both finite elasticity and M-finite thickness are minimally inferable from positive data. Here we check that the classes definable by length-bounded EFS's with at most $n$ axioms have both finite elasticity and M-finite thickness for $n \geq 1$. This result is a natural extension of the above mentioned results due to Shinohara[45].

This chapter is based on Mukouchi&Arikawa[30] and Mukouchi[31].

## 6.1. EFS's and Their Inferability

In this thesis we briefly recall EFS's. For detailed definitions and properties of EFS's, please refer to Smullyan[48], Arikawa[4], Arikawa et al.[5, 6] and Yamamoto[52].

Let $\Sigma$, $X$ and $\Pi$ be mutually disjoint nonempty sets. We assume that $\Sigma$ and $\Pi$ are finite, and fix them throughout this chapter. Elements in $\Sigma$, $X$ and $\Pi$ are called constant symbols, variables and predicate symbols, respectively. By $p, q, p_1, p_2, \cdots$, we denote predicate symbols. Each predicate symbol is associated with a positive integer which we call an *arity*.

**Definition 6.1.** A *term*, or a *pattern*, is an element in $(\Sigma \cup X)^+$, that is, it is a nonnull string over $(\Sigma \cup X)$. By $\pi, \pi_1, \pi_2, \cdots$, we denote terms. A term $\pi$ is said to be *ground*, if $\pi \in \Sigma^+$. By $w, w_1, w_2, \cdots$, we denote ground terms.

An *atomic formula* (*atom*, for short) is an expression of the form $p(\pi_1, \cdots, \pi_n)$, where $p$ is a predicate symbol with arity $n$, and $\pi_1, \cdots, \pi_n$ are terms. By $A, B, A_1, A_2, \cdots$, we denote atoms. An atom $p(\pi_1, \cdots, \pi_n)$ is said to be ground, if $\pi_1, \cdots, \pi_n$ are ground terms.

We define well-formed formulas and clauses in the ordinary ways (cf. Lloyd[22]).

**Definition 6.2.** A *definite clause* is a clause of the form

$$A \leftarrow B_1, \cdots, B_n,$$

where $n \geq 0$, and $A, B_1, \cdots, B_n$ are atoms. The atom $A$ above is called the *head* of the clause, and the sequence $B_1, \cdots, B_n$ is called the *body* of the clause. By $C, D, C_1, C_2, \cdots$, we denote definite clauses. Then an *EFS* is a finite set of definite clauses, each of which is called an *axiom*.

A *substitution* is a homomorphism from terms to terms which maps each symbol $a \in \Sigma$ to itself.

In the world of EFS's, the *Herbrand base* ($HB$, for short) is the set of all ground atoms. A subset $I$ of $HB$ is called an *Herbrand interpretation.* We also define *Herbrand model*, and *the least Herbrand model* in the ordinary ways[22].

For an EFS $\Gamma$, the least Herbrand model is denoted by $M(\Gamma)$. For an EFS $\Gamma$ and a predicate symbol $p$ with arity $n$, we define the set of $n$-tuples of ground terms as follows:

$$L(\Gamma, p) = \{(w_1, \cdots, w_n) \in (\Sigma^+)^n \mid p(w_1, \cdots, w_n) \in M(\Gamma)\}.$$

In case the arity of $p$ is 1, i.e. $p$ is unary, we regard $L(\Gamma, p)$ as a language over $\Sigma$.

**Definition 6.3.** A clause $C$ is said to be *provable* from an EFS $\Gamma$ (abbreviated to $\Gamma \vdash C$), if $C$ is obtained from $\Gamma$ by finitely many applications of substitutions and modus ponens. That is, we define the relation $\vdash$ inductively as follows:

(i) If $C \in \Gamma$, then $\Gamma \vdash C$.

(ii) If $\Gamma \vdash C$, then for any substitution $\theta$, $\Gamma \vdash C\theta$.

(iii) If $\Gamma \vdash A \leftarrow B_1, \cdots, B_n, B_{n+1}$ and $\Gamma \vdash B_{n+1}$, then $\Gamma \vdash A \leftarrow B_1, \cdots, B_n$.

The *provable set of $\Gamma$* (abbreviated to $PS(\Gamma)$) is the set of all ground atoms that are provable from $\Gamma$, that is,

$$PS(\Gamma) = \{A \in HB \mid \Gamma \vdash A\}.$$

Now we put a syntactical restriction on EFS's, because the least Herbrand model $M(\Gamma)$ for an unrestricted EFS $\Gamma$ may not be recursive, that is, for a given ground atom $A$, we can not recursively decide whether $A \in M(\Gamma)$ or not.

For a term $\pi$, $\|\pi\|$ denotes the length of $\pi$, and $o(x, \pi)$ denotes the number of all occurrences of a variable $x$ in $\pi$. For an atom $p(\pi_1, \cdots, \pi_n)$, we define the length of the atom and the number of variable's occurrences in the atom as follows:

$$\begin{aligned}\|p(\pi_1, \cdots, \pi_n)\| &= \|\pi_1\| + \cdots + \|\pi_n\|, \\ o(x, p(\pi_1, \cdots, \pi_n)) &= o(x, \pi_1) + \cdots + o(x, \pi_n).\end{aligned}$$

**Definition 6.4.** A clause $A \leftarrow B_1, \cdots, B_n$ is said to be *length-bounded*, if

$$\|A\theta\| \geq \|B_1\theta\| + \cdots + \|B_n\theta\|$$

for any substitution $\theta$.

An EFS $\Gamma$ is said to be length-bounded, if all axioms of $\Gamma$ are length-bounded.

The notion of length-bounded clauses is characterized by the following Lemma 6.1.

