Characterization of Pattern Languages

Mukouchi, Yasuhito
Department of Information Systems, Kyushu University

出版情報：RIFIS Technical Report. 41, 1991-06-30. Research Institute of Fundamental Information Science, Kyushu University
バージョン：

RIFIS-TR-CS-41

RIFIS Technical Report

Characterization of Pattern Languages

Yasuhito Mukouchi

June 30, 1991

Research Institute of Fundamental Information Science
Kyushu University 33
Fukuoka 812, Japan

E-mail: miyano@rifis.sci.kyushu-u.ac.jp Phone: 092 (641)1101 Ex. 4471

Characterization of Pattern Languages

YASUHITO MUKOUCHI
Department of Information Systems,
Kyushu University 39, Kasuga 816, Japan
Phone: 092-641-1101 Ext. 4484 Fax: 092-611-2668
E-mail: mukouchi@rifis.sci.kyushu-u.ac.jp

Abstract. A pattern is a finite string of constant symbols and variable symbols. The language of a pattern is the set of all strings obtained by substituting any non-null constant string for each variable symbol in the pattern. A pattern language was introduced by Angluin in 1979 as a concrete class of languages which is inferable from positive data. In this paper, we consider the decision problem, whether for given two patterns there is a containment relation between their pattern languages, which was posed by Angluin and remains open. We give some sufficient conditions to make this problem decidable. We introduce the notions of a generalization, an instance, a minimal generalization and a maximal instance for a set of patterns. We also show that a set of patterns with a same length forms a lattice under the operations which take a minimal generalization and a maximal instance. Further, we characterize the above open problem using the minimal generalization.

1. Introduction

A pattern is a finite string of constant symbols and variable symbols. The language of a pattern is the set of all strings obtained by substituting any non-null constant string for each variable symbol in the pattern. A pattern language was introduced by Angluin[1, 2] in 1979 as a concrete class of languages which is inferable from positive data.

Angluin[1] showed the followings: the class of pattern languages is incomparable with the classes of regular languages and context-free languages in Chomsky hierarchy. It is not closed under any of these operations: union, complementation, intersection, Kleene plus, homomorphism, and inverse homomorphism. However it is closed under concatenation and reversal.

In this paper, we are concerned with solving the containment problem between pattern languages and their union. For this purpose we introduce the notions of a generalization, an instance, a minimal generalization and a maximal instance for a set of patterns and investigate their properties. We consider the decision problem whether for given two patterns there is a containment relation between their pattern languages, which was posed by Angluin[1] and remains open.

Reynolds[4] and Plotkin[3] introduced the notions of a generalization and an instance for an atomic formula, a clause and a set of clauses. They also showed that these operations form a lattice. Patterns can be regarded to have a concatenation function which satisfies the associative law. Hence a minimal generalization of a set of patterns is not always a pattern but a set of patterns, and a maximal instance may be an infinite set.

In Section 2, we introduce the notions of a generalization, an instance, a minimal generalization and a maximal instance for a set of patterns. A minimal generalization is computable for a finite set, but the computability of a maximal instance is not known so far.

In Section 3, we study the properties of a minimal generalization and a maximal instance. As a main result, we show that a set of patterns with a same length forms a lattice under the operations which take a minimal generalization and a maximal instance.

In Section 4, we deal with pattern languages, especially the above mentioned containment problem. If a pattern π is a generalization of a pattern τ, then the pattern language of π contains that of τ, but the converse is not always true. Angluin[1] showed that the converse is true when the lengths of π and τ are equal to each other or when π is a one-variable pattern. Here, we give a sufficient condition on π and an alphabet Σ which makes the converse true. Further more, we give another result on a containment relation between a pattern language and a union of some pattern languages.

In Section 5, we discuss the set of patterns not satisfying the above converse. If this set is computable, the above open problem can be solved. We show that a sequence of minimal generalizations of the finite subset, consisting of patterns less than n long, of the pattern language of τ converges to the set $D(\tau, \Sigma)$ as n grows. $D(\tau, \Sigma)$ is a set of minimal patterns of which pattern languages contains that of τ respectively. We can show it converges, but we do not know how to decide whether it is converged or not for given n. To give a sufficient condition to determine $D(\tau, \Sigma)$, we introduce the notion of an expansion of a pattern.

In Section 6, we discuss the containment problem for regular pattern languages. A regular pattern is a pattern in which each variable does not occur more than once. For the regular pattern languages, we show that the above converse is always true. Shinohara[5] showed the same result in case of extended regular pattern languages, where he allowed erasing substitutions.

2. Preliminaries

We denote an alphabet, i.e., a non-empty finite set of constant symbols, by Σ and a countable set of variable symbols by X. We assume that Σ and X are mutually disjoint. Constant symbols are denoted by $a, b, c, d, a_1, a_2, \cdots$ and variable symbols by $x, y, z, x_1, x_2, \cdots$.

We call a non-null word over $\Sigma \cup X$ a pattern and a non-null word over Σ a ground pattern. Patterns are denoted by $\pi, \tau, \pi_1, \pi_2, \cdots$, ground patterns by $u, v, w, u_1, u_2, \cdots$ and the length of π by $|\pi|$. We denote a set of patterns by $S, S_1, S_2, \cdots$. We do not consider the empty set for patterns, if not specified.

Let A be a set of symbols. We denote a set of all non-null words over A by A^+, a set of words of length n over A by A^n, and a set added null word to A^+ by A^*.

Definition 2.1.

$$\mathrm{var}(\pi) = \{x \in X \mid x \textit{ occurs in } \pi\}, \qquad \mathrm{alph}(\pi) = \{a \in \Sigma \mid a \textit{ occurs in } \pi\}.$$

Definition 2.2. *π is called a canonical pattern iff* $\mathrm{var}(\pi) = \{x_1, \ldots, x_k\}$ $(k = \sharp\mathrm{var}(\pi))$ *and the leftmost occurrence of variable x_{i+1} in π is to the right of the leftmost occurrence of x_i in π $(i = 1, \ldots, k-1)$.*

Definition 2.3. *A substitution is a homomorphism from patterns to patterns that maps each constant symbol $a \in \Sigma$ to itself. For a substitution θ, we denote the image of π by $\pi\theta$.*

We write a substitution in the form of a set $\{x_1 := \tau_1, \cdots, x_n := \tau_n\}$. Substitutions are denoted by $\theta, \sigma, \delta, \cdots$.

