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THE RECURRENCE OF DYNAMIC FUZZY SYSTEMS

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Abstract : This paper analyses a recurrent behavior of dynamic fuzzy systems defined by fuzzy relations on a Euclidean space. By introducing a recurrence for crisp sets, we prove probability-theoretical properties for the fuzzy systems. In the contractive case in [5], the existence of the maximum recurrent set is proved. As another case, we introduce a monotonicity for fuzzy relations, which is extended from the linear structure in [11]. In the monotone case we prove the existence of the arcwise connected maximal recurrent sets.

Keyword : Recurrence; dynamic fuzzy systems; fuzzy relations; contraction; monotonicity; superharmonic property.

1. Introduction and notations

Limit theorems of a sequence of fuzzy sets defined successively by fuzzy relations are first studied by Bellman and Zadeh [1]. They considered a sequence of fuzzy numbers in a finite space and solved a fuzzy relational equation written in matrix form. Kurano et al. [5] and Yoshida et al. [11], under a contractive condition, studied the limiting behavior of fuzzy sets defined by the dynamic fuzzy system with a compact space. We, in [5], proved the existence and uniqueness of the solution for the fuzzy relational equation, and in [11], developed, under a linear structure, a potential theory of fuzzy relations on the positive orthant of a Euclidean space.

Our objective is to study maximal recurrence of the dynamic fuzzy systems defined by fuzzy relations on a Euclidean space. We introduce, for every level $\alpha \in (0,1]$, a recurrence for crisp sets and we call it α-recurrence. In Section 2 we prove, on the α-recurrent crisp sets, various probability-theoretical properties in the class of fuzzy sets satisfying a fuzzy relational inequality, which is a generalization of the fuzzy relational equations in [5] and which is also satisfied by optimal fuzzy goals in fuzzy dynamic programming of [1], [2], [10]. Further we establish the balayage theorem, which is well-known regarding Markov chains, for the dynamic fuzzy system. In Section 3 we introduce α-recurrence and represent the union of all α-recurrent sets by the fuzzy relation. In Section 4 we deal with the contractive case in [5]. We give an explicit solution of the fuzzy relational equation in [5] and we prove that the α-cut of the solution is the maximum α-recurrent set. In Section 5 we introduce a certain monotonicity for the fuzzy relation, which is a natural extension of one-dimensional fuzzy relations with the linear structure in [11]. Then we prove that at most countable maximal α-recurrent sets exist and that each maximal α-recurrent set is arcwise connected. In Section 6 numerical examples are given to illustrate our idea.

In the remainder of this section, we describe the notations for dynamic fuzzy systems defined by fuzzy relations on finite-dimensional Euclidean spaces and give some fundamental results for stopping times from Yoshida [10].

Let S be a metric space. We write a fuzzy set on S by its membership function $\tilde{s} : S \mapsto [0,1]$ and an ordinary set $A(\subset S)$ by its indicator function $1_A : S \mapsto \{0,1\}$. The α-cut $\tilde{s}_\alpha$ is defined by

$$\tilde{s}_\alpha := \{x \in S \mid \tilde{s}(x) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{s}_0 := \mathrm{cl}\{x \in S \mid \tilde{s}(x) > 0\},$$

where cl denotes the closure of a set. $\mathcal{F}(S)$ denotes the set of all fuzzy sets $\tilde{s}$ on S satisfying the following conditions (F.i) and (F.ii) :

(F.i) $\tilde{s}_\alpha \in \mathcal{E}(S)$ for $\alpha \in [0,1]$;

(F.ii) $\bigcap_{\alpha'<\alpha} \tilde{s}_{\alpha'} = \tilde{s}_\alpha$ for $\alpha \in (0,1]$,

where

$$\mathcal{E}(S) := \left\{ A \mid A = \bigcup_{n=0}^{\infty} C_n,\ C_n \text{ are closed subsets of } S\ (n = 0,1,2,\cdots) \right\}.$$

We also define

$$\mathcal{G}(S) := \{ \text{ fuzzy sets } \tilde{s} \text{ on } S \mid \text{there exists } \{\tilde{s}_n\}_{n\in\mathbf{N}} \subset \mathcal{F}(S) \text{ satisfying } \tilde{s} = \bigvee_{n\in\mathbf{N}} \tilde{s}_n \},$$

where $\mathbf{N} := \{0,1,2,3,\cdots\}$ and for a sequence of fuzzy sets $\{\tilde{s}_n\}_{n\in\mathbf{N}}$ on S we define

$$\bigwedge_{n\in\mathbf{N}} \tilde{s}_n(x) := \inf_{n\in\mathbf{N}} \tilde{s}_n(x) \quad \text{and} \quad \bigvee_{n\in\mathbf{N}} \tilde{s}_n(x) := \sup_{n\in\mathbf{N}} \tilde{s}_n(x) \qquad x \in S.$$

Let a time space by $\mathbf{N}$ and put $\overline{\mathbf{N}} := \mathbf{N} \cup \{\infty\}$. Let a state space E be a finite-dimensional Euclidean space. We put a path space by $\Omega := \prod_{k=0}^{\infty} E$ and we write a sample path by $\omega = (\omega(0), \omega(1), \omega(2), \cdots) \in \Omega$. We define a map $X_n(\omega) := \omega(n)$ and a shift $\theta_n(\omega) := (\omega(n), \omega(n+1), \omega(n+2), \cdots)$ for $n \in \mathbf{N}$ and $\omega = (\omega(0), \omega(1), \omega(2), \cdots) \in \Omega$. We put σ-fields by $\mathcal{M}_n := \sigma(X_0, X_1, \cdots, X_n)$ [1] for $n \in \mathbf{N}$ and $\mathcal{M} := \sigma(\bigcup_{n\in\mathbf{N}} \mathcal{M}_n)$ [2]. Let Δ be not a point of E and put $E_\Delta := E \cup \{\Delta\}$. We can extend the state space E to E_Δ, setting $\tilde{s}(\Delta) := 0$ for $\tilde{s} \in \mathcal{G}(E_\Delta)$ and $X_\infty(\omega) := \Delta$ for $\omega \in \Omega$ ([10, Section 2]). Let $\tilde{q}$ be an upper semi-continuous binary relation on $E \times E$ satisfying the following normality condition :

$$\sup_{x\in E} \tilde{q}(x,y) = 1 \ (y \in E) \quad \text{and} \quad \sup_{y\in E} \tilde{q}(x,y) = 1 \ (x \in E).$$

We call $\tilde{q}$ a fuzzy relation. We define a fuzzy expectation : For an initial state $x \in E$ and an $\mathcal{M}$-measurable fuzzy set $h \in \mathcal{F}(\Omega)$,

$$E_x(h) := \int_{\{\omega\in\Omega:\omega(0)=x\}} h(\omega)\ \mathrm{d}\tilde{P}(\omega),$$

[1] It denotes the smallest σ-field on Ω relative to which $X_0, X_1, \cdots, X_n$ are measurable.

[2] It denotes the smallest σ-field generated by $\bigcup_{n\in\mathbf{N}} \mathcal{M}_n$.

where $\tilde{P}$ is the following possibility measure :

$$\tilde{P}(\Lambda) := \sup_{\omega\in\Lambda} \bigwedge_{n\in\mathbf{N}} \tilde{q}(X_n\omega, X_{n+1}\omega) \qquad \Lambda \in \mathcal{M}$$

and $\displaystyle{\int\!\!\!\!-}\, \mathrm{d}\tilde{P}$ denotes Sugeno integral (Sugeno [9]). Then the fuzzy expectation has the following property.

Lemma 1.1 ([10, Section 3]). *For an $\mathcal{M}$-measurable sequence $\{h_n\}_{n\in\mathbf{N}} \subset \mathcal{G}(\Omega)$, it holds that*

$$\bigvee_{n\in\mathbf{N}} E_x(h_n) = E_x(\bigvee_{n\in\mathbf{N}} h_n) \qquad x \in E.$$

We need the first entry times (the first hitting times) of a set, which is adapted to the dynamic fuzzy system $X := \{X_n\}_{n\in\mathbf{N}}$, in order to define a recurrence of sets in Section 3. We define

$$\mathcal{E} := \{A \mid A \in \mathcal{E}(E) \text{ and } E \setminus A \in \mathcal{E}(E)\}$$

and we call a map $\tau : \Omega \mapsto \overline{\mathbf{N}}$ an $\mathcal{E}$-stopping time if

$$\{\tau = n\} \in \mathcal{M}_n \cap \mathcal{E}(\Omega) \qquad n \in \mathbf{N}.$$

For example, a constant stopping time i.e. $\tau = n_0$ for some $n_0 \in \mathbf{N}$, is an $\mathcal{E}$-stopping time. For $A \in \mathcal{E}$ we put

$$\tau_A(\omega) := \inf\{n \in \mathbf{N} \mid X_n(\omega) \in A\} \quad \omega \in \Omega;$$

$$\sigma_A(\omega) := \inf\{n \in \mathbf{N} \mid n \geq 1, X_n(\omega) \in A\} \quad \omega \in \Omega,$$

where the infimums of the empty set are understood to be $+\infty$. Then the first entry time τ_A of A and the first hitting time σ_A of A are also $\mathcal{E}$-stopping times ([10, Lemma 1.5]).

