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BALANCED FORMULAS, MINIMAL FORMULAS AND THEIR PROOFS

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Abstract

According to the formulas-as-types notion, an implicational formula can be identified with a type of a λ-term which represents a proof of the formula in implicational fragment of intuitionistic logic. A formula is balanced iff no type variable occurs more than twice in it. It is known that balanced formulas have unique proofs. In this paper, it is shown that closed λ-terms in β-normal form having balanced types are BCK-λ-terms in which each variable occurs at most once. A formula is BCK-minimal iff it is BCK-provable and it is not a non-trivial substitution instance of other BCK-provable formula. It is also shown that the set BCK-minimal formulas is identical to the set of principal type-schemes of BCK-λ-terms in β-η-normal form.

1 Introduction

We study the structure of normal proof figures of implicational formulas provable in BCK-logic, in which each assumption can be used at most once.

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According to the formulas-as-types notion [11], the set of BCK-formulas is identical to the set of types of closed BCK-λ-terms. So we use the words 'types' and 'formulas' with the same meaning.

In [13], Komori raised some conjectures among which is the following.

> "If a formula is BCK-minimal then its normal form proof is unique."

Here a BCK-minimal formula is a BCK-provable formula which is not a non-trivial substitution instance of other BCK-provable formula. This conjecture was proved independently by Wronski [17], by Hirokawa [7] and by Tatsuta [16]. Wronski proved the conjecture as a corollary of the coherence theorem in cartesian closed category. Babaev and Soloviev [1] and Mints [15] formulated the theorem as follows and gave simple proofs for it.

> "If a formula is balanced then its normal form proof is unique."

Here a formula is balanced iff no type variable occurs more than twice. Balanced formulas are called as formulas with one-two-property in [6, 12]. We denote the set of balanced formulas as $F_{1,2}$. The author learned a result by Jaskowski [12] from P. Idzjak which states that

$$BCK \cap F_{1,2} = LJ \cap F_{1,2}.$$

By analysis of the type assignment figures, we prove that if a λ-term in β-normal form has balanced type, then it is a BCK-λ-term. This gives a direct proof for Jaskowski's result.

The problem by Komori, the proof by Wronski and Tatsuta and the coherence theorem concerns the uniqueness of proof figure not in β-normal form but in β-η-normal form. A pricese statement of Wronski's answer is as follows.

> "If α is a BCK-minimal then the closed BCK-λ-term in β-η-normal form which has α as its type is unique."

On the other hand, the author's solution [7] for Komori's problem is as follows.

"If α is a BCK-minimal then the closed BCK-λ-term in β-normal form which has α as its type is unique."

We prove that a type-scheme α of a closed BCK-λ-term is BCK-minimal iff α is a principal type-scheme of some closed BCK-λ-term in β-η-normal form. This clarifies the difference between β-normal forms and β-η-normal form.

2 Balanced types and BCK-λ-terms

We use the terminology in Hindley [4] for type assignment figures to λ-terms. We say TA-figures instead of type assignment figures. Types are constructed from type variables and '$\to$'. We use the letters $a, b, c, \cdots$ for type variables, $\alpha, \beta, \gamma, \cdots$ for types, $x, y, z, \cdots$ for term variables and $L, M, N, \cdots$ for λ-terms. The set of type variables in a type α is denoted by $var(\alpha)$. The set of free variables in a λ-term M is denoted by $FV(M)$. We write $B \vdash M : \alpha$ for type assignment α to λ-term M from an assumption set $B = \{x_1 : \alpha_1, \cdots, x_n : \alpha_n\}$. Note that the set $\{x_1, \cdots, x_n\}$ of the subjects of B is identical to $FV(M)$. When (B, α) is not a non-trivial substitution of any (B', α') such that $B' \vdash M : \alpha'$, it is said to be a principal type assignment and the pair (B, α) is called a principal-pair. In such a case, we write $B || \!- M : \alpha$. When M is a closed-λ-term, α is called a principal-type-scheme of M. Given a set of closed λ-terms T, $ts(T)$ and $pts(T)$ denotes the set of type-schemes of terms in T and the set of principal type-schemes of terms in T.

A BCK-λ-term is a λ-term in which each variable occurs at most once. We consider only BCK-λ-terms in β-normal form.

Definition 1 (BCK-λ-term) The set of BCK-λ-terms is defined inductively as follows.

