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Abstract. The Vapnik-Chervonenkis (VC) dimension is known to be the crucial measure of the polynomial-sample learnability in the PAC-learning model. This paper investigates the complexity of computing VC-dimension of a concept class over a finite learning domain. We consider a decision problem called the discrete VC-dimension problem which is, for a given matrix representing a concept class $\mathcal{F}$ and an integer K, to determine whether the VC-dimension of $\mathcal{F}$ is greater than K or not. We prove that (1) the discrete VC-dimension problem is polynomial-time reducible to the satisfiability problem of length J with $O(\log^2 J)$ variables, and (2) for every constant C, the satisfiability problem in conjunctive normal form with m clauses and $C \log^2 m$ variables is polynomial-time reducible to the discrete VC-dimension problem. These results can be interpreted, in some sense, that the problem is "complete" for the class of $n^{O(\log n)}$ time computable sets.

1 Introduction

The PAC learnability due to Valiant [8] is to estimate the feasibility of learning a concept probably approximately correctly, from a reasonable amount of examples (polynomial-sample), within a reasonable amount of time (polynomial-time). It is well-known that the Vapnik-Chervonenkis Dimension (VC-dimension) which is a combinatorial parameter of a concept class plays the key role to determine whether the concept class is polynomial-sample learnable or not [2, 3, 5].

This paper settles a complexity issue on VC-dimension of a concept class over a finite learning domain. We remark that the complexity of computing VC-dimension is of independent interest from the polynomial-time learnability, since it is not directly related to the running time of learning algorithms.

Linial et al. [3] showed that the VC-dimension of a concept class over a finite learning domain can be computed in $n^{O(\log n)}$ time, where n is the size of a given matrix which represents the concept class. Nienhuys-Cheng and Polman [6] gave another $n^{O(\log n)}$-time algorithm, although they have not analyzed its running time. On the other hand, Linial et al. [3] pointed out that the decision version of the problem called the *discrete VC-dimension problem* may have some connection with the problem of finding a minimum dominating set in a tournament, which is shown by Megiddo and Vishkin [4] to be a kind of "complete" problem for the class of $n^{O(\log n)}$ time computable sets.

Along this line, we show that the discrete VC-dimension problem is also "complete" for the class of $n^{O(\log n)}$ time computable sets in the same sense. That is, we give the following two reductions: (1) The discrete VC-dimension problem is reducible in polynomial time to the satisfiability problem of a boolean formula of length J with $O(\log^2 J)$ variables. (2) On the other hand, for every constant C, the satisfiability problem in conjunctive normal form with m clauses and $C\log^2 m$ variables is polynomial-time reducible to the discrete VC-dimension problem. Therefore we can interpret that the discrete VC-dimension problem is one of the natural problems which seem to be neither NP-complete, nor in P.

2 Preliminaries

For a matrix M, let M_{ij} denote the element on row i and column j of M, and the size of M is the number of elements in M. The length of a boolean formula ψ, denoted by $|\psi|$, is the total number of variable occurrences in ψ. For a boolean formula ψ, we denote $[\psi, 1] = \psi$ and $[\psi, 0] = \neg\psi$. For any integers $i \geq 1$ and $t \geq 1$, let $b(i,t)$ denote the t-th binary digit of $(i-1)$, that is, $i = \sum_{t=1}^{\lceil \log i \rceil} 2^{t-1} \cdot b(i,t) + 1$. For example, $b(7,1) = 0$, $b(7,2) = 1$, and $b(7,3) = 1$.

Let U be a finite set called a *learning domain.* We call a subset f of U a *concept.* A concept f can be regarded as a function $f : U \to \{0,1\}$, where $f(x) = 1$ if x is in the concept and $f(x) = 0$ otherwise. A *concept class* is a nonempty set $\mathcal{F} \subseteq 2^U$. We represent a concept class $\mathcal{F}$ over a finite learning domain U, by a $|U| \times |\mathcal{F}|$ matrix M with $M_{ij} = f_j(x_i)$. Each column represents a concept in $\mathcal{F}$. For a $\{0,1\}$-valued matrix M, let $\mathcal{F}_M$ denote the concept class which M represents.