Define a map $P : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$P\tilde{s}(x) := E_x(\tilde{s}(X_1)) = \sup_{y\in E}\{\tilde{q}(x,y) \wedge \tilde{s}(y)\} \quad x \in E \qquad \text{for } \tilde{s} \in \mathcal{G}(E), \tag{1.1}$$

where we write binary operations $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$ for $a, b \in [0, 1]$. We call P a fuzzy transition defined by the fuzzy relation $\tilde{q}$. We also define n-steps fuzzy transitions $P_n : \mathcal{G}(E) \mapsto \mathcal{G}(E)$, $n \in \mathbf{N}$, by

$$P_n\tilde{s} := E_\cdot(\tilde{s}(X_n)) = \sup_{y\in E}\{\tilde{q}^n(\cdot, y) \wedge \tilde{s}(y)\} \quad \text{for } \tilde{s} \in \mathcal{G}(E),$$

where for $n \in \mathbf{N}$

$$\tilde{q}^1(x,y) := \tilde{q}(x,y) \text{ and } \tilde{q}^{n+1}(x,y) := \sup_{z\in E}\{\tilde{q}^n(x,z) \wedge \tilde{q}(z,y)\} \quad x, y \in E.$$

Further for an $\mathcal{E}$-stopping time τ, a fuzzy transition $P_\tau : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ is defined by

$$P_\tau \tilde{s} := E.(\tilde{s}(X_\tau)) \quad \text{for } \tilde{s} \in \mathcal{G}(E),$$

where $X_\tau := X_n$ on $\{\tau = n\}$, $n \in \overline{\mathbf{N}}$.

The fuzzy transition $\{P_n\}_{n\in\mathbf{N}}$ has the following property :

$$P_0 = I \text{ (identity)}, \quad P_1 = P \quad \text{and} \quad P_{m+n} = P_m P_n \ (m, n \in \mathbf{N}).$$

Further it also has a semi-group property with respect to $\mathcal{E}$-stopping times.

Lemma 1.2 ([10, Corollary 2.1]). *It holds that*

$$P_\sigma P_\tau = P_{\sigma+\tau\circ\theta_\sigma} \text{ on } \mathcal{G}(E) \quad \text{for finite } \mathcal{E}\text{-stopping times } \sigma \text{ and } \tau.$$

2. Transitive closures and P-superharmonic fuzzy sets

We define a partial order $\geq$ on $\mathcal{G}(E)$: For $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$

$$\tilde{s} \geq \tilde{r} \iff \tilde{s}(x) \geq \tilde{r}(x) \quad x \in E.$$

Definition ([10, Section 4]). A fuzzy set $\tilde{s}\,(\in \mathcal{G}(E))$ is called P-harmonic (P-superharmonic) provided that

$$\tilde{s} = P\tilde{s} \quad (\tilde{s} \geq P\tilde{s} \text{ resp.}).$$

Clearly a constant fuzzy set, $\tilde{s} = \beta$ for some $\beta \in [0,1]$, is P-superharmonic. We represent the fuzzy set by β simply.

In this section we investigate P-superharmonic property regarding fuzzy sets and we show the balayage theorem for P-superharmonic fuzzy sets. Using the results, we give a simple characterization for hitting possibilities of a set $A(\in \mathcal{E})$ by transitive closures. First we prove preliminary lemmas for P-superharmonic fuzzy sets, which are well-known property in the classical probability theory ([8]).

Lemma 2.1.

(i) *If $\tilde{s}_1$ and $\tilde{s}_2$ are P-superharmonic, then $\tilde{s}_1 \wedge \tilde{s}_2$ is also P-superharmonic.*

(ii) *If $\{\tilde{s}_n\}_{n\in\mathbf{N}}$ is a sequence of P-harmonic (P-superharmonic) fuzzy sets, then $\bigvee_{n\in\mathbf{N}} \tilde{s}_n$ is also P-harmonic (P-superharmonic resp.).*

Proof. (i) We can easily check $\tilde{s}_1 \wedge \tilde{s}_2 \in \mathcal{G}(E)$, using [10, Lemma 1.1]. Since the fuzzy transition P preserves the order $\geq$ on $\mathcal{G}(E)$, we have

$$\tilde{s}_1 \geq P\tilde{s}_1 \geq P(\tilde{s}_1 \wedge \tilde{s}_2) \quad \text{and} \quad \tilde{s}_2 \geq P\tilde{s}_2 \geq P(\tilde{s}_1 \wedge \tilde{s}_2).$$

Therefore $\tilde{s}_1 \wedge \tilde{s}_2$ is P-superharmonic.

(ii) It is trivial that $\bigvee_{n\in\mathbf{N}} \tilde{s}_n \in \mathcal{G}(E)$. By Lemma 1.1,

$$\bigvee_{n\in\mathbf{N}} \tilde{s}_n \geq \bigvee_{n\in\mathbf{N}} P\tilde{s}_n = P(\bigvee_{n\in\mathbf{N}} \tilde{s}_n).$$

Therefore $\bigvee_{n\in\mathbf{N}} \tilde{s}_n$ is P-superharmonic. The P-harmonic case is similar. □

Lemma 2.2.

(i) *If $\tilde{s}$ is P-superharmonic, then*

$$P_\sigma \tilde{s} \geq P_\tau \tilde{s} \quad \textit{for all } \mathcal{E}\textit{-stopping times } \sigma \textit{ and } \tau \textit{ such that } \sigma \leq \tau.$$

(ii) *If $\tilde{s}$ is P-harmonic, then*

$$P_\sigma \tilde{s} = P_\tau \tilde{s} \quad \textit{for all } \mathcal{E}\textit{-stopping times } \sigma \textit{ and } \tau \textit{ such that } \sigma \leq \tau < \infty.$$

Proof. (i) We check this lemma along the proof of [8, Proposition II-1.9]. Let σ and τ be $\mathcal{E}$-stopping times such that $\sigma \leq \tau \leq \sigma + 1$. Let $\Lambda_n := \{\sigma = n\} \cap \{\tau = n+1\} \in \mathcal{M}_n$ and $\Gamma_n := \{\sigma = \tau = n\} \in \mathcal{M}_n$ for $n \in \mathbf{N}$. By [10, Theorem 2.1], for $n \in \mathbf{N}$

$$E_x(\tilde{s}(X_\sigma) \wedge 1_{\Lambda_n}) \geq E_x(P\tilde{s}(X_n) \wedge 1_{\Lambda_n}) = E_x(\tilde{s}(X_{n+1}) \wedge 1_{\Lambda_n}) = E_x(\tilde{s}(X_\tau) \wedge 1_{\Lambda_n}) \quad x \in E.$$

Using Lemma 1.1, we obtain

$$\begin{aligned} P_\sigma \tilde{s}(x) &= \bigvee_{n\in\mathbf{N}} (E_x(\tilde{s}(X_\sigma) \wedge 1_{\Lambda_n}) \vee E_x(\tilde{s}(X_\sigma) \wedge 1_{\Gamma_n})) \\ &\geq \bigvee_{n\in\mathbf{N}} (E_x(\tilde{s}(X_\tau) \wedge 1_{\Lambda_n}) \vee E_x(\tilde{s}(X_\tau) \wedge 1_{\Gamma_n})) \\ &= P_\tau \tilde{s}(x) \qquad x \in E. \end{aligned}$$

More generally, for $\mathcal{E}$-stopping times σ and τ such that $\sigma \leq \tau$,

$$P_\sigma \tilde{s} \geq P_{(\sigma+1)\wedge\tau} \tilde{s} \geq \cdots \geq P_{(\sigma+n)\wedge\tau} \tilde{s} \geq \cdots \quad \text{for } n \in \mathbf{N}. \tag{2.1}$$

Here, from [10, Lemma 1.1(i)], we have the following facts :

$$\{\tau \leq \sigma + n < \infty\} = \bigcup_{l,m\in\mathbf{N}:\, l\leq m+n} (\{\tau = l\} \cap \{\sigma = m\}) \in \mathcal{E}(\Omega);$$

$$\{\tau < \infty\} = \bigcup_{l\in\mathbf{N}} \{\tau = l\} \in \mathcal{E}(\Omega);$$

$$\bigcup_{n\in\mathbf{N}} \{\tau \leq \sigma + n < \infty\} = \{\tau < \infty\}.$$

By Lemma 1.1 and (2.1),

$$P_\sigma \tilde{s}(x) \geq \bigvee_{n\in\mathbf{N}} P_{(\sigma+n)\wedge\tau} \tilde{s}(x) \geq \bigvee_{n\in\mathbf{N}} E_x(\tilde{s}(X_\tau) \wedge 1_{\{\tau\leq\sigma+n<\infty\}}) = E_x(\tilde{s}(X_\tau) \wedge 1_{\{\tau<\infty\}}) = P_\tau \tilde{s}(x)$$

for $x \in E$. Therefore we get (i). We can check (ii) similarly. $\square$

We show the balayage theorem for the dynamic fuzzy system X. The theorem plays one of important roles to analyse recurrence for the fuzzy relation $\tilde{q}$ in Section 3.

