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出版情報：RIFIS Technical Report. 27, 1990-07-07. Research Institute of Fundamental Information Science, Kyushu University
バージョン：

RIFIS-TR-CS-27

RIFIS Technical Report

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July 7, 1990

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# Bounded Degree Maximal Subgraph Problems are in NC

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**Abstract**

We show that the problem of finding a maximal vertex-induced (resp., edge-induced) subgraph of maximum degree $k$ is in $NC^2$ for $k \geq 0$ (resp., $k \geq 1$). For these problems, we develope a method which exploits the NC algorithm for the maximal independent set problem.

## 1 Introduction

Karp [5] and Luby [6] have shown that the problem of finding a maximal independent set of a graph, called MIS, is solvable in NC, which is known to be the class of problems computable by PRAMs with polynomial number of processors in $O((\log n)^k)$ time for some $k \geq 0$ [12], [13], [14]. In this paper, we show that finding a maximal subset of vertices whose induced subgraph is of degree at most $k$ allows an NC algorithm for any $k \geq 0$. For the proof, we develope an elegant method which employs the NC algorithm for MIS devised by [5], [6]. We also show that the problem of finding a maximal set of edges which forms a subgraph of degree at most $k$ is in NC.

In general, a problem of this kind is stated as a maximal subgraph problem for a given property $\pi$, which is to find a maximal subset of vertices (resp., edges) which induces a subgraph satisfying $\pi$. For example, MIS is the problem for property "no two vertices are adjacent". This paper deals with the problem for property "maximum degree $k$".

It has been shown that most of the lexicographically first maximal (abbreviated to lfm) subgraphs problems are P-complete [10]. Therefore no NC algorithms exist for the lfm subgraph problems if P$\neq$NC. On the other hand, the problem of finding *any* maximal subgraph which satisfies a given property seems to allow NC algorithms for many properties. However, only a few are shown to be in NC. The followings are some of them: As we mentioned above MIS is the one. In [10], the maximal edge-induced forest problem and the maximal edge-induced bipartite problems are shown to be in NC. With some restrictions, the maximal edge-induced outerplanar subgraph problem [8] and the maximal vertex-induced acyclic subgraph problem restricted to directed graphs with degree at most 3 also allow NC algorithms [9]. The results in this paper add a new family of such problems.

## 2 Preliminaries and Definitions

A graph $G = (V, E)$ means an undirected graph without any multiple eges and self-loops. For a subset $U \subseteq V$, we define as $E[U] = \{\{u, v\} \in E \mid u, v \in U\}$. The graph $G[U] = (U, E[U])$ is called the *vertex-induced subgraph of* $U$. For a subset $F \subseteq E$, we define $V[F]$ to be the set of endpoints of the edges in $F$. We denote by $\langle F \rangle = (V[F], F)$ the graph formed from $F$ and call it the *edge-induced subgraph of* $F$. For a vertex $u$, the degree of $u$ is denoted by $deg_G(u)$. We denote by $deg(G) = \max\{w \mid deg_G(w)\}$.

Let $k \geq 0$ be any integer. The *maximum degree $k$ vertex-induced subgraph problem* (VIMS($k$)) is stated as follows:

VIMS($k$)
Instance: A graph $G = (V, E)$.
Problem: Find a maximal subset $U \subseteq V$ such that $G[U]$ is of degree at most $k$.

In a similar way, the *maximum degree $k$ edge-induced subgraph problem* (EIMS($k$)) is defined as follows:

EIMS($k$)
Instance: A graph $G = (V, E)$.
Problem: Find a maximal subset $F \subseteq E$ such that $\langle F \rangle$ is of degree at most $k$.

## 3 Finding Bounded Degree Maximal Subgraphs

**Theorem 1** *VIMS($k$) is in $NC^2$ for $k \geq 0$.*

*Proof.* We show an NC algorithm by employing the NC algorithm for MIS. Let $G = (V, E)$ be a graph for which we are finding a maximal subset $U$ of vertices whose induced subgraph $G[U]$ is of degree at most $k$.