Definition 2.4. *π is called a generalization of τ (or τ is an instance of π) iff there exists a θ such that $\pi\theta = \tau$, which is denoted by $\pi \geq \tau$ (or $\tau \leq \pi$). We also write $\tau < \pi$ (or $\pi > \tau$) iff $\tau \leq \pi$ and $\pi \not\leq \tau$.*

Clearly from the definition, if $\tau \leq \pi$, then $|\tau| \geq |\pi|$. Angluin[1] showed that the problem of deciding whether $\tau \leq \pi$ for given arbitrary patterns π and τ is NP-complete.

Definition 2.5. *π is called a generalization of S, a set of patterns, iff $\tau \leq \pi$ for each $\tau \in S$. π is called an instance of S iff $\pi \leq \tau$ for each $\tau \in S$.*

For given set of patterns S, a pattern x is a generalizations of S. Hence S always has its generalization, but S does not always have its instance. In fact, $S = \{ax, bx\}$ $(a \neq b)$ have no instance of it.

Definition 2.6. *π is called a minimal generalization (mg for short) of S iff π is a generalization of S and $\pi \not> \tau$ for each generalization τ of S.*

π is called a longest minimal generalization (lmg for short) of S iff π is a minimal generalization of S and $|\pi| \geq |\tau|$ for each minimal generalization τ of S.

We similarly define a maximal instance (mi for short) and a shortest maximal instance (smi for short).

Definition 2.7.

$$\begin{aligned} \mathrm{mg}\, S &= \{\tau \mid \tau \text{ is a canonical mg of } S\}, & \mathrm{lmg}\, S &= \{\tau \mid \tau \text{ is a canonical lmg of } S\}, \\ \mathrm{mi}\, S &= \{\tau \mid \tau \text{ is a canonical mi of } S\}, & \mathrm{smi}\, S &= \{\tau \mid \tau \text{ is a canonical smi of } S\}. \end{aligned}$$

Note that any two different patterns in $\mathrm{mg}\, S$ ($\mathrm{lmg}\, S$, $\mathrm{mi}\, S$ or $\mathrm{smi}\, S$) do not have the relation $\leq$.

Definition 2.8. *π is called a variant of τ (denoted by $\pi \simeq \tau$) iff they are identical except renaming of variables.*

For given arbitrary pattern π, there exists a unique canonical pattern which is a variant of π. Therefore, $(N, \geq)$ with

$$N = \{\pi \in (\Sigma \cup X)^+ \mid \pi \text{ is a canonical pattern}\}$$

is a partial ordering. We also define N_n, a finite subset of N, as follows:

$$N_n = \{\pi \in (\Sigma \cup X)^n \mid \pi \text{ is a canonical pattern}\}.$$

To form a lattice in the next section, we define a universal pattern Ω such that $\Omega \leq \pi$ for any pattern π, and put $N'_n = N_n \cup \{\Omega\}$.

It is clear from the definition that if π is a generalization of S, then

$$|\pi| \leq \min\{|\tau| \mid \tau \in S\}.$$

Since the number of patterns shorter than a fixed length is finite, there are finitely many mg's of S except for variants for given S.

Example 2.1. *Let $\Sigma = \{a, b, c, d, e, f\}$ and $S = \{abcd, ecbf\}$. As easily seen, all the canonical generalizations of S are x_1, x_1x_2, $x_1x_2x_3$, x_1bx_2, x_1cx_2 and $x_1x_2x_3x_4$. Therefore,* $\text{mg}\, S = \{x_1cx_2, x_1bx_2, x_1x_2x_3x_4\}$ *and* $\text{lmg}\, S = \{x_1x_2x_3x_4\}$.

On the other hand, from the definition, if π is an instance of S, then

$$|\pi| \geq \max\{|\tau| \mid \tau \in S\}.$$

For given S, the number of mi's of S is not always finite.

Example 2.2. *Let $\Sigma = \{a\}$, $\pi_1 = axxa$ and $\pi_2 = yy$. Then* $\text{mi}\,\{\pi_1, \pi_2\} = \{a^{2n} \mid n \geq 2\}$. *In fact, suppose that τ is an instance of $\{\pi_1, \pi_2\}$, and put $\tau = \pi_1\{x := \tau_1\} = \pi_2\{y := \tau_2\}$. Then $a\tau_1\tau_1a = \tau_2\tau_2$, and it follows that $a\tau_1 = \tau_1a = \tau_2$. Thus $\tau = a^{2n}$ $(n \geq 2)$. Since all instances of $\{\pi_1, \pi_2\}$ are ground, there is no ordering relation $\leq$ between them. Hence they are all mi's of S.*

We can prove the following lemma in a similar way to Plotkin[3].

Lemma 2.1. *Let $S = \{\pi_1, \ldots, \pi_n\}$ $(n \geq 1)$, π be an mg of S and $\pi\theta_i = \pi_i$ $(i = 1, \ldots, n)$. If $x, y \in \text{var}(\pi)$ and $x\theta_i = y\theta_i$ for any i, then $x = y$.*

Proof: This proof is done by contraposition. Suppose $x \neq y$. Let $\pi' = \pi\{y := x\}$. Then $\pi' < \pi$. Let $\pi = \pi[x, y, z_1, \ldots, z_m]$, where $x, y, z_1, \ldots, z_m$ $(m \geq 0)$ are all the distinct variables of π. Then for each i,

$$\begin{aligned} \pi_i = \pi\theta_i &= \pi[x, y, z_1, \ldots, z_m]\theta_i \\ &= \pi[x\theta_i, y\theta_i, z_1\theta_i, \ldots, z_m\theta_i] \\ &= \pi[x\theta_i, x\theta_i, z_1\theta_i, \ldots, z_m\theta_i] \quad (y\theta_i = x\theta_i \text{ by the hypothesis}) \\ &= \pi[x, x, z_1, \ldots, z_m]\theta_i \\ &= \pi\{y := x\}\theta_i \\ &= \pi'\theta_i. \end{aligned}$$

Thus π' is a generalization of S. Since $\pi' < \pi$, π is not an mg of S. ∎

3. Properties of mg's and mi's, and a lattice on N_n'

In this section, we discuss fundamental properties of a generalization and an instance of a set of patterns. We also show that a set of patterns N_n' forms a lattice.

Lemma 3.1. *(1) If π is a generalization of S, then there exists a $\tau \in \mathrm{mg}\, S$ such that $\tau \leq \pi$.*
(2) If π is an instance of S, then there exists a $\tau \in \mathrm{mi}\, S$ such that $\pi \leq \tau$.