Theorem 2.1. *Let $\tilde{s}$ be P-superharmonic and let a set $A \in \mathcal{E}$. Then $P_{\tau_A}\tilde{s}$ is the smallest P-superharmonic fuzzy set which dominates $\tilde{s} \wedge 1_A$.*

Proof. We check this theorem along the proof of [8, Theorem II-2.1] for the classical Markov chain. It is trivial that $P_{\tau_A}\tilde{s} = \tilde{s}$ on A. $P_{\tau_A}\tilde{s}$ is P-superharmonic since $PP_{\tau_A}\tilde{s} = P_{\sigma_A}\tilde{s} \le P_{\tau_A}\tilde{s}$ by Lemmas 1.2 and 2.2(i). Therefore $P_{\tau_A}\tilde{s}$ is P-superharmonic and dominates $\tilde{s} \wedge 1_A$. Further let $\tilde{r}$ be P-superharmonic such that $\tilde{r} \ge \tilde{s} \wedge 1_A$. Then

$$\tilde{r}(x) \ge P_{\tau_A}\tilde{r}(x) = E_x(\tilde{r}(X_{\tau_A}) \wedge 1_{\{\tau_A < \infty\}}) \ge E_x(\tilde{s}(X_{\tau_A}) \wedge 1_{\{\tau_A < \infty\}}) = P_{\tau_A}\tilde{s}(x) \quad x \in E.$$

Thus $P_{\tau_A}\tilde{s}$ has the desired property and so we get this theorem. $\square$

We define an operator $G := \bigvee_{n \in \mathbf{N}} P_n$ on $\mathcal{G}(E)$. Then we note that

$$PG1_{\{y\}}(x) = \bigvee_{n \ge 1} P_n 1_{\{y\}}(x) = \sup_{n \ge 1} \tilde{q}^n(x, y) \qquad x, y \in E.$$

This is called a transitive closure ([3, Section 3.3]). In this paper we also call PG a transitive closure. Now we need to investigate the operator G in order to analyse the transitive closure $PG := \bigvee_{n \ge 1} P_n$. We have the following properties regarding G.

Lemma 2.3 ([10, Lemma 4.1(ii)]). *Let $\tilde{s} \in \mathcal{G}(E)$. Then :*

(i) *It holds that*

$$G\tilde{s} = \tilde{s} \vee P(G\tilde{s});$$

(ii) *$G\tilde{s}$ is the smallest P-superharmonic dominating $\tilde{s}$.*

Lemma 2.4. *Let $\tilde{s} \in \mathcal{G}(E)$. Then $\tilde{s}$ is P-superharmonic if and only if*

$$\tilde{s} = G\tilde{s}. \tag{2.2}$$

Proof. Let $\tilde{s}$ be P-superharmonic. Then

$$\tilde{s} = \tilde{s} \vee P\tilde{s} \vee P_2\tilde{s} \vee \cdots \vee P_n\tilde{s} \quad \text{for all } n \in \mathbf{N}.$$

So we obtain (2.2). The converse proof is trivial. $\square$

For $A \in \mathcal{E}(E)$ we introduce an operator $I_A : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$I_A\tilde{s} := \tilde{s} \wedge 1_A \qquad \tilde{s} \in \mathcal{G}(E).$$

We define a sequence of hitting times $\{\sigma_A^n\}_{n\in\mathbf{N}}$ of a set $A(\in \mathcal{E})$ by

$$\sigma_A^n := \begin{cases} 0 & \text{if } n = 0 \\ \sigma_A^{n-1} + \sigma_A \circ \theta_{\sigma_A^{n-1}} & \text{if } n \geq 1. \end{cases}$$

Then σ_A^n means the first time to hit A after time σ_A^{n-1} ([8]). We investigate an entry possibility, P_{τ_A}, of A, and we give a simple and interesting characterization of a possibility, $P_{\sigma_A^n}$, to hit A first n times.

Proposition 2.1. *Let $A \in \mathcal{E}$. Then :*

(i) $P_{\tau_A}\tilde{s} = GI_A\tilde{s}$ *for P-superharmonic $\tilde{s}$;*

(ii) $P_{\sigma_A^n}\tilde{s} = (PGI_A)^n\tilde{s}$ *for P-superharmonic $\tilde{s}$ and $n \in \mathbf{N}$.*

Proof. (i) From Theorem 2.1 and Lemma 2.3(ii) we obtain

$$P_{\tau_A}\tilde{s} = G(\tilde{s} \wedge 1_A) = GI_A\tilde{s}.$$

(ii) We prove the equality by induction on $n \in \mathbf{N}$. It is trivial when $n = 0$. From (i), $P_{\sigma_A}\tilde{s} = PP_{\tau_A}\tilde{s} = PGI_A\tilde{s}$. So (ii) also holds for $n = 1$. Next for every $n \in \mathbf{N}$, $(PGI_A)^{n+1}\tilde{s}$ is P-superharmonic since $GI_A(PGI_A)^n\tilde{s}$ is P-superharmonic by Lemma 2.3(ii). Therefore $(PGI_A)^n\tilde{s}$ is P-superharmonic for all $n \in \mathbf{N}$. Let $n \in \mathbf{N}$. We suppose that (ii) holds for n. From (i) and the fact that $(PGI_A)^n\tilde{s}$ is P-superharmonic,

$$P_{\sigma_A^{n+1}}\tilde{s} = P_{\sigma_A}P_{\sigma_A^n}\tilde{s} = PP_{\tau_A}(PGI_A)^n\tilde{s} = PGI_A(PGI_A)^n\tilde{s} = (PGI_A)^{n+1}\tilde{s}.$$

Thus we obtain (ii) inductively. □

3. α-recurrent sets

Definition. Let $\alpha \in (0, 1]$. A set $A \in \mathcal{E}(E)$ is called α-recurrent provided :

(a) A is non-empty;

(b) $P_{\sigma_B^n}1 \geq \alpha$ on A for all $n \in \mathbf{N}$ and all non-empty $B \in \mathcal{E}$ satisfying $B \subset A$.

The α-recurrence of a set A means that a possibility to transit infinite times from any point of A to any point of A is greater than α.

Lemma 3.1. *Let $\beta(\in [0, 1])$ be a constant fuzzy set. It holds that*

$$G(\tilde{s} \wedge \beta) = G\tilde{s} \wedge \beta \quad \text{and} \quad PG(\tilde{s} \wedge \beta) = PG\tilde{s} \wedge \beta \quad \text{for } \tilde{s} \in \mathcal{G}(E). \tag{3.1}$$

Especially,

$$GI_A(\beta) = G1_A \wedge \beta \quad \text{and} \quad PGI_A(\beta) = PG1_A \wedge \beta \quad \text{for } A \in \mathcal{E}(E). \tag{3.2}$$

Proof. By induction we show

$$P^n(\tilde{s} \wedge \beta) = P^n\tilde{s} \wedge \beta \quad \text{for } \tilde{s} \in \mathcal{G}(E) \text{ and } n \in \mathbf{N}. \tag{3.3}$$

First (3.3) holds clearly when $n = 0$. Next we have (3.3) for $n = 1$ since

$$P(\tilde{s} \wedge \beta)(x) = \sup_{y \in E}(\tilde{q}(x,y) \wedge \tilde{s}(y) \wedge \beta) = \left(\sup_{y \in E} \tilde{q}(x,y) \wedge \tilde{s}(y)\right) \wedge \beta = P\tilde{s}(x) \wedge \beta \qquad x \in E.$$

Further let $n \in \mathbf{N}$. Assuming that (3.3) holds for n, we have

$$P^{n+1}(\tilde{s} \wedge \beta) = PP^n(\tilde{s} \wedge \beta) = P(P^n\tilde{s} \wedge \beta) = P^{n+1}\tilde{s} \wedge \beta.$$

Thus (3.3) holds for all $n \in \mathbf{N}$. Therefore we get (3.1). We also obtain (3.2), taking $\tilde{s} = 1_A$ ($A \in \mathcal{E}(E)$) in (3.1). □

We give simple necessary and sufficient criteria for α-recurrence by the transitive closure PG.

Proposition 3.1. *Let $\alpha \in (0,1]$ and let non-empty $A \in \mathcal{E}(E)$. Then the following statements are equivalent :*

(i) *A is α-recurrent;*

(ii) *$PG1_B \geq \alpha \wedge 1_A$ for non-empty $B \in \mathcal{E}(E)$ satisfying $B \subset A$;*

(iii) *$PG1_{\{y\}} \geq \alpha \wedge 1_A$ for $y \in A$.*

Proof. First we check

$$\{y\} \in \mathcal{E} \quad \text{for } y \in E. \tag{3.4}$$

Let $y \in E$. Then $\{y\} \subset \mathcal{E}(E)$. Put $B_m(y) := \{z \in E \mid d(y,z) \geq 1/m\}$ for $m = 1, 2, \cdots$, where d denotes a metric on E. From [10, Lemma 1.1], $E \setminus \{y\} = \bigcup_{m=1}^{\infty} B_m(y) \in \mathcal{E}(E)$. Therefore we obtain (3.4). Next we prove the equivalences of (i) — (iii).

(ii) $\Longrightarrow$ (i) : Let non-empty $B \in \mathcal{E}$ satisfying $B \subset A$. By induction we show

$$(PGI_B)^n 1 \geq \alpha \wedge 1_A \qquad \text{for } n \in \mathbf{N}. \tag{3.5}$$

Inequality (3.5) is trivial for $n = 1$. We assume that (3.5) holds for some $n \in \mathbf{N}$. From Lemma 3.1,

$$(PGI_B)^{n+1}1 = (PGI_B)^n(PG1_B) \geq (PGI_B)^n(\alpha \wedge 1_A) = (PGI_B)^n(\alpha) = (PGI_B)^n \wedge \alpha = \alpha \wedge 1_A.$$

So (3.5) holds for all $n \in \mathbf{N}$. Therefore we obtain (i), using Proposition 2.1(ii).

