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Eiji Kiriyama and Hiroakira Ono

Abstract. This paper treats decision problems for the intuitionistic logic without weakening rule $\mathrm{FL_{ec}}$. First, the cut elimination theorem for $\mathrm{FL_{ec}}$ will be shown. Using this fact and Kripke's method, it will be proved that the propositional $\mathrm{FL_{ec}}$ is decidable. On the other hand, the predicate $\mathrm{FL_{ec}}$ will be shown to be undecidable by reducing the decision problem to that of the intuitionistic predicate logic.

1 Introduction

In recent years, various studies have been done concerning logics lacking some or all of structural rules. (For more information, see [7].) This paper will be devoted to study of the decision problem of the logic $\mathrm{FL_{ec}}$, which is obtained from the intuitionistic logic by deleting the weakening rule. The decidability of the *propositional* $\mathrm{FL_{ec}}$ will be shown by using a method originally developed for relevant logics. Next, the undecidability of the *predicate* $\mathrm{FL_{ec}}$ will be proved by reducing its decision problem to that of the intuitionistic predicate logic.

It is well-known that both the classical and the intuitionistic predicate logics are undecidable, while their propositional fragments are decidable. On the other hand, it was shown that *predicate* logics lacking some structural rules are decidable when they have no contraction rule (see [5]). The standard way of proving these results is to show the cut elimination theorem, in the first place. Suppose that the cut elimination theorem holds for a logic L, which does not have the contraction rule. Then, it can be shown that every cut-free proof has such a property that each upper sequent of a given rule of inference is *simpler* than its lower sequent. From this fact it follows that every *decomposition-tree* of a given sequent is finite, and that the number of its decomposition-trees is also finite. This gives us a procedure of deciding whether a given sequent is provable in L or not.

On the other hand, when a given logic L contains the contraction rule but not the weakening rule, some difficulties will occur, even if L is a cut-free propositional logic. Recall here that in the case of the classical and the intuitionistic propositional logics, the usual procedure relies on the fact that we can restrict the number of occurrences of a formula in a given sequent, by virtue of the existence of both contraction and weakening rules. To overcome the difficulty, Kripke[6] introduced a new method of showing the decidability, which was extended later to that of relevant logics by Belnap and Wallace [1] (see also [3]).

In §2, we will first show the cut elimination theorem for $\mathrm{FL_{ec}}$. Then, by applying the similar method to Kripke's one mentioned in the above, the decidability of the propositional $\mathrm{FL_{ec}}$ will be proved. In §3, we will give a translation of a given sequent of the intuitionistic predicate logic LJ into a sequent of the predicate $\mathrm{FL_{ec}}$, by using the idea mentioned in the paper [7, §3] by the second author. Making use of this translation, we can reduce the decision problem of the predicate $\mathrm{FL_{ec}}$ to that of LJ, which is known to be undecidable.

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2 Decidability of the propositional logic $\mathbf{FL_{ec}}$

In this section, we will prove that the propositional part of the logic $\mathrm{FL_{ec}}$ is decidable. Roughly speaking, the logic $\mathrm{FL_{ec}}$ is obtained from the intuitionistic predicate logic by deleting only the weakening rule. Our language contains logical connectives $\supset, \vee, \wedge, \&$, quantifiers $\exists, \forall$, and logical constants 0, 1, $\perp$. To simplify our subsequent discussion, we will adopt the multiset notation in the definition of sequents. Here, we say two multisets Γ and Δ are equal, when Γ and Δ have the same members with the same multiplicity. In other words, two multisets $\{ A_1, \cdots, A_m \}$ and $\{ B_1, \cdots, B_n \}$ are equal, if m=n and the sequence $B_1, \cdots, B_n$ is obtained from $A_1, \cdots, A_m$ by permuting them. Now, a *sequent* of $\mathrm{FL_{ec}}$ is an expression of the form $\Gamma \rightarrow A$, where Γ is a finite (possibly empty) multiset of formulas and A is a formula (possibly empty). We will sometimes write $\Gamma \rightarrow B$ as $A_1, \cdots, A_m \rightarrow B$, when $\Gamma = \{ A_1, \cdots, A_m \}$, and $\{ C_1, \cdots, C_n \} \cup \Delta \rightarrow B$

as $C_1, \cdots, C_n, \Delta \rightarrow B$. Initial sequents and rules of inferences of $\mathrm{FL_{ec}}$ are defined as follows (C may be empty in the following):