(1) If x is a variable then x is a BCK-λ-term.

(2) If M and N are BCK-λ-term and $FV(M) \cap FV(N) = \emptyset$ then (MN) is a BCK-λ-term.

(3) If M is a BCK-λ-term and x is a variable then $(\lambda x.M)$ is a BCK-λ-term.

Definition 2 The core of a type α, denoted by $core(\alpha)$, is the rightmost type variable in α.

We refer [6] to the history of the one-two-property and the two-property.

Definition 3 (balanced types) A type is balanced iff each type variable in the type occurs at most twice. The set of balanced types is denoted by $F_{1,2}$

Balanced formulas are identical to formulas with one-two-property in [6, 12]. We dente by BCK and BCK-β the set of closed BCK-λ-terms and the set of closed BCK-λ-terms in β-normal form respectively. We write $ts(T)$ for the set of type-schemes of terms in T, and write $pts(T)$ for the set of principal type-schemes of terms in T. Since BCK-λ-terms have types [6], we have $pts(BCK) = pts(BCK\text{-}\beta)$. The fact that

$$pts(BCK\text{-}\beta) \subseteq F_{1,2}$$

was known to Belnap [2]. It is also known that

$$pts(BCI) = ts(BCI) \cap F_2.$$

(See [14, 6, 9].) Here BCI is the set closed BCI-λ-terms in which each variable occurs exactly once, and F_2 is the set of types with two-property,i.e., the set of types in which each type variable occurs twice. So it would be tempting to hope $pts(BCK) = ts(BCK) \cap F_{1,2}$. But this is not the case. Characterizations of $pts(BCK)$ and $pts(BCI)$ are shown in [9, 10].

Jaskowski [12] proved that BCK-logic is strong enough to prove the balanced formulas. He proved

$$ts(BCK) \cap F_{1,2} = IL \cap F_{1,2},$$

where IL is the set of types of closed-λ-terms or equivalently the set of provable implicational formulas in intuitionistic logic. This result means that if one-two-formula is provable in IL then it is provable in BCK-logic. In the rest of this section, we prove the following theorem which states that if a balanced is provable in IL then its proof is in BCK-logic.

Theorem 1 For any type $\alpha \in F_{1,2}$ and a closed λ-term M in β-normal form, if $\vdash M : \alpha$ then M is a BCK-λ-term.

Proof. We prove the following claim which includes the theorem as a special case where the assumption set is empty.

> Claim: Let M be a λ-term in β-normal form and $B \vdash M : \alpha$. If (B, α) has the one-two-property then M is a BCK-λ-term.

Here, a pair (B, α) of an assumption set $B = \{x_1 : \alpha_1, \cdots, x_n : \alpha_n\}$ and α is balanced iff $\alpha_1 \to \cdots \to \alpha_n \to \alpha$ is balanced. We prove this claim by induction on M.

1. $M = x$. Then x is a BCK-λ-term.

2. $M = \lambda x.N$. Then we have $\alpha = \beta \to \gamma$. A TA-figure for $B \vdash M : \alpha$ has the following form.

$$\dfrac{\begin{array}{c}[x:\gamma]\\ \vdots \\ N:\beta\end{array}}{\lambda x.N : \gamma \to \beta}$$

2.1 $x \in FV(N)$. Then we have $B \cup \{x : \gamma\} \vdash N : \beta$. Since $(B, \gamma \to \beta)$ has the one-two-property, so does $(B \cup \{x : \gamma\}, \beta)$. By induction hypothesis, N is a BCK-λ-term. Therefore $\lambda x.N$ is a BCK-λ-term.

2.2 $x \notin FV(N)$. Then we have $B \vdash N : \beta$. Since $(B, \gamma \to \beta)$ has the one-two-property, so does (B, β). Therefore N is a BCK-λ-term. So is $\lambda x.N$.

3. $M = xM_1 \cdots M_n (n \geq 1)$.

First we claim that $x \notin FV(M_i)$ for $i = 1, \cdots, n$. Assume that $x \in FV(M_i)$. Then a TA-figure for $B \vdash xM_1 \cdots M_n : \alpha$ has the following form.

$$\dfrac{x:\xi \quad \cdots \quad P_i\left\{\begin{array}{c}x:\xi\\ \vdots \\ M_i:\alpha_i\end{array}\right. \quad \cdots \quad M_n:\alpha_n}{xM_1\cdots M_i \cdots M_n : \alpha}$$

Here $\xi = \alpha_1 \to \cdots \to \alpha_i \to \cdots \to \alpha_n \to \alpha$. Let $b = core(\alpha)$. We derive a contradiction by case analysis depending on whether $b \in var(\alpha_i)$ or not.