Proof: These follow immediately from the definitions of the mg and mi. ∎

Lemma 3.2. *(1)* $\mathrm{mg}\,(S \cup \{\pi\}) \subseteq \bigcup_{\tau \in \mathrm{mg}\, S} \mathrm{mg}\,\{\pi, \tau\}$.
(2) $\mathrm{mi}\,(S \cup \{\pi\}) \subseteq \bigcup_{\tau \in \mathrm{mi}\, S} \mathrm{mi}\,\{\pi, \tau\}$.

Proof: (1) Let $\pi' \in \mathrm{mg}\,(S \cup \{\pi\})$. Since π' is a generalization of $(S \cup \{\pi\})$, π' is also a generalization of S. Therefore, there exists a $\tau \in \mathrm{mg}\, S$ such that $\tau \leq \pi'$ by Lemma 3.1. Since $\pi \leq \pi'$, π' is a generalization of $\{\tau, \pi\}$. Therefore, there exists a $\pi'' \in \mathrm{mg}\,\{\tau, \pi\}$ such that $\pi'' \leq \pi'$ by Lemma 3.1. Also, π'' is a generalization of $(S \cup \{\pi\})$, because $\pi \leq \pi''$, $\tau \leq \pi''$ and that τ is a generalization of S. Though $\pi'' \leq \pi'$ holds, $\pi'' < \pi'$ contradicts the fact that π' is an mg of $(S \cup \{\pi\})$. Hence $\pi'' = \pi'$, and so $\pi' \in \bigcup_{\tau \in \mathrm{mg}\, S} \mathrm{mg}\,\{\pi, \tau\}$. This completes the proof of (1).
The proof of (2) is similar to (1). ∎

Considering a case of $S = \{abcd, ecbf\}$ (cf. Example 2.1) and $\pi = aaaa$, we see that the converse containment relation in Lemma 3.2 does not hold in general case.

Lemma 3.3. $\phi \neq \mathrm{lmg}\, S \subseteq N_n$, *where* $n = \min\{|\pi| \mid \pi \in S\}$.

Proof: Since S has its generalization x, clearly $\mathrm{lmg}\, S \neq \phi$. Let τ be an lmg of S. Then it is sufficient to show $|\tau| = n$.
(i) Since τ is a generalization of S, it follows that $|\tau| \leq n$.
(ii) Since a pattern $x_1 \cdots x_n$ is a generalization of S, there exists a $\tau' \in \mathrm{mg}\, S$ such that $\tau' \leq x_1 \cdots x_n$ by Lemma 3.1, and hence $|\tau'| \geq |x_1 \cdots x_n| = n$. Since τ is an lmg, it follows that $|\tau| \geq |\tau'| \geq n$.
Hence we have $|\tau| = n$. ∎

As to an smi of S, we can just say that if it exists, its length is not shorter than $\max\{|\pi| \mid \pi \in S\}$. In fact, let $S = \{xay, xby\}$ $(a \neq b)$. Then $\mathrm{mi}\, S = \{x_1abx_2, x_1bax_2, x_1ax_2bx_3, x_1bx_2ax_3\}$ and $\mathrm{smi}\, S = \{x_1abx_2, x_1bax_2\}$.

Lemma 3.4. *Let $S \subseteq N_n$. If π is a generalization of S with $|\pi| = n$, then there exists a $\tau \in \mathrm{lmg}\, S$ such that $\tau \leq \pi$ and $|\tau| = n$.*

Proof: Let π be a generalization of S with $|\pi| = n$. Then there exists a $\tau \in \mathrm{mg}\, S$ such that $\tau \leq \pi$. Since τ is a generalization of S, it follows that $|\tau| \leq n$. Also, since $\tau \leq \pi$, it follows that $|\tau| \geq |\pi| = n$. Hence we have $|\tau| = n$. Then it is clear that τ is an lmg of S. ∎

Theorem 3.5. *Let $\pi_1, \pi_2 \in N_n$. Then there exists a unique lmg of $\{\pi_1, \pi_2\}$ except for variants, and its length is n.*

Proof: By Lemma 3.3, $\phi \neq \text{lmg}\{\pi_1, \pi_2\} \subseteq N_n$. Now, suppose that there exist $\tau_1, \tau_2 \in \text{lmg}\{\pi_1, \pi_2\}$ such that $\tau_1 \neq \tau_2$, and put $\pi_i = \tau_1\theta_i = \tau_2\sigma_i$ $(i = 1, 2)$. We denote the j-th symbol of π by $\pi[j]$. Let k be the position at which τ_1 differs from τ_2 for the first time. Then $\tau_i[k] \in X$ $(i = 1, 2)$, because $\tau_i[k] \in \Sigma$ $(i = 1$ or $2)$ implies $\pi_1[k] = \pi_2[k] \in \Sigma$, and it follows that $\tau_1[k] = \tau_2[k] \in \Sigma$. Now, put $x_{k_1} = \tau_1[k]$ and $x_{k_2} = \tau_2[k]$. Then $x_{k_1}\theta_i = x_{k_2}\sigma_i$ $(i = 1, 2)$. We can assume $k_1 < k_2$ without loss of generality. Since τ_1 and τ_2 are canonical patterns, there exists a $k'(< k)$ such that $\tau_1[k'] = \tau_2[k'] = x_{k_1}$. Since $\tau_1[k']\theta_i = \tau_2[k']\sigma_i$, it follows that $x_{k_1}\theta_i = x_{k_1}\sigma_i$ $(i = 1, 2)$. Therefore, $x_{k_1}\sigma_i = x_{k_2}\sigma_i$ $(i = 1, 2)$ holds, and hence $x_{k_1} = x_{k_2}$ by Lemma 2.1, which contradicts $x_{k_1} \neq x_{k_2}$. ∎

About an smi, we have the following theorem:

Theorem 3.6. *Let $\pi_1, \pi_2 \in N_n$. If there exists an instance of $\{\pi_1, \pi_2\}$, and its length is n, then there exists a unique smi of $\{\pi_1, \pi_2\}$ except for variants, and its length is n.*

Proof: By Lemma 3.1, $\text{smi}\{\pi_1, \pi_2\} \neq \phi$, and clearly the length of an smi of $\{\pi_1, \pi_2\}$ is n. Now, suppose that there exist $\tau_1, \tau_2 \in \text{smi}\{\pi_1, \pi_2\}$ such that $\tau_1 \neq \tau_2$. Since π_i is a generalization of $\{\tau_1, \tau_2\}$, there exists a $\pi'_i \in \text{lmg}\{\tau_1, \tau_2\}$ such that $\pi'_i \leq \pi_i$ by Lemma 3.4 $(i = 1, 2)$. However, $\pi'_1 = \pi'_2$ holds by Theorem 3.5. Put $\pi = \pi'_1 = \pi'_2$. Then π is an instance of $\{\pi_1, \pi_2\}$, and $\tau_i \leq \pi$ holds $(i = 1, 2)$. Since $\tau_1 \neq \tau_2$, it follows that $\tau_1 < \pi$ or $\tau_2 < \pi$. This contradicts that both of τ_1 and τ_2 are mi's of $\{\pi_1, \pi_2\}$. ∎