**Lemma 6.1** (Arikawa et al.[5, 6]). *A clause $A \leftarrow B_1, \cdots, B_n$ is length-bounded, if and only if $\|A\| \geq \|B_1\| + \cdots + \|B_n\|$ and $o(x, A) \geq o(x, B_1) + \cdots + o(x, B_n)$ hold for any variable $x$.*

From now on, we only consider length-bounded EFS's. For length-bounded EFS's, the following Theorem 6.2 and 6.3 hold.

**Theorem 6.2** (Arikawa et al.[5, 6], Yamamoto[52]). *For any length-bounded EFS $\Gamma$, the least Herbrand model $M(\Gamma)$ is recursive, that is, for any ground atom $A$, whether $A \in M(\Gamma)$ or not is recursively decidable.*

**Theorem 6.3** (Arikawa et al.[5, 6], Yamamoto[52]). *For any length-bounded EFS $\Gamma$, $M(\Gamma)$ $= PS(\Gamma)$ holds.*

By the above Theorem 6.3, we need not to distinguish $M(\Gamma)$ from $PS(\Gamma)$.

Furthermore the following Theorem 6.4 shows the power of length-bounded EFS's.

**Theorem 6.4** (Arikawa et al.[5, 6]). *A language $L \subseteq \Sigma^+$ is context-sensitive, if and only if $L$ is definable by a length-bounded EFS.*

We denote by $\mathcal{LB}^{[\leq n]}$ the class of all length-bounded EFS's with at most $n$ axioms. Then $M(\mathcal{LB}^{[\leq n]})$ denotes the class of the least Herbrand models of EFS's in $\mathcal{LB}^{[\leq n]}$, and $L(\mathcal{LB}^{[\leq n]})$ denotes the class of all languages defined by EFS's in $\mathcal{LB}^{[\leq n]}$ with a fixed unary predicate symbol $p \in \Pi$.

Here we note that for a given EFS $\Gamma \in \mathcal{LB}^{[\leq n]}$ and a given predicate symbol $p$, whether $L(\Gamma, p) = \phi$ or not is *not* recursively decidable. Thus in this chapter, we do not care the case where the empty concept is chosen as a target concept, when we are discussing inferability from positive data.

**Definition 6.5** (Shinohara[45]). Let $T \subseteq HB$ be a nonempty finite set. Then an EFS $\Gamma$ is said to be *reduced w.r.t.* $T$, if $T \subseteq M(\Gamma)$ and $T \not\subseteq M(\Gamma')$ hold for any $\Gamma' \subsetneq \Gamma$. For a class $\mathcal{EC}$ of EFS's and a nonempty finite set $T \subseteq HB$, let $RED(T, \mathcal{EC}) = \{\Gamma \in \mathcal{EC} \mid \Gamma$ is reduced w.r.t. $T\}$.

**Lemma 6.5** (Shinohara[45]). *Let $\Gamma$ be a length-bounded EFS, and let $C = A \leftarrow B_1, \cdots, B_n$ be a clause such that $\Gamma \vdash C$. Then the head of every axiom used to prove $C$ is not longer than the head $A$ of $C$.*

*Proof.* This proof is given by mathematical induction on the number of applications of inference rule. We note that all clauses that are provable from $\Gamma$ are length-bounded by Definition 6.3 and 6.4.

(I) In case we can prove a clause $C$ by only one application of inference rule. The clause $C$ itself is an axiom of $\Gamma$.

(II) Otherwise. The rule lastly used to prove $C$ should be an application of a substitution or modus ponens. (i) In case $C = D\theta$ for some provable clause $D$ and some substitution $\theta$. The head of $D$ is not longer than the head of $C$. Thus, by induction hypothesis, we have the assertion. (ii) In case $C = A \leftarrow B_1, \cdots, B_n$, where $\Gamma \vdash D = A \leftarrow B_1, \cdots, B_n, B_{n+1}$ and $\Gamma \vdash B_{n+1}$. The head $A$ of $D$ and $B_{n+1}$ are not longer than the head $A$ of $C$. Thus, by induction hypothesis we have the assertion.

Hence we have the lemma. ∎

**Lemma 6.6** (Shinohara[45]). *For any $n \geq 1$ and any nonempty finite set $T \subseteq HB$, $\sharp RED(T, \mathcal{LB}^{[\leq n]})$ is finite.*

*Proof.* Let $T = \{A_1, \cdots, A_k\}$ be a finite subset of $HB$, and let $\Gamma \in RED(T, \mathcal{LB}^{[\leq n]})$ be a length-bounded EFS.

Here suppose that there is an axiom $C = A \leftarrow B_1, \cdots, B_n \in \Gamma$ such that $\|A\| > \max\{\|A_1\|, \cdots, \|A_k\|\}$. By Lemma 6.5 and Theorem 6.3, we see that $T \subseteq M(\Gamma \setminus \{C\})$, which contradicts the assumption that $\Gamma$ is reduced w.r.t. $T$.

Thus for any axiom $A \leftarrow B_1, \cdots, B_n \in \Gamma$, $\|A\| \leq \max\{\|A_1\|, \cdots, \|A_k\|\}$ holds. Since the number of patterns shorter than a fixed length is finite, it follows that the number of length-bounded clauses whose heads are shorter than a fixed length is finite. Therefore there are finitely many length-bounded EFS's with at most $n$ axioms that are reduced w.r.t. $T$. ∎

**Theorem 6.7** (Shinohara[45]). *For any $n \geq 1$, the classes $M(\mathcal{LB}^{[\leq n]})$ and $L(\mathcal{LB}^{[\leq n]})$ have finite elasticity, respectively. Therefore $M(\mathcal{LB}^{[\leq n]})$ and $L(\mathcal{LB}^{[\leq n]})$ are inferable in the limit from positive data, respectively.*

*Proof.* First, we show that $M(\mathcal{LB}^{[\leq n]})$ has finite elasticity for any $n \geq 1$. This proof is given by mathematical induction on $n$.

