On Maximum Uniform Partition of Line Graphs

Shiraishi, Shuji
Department of Applied Mathematics Fukuoka University

出版情報：RIFIS Technical Report. 32, 1990-10-22. 九州大学理学部附属基礎情報学研究施設
バージョン：

RIFIS-TR-CS-32

RIFIS Technical Report

On Maximum Uniform Partition of Line Graphs

Shuji Shiraishi

October 22, 1990

Research Institute of Fundamental Information Science
Kyushu University 33
Fukuoka 812, Japan
E-mail: a71415g@kyu-cc.cc.kyushu-u.ac.jp Phone: 092 (871)6631 Ex. 6125

On Maximum Uniform Partition of Line Graphs

SHUJI SHIRAISHI

Department of Applied Mathematics

Fukuoka University

Nanakuma, Jonan-ku, Fukuoka 814-01, Japan

Abstract

We show that (A) If $G = (V, E)$ is an Euler circuit, then the number of the maximum uniform partition of the line graph $L(G)$ is $(1/4)\sum_{v\in V} d_v^2(-1)(-1$ is added when $|E|$ is odd), where d_v is the degree of v. (B) If G is not an Euler circuit, then the number of the maximum uniform partition of $L(G)$ is $(1/4)\sum_{v\in V} d_v^2 - l$, where l is the number of vertices of odd degree.

Keywords: maximum uniform partition, line graph, Euler tour technique.

A *uniform partition* of a graph $G = (V, E)$ is a partition of V into two disjoint subsets V_0, V_1 such that $|V_0| = |V_1|$ or $|V_0| = |V_1| + 1$. We denote by (V_0, V_1) the uniform partition of G and (V_0, V_1) represents the set of all edges in G join V_0 with V_1. A uniform partition (V_0, V_1) of G is *maximum* if $|(V_0, V_1)| \geq |(V_0', V_1')|$ for any other uniform partition (V_0', V_1') of G. The *line graph* of G, denoted $L(G)$ is defined as follows: The vertices of $L(G)$ are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are. An *Euler graph* is a connected graph in which at most two vertices are of odd degree. An *Euler tour* in a graph is a circuit (path) that traverses every edge exactly once.

In general, the number of the maximum partition of a graph is not known well. In the present paper, we derive the number of the maximum uniform partition of the line graph exactly by the application of algorithmic reasoning. The algorithm used in the Theorem 1 is in NC[1,2]. The main result is the following:

THEOREM(A) *If G is an Euler circuit, then the number of the maximum uniform partition of the line graph $L(G)$ is $(1/4)\sum_{v \in V} d_v^2(-1)$($-1$ is added when m is odd), where d_v is the degree of v.* (B)*If G is not an Euler circuit, then the number of the maximum uniform partition of $L(G)$ is $(1/4)\sum_{v \in V} d_v^2 - l$, where l is the number of vertices of odd degree.*

PROOF. We consider the following maximum uniform partition algorithm using the Euler tour technique.

MAXIMUM UNIFORM PARTITION ALGORITHM

INPUT: A connected graph $G = (V, E)$.

OUTPUT: A maximum uniform partition (V_0, V_1) of the line graph $L(G)$.

1. Compute the degree of each vertex in G.

2. If G is not Euler graph, make G a graph of even degrees G^* by connecting all odd degree vertices to a dummy vertex v^*.

3. Find an Euler circuit in G^*.

4. Label the edges of G^* from v^* by 1 and 0 alternately following the route determined by its Euler circuit.

5. Let $V_0 = \{e | \text{edge } e \text{ labeled } 0 \text{ in } G\}$ and $V_1 = \{e | \text{edge } e \text{ labeled } 1 \text{ in } G\}$.

We remark that if G is an Euler graph, we start at any vertex v when labeling.

Let $E(v, label) = \{\{v, x\} | \{v, x\}$ in G^**and* $\{v, x\}$ *has label*$(= 0$ or $1)\}$. Then the following two facts and fact 3 guarantee that Step 5 of the algorithm is correct.

FACT 1. *If v is a start vertex, then $E(v, 0) = E(v, 1)$ or $E(v, 0) + 2 = E(v, 1)$, else $E(v, 0) = E(v, 1)$ or $E(v, 0) = E(v, 1) \pm 1$.*

FACT 2. $|V_0| = |V_1|$ *or* $|V_0| + 1 = |V_1|$.

These two facts are obtained by $1 - 0$ labeling of Step 4. $E(v, 0) + 2 = E(v, 1)$ holds when G is an Euler circuit with an odd number edges.

FACT 3[3].*G is a line graph iff the edges of G can be partitioned into complete subgraphs in such a way that no vertex lies in more than two of the subgraphs.*

Each vertex of G corresponds to a complete subgraph of $L(G)$, so that by Fact 1(except for $E(v,0)+2=E(v,1)$) each such subgraph is partitioned uniformly, and $L(G)$ itself is also partitioned uniformly by Fact 2. By Fact 3, each such complete subgraph is edge disjoint, therefore the maximum uniform partition is guaranteed. If $E(v,0)+2=E(v,1)$ holds, the corresponding complete subgraph is not partitioned uniformly. However the maximum uniform partition holds. It is easy to show.

By the definition of the line graph, each edge in $E(v,0)$ joins all edges in $E(v,1)$, so that the number of the joins about the vertex v is $|E(v,0)||E(v,1)|$. Therefore the number $|(V_0,V_1)|$ of the maximum uniform partition of $L(G)$ is $\sum_{v\in V}|E(v,0)||E(v,1)|$.

If $G=(V,E)$ is an Euler circuit with an even number of edges , then for any vertex v in G, $|E(v,0)|=|E(v,1)|=d_v/2$ and therefore $|(V_0,V_1)|=(1/4)\sum_{v\in V}d_v^2$. Otherwise $|E|$ is odd, then it is the same as the even number case except for exactly one vertex v. Since $|E(v,0)|+2=|E(v,1)|$ and $|E(v,0)|+|E(v,1)|=d_v$ hold, we get $|(V_0,V_1)|=(1/4)\sum_{v\in V}d_v^2-1$.

If G is not an Euler circuit and v is the odd degree, then $|E(v,0)||E(v,1)|=(d_v-1)/2\cdot(d_v+1)/2$ holds. Therefore we get $|(V_0,V_1)|=(1/4)\sum_{v\in V}d_v^2-l$, where l is the number of vertices of odd degree. □

REFERENCES

1. M. Atallah and U. Vishkin, Finding Euler tours in parallel, *J.Comput. System Sci.* **29**, (1984), 330-337.

2. A. Israeli and Y. Shiloach, An improved parallel algorithm for maximal matching, *Inform. Process. Lett.* **22**,(1986),57-60.

3. F. Harary, *Graph Theory*, Addison-Wesley,Reading,MA,1969