The Recurrence of Dynamic Fuzzy Systems

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Abstract: This paper analyses a recurrent behavior of dynamic fuzzy systems defined by fuzzy relations on a Euclidean space. By introducing a recurrence for crisp sets, we prove probability-theoretical properties for the fuzzy systems. In the contractive case in [5], the existence of the maximum recurrent set is proved. As another case, we introduce a monotonicity for fuzzy relations, which is extended from the linear structure in [11]. In the monotone case we prove the existence of the arcwise connected maximal recurrent sets.

Keyword: Recurrence; dynamic fuzzy systems; fuzzy relations; contraction; monotonicity; superharmonic property.

1. Introduction and notations

Limit theorems of a sequence of fuzzy sets defined successively by fuzzy relations are first studied by Bellman and Zadeh [1]. They considered a sequence of fuzzy numbers in a finite space and solved a fuzzy relational equation written in matrix form. Kurano et al. [5] and Yoshida et al. [11], under a contractive condition, studied the limiting behavior of fuzzy sets defined by the dynamic fuzzy system with a compact space. We, in [5], proved the existence and uniqueness of the solution for the fuzzy relational equation, and in [11], developed, under a linear structure, a potential theory of fuzzy relations on the positive orthant of a Euclidean space.

Our objective is to study maximal recurrence of the dynamic fuzzy systems defined by fuzzy relations on a Euclidean space. We introduce, for every level \( \alpha \in (0, 1] \), a recurrence for crisp sets and we call it \( \alpha \)-recurrence. In Section 2 we prove, on the \( \alpha \)-recurrent crisp sets, various probability-theoretical properties in the class of fuzzy sets satisfying a fuzzy relational inequality, which is a generalization of the fuzzy relational equations in [5] and which is also satisfied by optimal fuzzy goals in fuzzy dynamic programming of [1], [2], [10]. Further we establish the balayage theorem, which is well-known regarding Markov chains, for the dynamic fuzzy system. In Section 3 we introduce \( \alpha \)-recurrence and represent the union of all \( \alpha \)-recurrent sets by the fuzzy relation. In Section 4 we deal with the contractive case in [5]. We give an explicit solution of the fuzzy relational equation in [5] and we prove that the \( \alpha \)-cut of the solution is the maximum \( \alpha \)-recurrent set. In Section 5 we introduce a certain monotonicity for the fuzzy relation, which is a natural extension of one-dimensional fuzzy relations with the linear structure in [11]. Then we prove that at most countable maximal \( \alpha \)-recurrent sets exist and that each maximal \( \alpha \)-recurrent set is arcwise connected. In Section 6 numerical examples are given to illustrate our idea.
In the remainder of this section, we describe the notations for dynamic fuzzy systems defined by fuzzy relations on finite-dimensional Euclidean spaces and give some fundamental results for stopping times from Yoshida [10].

Let \( S \) be a metric space. We write a fuzzy set on \( S \) by its membership function \( \tilde{s} : S \rightarrow [0, 1] \) and an ordinary set \( A \subseteq S \) by its indicator function \( 1_A : S \rightarrow \{0, 1\} \). The \( \alpha \)-cut \( \tilde{s}_\alpha \) is defined by

\[
\tilde{s}_\alpha := \{ x \in S \mid \tilde{s}(x) \geq \alpha \} \quad (\alpha \in (0, 1]) \quad \text{and} \quad \tilde{s}_0 := \text{cl}\{ x \in S \mid \tilde{s}(x) > 0 \},
\]

where \( \text{cl} \) denotes the closure of a set. \( \mathcal{F}(S) \) denotes the set of all fuzzy sets \( \tilde{s} \) on \( S \) satisfying the following conditions (F.i) and (F.ii):

(F.i) \( \tilde{s}_\alpha \in \mathcal{E}(S) \) for \( \alpha \in [0, 1] \);

(F.ii) \( \bigcap_{\alpha' < \alpha} \tilde{s}_{\alpha'} = \tilde{s}_\alpha \) for \( \alpha \in (0, 1) \),

where

\[
\mathcal{E}(S) := \left\{ A \mid A = \bigcup_{n=0}^{\infty} C_n, C_n \text{ are closed subsets of } S \ (n = 0, 1, 2, \cdots) \right\}.
\]

We also define

\[
\mathcal{G}(S) := \{ \text{fuzzy sets } \tilde{s} \text{ on } S \mid \text{there exists } \{ \tilde{s}_n \}_{n \in \mathbb{N}} \subseteq \mathcal{F}(S) \text{ satisfying } \tilde{s} = \bigvee_{n \in \mathbb{N}} \tilde{s}_n \},
\]

where \( \mathbb{N} := \{0, 1, 2, 3, \cdots\} \) and for a sequence of fuzzy sets \( \{ \tilde{s}_n \}_{n \in \mathbb{N}} \) on \( S \) we define

\[
\bigwedge_{n \in \mathbb{N}} \tilde{s}_n(x) := \inf_{n \in \mathbb{N}} \tilde{s}_n(x) \quad \text{and} \quad \bigvee_{n \in \mathbb{N}} \tilde{s}_n(x) := \sup_{n \in \mathbb{N}} \tilde{s}_n(x) \quad \forall x \in S.
\]

Let a time space by \( \Omega := \prod_{n=0}^{\infty} E \) and we write a sample path by \( \omega = (\omega(0), \omega(1), \omega(2), \cdots) \in \Omega \). We define a map \( X_n(\omega) := \omega(n) \) and a shift \( \theta_n(\omega) := (\omega(n), \omega(n+1), \omega(n+2), \cdots) \) for \( n \in \mathbb{N} \) and \( \omega = (\omega(0), \omega(1), \omega(2), \cdots) \in \Omega \). We put \( \sigma \)-fields by \( \mathcal{M}_n := \sigma(X_0, X_1, \cdots, X_n) \) for \( n \in \mathbb{N} \) and \( \mathcal{M} := \sigma(\bigcup_{n \in \mathbb{N}} \mathcal{M}_n) \). Let \( \Delta \) be not a point of \( E \) and put \( E_\Delta := E \cup \{\Delta\} \). We can extend the state space \( E \) to \( E_\Delta \), setting \( \tilde{s}(\Delta) := 0 \) for \( \tilde{s} \in \mathcal{G}(E_\Delta) \) and \( X_\omega(\omega) := \Delta \) for \( \omega \in \Omega \) ([10, Section 2]). Let \( \tilde{q} \) be an upper semi-continuous binary relation on \( E \times E \) satisfying the following normality condition:

\[
\sup_{x \in E} \tilde{q}(x, y) = 1 \quad (y \in E) \quad \text{and} \quad \sup_{y \in E} \tilde{q}(x, y) = 1 \quad (x \in E).
\]

We call \( \tilde{q} \) a fuzzy relation. We define a fuzzy expectation : For an initial state \( x \in E \) and an \( \mathcal{M} \)-measurable fuzzy set \( h \in \mathcal{F}(\Omega) \),

\[
E_\omega(h) := \int_{\{\omega(\omega(0)) = x\}} h(\omega) \, d\tilde{P}(\omega),
\]

\[1\]It denotes the smallest \( \sigma \)-field on \( \Omega \) relative to which \( X_0, X_1, \cdots, X_n \) are measurable.

\[2\]It denotes the smallest \( \sigma \)-field generated by \( \bigcup_{n \in \mathbb{N}} \mathcal{M}_n \).
where \( \tilde{P} \) is the following possibility measure:

\[
\tilde{P}(\Lambda) := \sup_{\omega \in \Lambda} \bigwedge_{n \in \mathbb{N}} q(X_n\omega, X_{n+1}\omega) \quad \Lambda \in \mathcal{M}
\]

and \( \int d\tilde{P} \) denotes Sugeno integral (Sugeno [9]). Then the fuzzy expectation has the following property.