For subsets $W$ and $U$ of vertices with $W \cap U = \emptyset$, let $E_U^W = \{\{v, w\} \mid$ there is $u \in U$ with $w \neq v$ such that $u, w \in W$ and $\{v, u\} \in E$, $\{w, u\} \in E\}$. Then let $H_U^W = (W, E[W] \cup E_U^W)$. The required set $U$ of vertices is computed together with a set $W$ of vertices $W$ such that $W \cap U = \emptyset$. Initially let $W = V$ and $U = \emptyset$. At each iteration of the algorithm, a maximal independent set $I$ of $H_U^W$ is computed and added to $U$ while vertices which make the degree of some vertex greater than $k$ are deleted from $W$ together with $I$. This is iterated $k^2$ times. Formally the algorithm is described as follows:

**begin** /* $G = (V, E)$ is an input */
  $W \leftarrow V$; $U \leftarrow \emptyset$;
  **for** $i \leftarrow 1$ **to** $k^2$ **do**
    **begin**
      Find a maximal independent set $I$ of $H_U^W$;
      $U \leftarrow U \cup I$;
      $W \leftarrow W - I$;
      $W \leftarrow W - \{w \in W \mid deg(G[U \cup \{w\}]) > k\}$
    **end**
**end**

We show that this algorithm computes a maximal subset $U$ whose induced subgraph is of degree at most $k$.

Let $W_0 = V$ and $U_0 = \emptyset$. Then the graph $H_{U_0}^{W_0}$ is the same as $G = (V, E)$. Therefore in the first iteration, a maximal independent set of $G$ is computed at line 5. For $i = 1, ..., k^2$, let $U_i$, $I_i$ and $W_i$ be the contents of variables $U$, $I$ and $W$ at the end of $i$th iteration, respectively. Obviously, $W_i \cap U_i = \emptyset$ for $i = 0, ..., k^2$. We assume that the induced subgraph $G[U_{i-1}]$ is of degree at most $k$.

Let $\{w, u\}$ be an edge in $E$ with $w \in W_i$ and $u \in U_i$. Line 8 deletes every vertex which is adjacent to more than $k$ vertices in $U_i$ or adjacent to a vertex $v$ in $U_i$ with $deg_{G[U_i]}(v) = k$. Therefore $u$ is adjacent to at most $k$ vertices in $U_i$ and $deg_{G[U_i \cup \{w\}]}(u) \leq k$. Hence, for each $w$ in $W_i$, we see that

$$A_i(w) = \sum_{u \in U_i \text{ with } \{w,u\} \in E} deg_{G[U_i \cup \{w\}]}(u) \leq k^2.$$

To show that $W$ becomes empty after $k^2$ iterations, it suffices to prove that each $w$ in $W_i$ satisfies

$$A_i(w) > A_{i-1}(w)$$

for $i = 1, ..., k^2$. Since $w$ is not in the maximal independent set $I_i$ of $H_{U_{i-1}}^{W_{i-1}}$ computed by line 5, $w$ is adjacent to a vertex $v$ in $I_i \subseteq W_{i-1}$ via an edge $\{w, v\}$ in $E[W_{i-1}]$ or $E_{U_{i-1}}^{W_{i-1}}$.

*Case 1.* If $\{w, v\} \in E[W_{i-1}]$, then $\{w, v\}$ is an edge in $G[U_i \cup \{w\}]$. Hence $deg_{G[U_i \cup \{w\}]}(v) \geq 1$. Since $v \in U_i$, $v \notin U_{i-1}$ and $\{w, v\} \in E$, we see that $A_i(w) \geq A_{i-1}(w) + deg_{G[U_i \cup \{w\}]}(v) > A_{i-1}(w)$.

*Case 2.* If $\{w, v\} \in E_{U_{i-1}}^{W_{i-1}}$, then there is a vertex $u \in U_{i-1}$ with $\{w, u\} \in E$ and $\{v, u\} \in E$. Since $v \in W_{i-1}$, $W_{i-1} \cap U_{i-1} = \emptyset$ and $w \neq v$, we see $v \notin U_{i-1} \cup \{w\}$. Hence $\{v, u\}$ is not an edge in $G[U_{i-1} \cup \{w\}]$. On the other hand, $v$ is in $U_i$ and $u$ is in $U_{i-1} \subseteq U_i$. Hence $\{v, u\}$ is an edge in $G[U_i \cup \{w\}]$. Therefore $deg_{G[U_i \cup \{w\}]}(u) > deg_{G[U_{i-1} \cup \{w\}]}(u)$. Since $u \in U_i$ and $\{w, u\} \in E$, we see that $A_i(w) > A_{i-1}(w)$.