(I) In case $n = 1$. Assume $A \in M(\Gamma)$ and $\sharp\Gamma = 1$. Then $\Gamma = \{B \leftarrow\}$, where $B$ is an atom such that $A = B\theta$ for some substitution $\theta$. By a similar discussion to that in Example 2.1, we can show that $M(\mathcal{LB}^{[\leq 1]})$ has finite thickness, and thus it has finite elasticity.

(II) In case $n \geq 2$. Assume that for any $m$ with $1 \leq m < n$, $M(\mathcal{LB}^{[\leq m]})$ has finite elasticity. Then suppose that $M(\mathcal{LB}^{[\leq n]})$ has *infinite elasticity*. Thus there are two infinite sequences $A_0, A_1, A_2, \cdots$ of ground atoms and $\Gamma_1, \Gamma_2, \cdots \in \mathcal{LB}^{[\leq n]}$ of length-bounded EFS's such that for any $i \geq 1$,

$$\{A_0, A_1, \cdots, A_{i-1}\} \subseteq M(\Gamma_i) \quad \text{but} \quad A_i \notin M(\Gamma_i) \tag{$*$}$$

Let $h$ be the function defined as follows:

$$h(i) = \min\{j \leq i \mid \Gamma_i \text{ is reduced w.r.t. } \{A_0, \cdots, A_j\} \text{ or } j = i\}.$$

We consider the following two cases.

(1) In case the infinite sequence $h(1), h(2), \cdots$ has a finite bound $j_0$ such that for any $i \geq 1$, $h(i) \leq j_0$. For any $i \geq j_0$, $\Gamma_i$ should be reduced w.r.t. $\{A_0, \cdots, A_{j_0}\}$. This means

that there are infinitely many EFS's in $M(\mathcal{LB}^{[\leq n]})$ that are reduced w.r.t. $\{A_0, \cdots, A_{j_0}\}$, which contradicts Lemma 6.6.

(2) Otherwise. There is an infinite sequence $i_1, i_2, \cdots$ with $i_1 < i_2 < \cdots$ such that $h(i_1) < h(i_2) < \cdots$.

**Claim:** For any $j \geq 1$, there is an EFS $\Gamma'_{i_j} \subsetneq \Gamma_{i_j}$ such that $\{A_0, \cdots, A_{h(i_j)-1}\} \subseteq M(\Gamma'_{i_j})$ but $A_{h(i_j)} \notin M(\Gamma'_{i_j})$.

*Proof of the claim.* (i) In case $\Gamma_{i_j}$ is reduced w.r.t. $\{A_0, \cdots, A_{h(i_j)}\}$. By definition, for any $\Gamma''_{i_j} \subsetneq \Gamma_{i_j}$, $\{A_0, \cdots, A_{h(i_j)}\} \not\subseteq M(\Gamma''_{i_j})$ holds. Since $\Gamma_{i_j}$ is not reduced w.r.t. $\{A_0, \cdots, A_{h(i_j)-1}\}$, there is an EFS $\Gamma'_{i_j} \subsetneq \Gamma_{i_j}$ such that $\{A_0, \cdots, A_{h(i_j)-1}\} \subseteq M(\Gamma'_{i_j})$, and it follows that $A_{h(i_j)} \notin M(\Gamma'_{i_j})$.

(ii) Otherwise. By definition of $h$, it is easy to see that $h(i_j) = i_j$. This means that $\Gamma_{i_j}$ is not reduced w.r.t. $\{A_0, \cdots, A_k\}$ for any $k$ with $0 \leq k < i_j$. Thus it is also not reduced w.r.t. $\{A_0, \cdots, A_{h(i_j)-1}\}$, and it follows that there is an EFS $\Gamma'_{i_j} \subsetneq \Gamma_{i_j}$ such that $\{A_0, \cdots, A_{h(i_j)-1}\} \subseteq M(\Gamma'_{i_j})$. On the other hand, by $(*)$, we have $A_{h(i_j)} = A_{i_j} \notin M(\Gamma_{i_j})$. Then, by monotonicity of the least Herbrand model, we see that for any $\Gamma''_{i_j} \subseteq \Gamma_{i_j}$, $A_{h(i_j)} = A_{i_j} \notin M(\Gamma''_{i_j})$. Hence we have the claim. □

By this claim, we see that two infinite sequences $A_0, A_{h(i_1)}, A_{h(i_2)}, \cdots$ and $M(\Gamma'_{i_1})$, $M(\Gamma'_{i_2}), \cdots$ satisfy that for any $j \geq 1$,

$$\{A_0, A_{h(i_1)}, \cdots, A_{h(i_{j-1})}\} \subseteq M(\Gamma'_{i_j}) \quad \text{but} \quad A_{h(i_j)} \notin M(\Gamma'_{i_j}).$$

This contradicts the assumption that $M(\mathcal{LB}^{[\leq n-1]})$ has finite elasticity, because for any $j \geq 1$, $\Gamma'_{i_j} \in \mathcal{LB}^{[\leq n-1]}$ holds.

Since we can show a contradiction in each case (1) or (2), we have the assertion for $n$. This concludes the proof for $M(\mathcal{LB}^{[\leq n]})$.

Next, we proceed to show that $L(\mathcal{LB}^{[\leq n]})$ has finite elasticity for any $n \geq 1$. Suppose that $L(\mathcal{LB}^{[\leq n]})$ has *infinite elasticity.* Then there are two infinite sequences $w_0, w_1, w_2, \cdots \in \Sigma^+$ and $\Gamma_1, \Gamma_2, \cdots \in \mathcal{LB}^{[\leq n]}$ such that for any $i \geq 1$,

$$\{w_0, w_1, \cdots, w_{i-1}\} \subseteq L(\Gamma_i) \quad \text{but} \quad w_i \notin L(\Gamma_i).$$

Therefore for any $i \geq 1$,

$$\{p(w_0), p(w_1), \cdots, p(w_{i-1})\} \subseteq M(\Gamma_i) \quad \text{but} \quad p(w_i) \notin M(\Gamma_i).$$

Thus $M(\mathcal{LB}^{[\leq n]})$ also has infinite elasticity, which contradicts the above result. Hence $L(\mathcal{LB}^{[\leq n]})$ has finite elasticity. ■

In Chapter 3, we have seen that the class of pattern languages is finitely inferable from complete data, but for any $n \geq 0$, this class is not inferable within $n$ mind changes from positive data (cf. Example 3.3 and Corollary 3.8). Furthermore the class of unions of at most $n$ pattern languages is shown not to be finitely inferable from complete data (cf. Example 3.4). This directly leads to the following Proposition 6.8.