Definition 1. We say that $\mathcal{F}$ *shatters* a set $S \subseteq U$ if for every subset $T \subseteq S$ there exists a concept $f \in \mathcal{F}$ which *cuts* T *out of* S, i.e., $T = S \cap f$. The *Vapnik-Chervonenkis dimension* of $\mathcal{F}$, denoted by VC-dim($\mathcal{F}$), is the maximum cardinality of a set which is shattered by $\mathcal{F}$.

Lemma 2. *[5] For any concept class $\mathcal{F}$, VC-dim$(\mathcal{F}) \leq \log|\mathcal{F}|$.*

By this lemma, Linial et al. [3] immediately claimed that a simple algorithm which enumerates all possible sets to be shattered shall terminate in $n^{O(\log n)}$ time, where n is the size of a given matrix.

Definition 3. [3] The *discrete VC-dimension problem* is, given a $\{0,1\}$-valued matrix M and integer $K \geq 1$, to determine whether VC-dim$\mathcal{F}_M \geq K$ or not.

Definition 4. [4] The classes $\mathrm{SAT}_{\log^k n}$ and $\mathrm{SAT}^{\mathrm{CNF}}_{\log^k n}$ for $k \geq 1$ are defined as follows:

(1) A set L is in $\mathrm{SAT}_{\log^k n}$ if there exists a Turing machine M, a polynomial $p(n)$, and a constant C, such that for every string I of length n, M converts I within $p(n)$ time into a boolean formula Ψ_I (whose length is necessarily less than $p(n)$) with at most $C\log^k n$ variables, so that $I \in L$ if and only if Ψ_I is satisfiable.
(2) The definition of $\mathrm{SAT}^{\mathrm{CNF}}_{\log^k n}$ is essentially the same as that of $\mathrm{SAT}_{\log^k n}$ except that the formula Ψ_I is in conjunctive normal form.

From the definitions, it is easy to see that for each $k \geq 1$,

$$P \subseteq \mathrm{SAT}^{\mathrm{CNF}}_{\log^k n} \subseteq \mathrm{SAT}_{\log^k n} \subseteq NP.$$

3 Discrete VC-dimension Problem is in $\mathrm{SAT}_{\log^2 n}$

In this section, we show that the discrete VC-dimension problem is polynomial-time reducible to the satisfiability problem of a boolean formula of length J with $O(\log^2 J)$ variables.

Theorem 5. *The discrete VC-dimension problem is in $SAT_{\log^2 n}$.*

Proof. Let M be an $m \times r$ matrix and K be an integer. By Lemma 2, we can assume that $K \le \log r$ without loss of generality. Moreover, we can also assume that $m = 2^l$ for some integer l; if $m < 2^l$ for $l = \lceil \log m \rceil$, then we enlarge M by duplicating the last row of M until the row size reaches 2^l. It is easy to see that the size of the enlarged matrix M' is less than twice as large as that of the original matrix M, and $\text{VC-dim}(\mathcal{F}_{M'}) = \text{VC-dim}(\mathcal{F}_M)$.

We now construct a boolean formula Ψ_M which contains $K \cdot l$ variables v_{kt} ($1 \le k \le K$, $1 \le t \le l$) as follows:

$$\begin{aligned}
\Psi_M &= \textstyle\bigwedge_{s=1}^{2^K} \bigvee_{j=1}^{r} \beta_{sj}, \\
\beta_{sj} &= \textstyle\bigwedge_{k=1}^{K} [\alpha_{kj}, b(s,k)] && (1 \le s \le 2^K,\ \ 1 \le j \le r), \\
\alpha_{kj} &= \textstyle\bigvee_{i \in \{i \mid M_{ij}=1\}} \psi_{ki} && (1 \le k \le K,\ 1 \le j \le r), \\
\psi_{ki} &= \textstyle\bigwedge_{t=1}^{l} [v_{kt}, b(i,t)] && (1 \le k \le K,\ 1 \le i \le m).
\end{aligned}$$

Note that the length of Ψ_M is

$$\begin{aligned}
|\Psi_M| &\le l \cdot m \cdot K \cdot r \cdot 2^K \\
&\le \log m \cdot m \cdot \log r \cdot r \cdot 2^{\log r} \\
&< n^2 \log^2 n,
\end{aligned}$$

where $n = m \cdot r$ is the size of the given matrix M. Also note that Ψ_M can be constructed in polynomial time with respect to n.