(iii) $\Longrightarrow$ (ii) : Let non-empty $B \in \mathcal{E}(E)$ satisfying $B \subset A$ and let $y \in B$. Then

$$PG1_B \geq PG1_{\{y\}} \geq \alpha \wedge 1_A.$$

Therefore we obtain (ii).

(i) $\Longrightarrow$ (iii) : It is trivial from Proposition 2.1(ii).

Thus we complete the proof. $\square$

We gives, by the fuzzy relation $\tilde{q}$, a representation of the union of all α-recurrent sets.

Theorem 3.1. *It holds that*

$$\bigcup_{A \in \mathcal{E}(E)\,:\,\alpha-\text{recurrent sets}} A = \left\{ x \in E \mid \sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha \right\} \qquad \textit{for } \alpha \in (0, 1].$$

Proof. Let $A \in \mathcal{E}(E)$ be α-recurrent. From Proposition 3.1, for $x \in A$

$$PG1_{\{x\}} \geq \alpha \wedge 1_A \geq \alpha \wedge 1_{\{x\}}.$$

Therefore

$$A \subset \{x \in E \mid PG1_{\{x\}} \geq \alpha \wedge 1_{\{x\}}\} = \left\{ x \in E \mid \sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha \right\}.$$

Conversely let $x \in E$ satisfy $\sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha$. Then $PG1_{\{x\}} \geq \alpha \wedge 1_{\{x\}}$. From Proposition 3.1, $\{x\}$ is α-recurrent. Therefore

$$\{x\} \subset \bigcup_{A \in \mathcal{E}(E)\,:\,\alpha-\text{recurrent sets}} A.$$

Thus we obtain this theorem. $\square$

4. The contractive case

In this section we consider the contractive case in [5] and we give the maximum α-recurrent set for the dynamic fuzzy system X.

Let E_c be a compact subset of E. We deal with a dynamic fuzzy system restricted on the compact space E_c according to [5]. Let $\mathcal{C}(E_c)$ be the set of all closed subsets of E_c and let ρ be the Hausdorff metric on $\mathcal{C}(E_c)$. Let $\mathcal{F}^0(E_c)$ be the set of all fuzzy sets $\tilde{s}$ on E_c which are upper semi-continuous and satisfy $\sup_{x \in E_c} \tilde{s}(x) = 1$. Then we note $\mathcal{F}^0(E_c) \subset \mathcal{F}(E_c)$. Let $\tilde{p}_0 \in \mathcal{F}^0(E_c)$ be a fuzzy set. Define a sequence of fuzzy sets $\{\tilde{p}_n\}_{n=0}^{\infty}$ by

$$\tilde{p}_{n+1}(y) = \sup_{x \in E_c} \{\tilde{p}_n(x) \wedge \tilde{q}(x, y)\} \quad y \in E_c \quad \text{for } n \geq 0. \tag{4.1}$$

The fuzzy set $\tilde{p}_0$, in [5], is called an initial fuzzy state and the sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ is called a sequence of fuzzy states. The fuzzy relation $\tilde{q}$ is also restricted on $E_c \times E_c$ and it is assumed to be continuous on $E_c \times E_c$ and satisfy $\tilde{q}(x, \cdot) \in \mathcal{F}^0(E)$. Define a map $\tilde{r}_\alpha : \mathcal{C}(E_c) \to \mathcal{C}(E_c)$ ($\alpha \in (0,1)$) by

$$\tilde{r}_\alpha(D) := \begin{cases} \{y \mid \tilde{q}(x,y) \geq \alpha \ \text{ for some } x \in D\} & \text{for } \alpha > 0,\ D \in \mathcal{C}(E_c),\ D \neq \emptyset, \\ \mathrm{cl}\{y \mid \tilde{q}(x,y) > 0 \ \text{ for some } x \in D\} & \text{for } \alpha = 0,\ D \in \mathcal{C}(E_c),\ D \neq \emptyset, \\ E_c & \text{for } 0 \leq \alpha \leq 1,\ D = \emptyset. \end{cases}$$

In the sequel we assume the following contractive property for the fuzzy relation $\tilde{q}$ (see [5, Section 2]) : There exists a real number $\beta \in (0,1)$ satisfying

$$\rho(\tilde{r}_\alpha(A), \tilde{r}_\alpha(B)) \leq \beta\ \rho(A,B) \quad \text{for all } A, B \in \mathcal{C}(E_c) \text{ and all } \alpha \in (0,1).$$

Then we have proved a convergence of the sequence of fuzzy states $\{\tilde{p}_n\}_{n=0}^{\infty}$ defined by (4.1).

Lemma 4.1 ([5, Theorem 1]).

(i) *There exists a unique fuzzy state $\tilde{p} \in \mathcal{F}^0(E_c)$ satisfying*

$$\tilde{p}(y) = \max_{x \in E_c} \{\tilde{p}(x) \wedge \tilde{q}(x,y)\} \qquad y \in E_c. \tag{4.2}$$

(ii) *The sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ converges to a unique solution $\tilde{p} \in \mathcal{F}^0(E_c)$ of (4.2) independently of the initial fuzzy state $\tilde{p}_0$. Namely,*

$$\lim_{n \to \infty} \tilde{p}_n = \tilde{p},$$

where the convergence means $\sup_{\alpha \in [0,1]} \rho(\tilde{p}_{n,\alpha}, \tilde{p}_\alpha) \to 0$ ($n \to \infty$) provided $\tilde{p}_{n,\alpha}, \tilde{p}_\alpha$ are α-cuts ($\alpha \in [0,1]$) for the fuzzy states $\tilde{p}_n, \tilde{p}$ respectively.

First we give a solution of (4.2).

Proposition 4.1. *The α-cut of the solution $\tilde{p}$ of (4.2) is*

$$\tilde{p}_\alpha = \mathrm{cl}\left\{x \in E_c \mid \sup_{n \geq 1} \tilde{q}^n(x,x) \geq \alpha\right\} \qquad \text{for } \alpha \in (0,1].$$

Proof. First we prove

$$\sup_{n \geq 1} \tilde{q}^n(x,x) \leq \tilde{p}(x) \qquad x \in E_c. \tag{4.3}$$

Let $\alpha \in (0,1]$ and $x \in E_c$ satisfy $\sup_{n \geq 1} \tilde{q}^n(x,x) \geq \alpha$. For each $\alpha' < \alpha$ there exists $n' \geq 1$ such that

$$x \in \tilde{r}_{\alpha'}^{n'}(\{x\}),$$

where $\tilde{r}^1_{\alpha'} := \tilde{r}_{\alpha'}$ and $\tilde{r}^{n+1}_{\alpha'} := \tilde{r}_{\alpha'}(\tilde{r}^n_{\alpha'})$ for $n \geq 1$. Then, by induction, we shall check

$$x \in \tilde{r}^{n'm}_{\alpha'}(\{x\}) \qquad \text{for all } m \geq 1. \tag{4.4}$$

(4.4) is trivial for $m = 1$. We assume that (4.4) holds for $m = 1, 2, \cdots, l$. From the definition,

$$x \in \tilde{r}^{n'l}_{\alpha'}(\{x\}) \subset \bigcup_{y \in \tilde{r}^{n'}_{\alpha'}(\{x\})} \tilde{r}^{n'l}_{\alpha'}(\{y\}) = \tilde{r}^{n'(l+1)}_{\alpha'}(\{x\}).$$

Therefore we obtain (4.4) inductively. On the other hand, considering a case of $\tilde{p}_0 := 1_{\{z\}}$ ($z \in E_c$) in (4.1), from Lemma 4.1(ii) and [5, Lemma 1],

$$\lim_{n \to \infty} \rho(\tilde{r}^n_{\alpha'}(\{z\}), \tilde{p}_{\alpha'}) = 0 \qquad \text{for all } z \in E_c. \tag{4.5}$$

From (4.4) and (4.5), we obtain $x \in \tilde{p}_{\alpha'}$ for $\alpha' < \alpha$. Therefore we get $x \in \tilde{p}_\alpha$, using Lemma 4.1(i) and [5, Lemma 3(i,b)]. Thus we get (4.3).

Let $x \in E_c$. Next, considering a case of $\tilde{p}_0 := 1_{\{x\}}$ in (4.1), we can easily check

$$\tilde{q}^n(x, x) = \tilde{p}_n(x) \qquad \text{for all } n \geq 1.$$

Together with (4.3), we obtain

$$\tilde{p}_m(x) \leq \sup_{n \geq 1} \tilde{q}^n(x, x) \leq \tilde{p}(x) \quad x \in E_c \qquad \text{for all } m \geq 1.$$

By Lemma 4.1(ii), we get

$$\tilde{p}_\alpha = \text{cl}\left\{x \in E_c \mid \sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha\right\} \qquad \text{for all } \alpha \in (0, 1].$$

Therefore the proof is completed. □

Finally we prove that the closure of the union of all α-recurrent sets equals to α-cuts of the limit fuzzy state $\tilde{p}$. Now we compare (1.1) and (4.1). Using the inverse fuzzy relation $\hat{q}$ ([3, Section 3.2]):

$$\hat{q}(x, y) := \tilde{q}(y, x) \qquad x, y \in E_c,$$

we find that (4.1) follows

$$\tilde{p}_{n+1}(x) = \sup_{x \in E_c} \{\hat{q}(x, y) \wedge \tilde{p}_n(y)\} \quad x \in E_c \quad \text{for } n \geq 0.$$

Therefore we can apply the results in Sections 1 – 3 to a dynamic fuzzy system defined by the inverse fuzzy relation $\hat{q}$.