**Lemma 1.1** ([10, Section 3]). For an \( \mathcal{M} \)-measurable sequence \( \{h_n\}_{n \in \mathbb{N}} \subset \mathcal{G}(\Omega) \), it holds that

\[
\bigvee_{n \in \mathbb{N}} E_x(h_n) = E_x\left( \bigvee_{n \in \mathbb{N}} h_n \right) \quad x \in E.
\]

We need the first entry times (the first hitting times) of a set, which is adapted to the dynamic fuzzy system \( X := \{X_n\}_{n \in \mathbb{N}} \), in order to define a recurrence of sets in Section 3. We define

\[
\mathcal{E} := \{A \mid A \in \mathcal{E}(E) \text{ and } E \setminus A \in \mathcal{E}(E)\}
\]

and we call a map \( \tau : \Omega \mapsto \overline{\mathbb{N}} \) an \( \mathcal{E} \)-stopping time if

\[
\{\tau = n\} \in \mathcal{M}_n \cap \mathcal{E}(\Omega) \quad n \in \mathbb{N}.
\]

For example, a constant stopping time i.e. \( \tau = n_0 \) for some \( n_0 \in \mathbb{N} \), is an \( \mathcal{E} \)-stopping time. For \( A \in \mathcal{E} \) we put

\[
\tau_A(\omega) := \inf\{n \in \mathbb{N} \mid X_n(\omega) \in A\} \quad \omega \in \Omega;
\]

\[
\sigma_A(\omega) := \inf\{n \in \mathbb{N} \mid n \geq 1, X_n(\omega) \in A\} \quad \omega \in \Omega,
\]

where the infinums of the empty set are understood to be \(+\infty\). Then the first entry time \( \tau_A \) of \( A \) and the first hitting time \( \sigma_A \) of \( A \) are also \( \mathcal{E} \)-stopping times ([10, Lemma 1.5]).

Define a map \( P : \mathcal{G}(E) \mapsto \mathcal{G}(E) \) by

\[
P\tilde{s}(x) := E_x(\tilde{s}(X_1)) = \sup_{y \in E}\{q(x, y) \land \tilde{s}(y)\} \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E),
\]

where we write binary operations \( a \land b := \min\{a, b\} \) and \( a \lor b := \max\{a, b\} \) for \( a, b \in [0, 1] \).

We call \( P \) a fuzzy transition defined by the fuzzy relation \( \tilde{q} \). We also define \( n \)-steps fuzzy transitions \( P_n : \mathcal{G}(E) \mapsto \mathcal{G}(E) \), \( n \in \mathbb{N} \), by

\[
P_n\tilde{s} := E.(\tilde{s}(X_n)) = \sup_{y \in E}\{\tilde{q}^n(\cdot, y) \land \tilde{s}(y)\} \quad \text{for } \tilde{s} \in \mathcal{G}(E),
\]

where for \( n \in \mathbb{N} \)

\[
\tilde{q}^1(x, y) := \tilde{q}(x, y) \quad \text{and} \quad \tilde{q}^{n+1}(x, y) := \sup_{z \in E}\{\tilde{q}^n(x, z) \land \tilde{q}(z, y)\} \quad x, y \in E.
\]
Further for an $\mathcal{E}$-stopping time $\tau$, a fuzzy transition $P_\tau : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ is defined by

$$P_\tau \tilde{s} := E(\tilde{s}(X_\tau)) \quad \text{for } \tilde{s} \in \mathcal{G}(E),$$

where $X_\tau := X_n$ on $\{\tau = n\}, \ n \in \mathbb{N}$.

The fuzzy transition $\{P_n\}_{n \in \mathbb{N}}$ has the following property:

$$P_0 = I \quad \text{(identity),} \quad P_1 = P \quad \text{and} \quad P_{m+n} = P_m P_n \ (m, n \in \mathbb{N}).$$

Further it also has a semi-group property with respect to $\mathcal{E}$-stopping times.

**Lemma 1.2** ([10, Corollary 2.1]). It holds that

$$P_\sigma P_\tau = P_{\sigma + \tau \mathbb{1}} \text{ on } \mathcal{G}(E) \quad \text{for finite } \mathcal{E} \text{-stopping times } \sigma \text{ and } \tau.$$

## 2. Transitive closures and $P$-superharmonic fuzzy sets

We define a partial order $\geq$ on $\mathcal{G}(E)$: For $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$

$$\tilde{s} \geq \tilde{r} \iff \tilde{s}(x) \geq \tilde{r}(x) \quad x \in E.$$

**Definition** ([10, Section 4]). A fuzzy set $\tilde{s} \in \mathcal{G}(E)$ is called $P$-harmonic ($P$-superharmonic) provided that

$$\tilde{s} = P\tilde{s} \quad (\tilde{s} \geq P\tilde{s} \text{ resp.}).$$

Clearly a constant fuzzy set, $\tilde{s} = \beta$ for some $\beta \in [0, 1]$, is $P$-superharmonic. We represent the fuzzy set by $\beta$ simply.

In this section we investigate $P$-superharmonic property regarding fuzzy sets and we show the balayage theorem for $P$-superharmonic fuzzy sets. Using the results, we give a simple characterization for hitting possibilities of a set $A(\in \mathcal{E})$ by transitive closures. First we prove preliminary lemmas for $P$-superharmonic fuzzy sets, which are well-known property in the classical probability theory ([8]).

**Lemma 2.1.**

(i) If $\tilde{s}_1$ and $\tilde{s}_2$ are $P$-superharmonic, then $\tilde{s}_1 \wedge \tilde{s}_2$ is also $P$-superharmonic.

(ii) If $\{\tilde{s}_n\}_{n \in \mathbb{N}}$ is a sequence of $P$-harmonic ($P$-superharmonic) fuzzy sets, then $\bigvee_{n \in \mathbb{N}} \tilde{s}_n$ is also $P$-harmonic ($P$-superharmonic resp.).

**Proof.** (i) We can easily check $\tilde{s}_1 \wedge \tilde{s}_2 \in \mathcal{G}(E)$, using [10, Lemma 1.1]. Since the fuzzy transition $P$ preserves the order $\geq$ on $\mathcal{G}(E)$, we have

$$\tilde{s}_1 \geq P\tilde{s}_1 \geq P(\tilde{s}_1 \wedge \tilde{s}_2) \quad \text{and} \quad \tilde{s}_2 \geq P\tilde{s}_2 \geq P(\tilde{s}_1 \wedge \tilde{s}_2).$$
Therefore \( \hat{s}_1 \land \hat{s}_2 \) is \( P \)-superharmonic.

(ii) It is trivial that \( \bigvee_{n \in \mathbb{N}} \hat{s}_n \in \mathcal{G}(E) \). By Lemma 1.1,

\[
\bigvee_{n \in \mathbb{N}} \hat{s}_n \geq \bigvee_{n \in \mathbb{N}} P\hat{s}_n = P\left( \bigvee_{n \in \mathbb{N}} \hat{s}_n \right).
\]

Therefore \( \bigvee_{n \in \mathbb{N}} \hat{s}_n \) is \( P \)-superharmonic. The \( P \)-harmonic case is similar. \( \square \)

**Lemma 2.2.**

(i) If \( \hat{s} \) is \( P \)-superharmonic, then

\[
P_\sigma \hat{s} \geq P_\tau \hat{s} \quad \text{for all } \mathcal{E} \text{-stopping times } \sigma \text{ and } \tau \text{ such that } \sigma \leq \tau.
\]

(ii) If \( \hat{s} \) is \( P \)-harmonic, then

\[
P_\sigma \hat{s} = P_\tau \hat{s} \quad \text{for all } \mathcal{E} \text{-stopping times } \sigma \text{ and } \tau \text{ such that } \sigma \leq \tau < \infty.
\]

**Proof.** (i) We check this lemma along the proof of [8, Proposition 1.1.9]. Let \( \sigma \) and \( \tau \) be \( \mathcal{E} \)-stopping times such that \( \sigma \leq \tau \leq \sigma + 1 \). Let \( \Lambda_n := \{\sigma = n\} \cap \{\tau = n + 1\} \in \mathcal{M}_n \) and \( \Gamma_n := \{\sigma = \tau = n\} \in \mathcal{M}_n \) for \( n \in \mathbb{N} \). By [10, Theorem 2.1], for \( n \in \mathbb{N} \)

\[
E_x(\hat{s}(X_\sigma) \land 1_{\Lambda_n}) \geq E_x(P\hat{s}(X_n) \land 1_{\Lambda_n}) = E_x(\hat{s}(X_{n+1}) \land 1_{\Lambda_n}) = E_x(\hat{s}(X_\tau) \land 1_{\Lambda_n}) \quad x \in E.
\]

Using Lemma 1.1, we obtain

\[
P_\sigma \hat{s}(x) = \bigvee_{n \in \mathbb{N}} (E_x(\hat{s}(X_\sigma) \land 1_{\Lambda_n}) \lor E_x(\hat{s}(X_\sigma) \land 1_{\Gamma_n}))
\]

\[
\geq \bigvee_{n \in \mathbb{N}} (E_x(\hat{s}(X_\tau) \land 1_{\Lambda_n}) \lor E_x(\hat{s}(X_\tau) \land 1_{\Gamma_n}))
\]

\[
= P_\tau \hat{s}(x) \quad x \in E.
\]