Let $U = \{x_1, x_2, \ldots, x_m\}$ be the learning domain and $\mathcal{F}_M = \{f_1, f_2, \cdots, f_r\} \subseteq 2^U$ be the concept class which M represents. We will show that the formula Ψ_M is satisfiable if and only if $\mathcal{F}_M$ shatters a set $S \subseteq U$ of cardinality K.

For a formula ψ and a truth assignment σ to the variables of ψ, let $\sigma(\psi)$ denote the truth value of ψ evaluated under σ. We denote truth values by 0 and 1. For each assignment σ, we define a set $S_\sigma \subseteq U$ as follows:

$$S_\sigma = \{x_{\langle \sigma, k \rangle} \mid 1 \le k \le K\}, \quad \text{where } \langle \sigma, k \rangle = \sum_{t=1}^{l} 2^{t-1} \cdot \sigma(v_{kt}) + 1.$$

It should be noticed that the cardinality of S_σ is not always equal to K, since there may be two distinct k_1 and k_2 with $\langle \sigma, k_1 \rangle = \langle \sigma, k_2 \rangle$ in general.

We now show through a sequence of equivalences that an assignment σ satisfies Ψ_M if and only if $|S_\sigma| = K$ and S_σ is shattered by $\mathcal{F}_M$.

First, for any $k \in \{1, \ldots, K\}$ and any $i \in \{1, \ldots, m\}$,

$$
\begin{aligned}
&\sigma(\psi_{ki}) = 1 \\
&\iff \sigma([v_{kt}, b(i,t)]) = 1 \text{ for each } t \in \{1, \ldots, l\} \\
&\iff \sigma(v_{kt}) = \begin{cases} 1 & \text{if } b(i,t) = 1 \\ 0 & \text{if } b(i,t) = 0 \end{cases} \quad \text{for each } t \in \{1, \ldots, l\} \\
&\iff b(i,t) = \sigma(v_{kt}) \text{ for each } t \in \{1, \ldots, l\} \\
&\iff \sum_{t=1}^{l} 2^{t-1} \cdot b(i,t) = \sum_{t=1}^{l} 2^{t-1} \cdot \sigma(v_{kt}) \\
&\iff i = \langle \sigma, k \rangle.
\end{aligned}
$$

Next, for any $k \in \{1, \ldots, K\}$ and any $j \in \{1, \ldots, r\}$,

$$
\begin{aligned}
&\sigma(\alpha_{kj}) = 1 \\
&\iff \sigma(\psi_{ki}) = 1 \text{ and } M_{ij} = 1 \text{ for some } i \in \{1, \ldots, m\} \\
&\iff i = \langle \sigma, k \rangle \text{ and } x_i \in f_j \\
&\iff x_{\langle \sigma, k \rangle} \in f_j.
\end{aligned}
$$

For an integer $s \in \{1, \ldots, 2^K\}$, the s-th subset $S_\sigma^{[s]}$ of S_σ is defined by $S_\sigma^{[s]} = \{x_{\langle \sigma, k \rangle} \mid b(s,k) = 1,\ 1 \leq k \leq K\}$. For example, $S_\sigma^{[1]} = \emptyset$, $S_\sigma^{[5]} = \{x_{\langle \sigma, 3 \rangle}\}$ and $S_\sigma^{[6]} = \{x_{\langle \sigma, 1 \rangle}, x_{\langle \sigma, 3 \rangle}\}$. Then, for any $s \in \{1, \ldots, 2^K\}$ and any $j \in \{1, \ldots, r\}$,