We now show that $deg(G[U_i]) \leq k$. For a vertex $u$ in $U_{i-1}$, if $u$ is adjacent to a vertex $w$ in $I_i$ via an edge in $E$, then no other vertex in $I_i$ is adjacent to $u$ since $I_i$ is also an independent set with respect to $E_{U_{i-1}}^{W_{i-1}}$. Therefore the degree of $u$ in $G[U_{i-1} \cup I_i]$ remains at most $k$ since $deg(G[U_{i-1} \cup \{w\}]) \leq k$ by the algorithm. For a vertex $u$ in $I_i$, $deg_{G[U_{i-1} \cup I_i]}(u)$ is at most $k$ since $u$ is adjacent to at most $k$ vertices in $U_{i-1}$ and since $I_i$ is an independent set with respect to $E[W_{i-1}]$. Hence $deg_{G[U_{i-1} \cup I_i]}(u) \leq k$.

Since only vertices which violate the condition of maximum degree $k$ are deleted from $W$, the resulting set $U$ is a maximal subset inducing a subgraph of maximum degree $k$ when $W$ becomes empty.

Since MIS can be solved in $NC^2$ [6], it is not hard to see that the total algorithm can be implemented in $NC^2$.□

**Theorem 2** *EIMS(k) is in $NC^2$ for $k \geq 1$.*

*Proof.* For this problem, we use maximal matchings instead of maximal independent sets. The algorithm is similar to that in Theorem 1 and repeats the following procedure $2k$ times, where initially $Z = E$ and $F = \emptyset$.

```
1 begin
2     Find a maximal matching M of ⟨Z⟩;
3     F ← F ∪ M;
4     Z ← Z − M;
5     Z ← Z − {e ∈ Z | deg(⟨F ∪ {e}⟩]) > k}
6 end
```

Let $Z_0 = E$ and $F_0 = \emptyset$. In the same way as Theorem 1, let $F_i$, $M_i$ and $Z_i$ be the contents of $F$, $M$ and $Z$ just after the $i$th iteration.

For an edge $e = \{u, v\} \in Z_i$,

$$B_i(e) = deg_{\langle F_i \cup \{e\}\rangle}(u) + deg_{\langle F_i \cup \{e\}\rangle}(v) \leq 2k$$

holds since all edges making the degree greater than $k$ are deleted from $Z$ by line 5. To see that $Z$ becomes empty after $2k$ iterations, it suffices to show that

$$B_i(e) > B_{i-1}(e)$$

holds.

Since $e$ is not in $M_i$ and $M_i$ is a maximal matching in $\langle Z_{i-1}\rangle$, $e$ shares a vertex with some edge $e'$ in $M_i$. Without loss of generality, we may assume that $u$ is shared by $e$ and $e'$. Then $deg_{\langle F_i \cup \{e\}\rangle}(u)$ is greater than $deg_{\langle F_{i-1} \cup \{e\}\rangle}(u)$ since edge $e'$ is not contained in $\langle F_{i-1} \cup \{e\}\rangle$.

It is easy to see that $deg(\langle F_i\rangle) \leq k$ since $M_i$ is a matching of $\langle Z_{i-1}\rangle$ and since each edge $e$ in $M_i$ satisfies $deg(\langle F_{i-1} \cup \{e\}\rangle) \leq k$.

By the argument above we see that the resulting $F$ is a maximal set of edges such that $deg(\langle F\rangle) \leq k$. Since the problem of finding a maximal matching in a graph is also solvable in $NC^2$, we can see that EIMS($k$) is in $NC^2$.□

# 4 Concluding Remarks

A straightforward method to solve VIMS($k$) (resp., EIMS($k$)) is to use the polynomial-time greedy sequential algorithm that computes the lfm subset $U$ of vertices (resp., $F$ of edges) such that $deg(G[U])$ (resp., $deg(\langle F\rangle)$) is at most $k$ [3], [10].