By Theorem 3.5 and Theorem 3.6, we see that $(N'_n, \geq)$ forms a lattice, where least upper bound $\sqcup$ and greatest lower bound $\sqcap$ are defined as follows:

(1) $\pi \sqcup \tau =$ a canonical lmg of $\{\pi, \tau\}$,

(2) $$\pi \sqcap \tau = \begin{cases} \text{a canonical smi of } \{\pi, \tau\}, & \text{if there exists an instance of } \{\pi, \tau\} \text{ of length } n \\ \Omega, & \text{otherwise.} \end{cases}$$

As for the definition and the properties of a lattice, see a standard text such as Birkoff(1967).

Since $(N'_n, \geq)$ is a finite lattice, it is a complete lattice. But it is not a distributive lattice. In fact,

$$(ax_1b \sqcup x_1ab) \sqcap x_1x_1b = x_1x_1b \neq aab = (ax_1b \sqcap x_1x_1b) \sqcup (x_1ab \sqcap x_1x_1b).$$

4. Pattern languages and their containment relation

Angluin[1] introduced a pattern language in the form:

$$L(\pi) = \{w \in \Sigma^+ \mid w \leq \pi\}$$

The decision problem of a containment relation on pattern languages is to decide for given two patterns whether there is a containment relation between their pattern languages. This problem was posed by Angluin[1] and remains open.

Hereafter, we are mainly concerned with this problem. Particularly, in this section, we investigate the condition of τ, π and Σ to make the following equivalence hold:

$$\tau \leq \pi \Longleftrightarrow L(\tau) \subseteq L(\pi)$$

Angluin[1] showed that $(\Rightarrow)$ always holds for given τ and π, and that $(\Leftarrow)$ holds when $|\tau| = |\pi|$ or when π is a one-variable pattern.

Definition 4.1. $C(\pi) = L(\pi) \cap \Sigma^{|\pi|}$.

We give two basic lemmas on a containment relation of pattern languages.

Lemma 4.1. *Let $\sharp\Sigma \geq 2$. If $C(\tau) \subseteq L(\pi_1) \cup \cdots \cup L(\pi_n)$, then $\mathrm{alph}(\pi_1) \cap \cdots \cap \mathrm{alph}(\pi_n) \subseteq \mathrm{alph}(\tau)$. Therefore in case $n = 1$, if $C(\tau) \subseteq L(\pi)$, then $\mathrm{alph}(\pi) \subseteq \mathrm{alph}(\tau)$.*

Proof: This proof is done by contraposition. Let $\mathrm{alph}(\pi_1) \cap \cdots \cap \mathrm{alph}(\pi_n) \not\subseteq \mathrm{alph}(\tau)$. Hence there exists a $c \in \Sigma$ such that $c \in \mathrm{alph}(\pi_1) \cap \cdots \cap \mathrm{alph}(\pi_n) - \mathrm{alph}(\tau)$. Note that if $u \in L(\pi_1) \cup \cdots \cup L(\pi_n)$ then c occurs in u, because c occurs in each $\pi_1, \ldots, \pi_n$. Let w be a pattern substituted each variable symbol in τ with $a(\in \Sigma,\ a \neq c)$. Then $w \in C(\tau)$. But $w \notin L(\pi_1) \cup \cdots \cup L(\pi_n)$, because c does not occur in w. Hence we have $C(\tau) \not\subseteq L(\pi_1) \cup \cdots \cup L(\pi_n)$. ∎

Lemma 4.2. *Let $\sharp\Sigma \geq 3$ and $L(\tau) \subseteq L(\pi)$. If $w(\in \Sigma^+)$ is a substring of π, then w is also a substring of τ.*

Proof: Let c be any constant symbol different from the symbols at the both ends of w. Since $\sharp\Sigma \geq 3$, we can take such a symbol c. Let $n = |w|$, and τ' be a ground pattern substituted each variable symbol in τ with c^n. Since $\tau' \in L(\tau) \subseteq L(\pi)$, w is a substring of τ'. From the definitions of w and n, w and c^n are isolated from each other in τ'. Therefore, w is a substring of τ' part other than c^n, and thus w is a substring of τ. ∎

In case of $\sharp\Sigma \leq 2$, the following example shows that Lemma 4.2 does not hold in general.

Example 4.1. *Let $\Sigma = \{0, 1\}$, $\pi = x_1 01 x_2$ and $\tau = 00y11$. Considering the following two cases, we can confirm $L(\tau) \subseteq L(\pi)$:*
(i) $\tau\{y := 0^n\} = 000^n11 \in L(\pi)\ (n \geq 1)$,
(ii) $\tau\{y := 0^n1t\} = 000^n1t11 \in L(\pi)\ (n \geq 0,\ t \in \Sigma^*)$.
On the other hand, 01, a substring of π, is not a substring of τ.

Theorem 4.3. *If $\sharp\Sigma \geq \sharp(\mathrm{alph}(\tau) \cup \mathrm{alph}(\pi_1) \cup \cdots \cup \mathrm{alph}(\pi_n)) + \sharp\mathrm{var}(\tau)$, then the following three statements are equivalent: (1) $L(\tau) \subseteq L(\pi_1) \cup \cdots \cup L(\pi_n)$, (2) $C(\tau) \subseteq L(\pi_1) \cup \cdots \cup L(\pi_n)$, (3) there exists an i $(1 \leq i \leq n)$ such that $\tau \leq \pi_i$.*

Proof: From Angluin[1], (3) implies (1). Since $C(\tau) \subseteq L(\tau)$, (1) implies (2). Now, we show that (2) implies (3). Assume $\mathrm{var}(\tau) = \{x_1, \ldots, x_m\}$. Then we can take different constant symbols $c_1, \ldots, c_m$ not occurring in $\pi_1, \ldots, \pi_n$ from the assumption. Put $\theta = \{x_1 := c_1, \ldots, x_m := c_m\}$ and $w = \tau\theta$. Note that from the way to make θ, we can get τ from w by replacing all c_j in w with x_j respectively. Since $w = \tau\theta \in C(\tau)$, there exists a σ such that $w = \pi_i\sigma$ for some i $(1 \leq i \leq n)$. We assume $\sigma = \{y_1 := \alpha_1, \ldots, y_l := \alpha_l\}$. Since c_j does not occur in π_i, it follows that c_j's in $\pi_i\sigma(= w)$ are all substituted by σ. Let α'_k be a string obtained by replacing all c_j in α_k with x_j and put $\sigma' = \{y_1 := \alpha'_1, \ldots, y_l := \alpha'_l\}$. Then the string obtained by replacing c_j in $\pi_i\sigma(= w)$ with x_j is identical to $\pi_i\sigma'$. From the above observation, we see that $\tau = \pi_i\sigma'$ holds, and we obtain $\tau \leq \pi_i$. ∎

In case $n = 1$, we obtain the following theorem.