More generally, for \( \mathcal{E} \)-stopping times \( \sigma \) and \( \tau \) such that \( \sigma \leq \tau \),

\[
P_\sigma \hat{s} \geq P_{(\sigma+1) \land \tau} \hat{s} \geq \cdots \geq P_{(\sigma+n) \land \tau} \hat{s} \geq \cdots \quad \text{for } n \in \mathbb{N}.
\]

(2.1)

Here, from [10, Lemma 1.1(i)], we have the following facts:

\[
\{\tau \leq \sigma + n < \infty\} = \bigcup_{l,m \in \mathbb{N} : l \leq m+n} \{\{\tau = l\} \cap \{\sigma = m\}\} \in \mathcal{E}(\Omega);
\]

\[
\{\tau < \infty\} = \bigcup_{l \in \mathbb{N}} \{\tau = l\} \in \mathcal{E}(\Omega);
\]

\[
\bigcup_{n \in \mathbb{N}} \{\tau \leq \sigma + n < \infty\} = \{\tau < \infty\}.
\]

By Lemma 1.1 and (2.1),

\[
P_\sigma \hat{s}(x) \geq \bigvee_{n \in \mathbb{N}} P_{(\sigma+n) \land \tau} \hat{s}(x) \geq \bigvee_{n \in \mathbb{N}} E_x(\hat{s}(X_\tau) \land 1_{\{\tau \leq \sigma + n < \infty\}}) = E_x(\hat{s}(X_\tau) \land 1_{\{\tau < \infty\}}) = P_\tau \hat{s}(x)
\]

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for \( x \in E \). Therefore we get (i). We can check (ii) similarly. \( \Box \)

We show the balayage theorem for the dynamic fuzzy system \( X \). The theorem plays one of important roles to analyse recurrence for the fuzzy relation \( \hat{q} \) in Section 3.

**Theorem 2.1.** Let \( \tilde{s} \) be \( P \)-superharmonic and let a set \( A \in \mathcal{E} \). Then \( P_{r_A} \tilde{s} \) is the smallest \( P \)-superharmonic fuzzy set which dominates \( \tilde{s} \wedge 1_A \).

**Proof.** We check this theorem along the proof of [8, Theorem II-2.1] for the classical Markov chain. It is trivial that \( P_{r_A} \tilde{s} = \tilde{s} \) on \( A \). \( P_{r_A} \tilde{s} \) is \( P \)-superharmonic since \( PP_{r_A} \tilde{s} = P_{r_A} \tilde{s} \leq P_{r_A} \tilde{s} \) by Lemmas 1.2 and 2.2(i). Therefore \( P_{r_A} \tilde{s} \) is \( P \)-superharmonic and dominates \( \tilde{s} \wedge 1_A \). Further let \( \tilde{r} \) be \( P \)-superharmonic such that \( \tilde{r} \geq \tilde{s} \wedge 1_A \). Then

\[
\hat{r}(x) \geq P_{r_A} \hat{r}(x) = E_x(\hat{r}(X_{r_A}) \wedge 1_{\{r_A < \infty\}}) \geq E_x(\hat{s}(X_{r_A}) \wedge 1_{\{r_A < \infty\}}) = P_{r_A} \hat{s}(x) \quad x \in E.
\]

Thus \( P_{r_A} \tilde{s} \) has the desired property and so we get this theorem. \( \Box \)

We define an operator \( G := \bigvee_{n \in \mathbb{N}} P_n \) on \( \mathcal{G}(E) \). Then we note that

\[
PG1_{\{y\}}(x) = \bigvee_{n \geq 1} P_n1_{\{y\}}(x) = \sup_{n \geq 1} \tilde{q}^n(x, y) \quad x, y \in E.
\]

This is called a transitive closure ([3, Section 3.3]). In this paper we also call \( PG \) a transitive closure. Now we need to investigate the operator \( G \) in order to analyse the transitive closure \( PG := \bigvee_{n \geq 1} P_n \). We have the following properties regarding \( G \).

**Lemma 2.3** ([10, Lemma 4.1(ii)]). Let \( \hat{s} \in \mathcal{G}(E) \). Then :

(i) It holds that

\[
G\hat{s} = \hat{s} \vee P(G\hat{s});
\]

(ii) \( G\hat{s} \) is the smallest \( P \)-superharmonic dominating \( \hat{s} \).

**Lemma 2.4.** Let \( \hat{s} \in \mathcal{G}(E) \). Then \( \hat{s} \) is \( P \)-superharmonic if and only if

\[
\hat{s} = G\hat{s}. \tag{2.2}
\]

**Proof.** Let \( \hat{s} \) be \( P \)-superharmonic. Then

\[
\hat{s} = \hat{s} \vee P\hat{s} \vee P_2\hat{s} \vee \cdots \vee P_n\hat{s} \quad \text{for all } n \in \mathbb{N}.
\]

So we obtain (2.2). The converse proof is trivial. \( \Box \)

For \( A \in \mathcal{E}(E) \) we introduce an operator \( I_A : \mathcal{G}(E) \mapsto \mathcal{G}(E) \) by

\[
I_A\hat{s} := \hat{s} \wedge 1_A \quad \hat{s} \in \mathcal{G}(E).
\]
We define a sequence of hitting times \( \{\sigma^n_A\}_{n \in \mathbb{N}} \) of a set \( A \in \mathcal{E} \) by

\[
\sigma^n_A := \begin{cases} 
0 & \text{if } n = 0 \\
\sigma^{n-1}_A + \sigma_A \circ \theta_{\sigma^{n-1}_A} & \text{if } n \geq 1.
\end{cases}
\]

Then \( \sigma^n_A \) means the first time to hit \( A \) after time \( \sigma^{n-1}_A \) ([8]). We investigate an entry possibility, \( P_{\tau_A} \), of \( A \), and we give a simple and interesting characterization of a possibility, \( P_{\sigma^n_A} \), to hit \( A \) first \( n \) times.

**Proposition 2.1.** Let \( A \in \mathcal{E} \). Then:

(i) \( P_{\tau_A} \hat{s} = GIA \hat{s} \) for \( P \)-superharmonic \( \hat{s} \);

(ii) \( P_{\sigma^n_A} = (PGIA)^n \hat{s} \) for \( P \)-superharmonic \( \hat{s} \) and \( n \in \mathbb{N} \).

**Proof.** (i) From Theorem 2.1 and Lemma 2.3(ii) we obtain

\[ P_{\tau_A} \hat{s} = G(\hat{s} \wedge 1_A) = GIA \hat{s}. \]

(ii) We prove the equality by induction on \( n \in \mathbb{N} \). It is trivial when \( n = 0 \). From (i), \( P_{\sigma_A} \hat{s} = PP_{\tau_A} \hat{s} = PGIA \hat{s} \). So (ii) also holds for \( n = 1 \). Next for every \( n \in \mathbb{N} \), \( (PGIA)^n \hat{s} \) is \( P \)-superharmonic since \( GIA(PGIA)^n \hat{s} \) is \( P \)-superharmonic by Lemma 2.3(ii). Therefore \( (PGIA)^n \hat{s} \) is \( P \)-superharmonic for all \( n \in \mathbb{N} \). Let \( n \in \mathbb{N} \). We suppose that (ii) holds for \( n \). From (i) and the fact that \( (PGIA)^n \hat{s} \) is \( P \)-superharmonic,

\[
P_{\sigma^{n+1}_A} \hat{s} = P_{\sigma_A} P_{\sigma^n_A} \hat{s} = PP_{\tau_A} (PGIA)^n \hat{s} = PGIA(PGIA)^n \hat{s} = (PGIA)^{n+1} \hat{s}.
\]

Thus we obtain (ii) inductively. \( \square \)

3. \( \alpha \)-recurrent sets

**Definition.** Let \( \alpha \in (0, 1] \). A set \( A \in \mathcal{E}(E) \) is called \( \alpha \)-recurrent provided:

(a) \( A \) is non-empty;

(b) \( P_{\theta^A} \geq \alpha \) on \( A \) for all \( n \in \mathbb{N} \) and all non-empty \( B \in \mathcal{E} \) satisfying \( B \subseteq A \).

The \( \alpha \)-recurrence of a set \( A \) means that a possibility to transit infinite times from any point of \( A \) to any point of \( A \) is greater than \( \alpha \).