Initial sequents:

1) $A \rightarrow A$,
2) $0, \Gamma \rightarrow C$,
3) $\rightarrow 1$,
4) $\perp \rightarrow$.

Structural rules:

$$\frac{\Gamma \rightarrow C}{1, \Gamma \rightarrow C}(1w) \qquad \frac{\Gamma \rightarrow}{\Gamma \rightarrow \perp}(\perp w)$$

$$\frac{A, A, \Gamma \rightarrow C}{A, \Gamma \rightarrow C}(contraction) \qquad \frac{\Gamma \rightarrow A \quad A, \Delta \rightarrow C}{\Gamma, \Delta \rightarrow C}(cut)$$

Logical rules:

$$\frac{A, \Gamma \rightarrow B}{\Gamma \rightarrow A \supset B}(\rightarrow\supset) \qquad \frac{\Gamma \rightarrow A \quad B, \Delta \rightarrow C}{A \supset B, \Gamma, \Delta \rightarrow C}(\supset\rightarrow)$$

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B}(\rightarrow \vee 1) \qquad \frac{A, \Gamma \rightarrow C \quad B, \Gamma \rightarrow C}{A \vee B, \Gamma \rightarrow C}(\vee \rightarrow)$$

$$\frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \vee B}(\rightarrow \vee 2)$$

$$\frac{\Gamma \rightarrow A \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \wedge B}(\rightarrow \wedge) \qquad \frac{A, \Gamma \rightarrow C}{A \wedge B, \Gamma \rightarrow C}(\wedge 1 \rightarrow)$$

$$\frac{B, \Gamma \rightarrow C}{A \wedge B, \Gamma \rightarrow C}(\wedge 2 \rightarrow)$$

$$\frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A\&B}(\rightarrow \&) \qquad \frac{A, B, \Gamma \rightarrow C}{A\&B, \Gamma \rightarrow C}(\& \rightarrow)$$

$$\frac{\Gamma \rightarrow F(t)}{\Gamma \rightarrow \exists x F(x)}(\rightarrow \exists) \qquad \frac{F(a), \Gamma \rightarrow C}{\exists x F(x), \Gamma \rightarrow C}(\exists \rightarrow)$$

$$\frac{\Gamma \rightarrow F(a)}{\Gamma \rightarrow \forall x F(x)}(\rightarrow \forall) \qquad \frac{F(t), \Gamma \rightarrow C}{\forall x F(x), \Gamma \rightarrow C}(\forall \rightarrow)$$

In the above, t is any term, and a is any variable satisfying the eigenvariable condition. We remark that we can dispense with the exchange rule, as we adopt the multiset notation.

Theorem 2.1 *The cut elimination theorem holds for FL_{ec}.*

Proof. Our theorem can be obtained by modifying slightly the standard proof of the cut elimination theorem for LK (see e.g. [8]). To prove this theorem we introduce a new rule of inference, called the *weak-mix* rule;

$$\frac{\Gamma \to A \quad \Delta \to C}{\Gamma, \Delta^* \to C}.$$

Here, Δ must contain at least one A, and Δ^* is a multiset obtained from Δ by deleting at least one occurrence of A in it. Notice that Δ^* is not necessary the multiset obtained from Δ by deleting all occurrences of A in it. Therefore, Δ^* may contain some A's. The formula A is called the *weak-mix formula* of the above weak-mix rule.

It is clear that the cut rule is a special case of the weak-mix rule. On the other hand, each application of the weak-mix rule can be replaced by the cut rule, with the help of the contraction rule.