Theorem 4.4. *If $\sharp\Sigma \geq \max(2, \sharp\mathrm{var}(\tau) + \sharp\mathrm{alph}(\tau))$, then the following three statements are equivalent: (1) $L(\tau) \subseteq L(\pi)$, (2) $C(\tau) \subseteq L(\pi)$, (3) $\tau \leq \pi$.*

Proof: We can show that (1) implies (2) and that (3) implies (1) in a similar way to the above proof. We show that (2) implies (3). By Lemma 4.1, $C(\tau) \subseteq L(\pi)$ implies $\mathrm{alph}(\pi) \subseteq \mathrm{alph}(\tau)$. Therefore, the case of $n = 1$ in Theorem 4.3 holds. ∎

5. More about the containment relation

In this section, we consider a finite set of patterns π such that $L(\tau) \subseteq L(\pi)$ and $\tau \not\leq \pi$ for given τ. First, we define a set $D(\tau, \Sigma)$.

Definition 5.1.

$$D(\tau, \Sigma) = \left\{ \pi \in N \;\middle|\; \begin{array}{l} L(\tau) \subseteq L(\pi), \\ \forall \pi' < \pi, L(\tau) \not\subseteq L(\pi') \end{array} \right\}$$

It is clear from the definition that $\tau \in D(\tau, \Sigma)$, and that any two different patterns in $D(\tau, \Sigma)$ are not in the ordering relation $\leq$. Note that $D(\tau, \Sigma)$ is a finite set, because $\pi \in D(\tau, \Sigma)$ implies $|\pi| \leq |\tau|$. The following proposition is obvious.

Proposition 5.1. *$L(\tau) \subseteq L(\pi)$ iff there exists a $\pi' \in D(\tau, \Sigma)$ such that $\pi' \leq \pi$.*

If $D(\tau, \Sigma)$ is computable, then it is decidable for given two patterns whether there is a containment relation between their pattern languages. To characterize $D(\tau, \Sigma)$, we introduce a binary relation on sets of patterns.

Definition 5.2. *Let S_1 and S_2 be sets of patterns. We write $S_1 \preceq S_2$ (or $S_2 \succeq S_1$) iff for each $\tau \in S_2$, there exists a $\pi \in S_1$ such that $\pi \leq \tau$. We write $S_1 \preceq' S_2$ (or $S_2 \succeq' S_1$) iff for each $\tau \in S_1$, there exists a $\pi \in S_2$ such that $\pi \leq \tau$.*

We define H, a subset of 2^N, as follows:

$$H = \{S \subseteq N \mid \text{any different two patterns in } S \text{ are not in the ordering relation } \leq\}$$

It is clear from the definition that $(H, \succeq)$ satisfies the transitive law and the reflective law. It also satisfies the anti-symmetric law. In fact, let $S_1, S_2 \in H$, $S_1 \preceq S_2$ and $S_2 \preceq S_1$. Since $S_2 \preceq S_1$, for each $\pi \in S_1$, there exists a $\tau \in S_2$ such that $\tau \leq \pi$. Since $S_1 \preceq S_2$, there exists a $\pi' \in S_1$ such that $\pi' \leq \tau$. Therefore, $\pi' \leq \pi$ holds. Since $S_1 \in H$ and $\pi, \pi' \in S_1$, it follows that $\pi = \pi'$. Hence $\tau = \pi$, and so $S_1 \subseteq S_2$. Similarly we can show that $S_2 \subseteq S_1$, and hence $S_1 = S_2$. Therefore, $(H, \succeq)$ is a partial ordering. In a similar way, we can also show that $(H, \succeq')$ is a partial ordering.

From their definitions, we see that $S \preceq \mathrm{mg}\, S$, $S \preceq' \mathrm{mg}\, S$, $\mathrm{mi}\, S \preceq' S$ and that if $\mathrm{mi}\, S \neq \phi$ then $\mathrm{mi}\, S \preceq S$. We can easily prove the following lemma using Lemma 3.1.

Lemma 5.2. *If $S_1 \subseteq S_2$, then $\mathrm{mg}\, S_1 \preceq \mathrm{mg}\, S_2$ and $\mathrm{mi}\, S_2 \preceq' \mathrm{mi}\, S_1$.*

Proof: Let $S_1 \subseteq S_2$ and $\pi \in \mathrm{mg}\, S_2$. Since π is a generalization of S_2, π is also a generalization of S_1. Hence there exists a $\tau \in \mathrm{mg}\, S_1$ such that $\tau \leq \pi$ by Lemma 3.1, and it follows that $\mathrm{mg}\, S_1 \preceq \mathrm{mg}\, S_2$. Similarly we can show that $S_1 \subseteq S_2$ implies $\mathrm{mi}\, S_2 \preceq' \mathrm{mi}\, S_1$. ∎

Definition 5.3.

$$C_0(\pi) = L(\pi) \cap \Sigma^{|\pi|} \ (= C(\pi)),$$
$$C_i(\pi) = C_{i-1}(\pi) \cup \left(L(\pi) \cap \Sigma^{|\pi|+i}\right) \ (i \geq 1).$$

$C_i(\pi)$ is a finite subset of patterns in $L(\pi)$ not longer than $|\pi| + i$, and it satisfies the following relation:

$$C_0(\pi) \subseteq C_1(\pi) \subseteq \cdots \subseteq C_n(\pi) \subseteq \cdots \subseteq L(\pi)$$

Theorem 5.3. *A set of patterns* $\mathrm{mg}\, C_n(\pi)$ *converges as* $n \to \infty$.