Case (i) $b \in var(\alpha_i)$. Then b occurs twice in $x : \alpha_1 \to \cdots \to \alpha_i \to \cdots \to \alpha_n \to \alpha$ and occurs once in $xM_1 \cdots M_n : \alpha$. Thus b occurs three times in (B, α). This contradicts that (B, α) has the one-two-property.

Case (ii) $b \notin var(\alpha_i)$. Consider the sub-TA-figure P_i with the end-formula $M_i : \alpha_i$. Note that b occurs at the assumption $x : \alpha_1 \to \cdots \to \alpha_n \to \alpha$ and that b does occur at the end-formula $M_i : \alpha_i$. Traverse the sequence of TA-formulas from the assumption $x : \alpha_1 \to \cdots \to \alpha_n \to \alpha$ to $M_i : \alpha_i$. Then we would find a point where b disappear from the predicate of a TA-formula. The lowest TA-formulas, among that sequence, which contains b and below which b does not appear, is the minor premiss of an $(\to E)$. Let the TA-formula be $Q : \gamma$. Then P_i has the following form.

$$\dfrac{\dfrac{y : \gamma_1 \to \cdots \to \gamma_k \to \gamma \to \delta \quad R_1 : \gamma_1 \quad \cdots \quad R_k : \gamma_k}{yR_1 \cdots R_k : \gamma \to \delta} \qquad \begin{array}{c} x : \alpha_1 \to \cdots \to \alpha_n \to \alpha \\ \vdots \\ Q : \gamma \end{array}}{\begin{array}{c} yR_1 \cdots R_k Q : \delta \\ \vdots \\ M_i : \alpha_i \end{array}} (\to E)$$

Since b does not occur from $yR_1 \cdots R_k Q : \delta$ to $M_i : \alpha_i$, $y : \gamma_1 \to \cdots \to \gamma_k \to \gamma \to \delta$ is not discharged. Therefore $y \in FV(M_i)$. Thus $y \in FV(xM_1 \cdots M_n)$. Therefore b occurs in the core of $x : \alpha_1 \to \cdots \to \alpha_n \to \alpha$, in γ of $y : \gamma_1 \to \cdots \to \gamma_k \to \gamma \to \delta$ and in $xM_1 \cdots M_n : \alpha$. A contradiction. This completes the first claim.

Secondly we claim that M_i is a BCK-λ-term. It suffices to show that (B_i, α_i) has the one-two-property, where B_i is the assumption set for the sub-TA-figure whose end-formula is $M_i : \alpha_i$. If this is proved, we can apply induction hypothesis for (B_i, α_i) obtaining that M_i is a BCK-λ-term. Let b be a type variable in (B_i, α_i). Then we have the following inequalities where $\sharp b : \cdots$ means the number of occurrences of b in assumption sets.

$$\begin{aligned} \sharp b : (B_i, \alpha_i) &\le \sum_{j=1}^{n} \sharp b : (B_j, \alpha_j) \\ &\le \sharp b : ((\cup_{j=1}^{n} B_j) \cup \{x : \alpha_1 \to \cdots \alpha_n \to \alpha\}) \\ &\le \sharp b : B \\ &\le \sharp b : (B, \alpha) \\ &\le 2 \end{aligned}$$

Thus (B_i, α_i) has the one-two-property. This completes the second claim.

Thirdly we claim that $FV(M_i) \cap FV(M_j) = \emptyset$ for $i \neq j$. Assume that $z \in FV(M_i) \cap FV(M_j)$ for $i \neq j$. Let z be the leftmost free variable in M_i which occurs also in $FV(M_j)$. Let γ be the subject to z and $b = core(\gamma)$.