**Proposition 6.8.** *(a) For any $n \geq 1$ and any $m \geq 0$, the classes $M(\mathcal{LB}^{[\leq n]})$ and $L(\mathcal{LB}^{[\leq n]})$ are not inferable within $m$ mind changes from positive data.*

*(b) For any $n \geq 2$, the classes $M(\mathcal{LB}^{[\leq n]})$ and $L(\mathcal{LB}^{[\leq n]})$ are not finitely inferable from complete data.*

Concerning inferability with bounded mind changes from complete data, it is unknown at present whether or not the class $M(\mathcal{LB}^{[\leq n]})$ or $L(\mathcal{LB}^{[\leq n]})$ is inferable within $m$ mind changes from complete data for $n, m \geq 2$.

Then we adapt the definitions of inferability to the case of EFS's as follows, but as easily seen, the essential part is kept unchanged from Definition 2.4. In what follows, we assume that outputs from an IIM or an RIIM are EFS's.

**Definition 6.6.** For an atom $A$, pred$(A)$ denotes the predicate symbol of $A$. For a set $\Pi_0 \subseteq \Pi$ and a set $S$ of atoms, $S|_{\Pi_0}$ denotes the set of atoms restricted to $\Pi_0$, that is,

$$S|_{\Pi_0} = \{A \in S \mid \mathrm{pred}(A) \in \Pi_0\}.$$

A *predicate-restricted positive presentation* of a set $I \subseteq HB$ w.r.t. $\Pi_0 \subseteq \Pi$ is an infinite sequence $A_1, A_2, \cdots$ of elements in $HB|_{\Pi_0}$ such that $\{A_1, A_2, \cdots\} = I|_{\Pi_0}$. A *predicate-restricted complete presentation* of a set $I \subseteq HB$ w.r.t. $\Pi_0 \subseteq \Pi$ is an infinite sequence

$(A_1, t_1), (A_2, t_2), \cdots$ of elements in $HB|_{\Pi_0} \times \{+, -\}$ such that $\{A_i \mid t_i = +,\ i \geq 1\} = I|_{\Pi_0}$ and $\{A_i \mid t_i = -,\ i \geq 1\} = HB|_{\Pi_0} \setminus I|_{\Pi_0}$.

An IIM $M$ or an RIIM $M$ is said to *converge to an EFS $\Gamma$ for a presentation $\sigma$*, if there is an $n \geq 1$ such that for any $m \geq n$, $\overline{M}(\sigma[m]) = \Gamma$.

Let $\mathcal{EC}$ be a class of EFS's. For an EFS $\Gamma \in \mathcal{EC}$ and a predicate-restricted positive or complete presentation $\sigma$ of $M(\Gamma)$ w.r.t. $\Pi_0 \subseteq \Pi$, an IIM $M$ or an RIIM $M$ is said to *infer the EFS $\Gamma$ w.r.t. $\mathcal{EC}$ in the limit from $\sigma$*, if $M$ converges to an EFS $\Gamma' \in \mathcal{EC}$ with $M(\Gamma')|_{\Pi_0} = M(\Gamma)|_{\Pi_0}$ for $\sigma$.

A class $\mathcal{EC}$ is said to be *theoretical-term-freely inferable in the limit from positive data* (resp., *complete data*), if for any nonempty finite subset $\Pi_0$ of $\Pi$, there is an IIM $M$ which infers $\Gamma$ w.r.t. $\mathcal{EC}$ in the limit from $\sigma$ for any EFS $\Gamma \in \mathcal{EC}$ and any predicate-restricted positive presentation $\sigma$ (resp., any predicate-restricted complete presentation $\sigma$) of $M(\Gamma)$ w.r.t. $\Pi_0$.

A class $\mathcal{EC}$ is said to be *theoretical-term-freely and refutably inferable from positive data* (resp., *complete data*), if for any nonempty finite subset $\Pi_0$ of $\Pi$, there is an RIIM $M$ which satisfies the following condition: For any set $I \subseteq HB$ and any predicate-restricted positive presentation $\sigma$ (resp., any predicate-restricted complete presentation $\sigma$) of $I$ w.r.t. $\Pi_0$, (i) if there is an EFS $\Gamma \in \mathcal{EC}$ such that $M(\Gamma)|_{\Pi_0} = I|_{\Pi_0}$, then $M$ infers $\Gamma$ w.r.t. $\mathcal{EC}$ in the limit from $\sigma$, (ii) otherwise $M$ refutes the class $\mathcal{EC}$ from $\sigma$.

In a similar way, we also define theoretical-term-free and (refutably, reliably or strong-) minimal inferability from positive data or complete data.

Theoretical terms are supplementary predicates that are necessary for defining some goal predicates. In the above definition, the phrase 'theoretical-term-freely inferable' means that *from only the facts on the goal predicates*, an IIM or an RIIM generates some suitable predicates and infers an EFS which explains the goal predicates.

In a similar way to the proof of Theorem 6.7, we can show the following Corollary 6.9.