Theorem 4.1.

$$\tilde{p}_\alpha = \text{cl}\left\{x \in E_c \mid \sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha\right\} = \text{cl}\left(\bigcup_{A \in \mathcal{E}(E)\,:\,\alpha-\text{recurrent sets}} A\right) \qquad \textit{for } \alpha \in (0, 1]. \tag{4.6}$$

Further it is the maximum α-recurrent set for X.

Proof. From the definition of the inverse fuzzy relation $\hat{q}$, we can easily check

$$\hat{q}^n(x,x) = \tilde{q}^n(x,x) \quad x \in E_c,\ n \geq 1,$$

where, in the same way as $\{\tilde{q}^n\}_{n\geq 1}$ of Section 1, we define

$$\hat{q}^1(x,y) := \hat{q}(x,y) \quad \text{and} \quad \hat{q}^{n+1}(x,y) := \sup_{z\in E_c}\{\hat{q}^n(x,z) \wedge \hat{q}(z,y)\} \quad x,y \in E_c,\ n \geq 1.$$

From Proposition 4.1,

$$\tilde{p}_\alpha = \mathrm{cl}\left\{x \in E_c \mid \sup_{n\geq 1} \tilde{q}^n(x,x) \geq \alpha\right\} = \mathrm{cl}\left\{x \in E_c \mid \sup_{n\geq 1} \hat{q}^n(x,x) \geq \alpha\right\} \quad \text{for } \alpha \in (0,1].$$

This equality means that the closure of the union of all α-recurrent sets for the fuzzy relation $\tilde{q}$ equals to one for the inverse fuzzy relation $\hat{q}$, considering Theorem 3.2 for the dynamic fuzzy systems defined by the fuzzy relations $\tilde{q}$ and $\hat{q}$. Therefore we obtain (4.6). Finally (4.5) means that $\tilde{p}_\alpha$ is the maximum α-recurrent. □

5. The monotone case

In general, there does not always exist the maximum α-recurrent set for the dynamic fuzzy system X, however we can consider the existence of the maximal α-recurrent sets. In this section we deal with a case when the transition fuzzy relation $\tilde{q}$ has a certain monotone property (see Section 6 for numerical examples). Then we prove the existence of at most countable arcwise connected maximal α-recurrent sets.

In this section we use the notations in Sections 1 – 3. Further we introduce the following notations of α-cuts ([5, Section 2]) :

$$\tilde{q}_\alpha(x) := \{y \in E \mid \tilde{q}(x,y) \geq \alpha\} \quad \text{for } x \in E \text{ and } \alpha \in (0,1];$$

$$\tilde{q}_\alpha(A) := \bigcup_{x\in A} \tilde{q}_\alpha(x) \quad \text{for } A \in \mathcal{E}(E) \text{ and } \alpha \in (0,1];$$

$$\tilde{q}_0(A) := \mathrm{cl}(\bigcup_{\alpha>0} \tilde{q}_\alpha(A)) \quad \text{for } A \in \mathcal{E}(E).$$

For $\alpha \in (0,1]$ and $x \in E$ we define a sequence $\{\tilde{q}^m_\alpha(x)\}_{m=1,2,\cdots}$:

$$\tilde{q}^1_\alpha(x) := \tilde{q}_\alpha(x); \quad \text{and} \quad \tilde{q}^{m+1}_\alpha(x) := \tilde{q}_\alpha(\tilde{q}^m_\alpha(x)) \quad \text{for } m = 1,2,\cdots.$$

We also need some elementary notations in the finite dimensional Euclidean space E: $x+y$ denotes the sum of $x, y \in E$ and γx denotes the product of a real number γ and $x \in E$. We put $A + B := \{x + y \mid x \in A,\ y \in B\}$ for $A, B \in \mathcal{E}(E)$. Then we define a half line on E by

$$l(x,y) := \{\gamma(y-x) \mid \text{real numbers } \gamma \geq 0\} \quad \text{for } x,y \in E.$$

Definition. We call a transition fuzzy relation $\tilde{q}$ unimodal provided that $\tilde{q}_\alpha(x)$ are bounded closed convex subsets of E for all $\alpha \in (0,1]$ and all $x \in E$.

Definition. We call a unimodal transition fuzzy relation $\tilde{q}$ monotone provided that

$$\tilde{q}_\alpha(y) \subset \tilde{q}_\alpha(x) + l(x,y) \quad \text{for all } \alpha \in (0,1] \text{ and all } x,y \in E.$$

From now on we deal with only unimodal fuzzy relations $\tilde{q}$, which is monotone and continuous on $E \times E$. The monotonicity is a natural extension of one-dimensional models with the linear structure in [11] and means that the fuzzy relations $\tilde{q}$ keeps the partial order of fuzzy numbers (see (C.iii') in Section 6).

Lemma 5.1. *Assume that $\tilde{q}$ is monotone. Let $\alpha \in (0,1]$. If $x \in E$ satisfies $x \in \bigcup_{m=1}^{\infty} \tilde{q}_\alpha^m(x)$, then $x \in \tilde{q}_\alpha(x)$.*

Proof. Let $x(\in E)$ satisfy $x \notin \tilde{q}_\alpha(x)$. We put

$$C_+ := \bigcup_{y \in \tilde{q}_\alpha(x)} \{\tilde{q}_\alpha(x) + l(x,y)\}.$$

Since $\tilde{q}$ is monotone, we can easily check C_+ is convex and we have

$$\tilde{q}_\alpha^2(x) = \bigcup_{y \in \tilde{q}_\alpha(x)} \tilde{q}_\alpha(y) \subset C_+. \tag{5.1}$$

Here we show

$$\bigcup_{y \in C_+} \tilde{q}_\alpha(y) \subset C_+. \tag{5.2}$$

Let $z \in \bigcup_{y \in C_+} \tilde{q}_\alpha(y)$. Since $\tilde{q}$ is monotone, there exists $y_1 \in C_+$ such that $z \in \tilde{q}_\alpha(x) + l(x,y_1)$. So there exists $y_2 \in \tilde{q}_\alpha(x)$ such that $y_1 \in \tilde{q}_\alpha(x) + l(x,y_2)$. From the definitions, there exist $z_1 \in \tilde{q}_\alpha(x)$ and a real number $\gamma_1 \geq 0$ such that

$$z = z_1 + \gamma_1(y_1 - x) \tag{5.3}$$

and there exist $z_2 \in \tilde{q}_\alpha(x)$ and a real number $\gamma_2 \geq 0$ such that

$$y_1 = z_2 + \gamma_2(y_2 - x). \tag{5.4}$$

Since $\tilde{q}$ is unimodal, from (5.3) and (5.4) we obtain that

$$z = z_1 + \gamma_1(z_2 + \gamma_2(y_2 - x) - x) = z_1 + (\gamma_1 + \gamma_1\gamma_2)(\frac{\gamma_1 z_2 + \gamma_1\gamma_2 y_2}{\gamma_1 + \gamma_1\gamma_2} - x) \in C_+ \quad \text{if } \gamma_1 > 0$$

and that $z = z_1 \in C_+$ if $\gamma_1 = 0$. Thus we get (5.2). Therefore from (5.1) and (5.2)

$$\tilde{q}_\alpha^3(x) = \bigcup_{y \in \tilde{q}_\alpha^2(x)} \tilde{q}_\alpha(y) \subset \bigcup_{y \in C_+} \tilde{q}_\alpha(y) \subset C_+.$$

Thus using (5.2) inductively, we obtain

$$\bigcup_{m=1}^{\infty} \tilde{q}_{\alpha}^{m}(x) \subset C_{+}. \tag{5.5}$$

On the other hand we show $x \notin C_{+}$. If $x \in C_{+}$, then there exist $z, y \in \tilde{q}_{\alpha}(x)$ and a real number $\gamma \geq 0$ such that $x = z + \gamma(y - x)$. Therefore

$$x = \frac{z + \gamma y}{1 + \gamma} \in \tilde{q}_{\alpha}(x).$$

This contradicts the assumption on x at the beginning of this proof. Therefore we get $x \notin C_{+}$. Together (5.5), this implies

$$x \notin \bigcup_{m=1}^{\infty} \tilde{q}_{\alpha}^{m}(x).$$

Thus we obtain this lemma. □

When $\tilde{q}$ is monotone, Theorem 3.1 is reduced to the following representation (5.6), which is easy to calculate.

