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Abstract

We show that the problem of finding a maximal vertex-induced (resp., edge-induced) subgraph of maximum degree k is in NC^2 for $k \geq 0$ (resp., $k \geq 1$). For these problems, we develope a method which exploits the NC algorithm for the maximal independent set problem.

1 Introduction

Karp [5] and Luby [6] have shown that the problem of finding a maximal independent set of a graph, called MIS, is solvable in NC, which is known to be the class of problems computable by PRAMs with polynomial number of processors in $O((\log n)^k)$ time for some $k \geq 0$ [12], [13], [14]. In this paper, we show that finding a maximal subset of vertices whose induced subgraph is of degree at most k allows an NC algorithm for any $k \geq 0$. For the proof, we develope an elegant method which employs the NC algorithm for MIS devised by [5], [6]. We also show that the problem of finding a maximal set of edges which forms a subgraph of degree at most k is in NC.

In general, a problem of this kind is stated as a maximal subgraph problem for a given property π, which is to find a maximal subset of vertices (resp., edges) which induces a subgraph satisfying π. For example, MIS is the problem for property "no two vertices are adjacent". This paper deals with the problem for property "maximum degree k".

It has been shown that most of the lexicographically first maximal (abbreviated to lfm) subgraphs problems are P-complete [10]. Therefore no NC algorithms exist for the lfm subgraph problems if P$\neq$NC. On the other hand, the problem of finding *any* maximal subgraph which satisfies a given property seems to allow NC algorithms for many properties. However, only a few are shown to be in NC. The followings are some of them: As we mentioned above MIS is the one. In [10], the maximal edge-induced forest problem and the maximal edge-induced bipartite problems are shown to be in NC. With some restrictions, the maximal edge-induced outerplanar subgraph problem [8] and the maximal vertex-induced acyclic subgraph problem restricted to directed graphs with degree at most 3 also allow NC algorithms [9]. The results in this paper add a new family of such problems.

2 Preliminaries and Definitions

A graph $G = (V, E)$ means an undirected graph without any multiple eges and self-loops. For a subset $U \subseteq V$, we define as $E[U] = \{\{u, v\} \in E \mid u, v \in U\}$. The graph $G[U] = (U, E[U])$ is called the *vertex-induced subgraph of* U. For a subset $F \subseteq E$, we define $V[F]$ to be the set of endpoints of the edges in F. We denote by $\langle F \rangle = (V[F], F)$ the graph formed from F and call it the *edge-induced subgraph of* F. For a vertex u, the degree of u is denoted by $deg_G(u)$. We denote by $deg(G) = \max\{w \mid deg_G(w)\}$.

Let $k \geq 0$ be any integer. The *maximum degree k vertex-induced subgraph problem* (VIMS(k)) is stated as follows:

VIMS(k)
Instance: A graph $G = (V, E)$.
Problem: Find a maximal subset $U \subseteq V$ such that $G[U]$ is of degree at most k.

In a similar way, the *maximum degree k edge-induced subgraph problem* (EIMS(k)) is defined as follows:

EIMS(k)
Instance: A graph $G = (V, E)$.
Problem: Find a maximal subset $F \subseteq E$ such that $\langle F \rangle$ is of degree at most k.

3 Finding Bounded Degree Maximal Subgraphs

Theorem 1 *VIMS(k) is in* NC^2 *for* $k \geq 0$.

Proof. We show an NC algorithm by employing the NC algorithm for MIS. Let $G = (V, E)$ be a graph for which we are finding a maximal subset U of vertices whose induced subgraph $G[U]$ is of degree at most k.

For subsets W and U of vertices with $W \cap U = \emptyset$, let $E_U^W = \{\{v, w\} \mid$ there is $u \in U$ with $w \neq v$ such that $u, w \in W$ and $\{v, u\} \in E$, $\{w, u\} \in E\}$. Then let $H_U^W = (W, E[W] \cup E_U^W)$. The required set U of vertices is computed together with a set W of vertices W such that $W \cap U = \emptyset$. Initially let $W = V$ and $U = \emptyset$. At each iteration of the algorithm, a maximal independent set I of H_U^W is computed and added to U while vertices which make the degree of some vertex greater than k are deleted from W together with I. This is iterated k^2 times. Formally the algorithm is described as follows:

```
begin /* G = (V, E) is an input */
   W ← V; U ← ∅;
   for i ← 1 to k^2 do
      begin
         Find a maximal independent set I of H_U^W;
         U ← U ∪ I;
         W ← W − I;
         W ← W − {w ∈ W | deg(G[U ∪ {w}]) > k}
      end
end
```

We show that this algorithm computes a maximal subset U whose induced subgraph is of degree at most k.