**Definition 6.7.** For a class $\mathcal{EC}$ of EFS's and a set $\Pi_0 \subseteq \Pi$, let $M(\mathcal{EC})|_{\Pi_0} = \{M(\Gamma)|_{\Pi_0} \mid \Gamma \in \mathcal{EC}\}$.

**Corollary 6.9.** *For any $n \geq 1$ and any $\Pi_0 \subseteq \Pi$, the class $M(\mathcal{LB}^{[\leq n]})|_{\Pi_0}$ has finite elasticity. Therefore for any $n \geq 1$, the class $\mathcal{LB}^{[\leq n]}$ is theoretical-term-freely inferable in the limit from positive data.*

## 6.2. Refutable Inferability from Complete Data

In this section we show that for any $n \geq 0$, the class $\mathcal{LB}^{[\leq n]}$ is theoretical-term-freely and refutably inferable from *complete data*.

Here we note that for any $n \geq 1$, the class $\mathcal{LB}^{[\leq n]}$ is not theoretical-term-freely and refutably inferable from *positive data*, because it contains infinite concepts (cf. Lemma 4.10).

**Definition 6.8.** For two EFS's $\Gamma$ and $\Gamma'$, and a set $\Pi_0 \subseteq \Pi$, we write $\Gamma \equiv_{\Pi_0} \Gamma'$, if we can make $\Gamma'$ identical to $\Gamma$ by renaming predicate symbols other than those in $\Pi_0$ and by renaming variables in each axiom. We assume some canonical form of an EFS w.r.t. $\Pi_0$, and $\mathrm{canon}(\Gamma, \Pi_0)$ denotes the representative EFS for the set of EFS's $\{\Gamma' \mid \Gamma \equiv_{\Pi_0} \Gamma'\}$.

For an EFS $\Gamma$, $HPRED(\Gamma)$ (resp., $BPRED(\Gamma)$) denotes the set of all predicate symbols appearing in the heads (resp., the bodies) of the axioms of $\Gamma$, and $PRED(\Gamma)$ denotes the set $HPRED(\Gamma) \cup BPRED(\Gamma)$. We also define various sets and classes as follows:

$$\begin{aligned}
\mathcal{LB}^{[n]} &= \{\Gamma \mid \Gamma \text{ is a length-bounded EFS with } n \text{ axioms}\}, \\
\mathcal{LB}^{[n]}[\Pi_0] &= \{\mathrm{canon}(\Gamma, \Pi_0) \mid \Gamma \in \mathcal{LB}^{[n]}\}, \\
\mathcal{MLB}^{[n]}[\Pi_0] &= \{\Gamma \in \mathcal{LB}^{[n]}[\Pi_0] \mid BPRED(\Gamma) \subseteq HPRED(\Gamma)\}, \\
\mathcal{MLB}^{[n]}[\Pi_0](l) &= \left\{\Gamma \in \mathcal{MLB}^{[n]}[\Pi_0] \;\middle|\; \begin{array}{c}\text{the head's length of each axiom} \\ \text{of } \Gamma \text{ is not greater than } l.\end{array}\right\},
\end{aligned}$$

$$\mathcal{LB}^{[\leq n]} = \bigcup_{i=0}^{n} \mathcal{LB}^{[i]}, \qquad \mathcal{MLB}^{[\leq n]}[\Pi_0] = \bigcup_{i=0}^{n} \mathcal{MLB}^{[i]}[\Pi_0],$$

$$\mathcal{MLB}^{[\leq n]}[\Pi_0](l) = \bigcup_{i=0}^{n} \mathcal{MLB}^{[i]}[\Pi_0](l),$$

where $l, n \geq 0$, and $\Pi_0 \subseteq \Pi$.

We note that for any EFS's $\Gamma$ and $\Gamma'$, and any $\Pi_0 \subseteq \Pi$, whether $\Gamma \equiv_{\Pi_0} \Gamma'$ or not is recursively decidable, and that we can effectively obtain the EFS canon$(\Gamma, \Pi_0)$.

We prepare some basic lemmas.

**Lemma 6.10** (Shinohara[45]). *Let $\Gamma$ be a length-bounded EFS, and let $A \in M(\Gamma)$ be a ground atom. Then if $\Gamma$ has an axiom $C$ whose head is longer than $A$, then $A \in M(\Gamma \setminus \{C\})$ holds.*

*Proof.* Assume that $\Gamma$ has an axiom $C$ whose head is longer than $A$. Then we see by Lemma 6.5 that $(\Gamma \setminus \{C\}) \vdash A$, and it follows that $A \in M(\Gamma \setminus \{C\})$ by Lemma 6.3. ∎

**Lemma 6.11.** *For any EFS $\Gamma \in \mathcal{LB}^{[\leq n]}$ and any $\Pi_0 \subseteq \Pi$, there is an EFS $\Gamma' \in \mathcal{MLB}^{[\leq n]}[\Pi_0]$ such that $M(\Gamma')|_{\Pi_0} = M(\Gamma)|_{\Pi_0}$.*

*Proof.* Let $\Gamma \in \mathcal{LB}^{[\leq n]}$ and $\Gamma_1 = \text{canon}(\Gamma, \Pi_0)$. Then $M(\Gamma_1)|_{\Pi_0} = M(\Gamma)|_{\Pi_0}$ holds. Therefore if $BPRED(\Gamma_1) \subseteq HPRED(\Gamma_1)$, then $\Gamma_1$ itself is in $\mathcal{MLB}^{[\leq n]}[\Pi_0]$. Otherwise, let $\Gamma_2$ be the EFS which is obtained from $\Gamma_1$ by subtracting all axioms that have predicate symbols in $BPRED(\Gamma_1) \setminus HPRED(\Gamma_1)$ in their bodies, and let $\Gamma_3 = \text{canon}(\Gamma_2, \Pi_0)$. Then clearly, $M(\Gamma_3)|_{\Pi_0} = M(\Gamma)|_{\Pi_0}$ and $\Gamma_3 \in \mathcal{MLB}^{[\leq n]}[\Pi_0]$ hold. ∎