We derive a contradiction by case analysis depending on whether b occurs in α_i and in α_j or not.

$$\frac{x : \alpha_1 \to \cdots \to \alpha_n \to \alpha \quad \cdots \quad P_i \left\{ \begin{array}{c} z : \gamma \\ \vdots \\ M_i : \alpha_i \end{array} \right. \quad \cdots \quad P_j \left\{ \begin{array}{c} z : \gamma \\ \vdots \\ M_j : \alpha_j \end{array} \right. \quad \cdots}{x M_1 \cdots M_n : \alpha}$$

Case (i) $b \in var(\alpha_i)$ and $b \in var(\alpha_j)$. Then b occurs twice in $x : \alpha_1 \to \cdots \to \alpha_i \to \cdots \to \alpha_j \to \cdots \to \alpha_n \to \alpha$ and once in $z : \gamma$. By the first claim $x \neq z$. Therefore b occurs three times in (B, α). This is a contradiction.

Case (ii) b occurs only one of α_i and α_j. We can assume that $b \in var(\alpha_i)$ and $b \notin var(\alpha_j)$. Consider the sequence of TA-formulas from the assumption $z : \gamma$ to $M_j : \alpha_j$. Let $R : \beta$ be the lowest TA-formula along the sequence such that $b \in var(\beta)$ and b does not occur below $R : \beta$. Since M_j is in β-normal form, $R : \beta$ is the minor premiss of an $(\to E)$.

$$\cfrac{\cfrac{y : \beta_1 \to \cdots \to \beta_k \to \beta \to \delta \quad Q_1 : \beta_1 \quad \cdots \quad Q_k : \beta_k}{y Q_1 \cdots Q_k : \gamma \to \delta} \qquad \begin{array}{c} z : \gamma \\ \vdots \\ R : \beta \end{array}}{\begin{array}{c} y Q_1 \cdots Q_k R : \delta \\ \vdots \\ M_j : \alpha_j \end{array}} \ (\to E)$$

Since b does not occur from $yQ_1 \cdots Q_k R : \delta$ to $M_j : \alpha_j$, $y : \beta_1 \to \cdots \to \beta_k \to \beta \to \delta$ is not discharged. Therefore $y \in FV(M_j)$. By the first claim $x \neq y$ and $x \neq z$. By the second claim $y \neq z$. Therefore we have three occurrences of b in $y : \beta_1 \to \cdots \to \beta_k \to \beta \to \delta$, in $x : \alpha_1 \to \cdots \to \alpha_i \to \cdots \to \alpha_n \to \alpha$ and in $z : \gamma$. A contradiction.

Case (iii) $b \notin var(\alpha_i)$ and $b \notin var(\alpha_j)$. Let $R : \beta$ be the same TA-formula in case (ii). (See above figure.) Let $S : \nu$ be the lowest TA-formula, in the sequence of TA-formulas from $z : \gamma$ to $M_i : \alpha_i$, such that $b \in var(\nu)$ and b does not occur below $S : \nu$.

$$\cfrac{\cfrac{u : \xi_1 \to \cdots \to \xi_l \to \nu \to \mu \quad S_1 : \xi_1 \quad \cdots \quad S_l : \xi_l}{u S_1 \cdots S_l : \nu \to \mu} \qquad \begin{array}{c} z : \gamma \\ \vdots \\ S : \nu \end{array}}{\begin{array}{c} u S_1 \cdots S_l S : \mu \\ \vdots \\ M_i : \alpha_i \end{array}} \ (\to E)$$

Remember that z is the leftmost free variable in M_i which occurs also in $FV(M_j)$. Thus $u \neq y$. By the second claim, M_i and M_j are BCK-λ-terms. Therefore $z \neq u$ and $z \neq y$. Thus b occurs three times in $u : \xi_1 \to \cdots \to \xi_l \to \nu \to \mu$, $y : \beta_1 \to \cdots \to \beta_k \to \beta \to \delta$ and $z : \gamma$. A contradiction. This completes the third claim.

From the first, second and the third claim, $xM_1 \cdots M_n$ is a BCK-λ-term. ∎

3 BCK-minimal types and BCK-λ-terms in β-η-normal form

A BCK-minimal type is a type of a closed BCK-λ-term which is not a non-trivial substitution instance of any type of closed BCK-λ-term. The notion was defined by Komori in [13], where he conjectured that any BCK-minimal formula has the unique normal form proof. We can state the conjecture in terms of type assignment to BCK-λ-terms as follows.

> Let M and N be closed BCK-λ-terms in β-normal form and let α be a BCK-minimal type. If $\vdash M : \alpha$ and $\vdash N : \alpha$ then $M =_\eta N$.