Thus, to show our theorem, it is enough to prove that if a sequent is provable in $\mathrm{FL}_{\mathrm{ec}}$ (using some weak-mix rules in place of cut rules) then it is provable without a weak-mix rule. As usual, this can be obtained by showing that if a proof of a sequent S contains only one weak-mix, occurring as the last inference, then S is provable without a weal-mix. In fact, we can carry out the proof, by using double induction on the grade and the rank of a proof and considering the following four cases:

(1) Either $\Gamma \to A$ or $\Delta \to C$ is an initial sequent.

(2) Either $\Gamma \to A$ or $\Delta \to C$ is a lower sequent of a structural rule.

(3) Both $\Gamma \to A$ or $\Delta \to C$ are lower sequents of some logical rules such that principal formulas of both rules are just the weak-mix formula.

(4) Either $\Gamma \to A$ or $\Delta \to C$ is a lower sequent of a logical rule except Case (3).

□

We will show next that the propositional logic $\mathrm{FL}_{\mathrm{ec}}$ is decidable by applying the above result. We will use a method which is popular among relevant logicians (see §3 of [3]).

For our purpose, we will first introduce an auxiliary system $\mathrm{FL}'_{\mathrm{ec}}$. The system $\mathrm{FL}'_{\mathrm{ec}}$ has neither the cut rule nor the *explicit* contraction rule, but the contraction rule is incorporated into some logical rules. Initial sequents of $\mathrm{FL}'_{\mathrm{ec}}$ are the same as those of $\mathrm{FL}_{\mathrm{ec}}$, and structural rules of $\mathrm{FL}'_{\mathrm{ec}}$ are only $(1w)$ and $(\perp w)$. Among logical rules of $\mathrm{FL}'_{\mathrm{ec}}$, $(\to\supset), (\to \vee 1), (\to \vee 2)$ and $(\to \wedge)$ are the same as those of $\mathrm{FL}_{\mathrm{ec}}$, but other rules are slightly different from those of $\mathrm{FL}_{\mathrm{ec}}$, as shown in the following:

$$\frac{\Gamma \to A \quad B, \Delta \to C}{A \supset B, \Sigma \to C}(\supset\to) \qquad \frac{A, \Gamma \to C \quad B, \Gamma \to C}{A \vee B, \Pi \to C}(\vee \to)$$

$$\frac{A, \Gamma \to C}{A \wedge B, \Pi \to C}(\wedge 1 \to) \qquad \frac{B, \Gamma \to C}{A \wedge B, \Pi \to C}(\wedge 2 \to)$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Theta \to A\&B}(\to \&) \qquad \frac{A, B, \Gamma \to C}{A\&B, \Pi \to C}(\& \to)$$

Here, Σ, Π and Θ are multisets defined as follows. In the following, $\Gamma \cup \Delta$ denotes the multiset union of Γ and Δ, and $\#_\Sigma(D)$ denotes the multiplicity of a formula D in a multiset Σ.

i. Σ is a multiset obtained from $\Gamma \cup \Delta$ by deleting some duplicated formulas in it, which satisfies the following requirements:

(1) If the formula $A \supset B$ belongs to both Γ and Δ, then $\#_\Sigma(A \supset B) \geq \#_{\Gamma\cup\Delta}(A \supset B) - 2$. Otherwise, $\#_\Sigma(A \supset B) \geq \#_{\Gamma\cup\Delta}(A \supset B) - 1$.

(2) Let D be a formula in $\Gamma \cup \Delta$, other than $A \supset B$. Then, $\#_\Sigma(D) \geq \#_{\Gamma\cup\Delta}(D) - 1$ if D belongs to both Γ and Δ, and otherwise, $\#_\Sigma(D) = \#_{\Gamma\cup\Delta}(D)$.

ii. Let $\circ$ be any one of $\vee, \wedge$ and $\&$. Then, Π is either Γ or the multiset obtained from Γ by deleting one $A \circ B$, if possible.

iii. Θ is a multiset obtained from $\Gamma \cup \Delta$, which satisfies the following requirement:

(3) Let D be any formula in $\Gamma \cup \Delta$. Then, $\#_\Sigma(D) \geq \#_{\Gamma\cup\Delta}(D) - 1$ if D belongs to both Γ and Δ, and otherwise, $\#_\Sigma(D) = \#_{\Gamma\cup\Delta}(D)$.