Theorem 5.1. *Assume that $\tilde{q}$ is monotone. Let $\alpha \in (0,1]$. Then*

$$\bigcup_{A \in \mathcal{E}(E)\,:\,\alpha-\text{recurrent sets}} A = \{x \in E \mid \tilde{q}(x,x) \geq \alpha\}. \tag{5.6}$$

Proof. Let $x_1 \in \{x \in E \mid \tilde{q}(x,x) \geq \alpha\}$. Then $P1_{\{x_1\}}(x_1) \geq \alpha$. So $PG1_{\{x_1\}} \geq P1_{\{x_1\}} \geq \alpha 1_{\{x_1\}}$. Therefore $\{x_1\}$ is α-recurrent and so we obtain

$$\{x \in E \mid \tilde{q}(x,x) \geq \alpha\} \subset \bigcup_{A \in \mathcal{E}(E)\,:\,\alpha-\text{recurrent sets}} A.$$

Conversely let $A(\in \mathcal{E}(E))$ be α-recurrent. Let $x_1 \in A$. From Proposition 3.1,

$$PG1_{\{x_1\}} \geq \alpha 1_A \geq \alpha 1_{\{x_1\}}.$$

Therefore

$$x_1 \in \bigcup_{m=1}^{\infty} \tilde{q}_{\alpha'}^{m}(x_1) \quad \text{for all } \alpha' < \alpha.$$

From Lemma 3.2 we obtain

$$x_1 \in \tilde{q}_{\alpha'}(x_1) \quad \text{for all } \alpha' < \alpha.$$

Namely we get $\tilde{q}(x_1, x_1) \geq \alpha'$ for all $\alpha' < \alpha$. So we get $\tilde{q}(x_1, x_1) \geq \alpha$. Therefore $A \subset \{x \in E \mid \tilde{q}(x,x) \geq \alpha\}$. Thus we establish this theorem. □

We need the following assumption on $\tilde{q}$, which is technical but not so strong. It means that the function $\tilde{q}$ does not have flat areas as a curved surface (Section 6).

Assumption (A). For $\alpha \in (0,1)$,

$$\text{int}\,\{(x,y) \in E \times E \mid \tilde{q}(x,y) \geq \alpha\} = \{(x,y) \in E \times E \mid \tilde{q}(x,y) > \alpha\},$$

where int denotes the interior of a set.

Since $\tilde{q}$ is continuous, $\{x \in E \mid \tilde{q}(x,x) \geq \alpha\}$ is represented by a disjoint sum of at most countable arcwise connected closed sets ([4]), we represent it by

$$\{x \in E \mid \tilde{q}(x,x) \geq \alpha\} = \bigcup_{n \in \mathbf{N}(\alpha)} F_{\alpha,n} \quad \text{for } \alpha \in (0,1),$$

where $F_{\alpha,n}$ are arcwise connected closed subsets of E and we put the index set $\mathbf{N}(\alpha)$ $(\subset \mathbf{N})$.

Lemma 5.2. *We suppose Assumption (A). Let $\alpha \in (0,1)$ and $n \in \mathbf{N}(\alpha)$. Then $F_{\alpha,n}$ is α-recurrent.*

Proof. We write the interior of $F_{\alpha,n}$ by $F^o_{\alpha,n}$. First we prove that $F^o_{\alpha,n}$ is α-recurrent. Let $x_0 \in F^o_{\alpha,n}$. Let $c(x_0)$ be an arc in $F^o_{\alpha,n}$, which is connected from x_0 to a boundary point of $F_{\alpha,n}$. We consider along the arc $c(x_0)$. Then we show

$$c(x_0) \cap F^o_{\alpha,n} \subset \bigcup_{m \geq 1} \tilde{q}^m_\alpha(x_0). \tag{5.7}$$

Let x_1 be the first point arriving at the boundary of $\tilde{q}_\alpha(x_0)$ along $c(x_0)$. If either there do not exist such points or x_1 is a boundary point of $F_{\alpha,n}$, then $c(x_0) \subset \tilde{q}_\alpha(x_0)$ and clearly (5.7) holds. Therefore it is sufficient to consider a case of $x_1 \in F^o_{\alpha,n}$. Since $x_0 \in F^o_{\alpha,n}$, we have $x_0 \in (\tilde{q}_\alpha(x_0))^o$ and $d(x_0,x_1) > 0$ from Assumption (A). From $x_1 \in F^o_{\alpha,n} \cap c(x_0)$, we also define x_2 the first point arriving at the boundary of $\tilde{q}_\alpha(x_1)$ along $c(x_0)$. If either there do not exist such points or x_2 is a boundary point of $F_{\alpha,n}$, then similarly $c(x_0) \subset \tilde{q}_\alpha(x_1) \subset \tilde{q}^2_\alpha(x_0)$ and (5.7) holds. Therefore it is sufficient to consider a case of $x_2 \in F^o_{\alpha,n}$. Thus it is sufficient to check a sequence $\{x_l\}_{l=0,1,2,\cdots}$ which is defined successively in such a manner and which has the following three properties (Fig. 5.1) :

(a) $x_l \in F^o_{\alpha,n} \cap c(x_0)$ $(l = 0,1,2,\cdots)$;

(b) x_{l+1} is the boundary point of $\tilde{q}_\alpha(x_l)$ $(l = 0,1,2,\cdots)$;

(c) $d(x_l,x_{l+1}) > 0$ $(l = 0,1,2,\cdots)$.

Then there exists a limit point $x = \lim_{l\to\infty} x_l$ since $\tilde{q}_\alpha(x_0)$ is bounded and $c(x_0)$ is so. From the property (b) and Assumption (A), $\tilde{q}(x_l,x_{l+1}) = \alpha$ $(l = 0,1,2,\cdots)$. Using the continuity of $\tilde{q}$ and Assumption (A), we obtain $\tilde{q}(x,x) = \alpha$ and x is a boundary point of $F_{\alpha,n}$. Therefore (5.7) also holds for this case. Thus we obtain (5.7) in any cases. Since $x_0 (\in F^o_{\alpha,n})$ and the arc $c(x_0)$ are arbitrary in (5.7), we have

$$F^o_{\alpha,n} \subset \bigcup_{m \geq 1} \tilde{q}^m_\alpha(x) \qquad \text{for all } x \in F^o_{\alpha,n}. \tag{5.8}$$

This implies that $F^o_{\alpha,n}$ is α-recurrent for all $\alpha \in (0,1)$ and all $n \in \mathbf{N}(\alpha)$.

Next from the continuity of $\tilde{q}$ and (5.8), for all $\alpha \in (0,1)$ and $x \in F^o_{\alpha,n}$ we obtain

$$F_{\alpha,n} = \bigcap_{\alpha'<\alpha} F^o_{\alpha',n} \subset \bigcap_{\alpha'<\alpha} \bigcup_{m\geq 1} \tilde{q}^m_{\alpha'}(x) = \{y \in E \mid \sup_{m\geq 1} \tilde{q}^m(x,y) \geq \alpha\}.$$

Using this result and Proposition 2.1(ii), for $\alpha \in (0,1)$ and $x \in F_{\alpha,n} = \bigcap_{\alpha'<\alpha} F^o_{\alpha',n}$ we get

$$\begin{aligned}
F_{\alpha,n} &= \bigcap_{\alpha'<\alpha} F_{\alpha',n} \\
&\subset \bigcap_{\alpha'<\alpha} \{y \in E \mid \sup_{m\geq 1} \tilde{q}^m(x,y) \geq \alpha'\} \\
&= \{y \in E \mid \sup_{m\geq 1} \tilde{q}^m(x,y) \geq \alpha\} \\
&= \{y \in E \mid PGI_{\{y\}}1(x) \geq \alpha\} \\
&= \{y \in E \mid P_{\sigma_{\{y\}}}1(x) \geq \alpha\}.
\end{aligned}$$

This means that $F_{\alpha,n}$ is α-recurrent. □

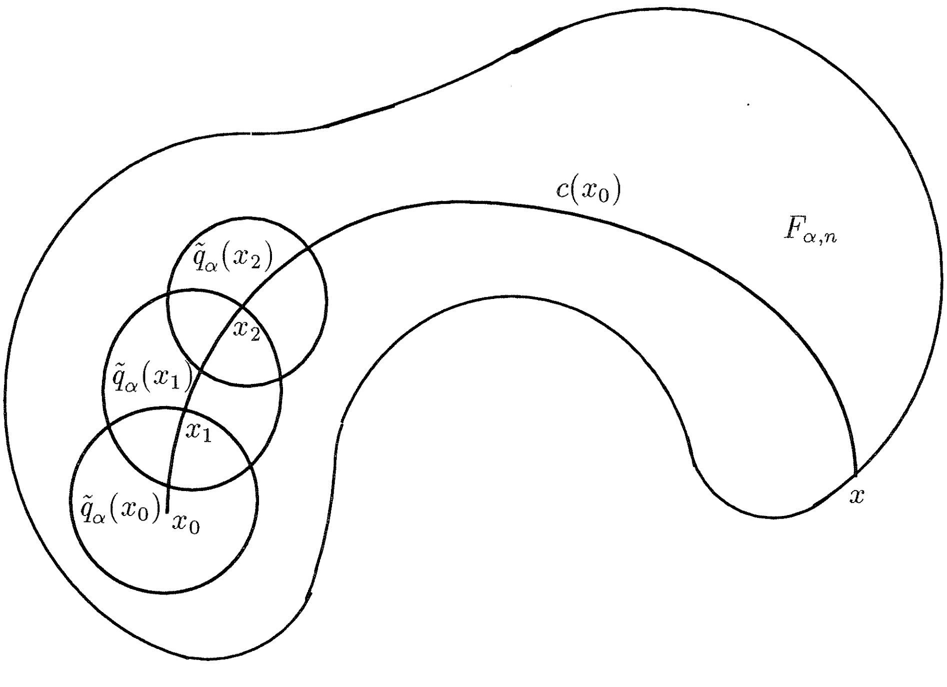

Fig. 5.1. The arcwise connected set $F_{\alpha,n}$ and the sequence $\{x_l\}_{l=0,1,2,\cdots}$.