Proof: Note that $\mathrm{mg}\, S \in H$ for any set of patterns S. By Lemma 5.2 and the definition of $\preceq$, we obtain

$$\mathrm{mg}\, C_0(\pi) \preceq \mathrm{mg}\, C_1(\pi) \preceq \cdots \preceq \mathrm{mg}\, C_n(\pi) \preceq \cdots \preceq \{x_1\},$$

and hence $\mathrm{mg}\, C_n(\pi)$ converges. ∎

Theorem 5.4. *Let* $S = \lim_{n\to\infty} \mathrm{mg}\, C_n(\tau)$. *Then for any pattern* τ, $D(\tau, \Sigma) = S$.

Proof: First, we show that $L(\tau) \subseteq L(\pi)$ if and only if there exists a $\tau' \in S$ such that $\tau' \leq \pi$. Then it is clear that $D(\tau, \Sigma) = S$ by Proposition 5.1, because $D(\tau, \Sigma), S \in H$.
(i) The 'if' part. Let $L(\tau) \subseteq L(\pi)$. Then $C_i(\tau) \subseteq L(\pi)$ holds, and it follows that π is a generalization of $C_i(\tau)$ for any $i(\geq 0)$. Therefore, there exists a $\tau_i' \in \mathrm{mg}\, C_i(\tau)$ such that $\tau_i' \leq \pi$ for any i. Hence there exists a $\tau' \in S$ such that $\tau' \leq \pi$.
(ii) The 'only if' part. Suppose that there exists a $\tau' \in S$ such that $\tau' \leq \pi$. Since $\mathrm{mg}\, C_i(\tau) \preceq S$, there exists a $\tau_i'' \in \mathrm{mg}\, C_i(\tau)$ such that $\tau_i'' \leq \tau'$ for any i. Therefore, τ' is a generalization of $C_i(\tau)$, and it follows that $C_i(\tau) \subseteq L(\tau')$ for any i. Hence we have $L(\tau) \subseteq L(\tau')$. Since $L(\tau') \subseteq L(\pi)$, we obtain $L(\tau) \subseteq L(\pi)$. ∎

Definition 5.4. *We say that* S, *a set of patterns, is an expansion of* τ *iff (1)* $\pi \leq \tau$ *for each* $\pi \in S$, *(2) any different two patterns in* S *are not in the ordering relation* $\leq$, *and (3)* $L(\tau) = \bigcup_{\pi \in S} L(\pi)$.

For example, let $\Sigma = \{a, b, c\}$ and $\tau = x$. Then $S = \{ax, bx, cx\}$ is an expansion of τ.

Theorem 5.5. *If there exist* n *and* S, *an expansion of* τ, *such that* $\mathrm{mg}\, C_n(\tau) = \mathrm{mg}\, S$, *then* $D(\tau, \Sigma) = \mathrm{mg}\, C_n(\tau)$.

Proof: Suppose $\pi \in \mathrm{mg}\, S$. Then π is a generalization of S, and it follows that $L(\tau) = \bigcup_{\tau' \in S} L(\tau') \subseteq L(\pi)$. Therefore, by Proposition 5.1 and Theorem 5.4, there exists a $\pi' \in D(\tau, \Sigma)$ such that $\pi' \leq \pi$. Hence $D(\tau, \Sigma) \preceq \mathrm{mg}\, S$, and so $\mathrm{mg}\, C_n(\tau) \preceq D(\tau, \Sigma) \preceq \mathrm{mg}\, S$. Since $\mathrm{mg}\, C_n(\tau) = \mathrm{mg}\, S$ holds from the assumption, it follows that $D(\tau, \Sigma) = \mathrm{mg}\, C_n(\tau)$. ∎

Example 5.1. *Let* $\pi = x_1 x_1 x_2$ *and* $\tau = 0x_1 10 x_1 x_1 1$. *These two patterns were given by Angluin[1] as an example which satisfies* $L(\tau) \subseteq L(\pi)$ *and* $\tau \not\leq \pi$ *when* $\Sigma = \{0, 1\}$. *First, we can confirm* $\mathrm{mg}\, C_0(\tau) = \{\tau, \pi\}$. *Let*

$$w_0 = \tau\{x_1 := 0\} = 0010001, \qquad w_1 = \tau\{x_1 := 1\} = 0110111,$$
$$\tau_0 = \tau\{x_1 := 0x\} = 00x100x0x1, \quad \tau_1 = \tau\{x_1 := 1x\} = 01x101x1x1.$$

Then $\{w_0, w_1, \tau_0, \tau_1\}$ *is an expansion of* τ. *We can also confirm* $\mathrm{mg}\, \{w_0, w_1, \tau_0, \tau_1\} = \{\tau, \pi\}$. *Therefore, we see* $D(\tau, \Sigma) = \{\tau, \pi\}$ *by Theorem 5.5.*

We can prove the following lemma using Lemma 3.2.

Lemma 5.6. *Let π be a canonical pattern and $S \subseteq L(\pi)$. If $\text{mg}\, S = \{\pi\}$, then $\text{mg}\, S' = \{\pi\}$ for each S' such that $S \subseteq S' \subseteq L(\pi)$.*

Proof: Let $\text{mg}\, S = \{\pi\}$ and $S \subseteq S' \subseteq L(\pi)$. Then $\text{mg}\, S \preceq \text{mg}\, S'$ by Lemma 5.2. Since $w \leq \pi$ for any $w \in S'$, it follows that $S' \preceq' \{\pi\}$, and hence $\text{mg}\, S' \preceq \text{mg}\, \{\pi\}$. Therefore, $\text{mg}\, S \preceq \text{mg}\, S' \preceq \text{mg}\, \{\pi\}$ holds. Then we have $\text{mg}\, S' = \{\pi\}$, because $\text{mg}\, S = \{\pi\}$ by the hypothesis and $\text{mg}\, \{\pi\} = \{\pi\}$. ∎

By this lemma, we see $D(\tau, \Sigma) = \{\tau\}$ if $\text{mg}\, C_i(\tau) = \{\tau\}$ for some i. In this case, $\tau \leq \pi$ iff $L(\tau) \subseteq L(\pi)$ by Proposition 5.1. In a special case, we have $D(\tau, \Sigma) = \text{mg}\, C_0(\tau) = \{\tau\}$ as in the following corollary.