We will write $\mathrm{FL}_{\mathrm{ec}} \vdash \Gamma \to C$ and $\mathrm{FL}'_{\mathrm{ec}} \vdash \Gamma \to C$, when a sequent $\Gamma \to C$ is provable in $\mathrm{FL}_{\mathrm{ec}}$ and $\mathrm{FL}'_{\mathrm{ec}}$, respectively. Now, we have the following lemma.

Lemma 2.2 *For any sequent $\Gamma \to C$, $FL_{ec} \vdash \Gamma \to C$ if and only if $FL'_{ec} \vdash \Gamma \to C$.*

Proof. Suppose that a proof of $\Gamma \to C$ in $\mathrm{FL}'_{\mathrm{ec}}$ is given. By supplementing some contractions to each logical rule other than $(\to\supset), (\to \vee 1), (\to \vee 2)$ and $(\to \wedge)$, we can get a proof of $\Gamma \to C$ in $\mathrm{FL}_{\mathrm{ec}}$.

Conversely, suppose that a proof P of $\Gamma \to C$ in $\mathrm{FL_{ec}}$ is given. By Theorem 2.1, we can suppose moreover that P is cut-free. We will construct a proof P$'$ of $\Gamma \to C$ in $\mathrm{FL'_{ec}}$ in the following way.

Suppose that an application of the contraction rule in P follows an application of a rule (R), which is either a logical rule or a structural rule except the contraction rule. We will interchange these applications as far as we can, or eliminate the contraction rule, as shown below. Here, we notice that two consecutive applications of the contraction rule are always interchangeable.

(1) Suppose that (R) is any of $(\perp w), (\to \wedge 1)$ and $(\to \wedge 2)$. Then, it is of the following form;

$$\cfrac{\cfrac{\vdots}{\cfrac{A, A, \Delta \to D}{A, A, \Delta \to D'}(R)}}{A, \Delta \to D'}(c),$$

where (c) is an abbreviation of (contraction). This can be transformed into the following form;

$$\cfrac{\cfrac{\vdots}{\cfrac{A, A, \Delta \to D}{A, \Delta \to D}(c)}}{A, \Delta \to D'}(R).$$

In the same way, we can interchange (R) and (c) when (R) is either $(\to\supset)$ or $(\to \wedge)$.

(2) Suppose that (R) is (1w). Then, it is of the following form;

$$\cfrac{\cfrac{\vdots}{\cfrac{\Delta \to D}{1, \Delta \to D}(1w)}}{\Sigma \to D}(c).$$

If the contraction is applied to a formula other than 1 then clearly (1w) and (c) are interchangeable. If the contraction is applied to 1 then Δ is equal to Σ (as multisets) and hence we can eliminate both (1w) and (c) in the above figure.

(3) Suppose that (R) is $(\wedge 1 \to)$. Then, it is of the following form;

$$\cfrac{\cfrac{\vdots}{\cfrac{A, \Delta \to D}{A \wedge B, \Delta \to D}(\wedge 1 \to)}}{\Sigma \to D}(c).$$

If the contraction is applied to two formulas in Δ, then clearly $(\wedge 1 \to)$ and (c) are interchangeable. But we should notice here that when one of them is just the principal formula $A \wedge B$, they are not interchangeable in general.

We can treat it in the same way when (R) is any of $(\wedge 2 \to)$, $(\& \to)$ and $(\vee \to)$.