**Lemma 6.12.** *Let $n \geq 1$ be an integer, let $\Pi_0 \subseteq \Pi$ be a set of predicates, let $T \subseteq HB|_{\Pi_0}$ be a nonempty finite set, and let $F \subseteq HB|_{\Pi_0}$ be a finite set. Assume that $(T, F)$ is not consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{[n-1]}[\Pi_0]$. Then for any EFS $\Gamma \in \mathcal{MLB}^{[n]}[\Pi_0] \setminus \mathcal{MLB}^{[n]}[\Pi_0](l)$, $(T, F)$ is not consistent with $M(\Gamma)|_{\Pi_0}$, where $l = \max\{\|A\| \mid A \in T\}$.*

*Proof.* Let $\Gamma \in \mathcal{MLB}^{[n]}[\Pi_0] \setminus \mathcal{MLB}^{[n]}[\Pi_0](l)$. Then there is a length-bounded clause $C \in \Gamma$ whose head's length is greater than $l$. Let $\Gamma_1 = \Gamma \setminus \{C\}$. Suppose that $(T, F)$ is consistent with $M(\Gamma)|_{\Pi_0}$. Since the head of $C$ is longer than every ground atom in $T$, it follows by Lemma 6.10 that $T \subseteq M(\Gamma_1)|_{\Pi_0}$. Furthermore, since $\Gamma_1 \subseteq \Gamma$, it follows that $M(\Gamma_1) \subseteq M(\Gamma)$ by monotonicity of the least Herbrand model. Therefore $F \subseteq HB|_{\Pi_0} \setminus M(\Gamma)|_{\Pi_0} \subseteq HB|_{\Pi_0} \setminus M(\Gamma_1)|_{\Pi_0}$ holds. Thus $(T, F)$ is consistent with $M(\Gamma_1)|_{\Pi_0}$. Let $\Gamma_2 = \text{canon}(\Gamma_1, \Pi_0)$. Then $M(\Gamma_2)|_{\Pi_0} = M(\Gamma_1)|_{\Pi_0}$ holds, and it follows that $(T, F)$ is consistent with $M(\Gamma_2)|_{\Pi_0}$. This contradicts the assumption, because $\Gamma_2 \in \mathcal{MLB}^{[n-1]}[\Pi_0]$. ∎

**Lemma 6.13.** *For any $l, n \geq 0$ and any $\Pi_0 \subseteq \Pi$, the set $\mathcal{MLB}^{[n]}[\Pi_0](l)$ is a uniformly and recursively generable finite set.*

*Proof.* As easily seen, for any EFS $\Gamma \in \mathcal{MLB}^{[n]}[\Pi_0](l)$, the number of predicate symbols appearing in $\Gamma$ is at most $n$ and the arities of those predicate symbols are at most $l$.

Put $PA(l, m) = \{\{q_1^{(j_1)}, \cdots, q_k^{(j_k)}\} \mid 0 \leq k \leq m \text{ and } 1 \leq j_1 \leq \cdots \leq j_k \leq l\}$, where for a predicate symbol $q_i^{(t)}$, the superscript $t$ represents its arity. Then put $PR(l, n, \Pi_0) = \{\Pi' \cup \Pi'' \mid \Pi' \subseteq \Pi_0 \text{ and } \Pi'' \in PA(l, n - \sharp\Pi')\}$. By the above observation, it is sufficient for us to generate EFS's $\Gamma$ with $PRED(\Gamma) \in PR(l, n, \Pi_0)$, because we do not distinguish two EFS's that are identical except for renaming of predicate symbols other than those in $\Pi_0$.

As easily seen, the above $PR(l, n, \Pi_0)$ is a uniformly and recursively generable finite set. Furthermore the set of all terms shorter than a fixed length is a uniformly and recursively generable finite set, where we do not distinguish two terms that are identical except for renaming of variables (cf. Example 2.1).

Roughly speaking, we recursively generate EFS's in $\mathcal{MLB}^{[n]}[\Pi_0](l)$ as follows: We combine sets of predicate symbols in $PR(l, n, \Pi_0)$ and terms whose lengths are not greater than $l$, rearrange variables in each axiom, make canonical form w.r.t. $\Pi_0$ of them and check whether each obtained EFS is in $\mathcal{MLB}^{[n]}[\Pi_0](l)$ or not by using Lemma 6.1. ■

**Theorem 6.14.** *For any $n \geq 0$, the class $\mathcal{LB}^{[\leq n]}$ is theoretical-term-freely and refutably inferable from complete data.*

*Proof.* Let us consider the procedure in Figure 6.1, where the procedure read_store is the same one as in Figure 2.1.

Let $\Pi_0$ be a nonempty finite subset of $\Pi$, and let $I_{\text{base}} \subseteq HB$ be a set of ground atoms. Then assume that we feed a predicate-restricted complete presentation $\sigma$ of $I_{\text{base}}$ w.r.t. $\Pi_0$ to the procedure.

(A) In case there is an EFS $\Gamma_{\text{base}} \in \mathcal{LB}^{[\leq n]}$ such that $I_{\text{base}}|_{\Pi_0} = M(\Gamma_{\text{base}})|_{\Pi_0}$.

**Claim:** In the procedure, for any $m$ with $0 \leq m \leq n$, if $T_m$ and $F_m$ are defined, then $(T_m, F_m)$ is *not* consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{[m]}[\Pi_0]$.

```
Procedure RIIM M(n, Π_0);
begin
    T = φ;    F = φ;
    read_store(T, F);
    while T = φ do begin  ............................................ (1)
        output the empty EFS;
        read_store(T, F);
    end;
    T_0 = T;    F_0 = F;
    for m = 1 to n do begin
        l_m = max{‖A‖ | A ∈ T_{m-1}};
        recursively generate MLB^[m][Π_0](l_m), and set it to S;
        for each Γ ∈ S do  ............................................ (2)
            while (T, F) is consistent with M(Γ)|_{Π_0} do begin  ........ (3)
                output Γ;
                read_store(T, F);
            end;
        T_m = T;    F_m = F;
    end;
    refute the class LB^[≤n] and stop;
end.
```

Figure 6.1: An RIIM for the class $\mathcal{LB}^{[\leq n]}$

*Proof of the claim.* This proof is given by mathematical induction on $m$.