$$
\begin{aligned}
&\sigma(\beta_{sj}) = 1 \\
&\iff \sigma([\alpha_{kj}, b(s,k)]) = 1 \text{ for each } k \in \{1, \ldots, K\} \\
&\iff \sigma(\alpha_{kj}) = \begin{cases} 1 & \text{if } b(s,k) = 1 \\ 0 & \text{if } b(s,k) = 0 \end{cases} \quad \text{for each } k \in \{1, \ldots, K\} \\
&\iff \begin{cases} x_{\langle \sigma, k \rangle} \in f_j & \text{if } b(s,k) = 1 \\ x_{\langle \sigma, k \rangle} \notin f_j & \text{if } b(s,k) = 0 \end{cases} \quad \text{for each } k \in \{1, \ldots, K\} \\
&\iff \{x_{\langle \sigma, k \rangle} \mid b(s,k) = 1,\ 1 \leq k \leq K\} \subseteq f_j \text{ and} \\
&\qquad \{x_{\langle \sigma, k \rangle} \mid b(s,k) = 0,\ 1 \leq k \leq K\} \subseteq U - f_j \\
&\iff S_\sigma \cap f_j = S_\sigma^{[s]} \text{ and} \\
&\qquad b(s,k_1) \neq b(s,k_2) \text{ implies } \langle \sigma, k_1 \rangle \neq \langle \sigma, k_2 \rangle \text{ for any } k_1, k_2 \in \{1, \ldots, K\}.
\end{aligned}
$$

Finally, we get the following equivalence:

$$
\begin{aligned}
&\sigma(\Psi_M) = 1 \\
&\iff \sigma\Big(\bigvee_{j=1}^{r} \beta_{sj}\Big) = 1 \text{ for any } s \in \{1, \ldots, 2^K\} \\
&\iff \text{for each } s \in \{1, \ldots, 2^K\}, \\
&\qquad \text{there exists } f_j \in \mathcal{F}_M \text{ with } S_\sigma \cap f_j = S_\sigma^{[s]} \text{ and} \\
&\qquad b(s,k_1) \neq b(s,k_2) \text{ implies } \langle \sigma, k_1 \rangle \neq \langle \sigma, k_2 \rangle \\
&\iff k_1 \neq k_2 \text{ implies } \langle \sigma, k_1 \rangle \neq \langle \sigma, k_2 \rangle, \text{ and} \\
&\qquad \text{for each } s \in \{1, \ldots, 2^K\} \text{ there exists } f_j \in \mathcal{F}_M \text{ with } S_\sigma \cap f_j = S_\sigma^{[s]} \\
&\iff |S_\sigma| = K \text{ and } S_\sigma \text{ is shattered by } \mathcal{F}_M
\end{aligned}
$$

Thus the formula Ψ_M is satisfiable if and only if VC-dim$\mathcal{F}_M \geq K$. □

4 Discrete VC-dimension Problem is $\mathrm{SAT}^{\mathrm{CNF}}_{\log^2 n}$-hard

This section shows that every set in $\mathrm{SAT}^{\mathrm{CNF}}_{\log^2 n}$ is reducible to the discrete VC-dimension problem in polynomial time, i.e., the problem is $\mathrm{SAT}^{\mathrm{CNF}}_{\log^2 n}$-hard.

Theorem 6. *Every $L \in SAT^{\mathrm{CNF}}_{\log^2 n}$ is polynomial-time reducible to the discrete VC-dimension problem.*

We use the following lemma in the proof of Theorem 6.

Lemma 7. *Let $\mathcal{F}$ be a concept class over a learning domain U, and S be a subset of U with $|S| = d \geq 2$. If S is shattered by $\mathcal{F}$, then for any two distinct x and y in S, the number of concepts which contain exactly one of either x or y is at least 2^{d-1}, i.e.,*