Most problems computed by greedy algorithms of this kind are known to be P-complete and therefore hardly efficiently parallelizable [1], [10], [11]. In fact, the lfm maximum degree $k$ vertex-induced subgraph problem is P-complete [10]. However, the situation is different for edge-induced subgraphs.

The class CC is defined to be the class of sets log-space reducible to C-CVP, the comparator circuit value problem [7]. A comparator circuit is a usual circuit such that it contains only comparators $C$ which are gates with two inputs $u$, $v$ and two outputs $uv$, $u + v$ and no duplication of the value of an output is allowed.

CC lies as NLOG⊆CC⊆P [4] and is closed under complement [7]. Currently, CC-complete problems are believed to be neither P-complete nor in NC.

Some CC-complete problems are reported in [7]. The lfm matching problem is one of them (stated as a work due to S.A. Cook in [7]). Since a matching is a subgraph of degree at most 1, it is natural to guess that the lfm maximum degree $k$ *edge-induced* subgraph problem, denoted

LF-EIMS($k$), is also CC-complete for all $k \geq 1$. We can show that this is the case. Since the proof techinque is the same as that for the lfm matching problem, we omit the proof.

**Theorem 3** *LF-EIMS(k) is CC-complete for* $k \geq 1$.

## References


[1] R. Anderson and E.W. Mayr, Parallelism and the maximal path problem, *Inf. Process. Lett.* **24** (1987) 121-126.

[2] E.W. Mayr and A. Subramanian, The complexity of circuit value and network stability, *Proc. 4th IEEE Conf. Structure in Complexity Theory* (1989) 114-123.

[3] S.A. Cook, A taxonomy of problems with fast parallel algoritms, *Inf. Contr.*, **64** (1985) 2-22.

[4] T. Feder, A new fiexed point approach for stable networks and stable marriages, *Proc. 21st ACM STOC* (1989) 513-522.

[5] R.M. Karp and A. Wigderson, A fast parallel algorithm for the maximal independent set problem, *J. Assoc. Comput. Mach.* **32** (1985) 762-773.

[6] M. Luby, A simple parallel algorithm for the maximal independent set problem, *Proc. 17th ACM STOC* (1985) 1-10.

[7] E.W. Mayr and A. Subramanian, The complexity of circuit value and network stability, *Proc. 4th Annual Conference on Structure in Complexity Theory* (1989) 114-123.

[8] S. Miyano, A parallelizable lexicographically first maximal subgraph problem, *Inf. Process. Lett.* **29** (1988) 75-78.

[9] S. Miyano, $\Delta_2^p$-complete lexicographically first maximal subgraph problems, *Proc. 13th Mathematical Foundation of Computer Science* (Lecture Notes in Computer Science Vol. 324) (Springer, Berlin, 1988) 454-462.

[10] S. Miyano, The lexicographically first maximal subgraph problems: P-completeness and NC algorithms, *Math. Systems Theory* **22** (1989) 47-73.

[11] S. Miyano, S. Shiraishi and T. Shoudai, A list of P-complete problems, RIFIS-TR-CS-17, Research Institute of Fundamental Information Science, Kyushu University, 1989.

[12] N. Pippenger, On simultaneous resource bounds, *Proc. 20th IEEE FOCS* (1979) 307-311.

[13] W.L. Ruzzo, On uniform circuit complexity, *J. Comput. System Sci.* **22** (1981) 365-383.

[14] L. Stockmeyer and U. Vishkin, Simulation of parallel random access machines by circuits, *SIAM J. Comput.* **13** (1984) 409-422.

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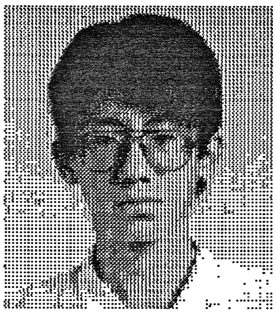

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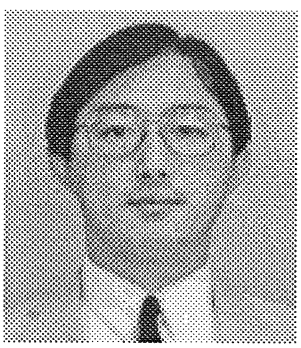

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