Corollary 5.7. *If $\sharp\Sigma \geq \max(2, \sharp\text{var}(\tau) + \sharp\text{alph}(\tau))$, then $D(\tau, \Sigma) = \{\tau\}$.*

Proof: Suppose that π is a generalization of $C_0(\tau)$. Then $C_0(\tau) \subseteq L(\pi)$, and $\tau \leq \pi$ by Theorem 4.4. Therefore, $\text{mg}\, C_0(\tau) = \{\tau\}$ and we obtain the result by Lemma 5.6. ∎

6. Regular pattern languages

Definition 6.1. *A regular pattern is a pattern in which each variable does not occur more than once.*

The following theorem shows that if π and τ are regular patterns, then the equivalence of (1) and (3) in Theorem 4.4 holds independently of π and τ.

Theorem 6.1. *Let π and τ be regular patterns and $\sharp\Sigma \geq 3$. Then $L(\tau) \subseteq L(\pi)$ if and only if $\tau \leq \pi$.*

Proof: The 'if' part is clear from Angluin[1]. So, we show the converse. Put $\pi = w_1 x_1 w_2 x_2 \cdots x_n w_{n+1}$ and $\tau = u_1 y_1 u_2 y_2 \cdots y_m u_{m+1}$ ($w_i, u_j \in \Sigma^*$). Let

c_1 be a constant symbol different from the first symbol of w_1,
c_i be a constant symbol different from both ends of w_i ($2 \leq i \leq n$) and
c_{n+1} be a constant symbol different from the last symbol of w_{n+1}.

Since $\sharp\Sigma \geq 3$, we can take such symbols c_j. Let $N = \max\{|w_i| \mid 1 \leq i \leq n+1\} + 1$. We see that u_1 starts with w_1 by considering τ substituted all variables in τ with c_1^N. We also see that u_{m+1} ends with w_{n+1} by considering τ substituted all variables in τ with c_{n+1}^N. Let $\pi' = x_1 w_2 x_2 \cdots w_n x_n$ and $\tau' = u_1' y_1 u_2 y_2 \cdots u_m y_m u_{m+1}'$ ($u_1 = w_1 u_1', u_{m+1} = u_{m+1}' w_{n+1}$). Then $L(\tau') \subseteq L(\pi')$ holds. Let k_2 be the position at which w_2 occurs in τ' for the first time, and k_i be the position at which w_i occurs in τ' after $(k_{i-1} + |w_{i-1}| + 1)$-st symbol for the first time ($3 \leq i \leq n$). We can show the existence of such k_i ($2 \leq i \leq n$) as follows:

(I) Such a k_2 exists, because w_2 must occur in τ' by Lemma 4.2

(II) Suppose that $k_2, \cdots, k_{i-1}$ exist and k_i does not exist. Let τ'' be a ground pattern substituted

all variables occurring between the first and the $(k_2 - 1)$-st symbol of τ' with c_2^N,
all variables between the $(k_2 + |w_2|)$-th and the $(k_3 - 1)$-st symbol of τ' with c_3^N,
⋮
all variables between the $(k_{i-2} + |w_{i-2}|)$-th and $(k_{i-1} - 1)$-st symbol of τ' with c_{i-1}^N,
all variables after $(k_{i-1} + |w_{i-1}|)$-th symbol with c_i^N.

Then $\tau'' \in L(\tau')$. But $\tau'' \notin L(\pi')$, because there exist no $w_1, \ldots, w_i$ which occur in τ'' in this order with at least one symbol put between them. This contradicts the assumption $L(\tau') \subseteq L(\pi')$.
It is clear that $\tau \leq \pi$ from the existence of such $k_2, \ldots, k_n$. ∎

In case $\sharp\Sigma \leq 2$, we see that Theorem 6.1 does not hold in general from Example 4.1. Shinohara[5] showed the same result in the case of extended regular pattern languages, where erasing substitutions, i.e., a substitution θ with $|x\theta| = 0$ for variable x, are allowed.
Now, we consider a containment relation between a regular pattern language and a union of some regular pattern languages.

Lemma 6.2. *Let $\tau, \pi_1, \ldots, \pi_n$ be regular patterns with a same length and $\Sigma \supsetneq \mathrm{alph}(\pi_1) \cup \cdots \cup \mathrm{alph}(\pi_n)$. If $C(\tau) \subseteq C(\pi_1) \cup \cdots \cup C(\pi_n)$, then there exists an i $(1 \leq i \leq n)$ such that $\tau \leq \pi_i$.*

Proof: Put $\tau = A_1 \cdots A_m$ and $\pi_1 = B_1^{(1)} \cdots B_m^{(1)}, \ldots, \pi_n = B_1^{(n)} \cdots B_m^{(n)}$ $(A_j, B_j^{(i)} \in \Sigma \cup X)$. Let $i_1, \ldots, i_{n'}$ be all the i such that

$$\forall j\ (1 \leq j \leq m),\ [A_j, B_j^{(i)} \in \Sigma \Rightarrow A_j = B_j^{(i)}].$$

For any i different from $i_1, \ldots, i_{n'}$, there exists a j such that $A_j, B_j^{(i)} \in \Sigma$ and $A_j \neq B_j^{(i)}$, and it follows that $C(\tau) \cap C(\pi_i) = \phi$. Therefore, $C(\tau) \subseteq C(\pi_{i_1}) \cup \cdots \cup C(\pi_{i_{n'}})$ holds. By renumbering, let $\pi_1, \ldots, \pi_{n'}$ be $\pi_{i_1}, \ldots, \pi_{i_{n'}}$. Hence

$$(\star) \qquad \forall i\ (1 \leq i \leq n'),\ \forall j\ (1 \leq j \leq m),\ [A_j, B_j^{(i)} \in \Sigma \Rightarrow A_j = B_j^{(i)}].$$

Let $j_1, \ldots, j_l$ be all the j such that $A_j \in X$, and $c \in \Sigma - \mathrm{alph}(\pi_1) \cup \cdots \cup \mathrm{alph}(\pi_n)$. Put $\tau' = \tau\{A_{j_1} := c, \ldots, A_{j_l} := c\}$. Since $\tau' \in C(\tau)$, there exists a k such that $\tau' \in L(\pi_k)$. From the way to select c, $B_{j_1}^{(k)}, \ldots, B_{j_l}^{(k)}$ are all variables. Therefore,

$$\forall j\ (1 \leq j \leq m),\ [A_j \in X \Rightarrow B_j^{(k)} \in X].$$

By a contraposition, we obtain

$$\forall j\ (1 \leq j \leq m),\ [B_j^{(k)} \in \Sigma \Rightarrow A_j \in \Sigma]$$

and

$$\forall j\ (1 \leq j \leq m),\ [B_j^{(k)} \in \Sigma \Rightarrow A_j = B_j^{(k)}]$$

by $(\star)$. Therefore, $\tau \leq \pi_k$ holds. ∎

Definition 6.2. *For a regular pattern $\pi = w_1 x_1 w_2 \cdots w_m x_m w_{m+1}$ $(w_i \in \Sigma^*)$ and $n \geq 1$,*

$$\mathrm{ext}(\pi, n) = \left\{ \tau \in N_n \;\middle|\; \begin{array}{l} \tau \textit{ is a regular pattern being a variant of} \\ w_1 \tau_1 w_2 \cdots w_m \tau_m w_{m+1} \quad (\tau_j \in X^+) \end{array} \right\}.$$