$$\left|\{f \in \mathcal{F} \mid f(x) \neq f(y)\}\right| \geq 2^{d-1}.$$

Proof. Let $\mathcal{F}_{\bar{x}y} = \{f \in \mathcal{F} \mid f(x) = 0,\ f(y) = 1\}$, and $\mathcal{F}_{x\bar{y}} = \{f \in \mathcal{F} \mid f(x) = 1,\ f(y) = 0\}$. Then $\{f \in \mathcal{F} \mid f(x) \neq f(y)\} = \mathcal{F}_{\bar{x}y} \cup \mathcal{F}_{x\bar{y}}$, and $\mathcal{F}_{\bar{x}y} \cap \mathcal{F}_{x\bar{y}} = \emptyset$. It is easy to see that if S is shattered by $\mathcal{F}$ then the set $S - \{x, y\}$ is shattered by both $\mathcal{F}_{\bar{x}y}$ and $\mathcal{F}_{x\bar{y}}$. By Lemma 2, $|S - \{x,y\}| \leq \log|\mathcal{F}_{\bar{x}y}|$ and $|S - \{x,y\}| \leq \log|\mathcal{F}_{x\bar{y}}|$. Thus $|\mathcal{F}_{\bar{x}y}| \geq 2^{d-2}$ and $|\mathcal{F}_{x\bar{y}}| \geq 2^{d-2}$, which yield $\left|\{f \in \mathcal{F} \mid f(x) \neq f(y)\}\right| = |\mathcal{F}_{\bar{x}y}| + |\mathcal{F}_{x\bar{y}}| \geq 2^{d-2} + 2^{d-2} = 2^{d-1}$. □

Proof of Theorem 6. Let $L \in \mathrm{SAT}^{\mathrm{CNF}}_{\log^2 n}$. Then there is a constant C_L and a polynomial $p_L(n)$ such that every string I of length n can be reduced in $p_L(n)$ time to a boolean formula in conjunctive normal form with at most $C_L \log^2 n$ variables, whose satisfiability coincides with the membership $I \in L$. Therefore we have only to show that, for any C, there is a polynomial-time reduction from the satisfiability problem in conjunctive normal form with at most $C \log^2 n$ variables to the discrete VC-dimension problem. Let $\Psi = E_1 \wedge \cdots \wedge E_m$ $(m \geq 2)$ be a boolean formula where each E_i is a disjunction and the total number of distinct variables occurring in Ψ is not greater than $C \log^2 m$. Without loss of generality, we can assume that m is a power of 2. We can also assume that the number of variables is exactly $C \log^2 m$, and let us rename them, for convenience, with double indices v_{st} $(1 \leq s \leq \log m,\ 1 \leq t \leq C \log m)$. We first construct a matrix M_Ψ which has $(m^C + 1)\log m$ rows and $m^2 + m(\log m - 1)$ columns, and then prove that VC-dim$(\mathcal{F}_{M_\Psi}) = 2\log m$ if and only if Ψ is satisfiable.

The learning domain U corresponding to Ψ is defined as $U = X \cup Y$ with $X \cap Y = \emptyset$, where $Y = \{y_u \mid 1 \leq u \leq \log m\}$ and $X = \{x_{sl} \mid 1 \leq s \leq \log m,\ 1 \leq l \leq m^C\}$. Let $X_s = \{x_{sl} \in X \mid 1 \leq l \leq m^C\}$ for each $s \in \{1, \ldots, \log m\}$, and let $X^{[k]} = \bigcup_{s \in \{s \mid b(k,s)=1\}} X_s$ for each $k \in \{1, \ldots, m\}$. The i-th subset $Y^{[i]}$ of Y is defined by $Y^{[i]} = \{y_u \in Y \mid b(i,u) = 1\}$ for each $i \in \{1, \ldots, m\}$.

The concept class $\mathcal{F} \subseteq 2^U$ is defined as the union of distinct subclasses $F_1, \ldots, F_m$, and G. Here, the structure of G depends only on the number m:

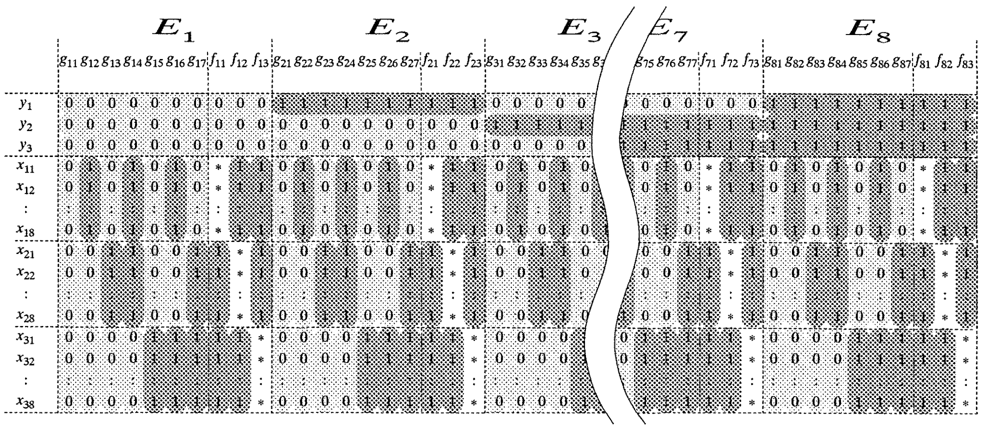

Fig. 1. Structure of the matrix M_Ψ reduced from a boolean formula $\Psi = E_1 \wedge E_2 \wedge \cdots \wedge E_8$ with $C = 1$. In this case, $K = 2\log 8 = 6$. The only elements marked '$*$' depend on the structure of each clause E_i in Ψ.

$$G = \{g_{ik} \mid 1 \le i \le m,\ 1 \le k \le m-1\}, \quad \text{where } g_{ik} = Y^{[i]} \cup X^{[k]}.$$

On the other hand, each concept in F_i reflects the structure of the clause E_i in Ψ:

$$F_i = \{f_{ij} \mid 1 \le j \le \log m\}, \quad \text{where } f_{ij} = Y^{[i]} \cup (X - X_j) \cup X_j^*(E_i) \quad \text{with}$$

$$X_j^*(E_i) = \left\{ x_{jl} \in X_j \;\middle|\; \begin{array}{l} E_i \text{ contains a positive literal } v_{jt} \text{ with } b(l,t) = 1, \text{ or} \\ E_i \text{ contains a negative literal } \neg v_{jt} \text{ with } b(l,t) = 0 \\ \text{for some } t \in \{1, \ldots, C\log m\} \end{array} \right\}.$$

Figure 1 illustrates the structure of the matrix M_Ψ.

Clearly the cardinality of learning domain, i.e., the row size of the matrix M representing $\mathcal{F}$ is

$$|U| = |X| + |Y| = m^C \cdot \log m + \log m = (m^C + 1)\log m,$$

and the cardinality of the concept class $\mathcal{F}$, i.e., the column size of M is

$$|\mathcal{F}| = |G| + |F_1| + \cdots + |F_m| = m(m-1) + m \cdot \log m.$$

Moreover, it is easy to see that M_Ψ can be constructed in polynomial time with respect to the length of given formula Ψ.

Now we prove that if the formula Ψ is satisfiable then VC-dim$(\mathcal{F}) = 2\log m$. For an assignment σ which satisfies Ψ, we consider the set $S_\sigma = Y \cup X_\sigma$ with

$$X_\sigma = \{x_{s,\langle\sigma,s\rangle} \in X \mid 1 \le s \le \log m\}, \quad \text{where } \langle\sigma,s\rangle = \sum_{t=1}^{C\log m} 2^{t-1} \cdot \sigma(v_{st}) + 1.$$

It is clear that $|S_\sigma| = |Y| + |X_\sigma| = 2\log m$. We will show that S_σ is shattered by $\mathcal{F}$, i.e., for every $T \subseteq S_\sigma$, there exists an $f \in \mathcal{F}$ with $S_\sigma \cap f = T$. Let $i_T = \sum_{y_u \in T\cap Y} 2^{u-1} + 1$. It is easy to see that $i_T \in \{1, \ldots, m\}$ and $T \cap Y = Y^{[i_T]}$. According to $T \cap X_\sigma = X_\sigma$ or not, we have the following two cases.

(1) In case of $T \cap X_\sigma \subsetneq X_\sigma$: Let $k_T = \sum_{x_{s,\langle\sigma,s\rangle} \in T \cap X_\sigma} 2^{s-1} + 1$. Then we can see that $k_T \in \{1, \ldots, m-1\}$ and $T \cap X_\sigma = X^{[k_T]} \cap X_\sigma$. Therefore the concept $g_{i_T,k_T} \in G \subseteq \mathcal{F}$ cuts T out of S_σ as follows:

$$\begin{aligned} g_{i_T,k_T} \cap S_\sigma &= (Y^{[i_T]} \cup X^{[k_T]}) \cap (Y \cup X_\sigma) = (Y^{[i_T]} \cap Y) \cup (X^{[k_T]} \cap X_\sigma) \\ &= (T \cap Y) \cup (T \cap X_\sigma) = T. \end{aligned}$$