(4) Suppose that (R) is $(\supset\to)$. Then, it is of the following form;

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Delta \to A\end{matrix} \quad \begin{matrix}\vdots \\ B, \Pi \to D\end{matrix}}{A \supset B, \Delta, \Pi \to D}(\supset\to)}{\Sigma \to D}(c)\;.$$

If the contraction is applied to two formulas in $\Delta \cup \Pi$, then it is easy to see that $(\supset\to)$ and (c) are interchangeable. But, in other cases, they are not interchangeable in general. When (R) is $(\to \&)$, we can treat it in the same way.

By repeating this process, we can get a proof P_1, in which every (consecutive) applications of the contraction rule follows either an initial sequent or one of rules treated in (3) and (4) in the above. It is easily seen that only initial sequents of the form $0, \Delta \to D$ can be followed by the contraction rule. So let us consider the following case;

$$\dfrac{0, \Delta \to D}{\Sigma \to D}(c)\;.$$

Then it is clear that Σ contains at least one 0, and hence $\Sigma \to D$ is also an initial sequent. Therefore, we can eliminate the upper sequent from the proof. As for the latter case, we can replace each logical rule followed by consecutive applications of the contraction, by a single application of the corresponding logical rule of $\mathrm{FL}'_{\mathrm{ec}}$. In this way, we can get a proof P' of $\mathrm{FL}'_{\mathrm{ec}}$ whose end sequent is $\Gamma \to C$. □

We say that a sequent S' is a *contraction* of another sequent S, if S' is obtained from S by some applications of the contraction rule. For each proof P of $\mathrm{FL}'_{\mathrm{ec}}$, we will define the length of P inductively as follows. If P consists only of an initial sequent then the length of P is 1. If the last inference of P is of the form

$$\frac{S_1}{S} \qquad \left(\text{and } \frac{S_1 \quad S_2}{S}\right),$$

and the length of the proof of S_1 (and of S_2) is n_1 (and n_2), then the length of P is $n_1 + 1$ (and $n_1 + n_2 + 1$, respectively).

Lemma 2.3 (Curry's lemma) *If a sequent S' is a contraction of a sequent S and S has a proof (in $\mathrm{FL}'_{\mathrm{ec}}$) of the length n then S' has a proof of the length $\leq n$.*

Proof. Suppose that S has a proof of the length n. We will show our lemma by induction on n. Here we will show this only when S is the lower sequent of the following $(\to \&)$;

$$\frac{\begin{matrix}\vdots & & \vdots \\ \Gamma \to A & & \Delta \to B\end{matrix}}{\Sigma \to A\&B}(\to \&).$$

Moreover suppose that the lengths of the proofs of $\Gamma \to A$ and $\Delta \to B$ are n_1 and n_2, respectively, with $n = n_1 + n_2 + 1$. Then S' is of the form $\Sigma' \to A\&B$. Clearly, Σ' can be obtained from the multiset union of Γ and Δ by applying the contraction rule. Thus, there exist multisets Γ' and Δ' such that 1) Γ' and Δ' are obtained from Γ and Δ by applying the contraction rule and 2) Σ' is obtained from the multiset union of Γ' and Δ' by contracting only such formulas that have a single occurrence both in Γ' and Δ'. (Notice that such Γ' and Δ' are not always determined uniquely from Γ, Δ and Σ'.) By induction hypothesis, both $\Gamma' \to A$ and $\Delta' \to B$ have proofs of the length $\leq n_1$ and $\leq n_2$, respectively. Since $\Sigma' \to A\&B$ is proved by applying $(\to \&)$ by $\Gamma' \to A$ and $\Delta' \to B$, it has a proof of the length $\leq n$. □

Let α be any branch in a given proof of $\mathrm{FL}'_{\mathrm{ec}}$. If there are two sequents S_1 and S_2 in α such that 1) S_2 is below S_1 in α and 2) S_2 is a contraction of S_1, then we say that the branch α is *redundant*.