Let $W_0 = V$ and $U_0 = \emptyset$. Then the graph $H_{U_0}^{W_0}$ is the same as $G = (V, E)$. Therefore in the first iteration, a maximal independent set of G is computed at line 5. For $i = 1, ..., k^2$, let U_i, I_i and W_i be the contents of variables U, I and W at the end of ith iteration, respectively. Obviously, $W_i \cap U_i = \emptyset$ for $i = 0, ..., k^2$. We assume that the induced subgraph $G[U_{i-1}]$ is of degree at most k.

Let $\{w, u\}$ be an edge in E with $w \in W_i$ and $u \in U_i$. Line 8 deletes every vertex which is adjacent to more than k vertices in U_i or adjacent to a vertex v in U_i with $deg_{G[U_i]}(v) = k$. Therefore u is adjacent to at most k vertices in U_i and $deg_{G[U_i \cup \{w\}]}(u) \leq k$. Hence, for each w in W_i, we see that

$$A_i(w) = \sum_{u \in U_i \text{ with } \{w,u\} \in E} deg_{G[U_i \cup \{w\}]}(u) \leq k^2.$$

To show that W becomes empty after k^2 iterations, it suffices to prove that each w in W_i satisfies

$$A_i(w) > A_{i-1}(w)$$

for $i = 1, ..., k^2$. Since w is not in the maximal independent set I_i of $H_{U_{i-1}}^{W_{i-1}}$ computed by line 5, w is adjacent to a vertex v in $I_i \subseteq W_{i-1}$ via an edge $\{w, v\}$ in $E[W_{i-1}]$ or $E_{U_{i-1}}^{W_{i-1}}$.

Case 1. If $\{w, v\} \in E[W_{i-1}]$, then $\{w, v\}$ is an edge in $G[U_i \cup \{w\}]$. Hence $deg_{G[U_i \cup \{w\}]}(v) \geq 1$. Since $v \in U_i$, $v \notin U_{i-1}$ and $\{w, v\} \in E$, we see that $A_i(w) \geq A_{i-1}(w) + deg_{G[U_i \cup \{w\}]}(v) > A_{i-1}(w)$.

Case 2. If $\{w, v\} \in E_{U_{i-1}}^{W_{i-1}}$, then there is a vertex $u \in U_{i-1}$ with $\{w, u\} \in E$ and $\{v, u\} \in E$. Since $v \in W_{i-1}$, $W_{i-1} \cap U_{i-1} = \emptyset$ and $w \neq v$, we see $v \notin U_{i-1} \cup \{w\}$. Hence $\{v, u\}$ is not an edge in $G[U_{i-1} \cup \{w\}]$. On the other hand, v is in U_i and u is in $U_{i-1} \subseteq U_i$. Hence $\{v, u\}$ is an edge in $G[U_i \cup \{w\}]$. Therefore $deg_{G[U_i \cup \{w\}]}(u) > deg_{G[U_{i-1} \cup \{w\}]}(u)$. Since $u \in U_i$ and $\{w, u\} \in E$, we see that $A_i(w) > A_{i-1}(w)$.

We now show that $deg(G[U_i]) \leq k$. For a vertex u in U_{i-1}, if u is adjacent to a vertex w in I_i via an edge in E, then no other vertex in I_i is adjacent to u since I_i is also an independent set with respect to $E_{U_{i-1}}^{W_{i-1}}$. Therefore the degree of u in $G[U_{i-1} \cup I_i]$ remains at most k since $deg(G[U_{i-1} \cup \{w\}]) \leq k$ by the algorithm. For a vertex u in I_i, $deg_{G[U_{i-1} \cup I_i]}(u)$ is at most k since u is adjacent to at most k vertices in U_{i-1} and since I_i is an independent set with respect to $E[W_{i-1}]$. Hence $deg_{G[U_{i-1} \cup I_i]}(u) \leq k$.

Since only vertices which violate the condition of maximum degree k are deleted from W, the resulting set U is a maximal subset inducing a subgraph of maximum degree k when W becomes empty.