For example, let $\tau = ax_1bcx_2e$. Then

$$\begin{aligned}
&\mathrm{ext}(\tau, 1) = \cdots = \mathrm{ext}(\tau, 5) = \phi, \\
&\mathrm{ext}(\tau, 6) = \{ax_1bcx_2e\}, \\
&\mathrm{ext}(\tau, 7) = \{ax_1x_2bcx_3e, ax_1bcx_2x_3e\}, \\
&\mathrm{ext}(\tau, 8) = \{ax_1x_2x_3bcx_4e, ax_1x_2bcx_3x_4e, ax_1bcx_3x_4x_5e\} \qquad \text{and so on.}
\end{aligned}$$

Lemma 6.3. *If $\tau \leq \pi$, then there exists a $\pi' \in \mathrm{ext}(\pi, |\tau|)$ such that $\tau \leq \pi'$.*

Proof: Let $\tau \leq \pi$. Then there exists a substitution θ such that $\pi\theta = \tau$. Assume $\pi = w_1 y_1 w_2 \cdots w_n y_n w_{n+1}$ ($w_i \in \Sigma^*$) and $\theta = \{y_1 := \alpha_1, \ldots, y_n := \alpha_n\}$. Note that $|\tau| = |w_1| + |\alpha_1| + |w_2| + \cdots + |w_n| + |\alpha_n| + |w_{n+1}|$. Let $n_i = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_i|$ ($1 \leq i \leq n$) and put $\sigma = \{y_1 := x_1 x_2 \cdots x_{n_1}, y_2 := x_{n_1+1} x_{n_1+2} \cdots x_{n_2}, \ldots, y_n := x_{n_{n-1}+1} x_{n_{n-1}+2} \cdots x_{n_n}\}$. Then clearly there exists a substitution δ such that $\theta = \sigma\delta$.

For example, if $\theta = \{y_1 := abc, y_2 := ef\}$, then $\sigma = \{y_1 := x_1 x_2 x_3, y_2 := x_4 x_5\}$ and $\delta = \{x_1 := a, x_2 := b, x_3 := c, x_4 := e, x_5 := f\}$.

Put $\pi' = \pi\sigma$. Then $\tau \leq \pi'$, because $\pi'\delta = \pi\theta = \tau$. Also, from the above observation, we see that $\pi' \in \mathrm{ext}(\pi, |\tau|)$. ∎

By Lemma 6.2, we can prove the following theorem, which has no restriction on lengths of their patterns.

Theorem 6.4. *Let $\tau, \pi_1, \ldots, \pi_n$ be regular patterns. If $\Sigma \supsetneq \mathrm{alph}(\pi_1) \cup \cdots \cup \mathrm{alph}(\pi_n)$, then the following three statements are equivalent: (1) $L(\tau) \subseteq L(\pi_1) \cup \cdots \cup L(\pi_n)$, (2) $C(\tau) \subseteq L(\pi_1) \cup \cdots \cup L(\pi_n)$, (3) there exists an i ($1 \leq i \leq n$) such that $\tau \leq \pi_i$.*

Proof: We can show that (1) implies (2) and that (3) implies (1) in a similar way to the proof of Theorem 4.3. Now, we show that (2) implies (3). Let $C(\tau) \subseteq L(\pi_1) \cup \cdots \cup L(\pi_n)$. Clearly,

$$\begin{aligned} C(\tau) &\subseteq (L(\pi_1) \cup \cdots \cup L(\pi_n)) \cap \Sigma^{|\tau|} \\ &= (L(\pi_1) \cap \Sigma^{|\tau|}) \cup \cdots \cup (L(\pi_n) \cap \Sigma^{|\tau|}). \end{aligned}$$

Since $L(\pi_i) \cap \Sigma^{|\tau|} = \bigcup_{\pi \in \mathrm{ext}(\pi_i, |\tau|)} C(\pi)$ by Lemma 6.3,

$$C(\tau) \subseteq \bigcup_{i=1}^{n} \bigcup_{\pi \in \mathrm{ext}(\pi_i, |\tau|)} C(\pi).$$

Put $S_i = \mathrm{ext}(\pi_i, |\tau|)$ ($i = 1, \ldots, n$) and $S = \bigcup_{i=1}^{n} S_i$. Then $C(\tau) \subseteq \bigcup_{\pi \in S} C(\pi)$. Since the lengths of all patterns in S are all equal to $|\tau|$ and $\Sigma \supsetneq \bigcup_{\pi \in S} \mathrm{alph}(\pi)$, there exists a $\pi \in S$ such that $\tau \leq \pi$ by Lemma 6.2. Therefore, there exists an i such that $\pi \in S_i$, and so $\pi \leq \pi_i$. Hence, we obtain $\tau \leq \pi_i$. ∎

7. Conclusion

In this paper, we have discussed the open problem posed by Angluin. Pattern languages were dealt with by her for the first time as a concrete class which is inferable from positive data. The class has deeply influenced the theories of inductive inference from positive data. As a future work, we are reconsidering the algorithm for inductive inference, based on the results we presented in this paper.

Acknowledgements

The author wishes to thank Prof. Masako Sato, University of Osaka Prefecture, for many suggestions and encouragement.

References

[1] Angluin, D., *Finding Patterns Common to a Set of Strings*, "Proceedings of the 11th Annual Symposium on Theory of Computing", 1979, pp.130–141.

[2] Angluin, D., *Inductive Inference of Formal Languages from Positive Data*, Information and Control **45** (1980), 117–135.

[3] Plotkin, G. D., *A Note on Inductive Generalization*, Machine Intelligence **5** (1970), 153–164.

[4] Reynolds, J. C., *Transformational Systems and the Algebraic Structure of Atomic Formulas*, Machine Intelligence **5** (1970), 135–152.

[5] Shinohara, T., *Polynomial Time Inference of Extended Regular Pattern Languages*, "RIMS Symposia on Software Science and Engineering, Kyoto, 1982, Proceedings, Lecture Notes in Computer Science", Springer-Verlag, 1982, pp.115–127.