Corollary 2.4 *If a sequent S is provable in FL'_{ec}, then there is a proof of S, which has no redundant branch.*

Proof. Take any proof P of S in $\mathrm{FL}'_{\mathrm{ec}}$, which has the smallest length among proofs of S. Suppose that there exists a branch α in P which is redundant. Then, α is of the following form, where S_2 is a contraction of S_1.

$$\left.\begin{matrix} n\left\{\begin{matrix}\vdots \\ S_1\end{matrix}\right. \\ \ddots \\ S_2 \end{matrix}\right\} n'(> n)$$
$$\begin{matrix}\ddots \\ S\end{matrix}$$

Suppose that S_1 is proved with the length n and S_2 is proved with the length $n'(> n)$. Then by Curry's lemma, S_2 can be proved with the length $m(\leq n)$. Let us consider the following proof P' of S.

$$\left.\begin{matrix}\vdots \\ S_2\end{matrix}\right\} m(\leq n)$$
$$\begin{matrix}\ddots \\ S\end{matrix}$$

It is clear that the proof P' has a smaller length than that of P. This is a contradiction. Thus, P is a proof of S in which no branch is redundant. □

In order to check whether a given sequent S is provable in $\mathrm{FL}'_{\mathrm{ec}}$ or not, we will try to find a proof of S in the following way. First, we will search for every such sequent that can be an upper sequent of some rules of inference of $\mathrm{FL}'_{\mathrm{ec}}$ whose lower sequent is S. Then, we will write each of them, just above the sequent S. We call this process, the *decomposition* of S. Next, we will decompose each sequent which we have obtained just now, and repeat it again. An exceptional case is when a sequent S_1 is obtained by the decomposition but the branch from S_1 to S becomes redundant. In such a case, we will omit the sequent S_1. Of course, we can not decompose a sequent which can not be a lower sequent of any rule of inference. By doing so, we can get a tree such that some sequent is attached to each of its points. Let us call it, the *decomposition-tree* of S.

In the following, we will show that the decomposition-tree of each sequent is finite. To show this, we will use the following König's lemma.

Proposition 2.5 (König's lemma) *A tree is finite if and only if both 1) there are only finitely many points connected directly to a given point (finite fork property) and 2) each branch is finite (finite branch property).*

Lemma 2.6 *Finite fork property holds for any decomposition-tree in* FL'_{ec}.

Proof. Suppose that a sequent S is given. We will show that the number of sequents which are obtained by the decomposition of S is finite. So, suppose first that S is the lower sequent of a given inference (R). We will show that the number $\mathrm{n_R}$ of sequents which can be upper sequents of (R) is finite. This is almost obvious, if (R) is one of rules of inference except $(\rightarrow \&)$ and $(\supset\rightarrow)$. (Recall here that $\mathrm{FL}'_{\mathrm{ec}}$ is a cut-free system.)

Now, suppose that (R) is the following $(\rightarrow \&)$

$$\frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Theta \rightarrow A\&B}(\rightarrow \&).$$

It is clear that $\mathrm{n_R}$ becomes maximum, when Σ consists of mutually different formulas. For each formula C in S, the decomposition of S by $(\rightarrow \&)$ has three possible cases, i.e., 1) only Γ contains C, 2) only Δ contains C, and 3) both Γ and Δ contain C. Thus, $\mathrm{n_R}$ is at most 3^n if Σ consists of n formulas. Since S is decomposed into two sequents in each case, there are at most $2 \cdot 3^n$ possible points just above S. By a similar consideration, we can show that $\mathrm{n_R}$ is finite when (R) is $(\supset\rightarrow)$. □

If the same formulas occur in Γ as in Δ, we say that two sequents $\Gamma \to A$ and $\Delta \to A$ are *cognate*. Then, we have the following (see [1], [3] and [6]).

Proposition 2.7 (Kripke's lemma) *If a sequence of cognate sequents is not redundant, it is finite.*

Since our system $\mathrm{FL}'_{\mathrm{ec}}$ is cut-free, the subformula property holds. Therefore, the number of cognation classes occurring in a given proof is finite. Moreover, the above Kripke's Lemma said that only a finite number of members of each cognation class will appear in a branch which is not redundant. Thus, we have the following.