**Lemma 3.1.** Let \( \beta \in [0, 1] \) be a constant fuzzy set. It holds that

\[
G(\hat{s} \wedge \beta) = G \hat{s} \wedge \beta \quad \text{and} \quad PG(\hat{s} \wedge \beta) = PG \hat{s} \wedge \beta \quad \text{for } \hat{s} \in \mathcal{G}(E).
\]

Especially,

\[
GIA(\beta) = G1_A \wedge \beta \quad \text{and} \quad PGI_A(\beta) = PG1_A \wedge \beta \quad \text{for } A \in \mathcal{E}(E).
\]
**Proof.** By induction we show

\[ P^n(\tilde{s} \wedge \beta) = P^n \tilde{s} \wedge \beta \quad \text{for } \tilde{s} \in \mathcal{G}(E) \text{ and } n \in \mathbb{N}. \quad (3.3) \]

First (3.3) holds clearly when \( n = 0 \). Next we have (3.3) for \( n = 1 \) since

\[ P(\tilde{s} \wedge \beta)(x) = \sup_{y \in E}(\tilde{q}(x, y) \wedge \tilde{s}(y) \wedge \beta) = \left( \sup_{y \in E}(\tilde{q}(x, y) \wedge \tilde{s}(y)) \right) \wedge \beta = P\tilde{s}(x) \wedge \beta \quad x \in E. \]

Further let \( n \in \mathbb{N} \). Assuming that (3.3) holds for \( n \), we have

\[ P^{n+1}(\tilde{s} \wedge \beta) = PP^n(\tilde{s} \wedge \beta) = P(P^n \tilde{s} \wedge \beta) = P^{n+1} \tilde{s} \wedge \beta. \]

Thus (3.3) holds for all \( n \in \mathbb{N} \). Therefore we get (3.1). We also obtain (3.2), taking \( \tilde{s} = 1_A \) \((A \in \mathcal{E}(E))\) in (3.1). \(\square\)

We give simple necessary and sufficient criteria for \( \alpha \)-recurrence by the transitive closure \( PG \).

**Proposition 3.1.** Let \( \alpha \in (0,1] \) and let non-empty \( A \in \mathcal{E}(E) \). Then the following statements are equivalent:

(i) \( A \) is \( \alpha \)-recurrent;

(ii) \( PG_{1B} \geq \alpha \wedge 1_A \) for non-empty \( B \in \mathcal{E}(E) \) satisfying \( B \subseteq A \);

(iii) \( PG_{1\{y\}} \geq \alpha \wedge 1_A \) for \( y \in A \).

**Proof.** First we check

\[ \{y\} \in \mathcal{E} \quad \text{for } y \in E. \quad (3.4) \]

Let \( y \in E \). Then \( \{y\} \subset \mathcal{E}(E) \). Put \( B_m(y) := \{ z \in E \mid d(y, z) \geq 1/m \} \) for \( m = 1, 2, \ldots \), where \( d \) denotes a metric on \( E \). From [10, Lemma 1.1], \( E \setminus \{y\} = \bigcup_{m=1}^\infty B_m(y) \in \mathcal{E}(E) \). Therefore we obtain (3.4). Next we prove the equivalences of (i) \(-\) (iii).

(ii) \(\implies\) (i) : Let non-empty \( B \in \mathcal{E} \) satisfying \( B \subseteq A \). By induction we show

\[ (PGIB)^n \geq \alpha \wedge 1_A \quad \text{for } n \in \mathbb{N}. \quad (3.5) \]

Inequality (3.5) is trivial for \( n = 1 \). We assume that (3.5) holds for some \( n \in \mathbb{N} \). From Lemma 3.1,

\[ (PGIB)^{n+1} = (PGIB)^n(PGIB) \geq (PGIB)^n(\alpha \wedge 1_A) = (PGIB)^n(\alpha) = (PGIB)^n \wedge \alpha = \alpha \wedge 1_A. \]

So (3.5) holds for all \( n \in \mathbb{N} \). Therefore we obtain (i), using Proposition 2.1(ii).
(iii) \implies (ii) : Let non-empty \( B \in \mathcal{E}(E) \) satisfying \( B \subseteq A \) and let \( y \in B \). Then
\[
P G 1_B \geq P G 1_{(y)} \geq \alpha \wedge 1_A.
\]
Therefore we obtain (ii).

(i) \implies (iii) : It is trivial from Proposition 2.1(ii).
Thus we complete the proof. \( \square \)

We gives, by the fuzzy relation \( \tilde{q} \), a representation of the union of all \( \alpha \)-recurrent sets.

**Theorem 3.1.** It holds that
\[
\bigcup_{A \in \mathcal{E}(E) : \alpha \text{-recurrent sets}} A = \left\{ x \in E \mid \sup_{n \geq 1} q^n(x, x) \geq \alpha \right\} \text{ for } \alpha \in (0, 1].
\]

**Proof.** Let \( A \in \mathcal{E}(E) \) be \( \alpha \)-recurrent. From Proposition 3.1, for \( x \in A \)
\[
P G 1_{(x)} \geq \alpha \wedge 1_A \geq \alpha \wedge 1_{(x)}.
\]
Therefore
\[
A \subseteq \{ x \in E \mid P G 1_{(x)} \geq \alpha \wedge 1_{(x)} \} = \left\{ x \in E \mid \sup_{n \geq 1} q^n(x, x) \geq \alpha \right\}.
\]
Conversely let \( x \in E \) satisfy \( \sup_{n \geq 1} q^n(x, x) \geq \alpha \). Then \( P G 1_{(x)} \geq \alpha \wedge 1_{(x)} \). From Proposition 3.1, \{x\} is \( \alpha \)-recurrent. Therefore
\[
\{x\} \subseteq \bigcup_{A \in \mathcal{E}(E) : \alpha \text{-recurrent sets}} A.
\]
Thus we obtain this theorem. \( \square \)

4. **The contractive case**

In this section we consider the contractive case in [5] and we give the maximum \( \alpha \)-recurrent set for the dynamic fuzzy system \( X \).

Let \( E_c \) be a compact subset of \( E \). We deal with a dynamic fuzzy system restricted on the compact space \( E_c \) according to [5]. Let \( \mathcal{C}(E_c) \) be the set of all closed subsets of \( E_c \) and let \( \rho \) be the Hausdorff metric on \( \mathcal{C}(E_c) \). Let \( \mathcal{F}^0(E_c) \) be the set of all fuzzy sets \( \tilde{s} \) on \( E_c \) which are upper semi-continuous and satisfy \( \sup_{x \in E_c} \tilde{s}(x) = 1 \). Then we note \( \mathcal{F}^0(E_c) \subseteq \mathcal{F}(E_c) \). Let \( \tilde{p}_0 \in \mathcal{F}^0(E_c) \) be a fuzzy set. Define a sequence of fuzzy sets \( \{\tilde{p}_n\}_{n=0}^\infty \) by
\[
\tilde{p}_{n+1}(y) = \sup_{x \in E_c} \{ \tilde{p}_n(x) \wedge \tilde{q}(x, y) \} \quad y \in E_c \text{ for } n \geq 0.
\]
(4.1)
The fuzzy set \( \tilde{p}_0 \), in [5], is called an initial fuzzy state and the sequence \( \{\tilde{p}_n\}_{n=0}^\infty \) is called a sequence of fuzzy states. The fuzzy relation \( \tilde{q} \) is also restricted on \( E_c \times E_c \) and it
is assumed to be continuous on $E_c \times E_c$ and satisfy $\tilde{q}(x, \cdot) \in \mathcal{F}^0(E)$. Define a map $\tilde{r}_\alpha : C(E_c) \to C(E_c)$ $(\alpha \in (0, 1))$ by

$$
\tilde{r}_\alpha(D) := \begin{cases} 
\{y \mid \tilde{q}(x, y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0, \ D \in C(E_c), \ D \neq \emptyset, \\
\text{cl}\{y \mid \tilde{q}(x, y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0, \ D \in C(E_c), \ D \neq \emptyset, \\
E_c & \text{for } 0 \leq \alpha \leq 1, \ D = \emptyset.
\end{cases}
$$

In the sequel we assume the following contractive property for the fuzzy relation $\tilde{q}$ (see [5, Section 2]): There exists a real number $\beta \in (0, 1)$ satisfying

$$
\rho(\tilde{r}_\alpha(A), \tilde{r}_\alpha(B)) \leq \beta \rho(A, B) \quad \text{for all } A, B \in C(E_c) \text{ and all } \alpha \in (0, 1).
$$

Then we have proved a convergence of the sequence of fuzzy states $\{\tilde{p}_n\}_{n=0}^\infty$ defined by (4.1).

Lemma 4.1 ([5, Theorem 1]).