Theorem 5.2. *We suppose Assumption (A). Let $\alpha \in (0,1)$. Then maximal α-recurrent sets for X are $F_{\alpha,n}$ $(n \in \mathbf{N}(\alpha))$.*

Proof. We show that α-recurrent sets $F_{\alpha,n}$ $(n \in \mathbf{N}(\alpha))$ are maximal. It is sufficient to prove that $F_{\alpha,n} \cup F_{\alpha,n'}$ is not α-recurrent, assuming that $\mathbf{N}(\alpha)$ has at least two elements

n, n' $(n \neq n')$. Now we suppose that there exist $n, n' \in \mathbf{N}(\alpha)$ $(n \neq n')$ such that $F_{\alpha,n} \cup F_{\alpha,n'}$ is α-recurrent. Then there exist sequences $\{x_m\}_{m=0,1,2,\cdots}$ and $\{x'_m\}_{m=0,1,2,\cdots}$ satisfying (a) – (d) :

(a) $x_0 \in F_{\alpha,n}$ and $\lim_{m\to\infty} x_m \in F_{\alpha,n'}$;

(b) $x'_0 \in F_{\alpha,n'}$ and $\lim_{m\to\infty} x'_m \in F_{\alpha,n}$;

(c) $x_{m+1} \in \tilde{q}_\alpha(x_m)$ $(m = 0, 1, 2, \cdots)$;

(d) $x'_{m+1} \in \tilde{q}_\alpha(x'_m)$ $(m = 0, 1, 2, \cdots)$.

We consider the following three cases :

Case when there exists a point $x_{m'}$ such that $x_{m'} \notin \tilde{q}_\alpha(x_{m'})$: Then we have

$$x_{m'} \notin \{x \mid \tilde{q}(x, x) \geq \alpha\}. \tag{5.9}$$

Since $F_{\alpha,n}$ and $F_{\alpha,n'}$ are α-recurrent sets, together with (a) – (d), there exists a path from $x_{m'}$ to itself through $F_{\alpha,n}$ and $F_{\alpha,n'}$, keeping a level of $\tilde{q}$ greater than α. Therefore $\{x_{m'}\} \cup F_{\alpha,n} \cup F_{\alpha,n'}$ becomes α-recurrent. By Theorem 5.1, this fact contradicts (5.9).

Case when there exists a point $x'_{m'}$ such that $x'_{m'} \notin \tilde{q}_\alpha(x'_{m'})$: We can derive a contradiction in the same way as the previous case.

Case when $x_m \in \tilde{q}_\alpha(x_m)$ $(m = 0, 1, 2, \cdots)$ and $x'_m \in \tilde{q}_\alpha(x'_m)$ $(m = 0, 1, 2, \cdots)$: From the assumption that $F_{\alpha,n} \cup F_{\alpha,n'}$ is α-recurrent, there exists m' such that $x_{m'} \in F_{\alpha,n}$ and $x_{m'+1} \in F_{\alpha,n'}$. Therefore

$$x_{m'+1} \in \tilde{q}_\alpha(x_{m'}) \cap \tilde{q}_\alpha(x_{m'+1}). \tag{5.10}$$

There exists a point $y \notin F_{\alpha,n} \cup F_{\alpha,n'}$ such that $y = \lambda x_{m'} + (1 - \lambda)x_{m'+1}$ $(0 < \lambda < 1)$ since $F_{\alpha,n}$ and $F_{\alpha,n'}$ are arcwise connected, closed and disjoint. Then we may take

$$y \notin \{x \mid \tilde{q}(x, x) \geq \alpha\}. \tag{5.11}$$

On the other hand, since $\tilde{q}$ is monotone, we have

$$\tilde{q}_\alpha(x_{m'+1}) \subset \tilde{q}_\alpha(y) + l(y, x_{m'+1}) = \tilde{q}_\alpha(y) + l(x_{m'}, x_{m'+1}) \tag{5.12}$$

and

$$\tilde{q}_\alpha(x_{m'}) \subset \tilde{q}_\alpha(y) + l(y, x_{m'}) = \tilde{q}_\alpha(y) + l(x_{m'+1}, x_{m'}). \tag{5.13}$$

From (5.10), (5.12) and (5.13), we obtain

$$x_{m'+1} \in (\tilde{q}_\alpha(y) + l(x_{m'}, x_{m'+1})) \cap (\tilde{q}_\alpha(y) + l(x_{m'+1}, x_{m'})) = \tilde{q}_\alpha(y). \tag{5.14}$$

Further since $\tilde{q}_\alpha(x_{m'})$ is convex, from $x_{m'} \in \tilde{q}_\alpha(x_{m'})$ and (5.10), we have

$$y = \lambda x_{m'} + (1-\lambda)x_{m'+1} \in \tilde{q}_\alpha(x_{m'}). \tag{5.15}$$

From (5.14) and (5.15), we get

$$x_{m'+1} \in \tilde{q}_\alpha(y) \quad \text{and} \quad y \in \tilde{q}_\alpha(x_{m'}).$$

Since $F_{\alpha,n}$ and $F_{\alpha,n'}$ are α-recurrent sets, together with (a) – (d), there exists a path from y to itself through $F_{\alpha,n}$ and $F_{\alpha,n'}$, keeping a level of $\tilde{q}$ greater than α. By Theorem 5.1, this fact also contradicts (5.11).

Therefore $F_{\alpha,n}$ $(n \in \mathbf{N}(\alpha))$ are maximal α-recurrent. By Theorem 5.1, we obtain that maximal α-recurrent sets are only $F_{\alpha,n}$ $(n \in \mathbf{N}(\alpha))$. □

Remark. When $\alpha = 1$, Theorem 5.2 does not hold in general. We consider the following non-contractive numerical example : Let a one-dimensional state space $E = \mathbf{R}$ (the set of all real numbers). We give a fuzzy relation by

$$\tilde{q}(x,y) = (1 - |y-x|) \vee 0, \quad x, y \in \mathbf{R}.$$

Then we have

$$\{x \in \mathbf{R} \mid \tilde{q}(x,x) = 1\} = \mathbf{R}.$$

Further we can easily check that every one point set $\{x\}$ $(x \in \mathbf{R})$ are maximal 1-recurrent sets since $\{x\} = \tilde{q}_1(x)$ $(x \in \mathbf{R})$.

6. Numerical examples

Let a one-dimensional state space $E = \mathbf{R}$. We consider one-dimensional numerical examples. In Section 5 we have assumed the following conditions (C.i) — (C.iv):

(C.i) $\tilde{q}$ is continuous on $E \times E$;

(C.ii) $\tilde{q}$ is unimodal;

(C.iii) $\tilde{q}$ is monotone;

(C.iv) $\tilde{q}$ satisfies Assumption (A).

When $E = \mathbf{R}$, $\mathcal{F}^0(\mathbf{R})$ means all fuzzy numbers on $\mathbf{R}$. From (C.ii), $\tilde{q}_\alpha(x)$ are bounded closed intervals of $\mathbf{R}$ $(\alpha \in (0,1], x \in \mathbf{R})$. So we write $\tilde{q}_\alpha(x) = [\min \tilde{q}_\alpha(x), \max \tilde{q}_\alpha(x)]$, where $\min A$ ($\max A$) denotes the minimum (maximum resp.) point of a interval $A \subset \mathbf{R}$. Then (C.iii) is equivalent to the following (C.iii') :

(C.iii') $\min \tilde{q}_\alpha(\cdot)$ and $\max \tilde{q}_\alpha(\cdot)$ are non-decreasing functions on $\mathbf{R}$ for all $\alpha \in (0,1]$.

Next we consider the following partial order $\preceq$ on $\mathcal{F}^0(\mathbf{R})$ (Nanda [6]) : For $\tilde{s}, \tilde{r} \in \mathcal{F}^0(\mathbf{R})$,

$$\tilde{s} \preceq \tilde{r} \quad \text{means that} \quad \min \tilde{s}_\alpha \le \min \tilde{r}_\alpha \text{ and } \max \tilde{s}_\alpha \le \max \tilde{r}_\alpha \text{ for all } \alpha \in (0,1].$$

Then we can easily find that (C.iii) is equivalent to the following (C.iii") :

(C.iii") If $\tilde{s}, \tilde{r} \in \mathcal{F}^0(\mathbf{R})$ satisfy $\tilde{s} \preceq \tilde{r}$, then $Q(\tilde{s}) \preceq Q(\tilde{r})$,

where $Q : \mathcal{F}^0(\mathbf{R}) \mapsto \mathcal{F}^0(\mathbf{R})$, see (4.1), is defined by

$$Q\tilde{s}(y) = \max_{x \in \mathbf{R}} \{\tilde{s}(x) \wedge \tilde{q}(x,y)\}, \quad y \in \mathbf{R} \quad \text{for } \tilde{s} \in \mathcal{F}^0(\mathbf{R}).$$

(C.iii") means that Q preserves the monotonicity on $\mathcal{F}^0(\mathbf{R})$ with respect to the order $\preceq$. Finally (C.iv) means that the α-level sets $\{x \in \mathbf{R} \mid \tilde{q}(x,x) = \alpha\}$ ($\alpha \in (0,1)$) are drawn by not areas but curved lines. The linear case of [11, Fig. 2] clearly satisfies the above conditions (C.i) – (C.iv), taking the state space $E = (0, \infty)$.

We give an example of monotone fuzzy relations, which is not contractive and does not have the linear structure in [11]. Then we calculate its maximal α-recurrent sets.