(I) In case $m = 0$. It is clear because $T_0$ is nonempty and the least Herbrand model of the empty EFS is empty.

(II) In case $m \geq 1$. We assume the claim for $m-1$, and assume that $T_m$ and $F_m$ are defined. Then we see that $T_{m-1}$ and $F_{m-1}$ are also defined, and by the induction hypothesis, $(T_{m-1}, F_{m-1})$ is *not* consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{[m-1]}[\Pi_0]$. Therefore by Lemma 6.12, $(T_{m-1}, F_{m-1})$ is *not* consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{[m]}[\Pi_0] \setminus \mathcal{MLB}^{[m]}[\Pi_0](l_m)$. Thus $(T_m, F_m)$ is *not* consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{[m]}[\Pi_0] \setminus \mathcal{MLB}^{[m]}[\Pi_0](l_m)$, because $T_{m-1} \subseteq T_m$ and $F_{m-1} \subseteq F_m$ hold.

Furthermore, since the for-loop (2) terminates, we see that $(T_m, F_m)$ is also *not* consistent with $M(\Gamma)|_{\Pi_0}$ for any EFS $\Gamma \in \mathcal{MLB}^{[m]}[\Pi_0](l_m)$.

Hence we have the claim for $m$. $\square$

By Lemma 6.11, there is an EFS $\Gamma \in \mathcal{MLB}^{[\leq n]}[\Pi_0]$ such that $M(\Gamma)|_{\Pi_0} = M(\Gamma_{\text{base}})|_{\Pi_0}$ ($= I_{\text{base}}|_{\Pi_0}$). Therefore we see by the above claim that $T_n$ and $F_n$ are never defined. By Lemma 4.13, this means that the procedure outputs an EFS $\Gamma$ with $M(\Gamma)|_{\Pi_0} = I_{\text{base}}|_{\Pi_0}$ and never terminates the while-loop (1) or (3).

(B) In case there is no EFS $\Gamma \in \mathcal{LB}^{[\leq n]}$ such that $I_{\text{base}}|_{\Pi_0} = M(\Gamma)|_{\Pi_0}$. By using Lemma 4.13 $n$ times, we see that the procedure refutes the class $\mathcal{LB}^{[\leq n]}$ from $\sigma$. ■

By Corollary 4.5, we see that if the number of axioms is not bounded by a constant number, then this class is not refutably inferable, because it contains all finite concepts on $HB|_{\Pi_0}$.

The following Corollary 6.15 is obvious from Theorem 6.14.

**Corollary 6.15.** *For any $n \geq 0$, the classes $M(\mathcal{LB}^{[\leq n]})$ and $L(\mathcal{LB}^{[\leq n]})$ are refutably inferable from complete data, respectively.*

## 6.3. Minimal Inferability from Positive Data

In this section we show that for any $n \geq 1$, the class $\mathcal{LB}^{[\leq n]}$ is theoretical-term-freely and refutably minimally inferable from *positive data*. This proceeds by showing that for any $n \geq 1$ and any $\Pi_0 \subseteq \Pi$, $M(\mathcal{LB}^{[\leq n]})|_{\Pi_0}$ has M-finite thickness.

We prepare some lemmas.

**Definition 6.9.** For a nonempty finite set $T \subseteq HB$ and an EFS $\Gamma$, let *TR-RED*$(T, \Gamma) = \{\Gamma' \subseteq \Gamma \mid \Gamma' \text{ is reduced w.r.t. } T\}$.

For any nonempty finite set $T \subseteq HB$ and any EFS $\Gamma$, if $T \subseteq M(\Gamma)$, then *TR-RED*$(T, \Gamma)$ is a nonempty finite set.

**Lemma 6.16** (Sato&Moriyama[39])**.** *For any $n \geq 1$ and any $\Pi_0 \subseteq \Pi$, the class $M(\mathcal{LB}^{[\leq n]})|_{\Pi_0}$ satisfies MEF-condition.*

*Proof.* Suppose that $M(\mathcal{LB}^{[\leq n]})|_{\Pi_0}$ does not satisfy MEF-condition. By Definition 5.2, there are a nonempty finite set $T \subseteq HB$ and an EFS $\Gamma_0 \in \mathcal{LB}^{[\leq n]}$ such that

(i) $T \subseteq M(\Gamma_0)|_{\Pi_0}$, and

(ii) for any EFS $\Gamma \in \mathcal{LB}^{[\leq n]}$, $M(\Gamma)|_{\Pi_0}$ is not a minimal concept of $T$ within $M(\mathcal{LB}^{[\leq n]})|_{\Pi_0}$ such that $M(\Gamma)|_{\Pi_0} \subseteq M(\Gamma_0)|_{\Pi_0}$.

Let $\Gamma_{j_1} \in$ *TR-RED*$(T, \Gamma_0)$ be an EFS, and define $\Gamma_{j_i}$'s ($i \geq 2$) inductively by the following stages:

Stage $i$ ($\geq 2$):

Since $M(\Gamma_{j_{i-1}})|_{\Pi_0} \subseteq M(\Gamma_0)|_{\Pi_0}$, we see by (ii) that $M(\Gamma_{j_{i-1}})|_{\Pi_0}$ is not a minimal concept of $T$ within $M(\mathcal{LB}^{[\leq n]})|_{\Pi_0}$. Therefore there is an EFS $\Gamma \in \mathcal{LB}^{[\leq n]}$ such that $T \subseteq M(\Gamma)|_{\Pi_0} \subsetneq M(\Gamma_{j_{i-1}})|_{\Pi_0}$. Let $\Gamma_{j_i} \in$ *TR-RED*$(T, \Gamma)$ be an EFS, and goto Stage $i+1$.