(2) In case of $T \cap X_\sigma = X_\sigma$: Since σ satisfies Ψ, the disjunction E_{i_T} in Ψ is also satisfied by σ. That means E_{i_T} contains either positive literal v_{st} with $\sigma(v_{st}) = 1$, or negative literal $\neg v_{st}$ with $\sigma(v_{st}) = 0$, for some s and t. Let us take such an s (not necessarily unique), and let $j_T = s$. Then by the definition of $\langle\sigma, j_T\rangle$, we see $b(\langle\sigma, j_T\rangle, t) = \sigma(v_{j_T,t})$ for each t. Thus $x_{j_T,\langle\sigma,j_T\rangle}$ is included in $X^*_{j_T}(E_{i_T})$, and moreover, $X^*_{j_T}(E_{i_T}) \cap X_\sigma = \{x_{j_T,\langle\sigma,j_T\rangle}\}$. Therefore the concept $f_{i_T,j_T} \in F_{i_T} \subseteq \mathcal{F}$ cuts T out of S_σ as follows:

$$\begin{aligned} f_{i_T,j_T} \cap S_\sigma &= (Y^{[i_T]} \cup (X - X_{j_T}) \cup X^*_{j_T}(E_{i_T})) \cap (Y \cup X_\sigma) \\ &= (Y^{[i_T]} \cap Y) \cup ((X - X_{j_T}) \cap X_\sigma) \cup (X^*_{j_T}(E_{i_T}) \cap X_\sigma) \\ &= (T \cap Y) \cup (X_\sigma - \{x_{j_T,\langle\sigma,j_T\rangle}\}) \cup \{x_{j_T,\langle\sigma,j_T\rangle}\} \\ &= (T \cap Y) \cup X_\sigma = (T \cap Y) \cup (T \cap X_\sigma) = T. \end{aligned}$$

In each case, T is shown to be cut out of S_σ by some concept in $\mathcal{F}$. Therefore S_σ is shuttered by $\mathcal{F}$.

Now we show the converse. Suppose that VC-dim$(\mathcal{F}) = 2\log m$. Then there is a set $S \subseteq U$ of cardinality $2\log m$ which is shattered by $\mathcal{F}$.

Claim 1 *S contains exactly one element from each X_s $(1 \le s \le \log m)$, and all elements from Y.*

Proof of Claim 1. Case $m = 2$: The learning domain is $U = \{y_1\} \cup X_1$ and the concept class is $\mathcal{F} = \{f_{11}, g_{11} f_{21}, g_{21}\}$. Since $|\mathcal{F}| = 4$ and $g_{11}(x) = g_{21}(x) = 0$ for any $x \in X_1$, no two elements from X_1 can be included in S which is to be shattered by $\mathcal{F}$. Moreover, since $|Y| = |\{y_1\}| = 1$, the claim holds.

Case $m \ge 3$: Let $s \in \{1, \ldots, \log m\}$ be fixed arbitrarily, and x_1, x_2 be distinct elements in X_s. Suppose that S contains both x_1 and x_2. Then by Lemma 7,

$$\left|\{h \in \mathcal{F} \mid h(x_1) \neq h(x_2)\}\right| \ge 2^{2\log m - 1} = \frac{m^2}{2}.$$

On the other hand, let us consider a concept $h \in \mathcal{F}$ with $h(x_1) \neq h(x_2)$. Since $g_{ik}(x_1) = g_{ik}(x_2) = b(k, s)$ for any $g_{ik} \in G$, the concept h is not in G. Moreover, since $f_{ij}(x_1) = f_{ij}(x_2) = 1$ for any $f_{ij} \in F_1 \cup \cdots \cup F_m$ with $j \neq s$, thus h must be one of the concepts from $\{f_{1s}, f_{2s}, \ldots, f_{ms}\}$. Therefore

$$\left|\{h \in \mathcal{F} \mid h(x_1) \neq h(x_2)\}\right| \le \left|\{f_{1s}, f_{2s}, \ldots, f_{ms}\}\right| = m,$$

which yields a contradiction since $\frac{m^2}{2} > m$ for any $m \ge 3$. Thus S can contain at most one element from X_s for each $s \in \{1, \ldots, \log m\}$. Since $|S| = 2\log m$ and $|Y| = \log m$, the set S must contain exactly one element from each X_s and all elements from Y. □