Example 6.1 (monotone case). We give a fuzzy relation by

$$\tilde{q}(x,y) = (1 - |y - x^3|) \vee 0, \quad x, y \in \mathbf{R}.$$

Then $\tilde{q}(x,y)$ satisfies the conditions (C.i) – (C.iv) (see Figure 6.1 for the fuzzy relation $\tilde{q}(x,y)$ and Figure 6.2 for the $\frac{3}{4}$-level sets).

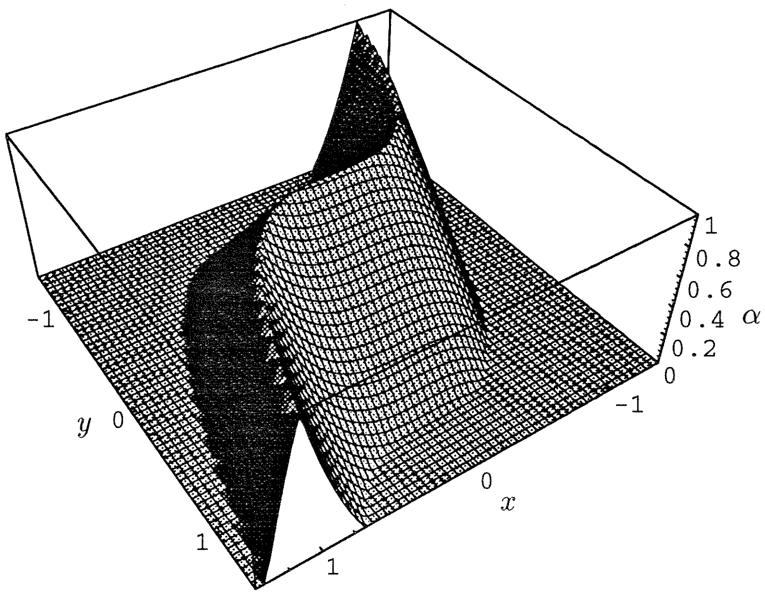

Fig. 6.1 : The monotone fuzzy relation $\tilde{q}(x,y)$.

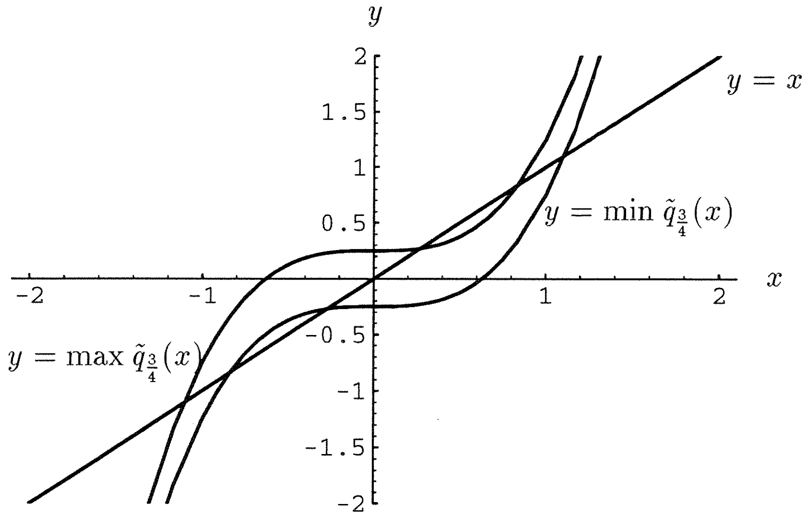

Fig. 6.2. The $\frac{3}{4}$-level sets $\{(x,y) \mid \tilde{q}(x,y) = \frac{3}{4}\}$.

Then we have

$$\tilde{q}(x,x) = (1 - |x - x^3|) \vee 0, \quad x \in \mathbf{R}.$$

Therefore $\mathbf{N}(\frac{3}{4}) = \{0,1,2\}$ and

$$\begin{aligned}&\left\{x \in \mathbf{R} \mid \tilde{q}(x,x) \geq \tfrac{3}{4}\right\} = F_{\frac{3}{4},0} \cup F_{\frac{3}{4},1} \cup F_{\frac{3}{4},2} \\ &\approx [-1.10716, -0.837565] \cup [-0.269594, 0.269594] \cup [0.837565, 1.10716].\end{aligned}$$

By Theorem 5.2, the maximal $\frac{3}{4}$-recurrent sets are given by three intervals

$$\begin{aligned} F_{\frac{3}{4},0} &\approx [-1.10716, -0.837565], \\ F_{\frac{3}{4},1} &\approx [-0.269594, 0.269594], \\ F_{\frac{3}{4},2} &\approx [0.837565, 1.10716]. \end{aligned}$$

Finally we consider the following numerical example, which is not monotone.

Example 6.2 (non-monotone case). We consider a fuzzy relation

$$\tilde{q}(x,y) = \max\left\{\left(1 - 2\left|y - \frac{1}{4}x\right|\right) \vee 0, \min\left\{\left(1 - \frac{1}{4}\left|y - \frac{1}{4}x\right|\right) \vee 0, \frac{3}{2}|x| \wedge 1\right\}\right\}, \quad x, y \in \mathbf{R}.$$

Then $\tilde{q}(x,y)$ satisfies the conditions (C.i), (C.ii) and (C.iv) except for (C.iii) (see Figure 6.3 for the fuzzy relation $\tilde{q}(x,y)$ and Figure 6.4 for the $\frac{3}{4}$-level sets).

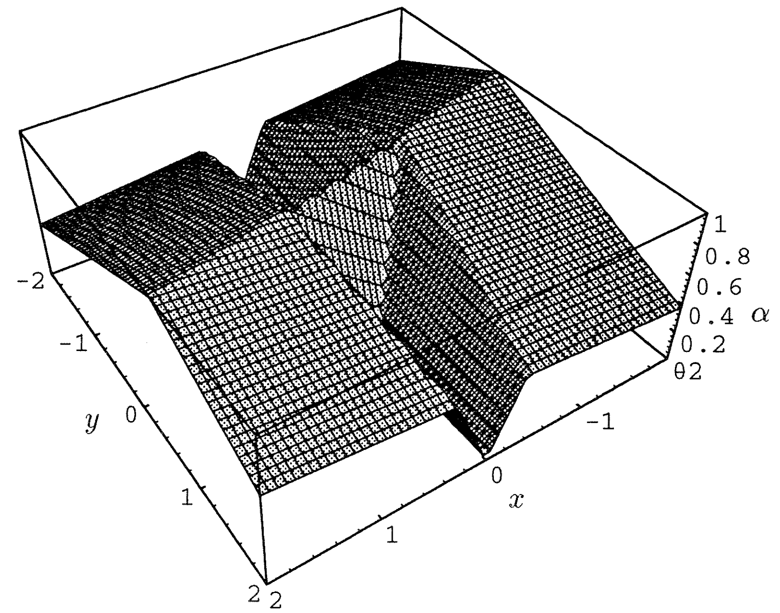

Fig. 6.3 : The non-monotone fuzzy relation $\tilde{q}(x, y)$.

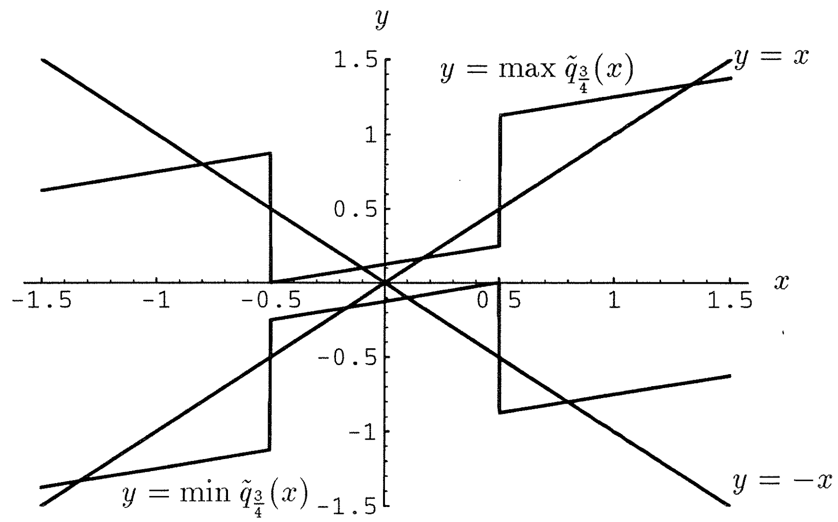

Fig. 6.4. The $\frac{3}{4}$-level set $\{(x, y) \mid \tilde{q}(x, y) = \frac{3}{4}\}$.

Then

$$\left\{x \in \mathbf{R} \mid \tilde{q}(x, x) \geq \frac{3}{4}\right\} = \left\{x \in \mathbf{R} \mid \sup_{n \geq 1} \tilde{q}^n(x, x) \geq \frac{3}{4}\right\} = \left[-\frac{4}{3}, -\frac{1}{2}\right] \cup \left[-\frac{1}{6}, \frac{1}{6}\right] \cup \left[\frac{1}{2}, \frac{4}{3}\right].$$

We can easily check the maximal $\frac{3}{4}$-recurrent sets are

$$\left[-\frac{1}{6}, \frac{1}{6}\right] \quad \text{and} \quad \left[-\frac{4}{3}, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{4}{3}\right].$$

Therefore, in non-monotone case, Theorem 5.2 does not hold in general.

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