(i) There exists a unique fuzzy state $\tilde{p} \in \mathcal{F}^0(E_c)$ satisfying

$$
\tilde{p}(y) = \max_{x \in E_c}\{\tilde{p}(x) \land \tilde{q}(x, y)\} \quad y \in E_c. \tag{4.2}
$$

(ii) The sequence $\{\tilde{p}_n\}_{n=0}^\infty$ converges to a unique solution $\tilde{p} \in \mathcal{F}^0(E_c)$ of (4.2) independently of the initial fuzzy state $\tilde{p}_0$. Namely,

$$
\lim_{n \to \infty} \tilde{p}_n = \tilde{p},
$$

where the convergence means $\sup_{\alpha \in [0, 1]} \rho(\tilde{p}_{n, \alpha}, \tilde{p}_\alpha) \to 0$ $(n \to \infty)$ provided $\tilde{p}_{n, \alpha}, \tilde{p}_\alpha$ are $\alpha$-cuts $(\alpha \in [0, 1])$ for the fuzzy states $\tilde{p}_n, \tilde{p}$ respectively.

First we give a solution of (4.2).

Proposition 4.1. The $\alpha$-cut of the solution $\tilde{p}$ of (4.2) is

$$
\tilde{p}_\alpha = \text{cl}\left\{x \in E_c \mid \sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha \right\} \quad \text{for } \alpha \in (0, 1).
$$

Proof. First we prove

$$
\sup_{n \geq 1} \tilde{q}^n(x, x) \leq \tilde{p}(x) \quad x \in E_c. \tag{4.3}
$$

Let $\alpha \in (0, 1]$ and $x \in E_c$ satisfy $\sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha$. For each $\alpha' < \alpha$ there exists $n' \geq 1$ such that

$$
x \in \tilde{r}_{\alpha'}(\{x\}),
$$

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where \( \tilde{r}_{\alpha'}^n := \tilde{r}_{\alpha'} \) and \( \tilde{p}_{\alpha'}^{n+1} := \tilde{r}_{\alpha'}(\tilde{p}_{\alpha'}^n) \) for \( n \geq 1 \). Then, by induction, we shall check

\[
x \in \tilde{r}_{\alpha'}^{nm}(\{x\}) \quad \text{for all } m \geq 1.
\] (4.4)

(4.4) is trivial for \( m = 1 \). We assume that (4.4) holds for \( m = 1, 2, \cdots, l \). Then, by induction, we shall check

\[
x \in \tilde{r}_{\alpha'}^{nl}(\{x\}) \subseteq \bigcup_{y \in \tilde{r}_{\alpha'}^{nl}(\{x\})} \tilde{r}_{\alpha'}^{nl}(\{y\}) = \tilde{r}_{\alpha'}^{nl(l+1)}(\{x\}).
\]

Therefore we obtain (4.4) inductively. On the other hand, considering a case of \( \tilde{p}_0 := 1_{\{x\}} \) \((z \in E_c)\) in (4.1), from Lemma 4.1(i) and [5, Lemma 1],

\[
\lim_{n \to \infty} \rho(\tilde{p}_{\alpha'}^{n}(\{z\}), \tilde{p}_{\alpha'}^{n}) = 0 \quad \text{for all } z \in E_c.
\] (4.5)

From (4.4) and (4.5), we obtain \( x \in \tilde{p}_{\alpha'} \), for \( \alpha' < \alpha \). Therefore we get \( x \in \tilde{p}_{\alpha} \), using Lemma 4.1(i) and [5, Lemma 3(i,b)]. Thus we get (4.3).

Let \( x \in E_c \). Next, considering a case of \( \tilde{p}_0 := 1_{\{x\}} \) in (4.1), we can easily check

\[
\tilde{q}^n(x, x) = \tilde{p}_n(x) \quad \text{for all } n \geq 1.
\]

Together with (4.3), we obtain

\[
\tilde{p}_n(x) \leq \sup_{n \geq 1} \tilde{q}^n(x, x) \leq \tilde{p}(x) \quad x \in E_c \quad \text{for all } m \geq 1.
\]

By Lemma 4.1(ii), we get

\[
\tilde{p}_{\alpha} = \text{cl} \left\{ x \in E_c \mid \sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha \right\} \quad \text{for all } \alpha \in (0, 1].
\]

Therefore the proof is completed. \( \Box \)

Finally we prove that the closure of the union of all \( \alpha \)-recurrent sets equals to \( \alpha \)-cuts of the limit fuzzy state \( \tilde{p} \). Now we compare (1.1) and (4.1). Using the inverse fuzzy relation \( \hat{q} \) ([3, Section 3.2]):

\[
\hat{q}(x, y) := \tilde{q}(y, x) \quad x, y \in E_c,
\]

we find that (4.1) follows

\[
\tilde{p}_{n+1}(x) = \sup_{x \in E_c} \{ \hat{q}(x, y) \land \tilde{p}_n(y) \} \quad x \in E_c \quad \text{for } n \geq 0.
\]

Therefore we can apply the results in Sections 1 – 3 to a dynamic fuzzy system defined by the inverse fuzzy relation \( \hat{q} \).

\textbf{Theorem 4.1.}

\[
\tilde{p}_{\alpha} = \text{cl} \left\{ x \in E_c \mid \sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha \right\} = \text{cl} \left( \bigcup_{A \in \mathcal{E}(E) : \alpha \text{-recurrent sets}} A \right) \quad \text{for } \alpha \in (0, 1].
\] (4.6)

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Further it is the maximum $\alpha$-recurrent set for $X$. 

**Proof.** From the definition of the inverse fuzzy relation $\hat{q}$, we can easily check 

$$\hat{q}^n(x, x) = \hat{q}^n(x, x) \quad x \in E_c, \ n \geq 1,$$

where, in the same way as $\{\hat{q}^n\}_{n \geq 1}$ of Section 1, we define 

$$\hat{q}^1(x, y) := \hat{q}(x, y) \quad \text{and} \quad \hat{q}^{n+1}(x, y) := \sup_{z \in E_c} \{\hat{q}^n(x, z) \wedge \hat{q}(z, y)\} \quad x, y \in E_c, \ n \geq 1.$$

From Proposition 4.1, 

$$\bar{p}_\alpha = \text{cl} \left\{ x \in E_c \mid \sup_{n \geq 1} \hat{q}^n(x, x) \geq \alpha \right\} = \text{cl} \left\{ x \in E_c \mid \sup_{n \geq 1} \hat{q}^n(x, x) \geq \alpha \right\} \quad \text{for} \ \alpha \in (0, 1].$$

This equality means that the closure of the union of all $\alpha$-recurrent sets for the fuzzy relation $\hat{q}$ equals to one for the inverse fuzzy relation $\hat{q}$, considering Theorem 3.2 for the dynamic fuzzy systems defined by the fuzzy relations $\hat{q}$ and $\hat{q}$. Therefore we obtain (4.6). Finally (4.5) means that $\bar{p}_\alpha$ is the maximum $\alpha$-recurrent. \[\square\]

5. The monotone case

In general, there does not always exist the maximum $\alpha$-recurrent set for the dynamic fuzzy system $X$, however we can consider the existence of the maximal $\alpha$-recurrent sets. In this section we deal with a case when the transition fuzzy relation $\hat{q}$ has a certain monotone property (see Section 6 for numerical examples). Then we prove the existence of at most countable arcwise connected maximal $\alpha$-recurrent sets.

In this section we use the notations in Sections 1 – 3. Further we introduce the following notations of $\alpha$-cuts ([5, Section 2]):

$$\hat{q}_\alpha(x) := \{y \in E \mid \hat{q}(x, y) \geq \alpha\} \quad \text{for} \ x \in E \ \text{and} \ \alpha \in (0, 1];$$

$$\hat{q}_\alpha(A) := \bigcup_{x \in A} \hat{q}_\alpha(x) \quad \text{for} \ A \in \mathcal{E}(E) \ \text{and} \ \alpha \in (0, 1];$$

$$\hat{q}_0(A) := \text{cl} \left( \bigcup_{\alpha > 0} \hat{q}_\alpha(A) \right) \quad \text{for} \ A \in \mathcal{E}(E).$$

For $\alpha \in (0, 1]$ and $x \in E$ we define a sequence $\{\hat{q}^m_\alpha(x)\}_{m=1,2,\ldots}$:

$$\hat{q}^1_\alpha(x) := \hat{q}_\alpha(x); \quad \text{and} \quad \hat{q}^{m+1}_\alpha(x) := \hat{q}_\alpha(\hat{q}^m_\alpha(x)) \quad \text{for} \ m = 1, 2, \ldots.$$

We also need some elementary notations in the finite dimensional Euclidean space $E$: $x + y$ denotes the sum of $x, y \in E$ and $\gamma x$ denotes the product of a real number $\gamma$ and $x \in E$. We put $A + B := \{x + y \mid x \in A, y \in B\}$ for $A, B \in \mathcal{E}(E)$. Then we define a half line on $E$ by

$$l(x, y) := \{\gamma(y - x) \mid \text{real numbers} \ \gamma \geq 0\} \quad \text{for} \ x, y \in E.$$
Definition. We call a transition fuzzy relation $\tilde{q}$ unimodal provided that $\tilde{q}_a(x)$ are bounded closed convex subsets of $E$ for all $a \in (0, 1]$ and all $x \in E$.