This conjecture was proved independently by Wronski [17], by Hirokawa [7] and by Tatsuta [16]. Wronski derived it from the following theorem by Mints [15].

Theorem 2 ([15]) Let M and N be closed λ-terms in β-normal form and let α be a balanced type. If $\vdash M : \alpha$ and $\vdash N : \alpha$ then $M =_\eta N$.

We [7] proved the conjecture as a corollary of the following theorem.

Theorem 3 ([7]) Let M and N be closed λ-term in β-normal form and α be a type. If $||\!-M : \alpha$ and $||\!-N : \alpha$ then $M = N$.

Note that if M has a BCK-minimal type α as its type-scheme, then α is a principal type-scheme of M. Thus Komori's conjecture is a corollary of the above theorem. Moreover η-convertibility in the Komori's conjecture is replaced by equality. In [7], we left the following conjecture concerning to the BCK-minimality and η-normal form.

A type α is BCK-minimal iff α is a principal type-scheme of some closed BCK-λ-term in β-η-normal form.

Note that $pts(BCK\text{-}\beta\text{-}\eta) \overset{\subset}{\neq} ts(BCK\text{-}\beta)$. In fact, $\alpha = (a \to b) \to a \to b$ is a principal type-scheme of BCK-λ-term $\lambda xy.xy$. By Theorem 3, $\lambda xy.xy$ is a unique BCK-λ-term in β-normal form. Since $\lambda xy.xy$ is not in η-normal form, $\alpha \notin pts(BCK\text{-}\beta\text{-}\eta)$.

Theorem 4 (Characterization of BCK-minimal types)
$pts(BCK\text{-}\beta\text{-}\eta) = BCK$-minimal.

Remark 1 We proved, in [9], $pts(BCI\text{-}\beta\text{-}\eta) = BCI\text{-}minimal$. The proof of $pts(BCK\text{-}\beta\text{-}\eta) \supseteq BCK\text{-}minimal$ is essentially the same to that of $pts(BCI\text{-}\beta\text{-}\eta) \supseteq BCI\text{-}minimal$. But the proof for the opposite inclusion is different to that for BCI-λ-terms in [9].

Proof of $pts(BCK\text{-}\beta\text{-}\eta) \supseteq BCK$-minimal. Let α be a BCK-minimal type. Then there is a closed BCK-λ-term M in β-normal form such that $||\!-M : \alpha$. Let P be a TA-figure for $||\!-M : \alpha$. Assume that M is not in η-normal form. Then M contains some subterm $\lambda x.Nx$ such that $x \notin FV(N)$. Since M is in β-normal form, so is $\lambda x.Nx$. Therefore N has the form $N = yN_1 \cdots N_n$ and P has the following form.

$$P_1\left\{\left.\begin{array}{c}\dfrac{\dfrac{y:\alpha_1\to\cdots\to\alpha_n\to\beta\to\gamma\quad N_1:\alpha_1\quad\cdots\quad N_n:\alpha_n}{yN_1\cdots N_n:\beta\to\gamma}\qquad x:\beta}{\dfrac{yN_1\cdots N_n x:\gamma}{\lambda x.yN_1\cdots N_n x:\beta\to\gamma}}\end{array}\right\}P_2\right.$$
$$\vdots$$
$$M:\alpha$$

Let P_1 be the sub-TA-figure with the end-formula $yN_1 \cdots N_n : \beta \to \gamma$ and let P_2 be the sub-TA-figure with the end-formula $\lambda x.yN_1 \cdots N_n x : \beta \to \gamma$. In P_1 replace $\beta \to \gamma$ by a new type variable c. Let P_1^* be the result of P_1. Note that in the rest of P_2 the predicate $\beta \to \gamma$ of $\lambda x.yN_1 \cdots N_n x : \beta \to \gamma$ does not connect to the predicate of any major premiss of $(\to I)$. This is proved by induction on $\vdash M : \alpha$. Below the type-assignment formula $\lambda x.yN_1 \cdots N_n x :$

$\beta \to \gamma$ in P, replace the connection of $\beta \to \gamma$ by c and replace the occurrence of the subterm $\lambda x.yN_1 \cdots N_n x$ by $yN_1 \cdots N_n$. After that rewriting, replace P_2 by P_1^*. Then we obtain a TA-figure for $\vdash M[\lambda x.yN_1 \cdots N_n x / yN_1 \cdots N_n] : \delta$ where $\alpha = \delta[c := \beta \to \gamma]$. This contradicts the minimality of α. Therefore M is in η-normal form. ∎

To prove $pts(BCK\text{-}\beta\text{-}\eta) \subseteq BCK\text{-}minimal$ we need the following lemma which is a special case of the subject expansion theorem [4].