It is clear that $\Gamma_{j_1}, \Gamma_{j_2}, \cdots$ are all distinct and reduced w.r.t. $T$, which contradicts Lemma 6.6. ∎

**Lemma 6.17** (Sato&Moriyama[39]). *For any $n \geq 1$ and any $\Pi_0 \subseteq \Pi$, the class $M(\mathcal{LB}^{[\leq n]})|_{\Pi_0}$ satisfies MFF-condition.*

*Proof.* Let $T \subseteq HB$ be a nonempty finite set such that $T \subseteq M(\Gamma)|_{\Pi_0}$ for some EFS $\Gamma \in \mathcal{LB}^{[\leq n]}$. Then, by Lemma 6.16, there is an EFS $\Gamma' \in \mathcal{LB}^{[\leq n]}$ such that $M(\Gamma')|_{\Pi_0}$ is a minimal concept of $T$ within $M(\mathcal{LB}^{[\leq n]})|_{\Pi_0}$ with $M(\Gamma')|_{\Pi_0} \subseteq M(\Gamma)|_{\Pi_0}$. If $\Gamma'$ is not reduced w.r.t. $T$, then there is an EFS $\Gamma'' \in RED(T, \mathcal{LB}^{[\leq n]})$ such that $\Gamma'' \subsetneq \Gamma'$. However $M(\Gamma')|_{\Pi_0} = M(\Gamma'')|_{\Pi_0}$ holds, because $M(\Gamma')|_{\Pi_0}$ is a minimal concept of $T$ within $M(\mathcal{LB}^{[\leq n]})|_{\Pi_0}$. This means that $\sharp\{M(\Gamma)|_{\Pi_0} \mid M(\Gamma)|_{\Pi_0}$ is a minimal concept of $T$ within $M(\mathcal{LB}^{[\leq n]})|_{\Pi_0}\}$ is not greater than $\sharp RED(T, \mathcal{LB}^{[\leq n]})$, which is finite by Lemma 6.6. Thus $M(\mathcal{LB}^{[\leq n]})|_{\Pi_0}$ satisfies MFF-condition. ∎

**Theorem 6.18.** *For any $n \geq 1$, the class $\mathcal{LB}^{[\leq n]}$ is theoretical-term-freely and refutably minimally inferable from positive data.*

*Proof.* We define the function $\mathrm{econs}_p$ as follows: For a finite set $T \subseteq HB$,

$$\mathrm{econs}_p(T) = \begin{cases} 1, & \text{if the number of distinct predicate symbols} \\ & \qquad\qquad \text{appearing in } T \text{ is not greater than } n, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that this function $\mathrm{econs}_p$ agrees with Definition 4.3 and it is recursively computable.

Thus, by Theorem 6.7, Lemma 6.16, Lemma 6.17 and Corollary 5.11, we see that $\mathcal{LB}^{[\leq n]}$ is theoretical-term-freely and refutably minimally inferable from positive data. ∎

By the above Theorem 6.18, we have the following Corollary 6.19 and 6.20.

**Corollary 6.19.** *For any $n \geq 1$, the class $M(\mathcal{LB}^{[\leq n]})$ is refutably minimally inferable from positive data.*

Since $L(\mathcal{LB}^{[\leq n]})$ contains the universal set $U = \Sigma^+$ as its member, we have the following Corollary 6.20.

**Corollary 6.20.** *For any $n \geq 1$, the class $L(\mathcal{LB}^{[\leq n]})$ is strong-minimally inferable from positive data.*

## 6.4. Discussion

We have shown that for any $n \geq 1$, the classes definable by length-bounded EFS's with at most $n$ axioms are refutably inferable from complete data as well as (refutably or strong-) minimally inferable from positive data.

In 1990, Shinohara[46] introduced a more generalized framework called *monotonic formal systems*. A monotonic formal system is a triplet $(U, E, M)$, where $U$ is a universal set, $E$ is a set of expressions and $M$ is a semantic mapping from $2^E$ to $2^U$ such that for any sets $\Gamma_1, \Gamma_2$ of expressions, $\Gamma_1 \subseteq \Gamma_2$ implies $M(\Gamma_1) \subseteq M(\Gamma_2)$. For example, we can deal with the class of context-sensitive languages in the framework of a monotonic formal system, where $U = \Sigma^+$, $E$ is the set of all context-sensitive productions, and for a context-sensitive grammar $G$, $M(G)$ is the context-sensitive language accepted by $G$. Using this framework, Shinohara[46] showed that the classes definable by monotonic formal systems with *bounded finite thickness* are inferable in the limit from positive data. By this results, the classes definable by weakly reducing EFS's[52] (or max-length bounded EFS's in [25]) or linear prolog programs[42] (or weakly reducing logic programs in [7]) with at most $n$ axioms, and the class of languages definable by context-sensitive grammars with at most $n$ productions are shown to be inferable in the limit from positive data for any $n \geq 1$.

We can apply the technique in the proof of Theorem 6.14 to showing that the classes definable by monotonic formal systems that satisfy certain conditions are refutably inferable from complete data. Furthermore we can show that the classes definable by monotonic formal systems with bounded finite thickness are also (refutably or strong-) minimally inferable from positive data. By these results, the above various classes are also shown to be refutably inferable from complete data as well as (refutably or strong-) minimally inferable from positive data.

As a future work, I am very interested in searching for an algorithm which infers EFS's or prolog programs taking full advantage of structures of programs or clauses, like the model inference system due to Shapiro[41].

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