Definition. We call a unimodal transition fuzzy relation $\tilde{q}$ monotone provided that

$$\tilde{q}_a(y) \subseteq \tilde{q}_a(x) + l(x, y)$$

for all $a \in (0, 1]$ and all $x, y \in E$.

From now on we deal with only unimodal fuzzy relations $\tilde{q}$, which is monotone and continuous on $E \times E$. The monotonicity is a natural extension of one-dimensional models with the linear structure in [11] and means that the fuzzy relations $\tilde{q}$ keeps the partial order of fuzzy numbers (see (C.iii') in Section 6).

Lemma 5.1. Assume that $\tilde{q}$ is monotone. Let $a \in (0, 1]$. If $x \in E$ satisfies $x \in \bigcup_{m=1}^{\infty} \tilde{q}_a^m(x)$, then $x \in \tilde{q}_a(x)$.

Proof. Let $x \in E$ satisfy $x \notin \tilde{q}_a(x)$. We put

$$C_+: = \bigcup_{y \in \tilde{q}_a(x)} \{\tilde{q}_a(x) + l(x, y)\}.$$ 

Since $\tilde{q}$ is monotone, we can easily check $C_+$ is convex and we have

$$\tilde{q}_a^2(x) = \bigcup_{y \in \tilde{q}_a(x)} \tilde{q}_a(y) \subseteq C_+.$$ 

(5.1)

Here we show

$$\bigcup_{y \in C_+} \tilde{q}_a(y) \subseteq C_+.$$ 

(5.2)

Let $z \in \bigcup_{y \in C_+} \tilde{q}_a(y)$. Since $\tilde{q}$ is monotone, there exists $y_1 \in C_+$ such that $z \subseteq \tilde{q}_a(x) + l(x, y_1)$. So there exists $y_2 \in \tilde{q}_a(x)$ such that $y_1 \subseteq \tilde{q}_a(x) + l(x, y_2)$. From the definitions, there exist $z_1 \in \tilde{q}_a(x)$ and a real number $\gamma_1 \geq 0$ such that

$$z = z_1 + \gamma_1(y_1 - x)$$ 

(5.3)

and there exist $z_2 \in \tilde{q}_a(x)$ and a real number $\gamma_2 \geq 0$ such that

$$y_1 = z_2 + \gamma_2(y_2 - x).$$ 

(5.4)

Since $\tilde{q}$ is unimodal, from (5.3) and (5.4) we obtain that

$$z = z_1 + \gamma_1(z_2 + \gamma_2(y_2 - x) - x) = z_1 + (\gamma_1 + \gamma_1 \gamma_2)\left(\frac{\gamma_1 \gamma_2 y_2}{\gamma_1 + \gamma_1 \gamma_2} - x\right) \in C_+ \text{ if } \gamma_1 > 0$$

and that $z = z_1 \in C_+$ if $\gamma_1 = 0$. Thus we get (5.2). Therefore from (5.1) and (5.2)

$$\tilde{q}_a^3(x) = \bigcup_{y \in \tilde{q}_a^2(x)} \tilde{q}_a(y) \subseteq \bigcup_{y \in C_+} \tilde{q}_a(y) \subseteq C_+.$$
Thus using (5.2) inductively, we obtain
\[
\bigcup_{m=1}^{\infty} \tilde{q}_m(x) \subseteq C_+.
\] (5.5)

On the other hand we show \(x \notin C_+\). If \(x \in C_+\), then there exist \(z, y \in \tilde{q}_\alpha(x)\) and a real number \(\gamma \geq 0\) such that \(x = z + \gamma(y - x)\). Therefore
\[
x = \frac{z + \gamma y}{1 + \gamma} \in \tilde{q}_\alpha(x).
\]
This contradicts the assumption on \(x\) at the beginning of this proof. Therefore we get \(x \notin C_+\). Together (5.5), this implies
\[
x \notin \bigcup_{m=1}^{\infty} \tilde{q}_m(x).
\]
Thus we obtain this lemma. \(\Box\)

When \(\tilde{q}\) is monotone, Theorem 3.1 is reduced to the following representation (5.6), which is easy to calculate.

**Theorem 5.1.** Assume that \(\tilde{q}\) is monotone. Let \(\alpha \in (0, 1]\). Then
\[
\bigcup_{A \in \mathcal{E}(E) : \alpha \text{-recurrent sets}} A = \{x \in E \mid \tilde{q}(x, x) \geq \alpha\}.
\] (5.6)

**Proof.** Let \(x_1 \in \{x \in E \mid \tilde{q}(x, x) \geq \alpha\}\). Then \(P_{1_{\{x_1\}}}(x_1) \geq \alpha\). So \(P_{G1_{\{x_1\}}}(x_1) \geq P_{1_{\{x_1\}}} \geq \alpha_{1_{\{x_1\}}}\). Therefore \(\{x_1\}\) is \(\alpha\)-recurrent and so we obtain
\[
\{x \in E \mid \tilde{q}(x, x) \geq \alpha\} \subseteq \bigcup_{A \in \mathcal{E}(E) : \alpha \text{-recurrent sets}} A.
\]
Conversely let \(A(\in \mathcal{E}(E))\) be \(\alpha\)-recurrent. Let \(x_1 \in A\). From Proposition 3.1,

\[
P_{G1_{\{x_1\}}} \geq \alpha_{1_A} \geq \alpha_{1_{\{x_1\}}}.
\]
Therefore
\[
x_1 \in \bigcup_{m=1}^{\infty} \tilde{q}_m(x_1) \quad \text{for all } \alpha' < \alpha.
\]
From Lemma 3.2 we obtain
\[
x_1 \in \tilde{q}_{\alpha'}(x_1) \quad \text{for all } \alpha' < \alpha.
\]
Namely we get \(\tilde{q}(x_1, x_1) \geq \alpha'\) for all \(\alpha' < \alpha\). So we get \(\tilde{q}(x_1, x_1) \geq \alpha\). Therefore \(A \subseteq \{x \in E \mid \tilde{q}(x, x) \geq \alpha\}\). Thus we establish this theorem. \(\Box\)

We need the following assumption on \(\tilde{q}\), which is technical but not so strong. It means that the function \(\tilde{q}\) does not have flat areas as a curved surface (Section 6).
Assumption (A). For $\alpha \in (0, 1)$,
\[
\text{int} \{ (x, y) \in E \times E \mid \tilde{q}(x, y) \geq \alpha \} = \{ (x, y) \in E \times E \mid \tilde{q}(x, y) > \alpha \},
\]
where int denotes the interior of a set.

Since $\tilde{q}$ is continuous, \( \{ x \in E \mid \tilde{q}(x, x) \geq \alpha \} \) is represented by a disjoint sum of at most countable arcwise connected closed sets ([4]), we represent it by
\[
\{ x \in E \mid \tilde{q}(x, x) \geq \alpha \} = \bigcup_{n \in \mathbb{N}(\alpha)} F_{\alpha, n} \quad \text{for} \ \alpha \in (0, 1),
\]
where $F_{\alpha, n}$ are arcwise connected closed subsets of $E$ and we put the index set $\mathbb{N}(\alpha) (\subset \mathbb{N})$.

**Lemma 5.2.** We suppose Assumption (A). Let $\alpha \in (0, 1)$ and $n \in \mathbb{N}(\alpha)$. Then $F_{\alpha, n}$ is $\alpha$-recurrent.