Lemma 1 Let M and N be closed BCK-λ-terms in β-normal form and $||\!-M : \alpha$. If N is η-reducible to M then $||\!-N : \alpha\theta$ for some substitution θ.

Proof. We perform a transformation to the TA-figure P for $||\!-M : \alpha$. It is the reverse transformation we applied in the proof of $pts(BCK - \beta - \eta) \supseteq BCK - minimal$. The proof is by induction on the number of η-reductions. It suffices to prove the case where N is η-reducible to M by one-step. Since N is η-reducible to M, N contains a η-redex $\lambda x.Qx$ such that $x \notin FV(Q)$. Since N is in β-normal form, Q has the form $Q = yQ_1 \cdots Q_n$. The contructum of $\lambda x.yQ_1 \cdots Q_n x$ is $yQ_1 \cdots Q_n$. Note that the occurrence $yQ_1 \cdots Q_n$ is not in a function part, i.e., $yQ_1 \cdots Q_n$ does not occur as $yQ_1 \cdots Q_n R$ in M. If the contructum occurred in that form, then η redex in N has the form $(\lambda y.Q_1 \cdots Q_n x)R$. This contradicts the β-normality of N. Therefore P has the following form.

$$\left.\begin{array}{c}\vdots\\ yQ_1 \cdots Q_n x : b\end{array}\right\} P_1$$
$$\begin{array}{c}\vdots\\ M : \alpha\end{array}$$

Here b is a type variable. Let P_1 be the sub-TA-figure whose end-formula is $yQ_1 \cdots Q_n x : b$. Let c and d be new type variables. First rewrite all the occurrences of b by $c \to d$. Let P_1^* be the result of P_1 by this rewriting and P^* be the result of P. Next replace this P_1^* by the following P_2. Finally in P^*, rewrite the occurrence of $yQ_1 \cdots Q_n$ below $y : yQ_1 \cdots Q_n : c \to d$ by

$\lambda x.yQ_1 \cdots Q_n x$. This becomes a TA-figure for $||\!-M : \alpha[b : c \to d]$. ∎

$$
\left.\begin{array}{c}
\dfrac{P_1^*\left\{\begin{array}{c}\vdots \\ yQ_1 \cdots Q_n : c \to d\end{array}\right. \qquad x : c}{\dfrac{yQ_1 \cdots Q_n x : d}{\lambda x.yQ_1 \cdots Q_n x : c \to d}}
\end{array}\right\} P_2
$$

$$
\vdots
$$

$$
N[yQ_1 \cdots Q_n / \lambda x.yQ_1 \cdots Q_n x] : \alpha[b := c \to d]
$$

Proof of pts(BCK-β-η) $\subseteq$ BCK-minimal Let $\alpha \in pts(BCK\text{-}\beta\text{-}\eta)$ and $||\!-M : \alpha$ for a closed BCK-λ-term M in β-η-normal form. To prove the minimality of α, assume $\alpha = \beta\theta_1$ for some substitution θ_1 and β where β is a type-scheme of a closed BCK-λ-term N in β-normal form, i.e., $\vdash N : \beta$. By the principal type-scheme theorem [4, 8], there is a type γ and a substitution θ_2 such that $||\!-N : \gamma$ and $\beta = \gamma\theta_2$. Therefore $= \gamma\theta_2\theta_1$. Since $\vdash N : \beta$, it follows that $\vdash N : \alpha$. Since α is a p.t.s. of closed BCK-λ-term in β-normal form, α is balanced. Thus we have $\vdash M : \alpha$ and $\vdash N : \alpha$ for balanced type α. By Theorem 2 we have $M =_\eta N$. Since M is in η-normal form, N is η-reducible to M. By Lemma 1 we have $\gamma = \alpha\theta$ for some substitution θ. Thus we have $\alpha = \gamma\theta_1\theta_2$ and $\gamma = \alpha\theta$. Therefore $\theta_1\theta_2$ and θ is trivial. Thus θ_1 is trivial. Thus α is BCK-minimal. ∎

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