Therefore for each $s \in \{1, \dots, \log m\}$, there is a unique $l = l(s) \in \{1, \dots, m^C\}$ such that $x_{s,l(s)} \in S$, and we can assume that $S = Y \cup X_{(l)}$, where $X_{(l)} = \{x_{s,l(s)} \mid 1 \leq s \leq \log m\}$. Let σ_S be an assignment corresponding to S with

$$\sigma_S(v_{st}) = b(l(s), t) \quad (1 \leq s \leq \log m,\ 1 \leq t \leq C \log m).$$

Now we show that σ_S satisfies all disjunctions E_i in Ψ. Let $i \in \{1, \dots, m\}$ be fixed arbitrarily. Since S is shattered by $\mathcal{F}$, for the subset $T_i = Y^{[i]} \cup X_{(l)}$ of S there is a concept $h_i \in \mathcal{F}$ with $S \cap h_i = T_i$. Since $S \cap h_i = (Y \cap h_i) \cup (X_{(l)} \cap h_i)$, the concept h_i must satisfy the following two conditions:

(1) $Y \cap h_i = Y^{[i]}$.
(2) $X_{(l)} \cap h_i = X_{(l)}$.

Note that no concept in G satisfies the condition (2), and no concept in $F_{i'}$ with $i' \neq i$ satisfies the condition (1). Therefore such an $h_i \in \mathcal{F}$ is in F_i, and thus we can assume $h_i = f_{ij}$ for some $j \in \{1, \dots, \log m\}$. The above condition (2) requires that f_{ij} contains all elements from $X_{(l)}$. Especially, remark that $x_{j,l(j)} \in X_{(l)}$ is included in f_{ij} for the above j. By the definition of f_{ij}, the element $x_{j,l(j)}$ is in $X_j^*(E_i)$. Thus the clause E_i satisfies either (a) or (b):

(a) E_i contains a positive literal v_{jt} with $b(l(j), t) = 1$.
(b) E_i contains a negative literal $\neg v_{jt}$ with $b(l(j), t) = 0$.

By the definition of σ_S, we see $\sigma_S(v_{jt}) = 1$ in case of (a), and $\sigma_S(v_{jt}) = 0$ in case of (b). In each case, $\sigma_S(E_i) = 1$. Therefore σ_S satisfies every disjunction E_i in Ψ. Thus Ψ is satisfiable. □

5 Conclusion

We showed that the discrete VC-dimension problem is in $\mathrm{SAT}_{\log^2 n}$ and $\mathrm{SAT}^{\mathrm{CNF}}_{\log^2 n}$-hard. Therefore we may interpret that the discrete VC-dimension problem is, in some sense, "complete" for the class of $n^{O(\log n)}$ time computable sets. It remains open that the discrete VC-dimension problem is in $\mathrm{SAT}^{\mathrm{CNF}}_{\log^2 n}$, or $\mathrm{SAT}_{\log^2 n}$-hard.

As a dual to the VC-dimension, Romanik [7] defined the *testing dimension* of a concept class $\mathcal{F}$ as the *minimum* cardinality of a set $S \subseteq U$ which is *not* shattered by $\mathcal{F}$. We can see that testing dimension problem is also in $\mathrm{SAT}_{\log^2 n}$, by a similar reduction in the proof of Theorem 5. It is open whether the problem is $\mathrm{SAT}^{\mathrm{CNF}}_{\log^2 n}$-hard or not. It is also interesting to evaluate the complexity of computing another various dimensions of the class of *multi-valued functions* introduced in [1].

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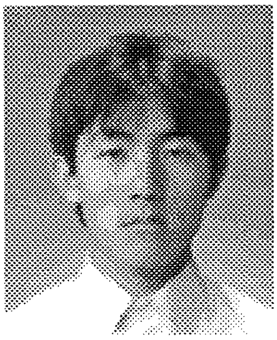
About the Author



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