**Proof.** We write the interior of $F_{\alpha, n}$ by $F_{\alpha, n}^o$. First we prove that $F_{\alpha, n}^o$ is $\alpha$-recurrent. Let $x_0 \in F_{\alpha, n}^o$. Let $c(x_0)$ be an arc in $F_{\alpha, n}^o$, which is connected from $x_0$ to a boundary point of $F_{\alpha, n}$. We consider along the arc $c(x_0)$. Then we show
\[
c(x_0) \cap F_{\alpha, n}^o \subset \bigcup_{m \geq 1} \tilde{q}_{\alpha}^m(x_0). \tag{5.7}
\]
Let $x_1$ be the first point arriving at the boundary of $\tilde{q}_\alpha(x_0)$ along $c(x_0)$. If either there do not exist such points or $x_1$ is a boundary point of $F_{\alpha, n}$, then $c(x_0) \subset \tilde{q}_\alpha(x_0)$ and clearly (5.7) holds. Therefore it is sufficient to consider a case of $x_1 \in F_{\alpha, n}$. Since $x_0 \in F_{\alpha, n}^o$, we have $x_0 \in (\tilde{q}_\alpha(x_0))^o$ and $d(x_0, x_1) > 0$ from Assumption (A). From $x_1 \in F_{\alpha, n}^o \cap c(x_0)$, we also define $x_2$ the first point arriving at the boundary of $\tilde{q}_\alpha(x_1)$ along $c(x_0)$. If either there do not exist such points or $x_2$ is a boundary point of $F_{\alpha, n}$, then similarly $c(x_0) \subset \tilde{q}_\alpha(x_1) \subset \tilde{q}_\alpha^2(x_0)$ and (5.7) holds. Therefore it is sufficient to consider a case of $x_2 \in F_{\alpha, n}^o$. Thus it is sufficient to check a sequence $\{ x_l \}_{l=0,1,2,\ldots}$ which is defined successively in such a manner and which has the following three properties (Fig. 5.1):

(a) $x_l \in F_{\alpha, n}^o \cap c(x_0)$ \((l = 0, 1, 2, \cdots)\);

(b) $x_{l+1}$ is the boundary point of $\tilde{q}_\alpha(x_l)$ \((l = 0, 1, 2, \cdots)\);

(c) $d(x_l, x_{l+1}) > 0$ \((l = 0, 1, 2, \cdots)\).

Then there exists a limit point $x = \lim_{l \to \infty} x_l$ since $\tilde{q}_\alpha(x_0)$ is bounded and $c(x_0)$ is so. From the property (b) and Assumption (A), $\tilde{q}(x_l, x_{l+1}) = \alpha$ \((l = 0, 1, 2, \cdots)\). Using the continuity of $\tilde{q}$ and Assumption (A), we obtain $\tilde{q}(x_0, x) = \alpha$ and $x$ is a boundary point of $F_{\alpha, n}$. Therefore (5.7) also holds for this case. Thus we obtain (5.7) in any cases. Since $x_0(\in F_{\alpha, n}^o)$ and the arc $c(x_0)$ are arbitrary in (5.7), we have
\[
F_{\alpha, n}^o \subset \bigcup_{m \geq 1} \tilde{q}_\alpha^m(x) \quad \text{for all} \ x \in F_{\alpha, n}^o. \tag{5.8}
\]
This implies that $F_{\alpha,n}^\alpha$ is $\alpha$-recurrent for all $\alpha \in (0,1)$ and all $n \in \mathbb{N}(\alpha)$.

Next from the continuity of $\tilde{q}$ and (5.8), for all $\alpha \in (0,1)$ and $x \in F_{\alpha,n}^\alpha$ we obtain

$$F_{\alpha,n} = \bigcap_{\alpha' < \alpha} F_{\alpha',n}^\alpha \subset \bigcap_{\alpha' < \alpha} \bigcup_{m \geq 1} \tilde{q}_{\alpha'}^m(x) = \{ y \in E | \sup_{m \geq 1} \tilde{q}_m(x, y) \geq \alpha \}.$$

Using this result and Proposition 2.1(ii), for $\alpha \in (0,1)$ and $x \in F_{\alpha,n} = \bigcap_{\alpha' < \alpha} F_{\alpha',n}^\alpha$ we get

$$F_{\alpha,n} = \bigcap_{\alpha' < \alpha} F_{\alpha',n}^\alpha \subset \bigcap_{\alpha' < \alpha} \{ y \in E | \sup_{m \geq 1} \tilde{q}_m(x, y) \geq \alpha' \}$$

$$= \{ y \in E | \sup_{m \geq 1} \tilde{q}_m(x, y) \geq \alpha \}$$

$$= \{ y \in E | PGI(y)_1(x) \geq \alpha \}$$

$$= \{ y \in E | PGI(y)_1(x) \geq \alpha \}.$$

This means that $F_{\alpha,n}$ is $\alpha$-recurrent. $\square$

---

**Fig. 5.1.** The arcwise connected set $F_{\alpha,n}$ and the sequence $\{x_i\}_{i=0,1,\ldots}$

**Theorem 5.2.** We suppose Assumption (A). Let $\alpha \in (0,1)$. Then maximal $\alpha$-recurrent sets for $X$ are $F_{\alpha,n}$ ($n \in \mathbb{N}(\alpha)$).

**Proof.** We show that $\alpha$-recurrent sets $F_{\alpha,n}$ ($n \in \mathbb{N}(\alpha)$) are maximal. It is sufficient to prove that $F_{\alpha,n} \cup F_{\alpha,n}'$ is not $\alpha$-recurrent, assuming that $\mathbb{N}(\alpha)$ has at least two elements.
n, n' (n ≠ n'). Now we suppose that there exist n, n' ∈ N(α) (n ≠ n') such that \( F_{α,n} \cup F_{α,n'} \) is α-recurrent. Then there exist sequences \( \{x_m\}_{m=0,1,2,\ldots} \) and \( \{x'_m\}_{m=0,1,2,\ldots} \) satisfying (a) – (d):

(a) \( x_0 \in F_{α,n} \) and \( \lim_{m→∞} x_m \in F_{α,n'} \);

(b) \( x'_0 \in F_{α,n'} \) and \( \lim_{m→∞} x'_m \in F_{α,n} \);

(c) \( x_{m+1} \in \tilde{q}_α(x_m) \) (\( m = 0, 1, 2, \ldots \));

(d) \( x'_{m+1} \in \tilde{q}_α(x'_m) \) (\( m = 0, 1, 2, \ldots \)).

We consider the following three cases:

**Case when there exists a point \( x_{m'} \) such that \( x_{m'} \notin \tilde{q}_α(x_{m'}) \):** Then we have

\[
x_{m'} \notin \{x \mid \tilde{q}(x, x) \geq α\}.
\] (5.9)

Since \( F_{α,n} \) and \( F_{α,n'} \) are α-recurrent sets, together with (a) – (d), there exists a path from \( x_{m'} \) to itself through \( F_{α,n} \) and \( F_{α,n'} \), keeping a level of \( \tilde{q} \) greater than \( α \). Therefore \( \{x_{m'}\} \cup F_{α,n} \cup F_{α,n'} \) becomes α-recurrent. By Theorem 5.1, this fact contradicts (5.9).

**Case when there exists a point \( x'_{m'} \) such that \( x'_{m'} \notin \tilde{q}_α(x'_{m'}) \):** We can derive a contradiction in the same way as the previous case.

**Case when \( x_m \in \tilde{q}_α(x_m) \) (\( m = 0, 1, 2, \ldots \)) and \( x'_m \in \tilde{q}_α(x'_m) \) (\( m = 0, 1, 2, \ldots \)):** From the assumption that \( F_{α,n} \cup F_{α,n'} \) is α-recurrent, there exists \( m' \) such that \( x_{m'} \in F_{α,n} \) and \( x_{m'+1} \in F_{α,n'} \). Therefore

\[
x_{m'+1} \in \tilde{q}_α(x_{m'}) \cap \tilde{q}_α(x_{m'+1}).
\] (5.10)

There exists a point \( y \notin F_{α,n} \cup F_{α,n'} \) such that \( y = λx_{m'} + (1 - λ)x_{m'+1} \) (\( 0 < λ < 1 \)) since \( F_{α,n} \) and \( F_{α,n'} \) are arcwise connected, closed and disjoint. Then we may take

\[
y \notin \{x \mid \tilde{q}(x, x) \geq α\}.
\] (5.11)

On the other hand, since \( \tilde{q} \) is monotone, we have

\[
\tilde{q}_α(x_{m'+1}) \subset \tilde{q}_α(y) + l(y, x_{m'+1}) = \tilde{q}_α(y) + l(x_{m'}, x_{m'+1})
\] (5.12)

and

\[
\tilde{q}_α(x_{m'}) \subset \tilde{q}_α(y) + l(y, x_{m'}) = \tilde{q}_α(y) + l(x_{m'+1}, x_{m'}).
\] (5.13)

From (5.10), (5.12) and (5.13), we obtain

\[
x_{m'+1} \in (\tilde{q}_α(y) + l(x_{m'}, x_{m'+1})) \cap (\tilde{q}_α(y) + l(x_{m'+1}, x_{m'})) = \tilde{q}_α(y).
\] (5.14)
Further since \( \tilde{q}_\alpha(x_{m'}) \) is convex, from \( x_{m'} \in \tilde{q}_\alpha(x_{m'}) \) and (5.10), we have

\[
y = \lambda x_{m'} + (1 - \lambda) x_{m'+1} \in \tilde{q}_\alpha(x_{m'}).
\]

From (5.14) and (5.15), we get

\[
x_{m'+1} \in \tilde{q}_\alpha(y) \quad \text{and} \quad y \in \tilde{q}_\alpha(x_{m'}).
\]

Since \( F_{\alpha,n} \) and \( F_{\alpha,n'} \) are \( \alpha \)-recurrent sets, together with (a) - (d), there exists a path from \( y \) to itself through \( F_{\alpha,n} \) and \( F_{\alpha,n'} \), keeping a level of \( \tilde{q} \) greater than \( \alpha \). By Theorem 5.1, this fact also contradicts (5.11).