Since MIS can be solved in NC^2 [6], it is not hard to see that the total algorithm can be implemented in NC^2.□

Theorem 2 *EIMS(k) is in NC^2 for $k \geq 1$.*

Proof. For this problem, we use maximal matchings instead of maximal independent sets. The algorithm is similar to that in Theorem 1 and repeats the following procedure $2k$ times, where initially $Z = E$ and $F = \emptyset$.

```
1 begin
2     Find a maximal matching M of ⟨Z⟩;
3     F ← F ∪ M;
4     Z ← Z − M;
5     Z ← Z − {e ∈ Z | deg(⟨F ∪ {e}⟩]) > k}
6 end
```

Let $Z_0 = E$ and $F_0 = \emptyset$. In the same way as Theorem 1, let F_i, M_i and Z_i be the contents of F, M and Z just after the ith iteration.

For an edge $e = \{u, v\} \in Z_i$,

$$B_i(e) = deg_{\langle F_i \cup \{e\}\rangle}(u) + deg_{\langle F_i \cup \{e\}\rangle}(v) \leq 2k$$

holds since all edges making the degree greater than k are deleted from Z by line 5. To see that Z becomes empty after $2k$ iterations, it suffices to show that

$$B_i(e) > B_{i-1}(e)$$

holds.

Since e is not in M_i and M_i is a maximal matching in $\langle Z_{i-1}\rangle$, e shares a vertex with some edge e' in M_i. Without loss of generality, we may assume that u is shared by e and e'. Then $deg_{\langle F_i \cup \{e\}\rangle}(u)$ is greater than $deg_{\langle F_{i-1} \cup \{e\}\rangle}(u)$ since edge e' is not contained in $\langle F_{i-1} \cup \{e\}\rangle$.

It is easy to see that $deg(\langle F_i\rangle) \leq k$ since M_i is a matching of $\langle Z_{i-1}\rangle$ and since each edge e in M_i satisfies $deg(\langle F_{i-1} \cup \{e\}\rangle) \leq k$.

By the argument above we see that the resulting F is a maximal set of edges such that $deg(\langle F\rangle) \leq k$. Since the problem of finding a maximal matching in a graph is also solvable in NC^2, we can see that EIMS(k) is in NC^2.□

4 Concluding Remarks

A straightforward method to solve VIMS(k) (resp., EIMS(k)) is to use the polynomial-time greedy sequential algorithm that computes the lfm subset U of vertices (resp., F of edges) such that $deg(G[U])$ (resp., $deg(\langle F\rangle)$) is at most k [3], [10].

Most problems computed by greedy algorithms of this kind are known to be P-complete and therefore hardly efficiently parallelizable [1], [10], [11]. In fact, the lfm maximum degree k vertex-induced subgraph problem is P-complete [10]. However, the situation is different for edge-induced subgraphs.

The class CC is defined to be the class of sets log-space reducible to C-CVP, the comparator circuit value problem [7]. A comparator circuit is a usual circuit such that it contains only comparators C which are gates with two inputs u, v and two outputs uv, $u + v$ and no duplication of the value of an output is allowed.

CC lies as NLOG⊆CC⊆P [4] and is closed under complement [7]. Currently, CC-complete problems are believed to be neither P-complete nor in NC.

Some CC-complete problems are reported in [7]. The lfm matching problem is one of them (stated as a work due to S.A. Cook in [7]). Since a matching is a subgraph of degree at most 1, it is natural to guess that the lfm maximum degree k *edge-induced* subgraph problem, denoted

LF-EIMS(k), is also CC-complete for all $k \geq 1$. We can show that this is the case. Since the proof techinque is the same as that for the lfm matching problem, we omit the proof.

Theorem 3 *LF-EIMS(k) is CC-complete for* $k \geq 1$.

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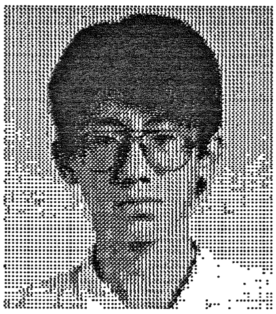

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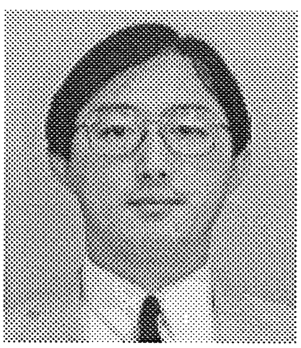

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