Therefore \( F_{\alpha,n} \) (\( n \in \mathbb{N}(\alpha) \)) are maximal \( \alpha \)-recurrent. By Theorem 5.1, we obtain that maximal \( \alpha \)-recurrent sets are only \( F_{\alpha,n} \) (\( n \in \mathbb{N}(\alpha) \)). \( \square \)

**Remark.** When \( \alpha = 1 \), Theorem 5.2 does not hold in general. We consider the following non-contractive numerical example: Let a one-dimensional state space \( E = \mathbb{R} \) (the set of all real numbers). We give a fuzzy relation by

\[
\tilde{q}(x, y) = (1 - |y - x|) \lor 0, \quad x, y \in \mathbb{R}.
\]

Then we have

\[
\{ x \in \mathbb{R} \mid \tilde{q}(x, x) = 1 \} = \mathbb{R}.
\]

Further we can easily check that every one point set \( \{ x \} \) (\( x \in \mathbb{R} \)) are maximal 1-recurrent sets since \( \{ x \} = \tilde{q}_1(x) \) (\( x \in \mathbb{R} \)).

6. Numerical examples

Let a one-dimensional state space \( E = \mathbb{R} \). We consider one-dimensional numerical examples. In Section 5 we have assumed the following conditions (C.i) — (C.iv):

(C.i) \( \tilde{q} \) is continuous on \( E \times E \);

(C.ii) \( \tilde{q} \) is unimodal;

(C.iii) \( \tilde{q} \) is monotone;

(C.iv) \( \tilde{q} \) satisfies Assumption (A).

When \( E = \mathbb{R} \), \( \mathcal{F}^0(\mathbb{R}) \) means all fuzzy numbers on \( \mathbb{R} \). From (C.ii), \( \tilde{q}_\alpha(x) \) are bounded closed intervals of \( \mathbb{R} \) (\( \alpha \in (0, 1], x \in \mathbb{R} \)). So we write \( \tilde{q}_\alpha(x) = [\min \tilde{q}_\alpha(x), \max \tilde{q}_\alpha(x)] \), where \( \min A \) (\( \max A \)) denotes the minimum (maximum resp.) point of a interval \( A \subset \mathbb{R} \). Then (C.iii) is equivalent to the following (C.iii'):

(C.iii') \( \min \tilde{q}_\alpha(\cdot) \) and \( \max \tilde{q}_\alpha(\cdot) \) are non-decreasing functions on \( \mathbb{R} \) for all \( \alpha \in (0, 1] \).

Next we consider the following partial order \( \preceq \) on \( \mathcal{F}^0(\mathbb{R}) \) (Nanda [6]): For \( \tilde{s}, \tilde{r} \in \mathcal{F}^0(\mathbb{R}) \),
\( \hat{s} \preceq \hat{r} \) means that \( \min \hat{s}_\alpha \leq \min \hat{r}_\alpha \) and \( \max \hat{s}_\alpha \leq \max \hat{r}_\alpha \) for all \( \alpha \in (0,1] \).

Then we can easily find that (C.iii) is equivalent to the following (C.iii'') :

(C.iii'') If \( \hat{s}, \hat{r} \in \mathcal{F}^0(\mathbb{R}) \) satisfy \( \hat{s} \preceq \hat{r} \), then \( Q(\hat{s}) \preceq Q(\hat{r}) \),

where \( Q : \mathcal{F}^0(\mathbb{R}) \to \mathcal{F}^0(\mathbb{R}) \), see (4.1), is defined by

\[
Q\hat{s}(y) = \max_{x \in \mathbb{R}} \{ \hat{s}(x) \wedge \hat{q}(x,y) \}, \quad y \in \mathbb{R} \quad \text{for} \quad \hat{s} \in \mathcal{F}^0(\mathbb{R}).
\]

(C.iii'') means that \( Q \) preserves the monotonicity on \( \mathcal{F}^0(\mathbb{R}) \) with respect to the order \( \preceq \). Finally (C.iv) means that the \( \alpha \)-level sets \( \{ x \in \mathbb{R} \mid \hat{q}(x,x) = \alpha \} \ (\alpha \in (0,1)) \) are drawn by not areas but curved lines. The linear case of [11, Fig. 2] clearly satisfies the above conditions (C.i) – (C.iv), taking the state space \( E = (0,\infty) \).

We give an example of monotone fuzzy relations, which is not contractive and does not have the linear structure in [11]. Then we calculate its maximal \( \alpha \)-recurrent sets.

**Example 6.1 (monotone case).** We give a fuzzy relation by

\[
\hat{q}(x,y) = (1-|y-x^3|) \lor 0, \quad x,y \in \mathbb{R}.
\]

Then \( \hat{q}(x,y) \) satisfies the conditions (C.i) – (C.iv) (see Figure 6.1 for the fuzzy relation \( \hat{q}(x,y) \) and Figure 6.2 for the \( \frac{3}{4} \)-level sets).

![Fig. 6.1: The monotone fuzzy relation \( \hat{q}(x,y) \).](image)
Fig. 6.2. The $\frac{3}{4}$-level sets \{(x, y) \mid \tilde{q}(x, y) = \frac{3}{4}\}.

Then we have

$$\tilde{q}(x, x) = (1 - |x - x^3|) \lor 0, \quad x \in \mathbb{R}.$$ 

Therefore $N(\frac{3}{4}) = \{0, 1, 2\}$ and

$$\{x \in \mathbb{R} \mid \tilde{q}(x, x) \geq \frac{3}{4}\} = F_{\frac{3}{4}, 0} \cup F_{\frac{3}{4}, 1} \cup F_{\frac{3}{4}, 2}$$

$$\approx [-1.10716, -0.837565] \cup [-0.269594, 0.269594] \cup [0.837565, 1.10716].$$

By Theorem 5.2, the maximal $\frac{3}{4}$-recurrent sets are given by three intervals

$$F_{\frac{3}{4}, 0} \approx [-1.10716, -0.837565],$$

$$F_{\frac{3}{4}, 1} \approx [-0.269594, 0.269594],$$

$$F_{\frac{3}{4}, 2} \approx [0.837565, 1.10716].$$

Finally we consider the following numerical example, which is not monotone.

**Example 6.2 (non-monotone case).** We consider a fuzzy relation

$$\tilde{q}(x, y) = \max \left\{ \left(1 - 2 \left| y - \frac{1}{4} x \right| \right) \lor 0, \min \left\{ \left(1 - \frac{1}{4} \left| y - \frac{1}{4} x \right| \right) \lor 0, \frac{3}{2} \left| x \right| \lor 1 \right\} \right\}, \quad x, y \in \mathbb{R}.$$ 

Then $\tilde{q}(x, y)$ satisfies the conditions (C.i), (C.ii) and (C.iv) except for (C.iii) (see Figure 6.3 for the fuzzy relation $\tilde{q}(x, y)$ and Figure 6.4 for the $\frac{3}{4}$-level sets).
Then
\[
\left\{ x \in \mathbb{R} \mid \bar{q}(x, x) \geq \frac{3}{4} \right\} = \left\{ x \in \mathbb{R} \mid \sup_{n \geq 1} \bar{q}^n(x, x) \geq \frac{3}{4} \right\} = \left[ -\frac{4}{3}, -\frac{1}{2} \right] \cup \left[ -\frac{1}{6}, \frac{1}{6} \right] \cup \left[ \frac{1}{2}, \frac{4}{3} \right].
\]
We can easily check the maximal $\frac{3}{4}$-recurrent sets are
\[
\left[ -\frac{1}{6}, \frac{1}{6} \right] \quad \text{and} \quad \left[ -\frac{4}{3}, -\frac{1}{2} \right] \cup \left[ \frac{1}{2}, \frac{4}{3} \right].
\]
Therefore, in non-monotone case, Theorem 5.2 does not hold in general.

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References