Lemma 2.8 *Finite branch property holds for any decomposition-tree in FL'_{ec}, if the tree has no redundant branch.*

Theorem 2.9 *The propositional logic FL_{ec} is decidable.*

Proof. It suffices to show that $\mathrm{FL}'_{\mathrm{ec}}$, which is equivalent to $\mathrm{FL}_{\mathrm{ec}}$, is decidable. Suppose that an arbitrary sequent S_0 is given. We construct the decomposition-tree of S_0. By Corollary 2.4, it is enough to consider the decomposition-tree without redundant branches. But, the decomposition-tree becomes finite, by using König's lemma with Lemmas 2.6 and 2.8. Then, we check whether the decomposition-tree contains such a subtree that every top sequent is an initial sequent of $\mathrm{FL}'_{\mathrm{ec}}$ or not. If it does, then S_0 is provable. Otherwise, S_0 is not provable. $\square$

3 Undecidability of the predicate logic $\mathbf{FL}_{\mathrm{ec}}$

We have shown in §2 that the propositional logic $\mathrm{FL}_{\mathrm{ec}}$ is decidable. On the other hand, we will show in this section that the predicate logic $\mathrm{FL}_{\mathrm{ec}}$ is undecidable. Our result relies on the following well-known fact (e.g. see [2]).

Proposition 3.1 *The intuitionistic predicate logic is undecidable.*

In the following, we will show that the decision problem of the predicate logic $\mathrm{FL}_{\mathrm{ec}}$ can be reduced to that of the intuitionistic predicate logic. To do so, we will introduce a formal system IL for the intuitionistic predicate logic, which is a variant of Gentzen's LJ. Roughly speaking, our system IL differs only from LJ in the definition of sequents. That is, a sequent of IL is defined in the same way as that of $\mathrm{FL}_{\mathrm{ec}}$, but it must consist of formulas containing neither constants 0, 1 nor the logical symbol &. As initial sequents of IL, we will take initial sequents 1) and 4) of $\mathrm{FL_{ec}}$. Structural rules of IL is the contraction rule, the cut rule and the following weakening rules;

$$\frac{\Gamma \to C}{A, \Gamma \to C}(w \to) \qquad\qquad \frac{\Gamma \to}{\Gamma \to C}(\to w).$$

Logical rules of IL are just logical rules of $\mathrm{FL_{ec}}$ other than $(\to \&)$ and $(\& \to)$. (We will treat the negation $\neg A$ of a formula A as an abbreviation of $A \supset \perp$.) Similarly to Theorem 2.1, we can show that the cut elimination theorem holds for IL.

Now for each formula A of IL, i.e., a formula containing neither constants 0,1 nor logical symbol &, we will define formulas $|A|^-$ and $|A|^+$ as follows:

(1) $|A|^- = 1 \wedge A, \quad |A|^+ = \perp \vee A \qquad$ if A is atomic,

(2) $|A \supset B|^- = 1 \wedge (|A|^+ \supset |B|^-), \quad |A \supset B|^+ = \perp \vee (|A|^- \supset |B|^+),$

(3) $|A \circ B|^- = |A|^- \circ |B|^-, \quad |A \circ B|^+ = |A|^+ \circ |B|^+ \qquad$ if $\circ \in \{\vee, \wedge\}$,

(4) $|QxA|^- = Qx|A|^-, \quad |QxA|^+ = Qx|A|^+ \qquad$ if $Q \in \{\exists, \forall\}$.

By induction on the length of A, we can easily show the following.

Lemma 3.2 *For any formula A of IL, both $|A|^- \to 1$ and $\perp \to |A|^+$ are provable in FL_{ec}.*

Lemma 3.3 *For any sequent $\Gamma \to A$ of IL, $IL \vdash \Gamma \to A$ implies $FL_{ec} \vdash |\Gamma|^- \to |A|^+$, where $|\Gamma|^-$ means $\{|B_1|^-, \cdots, |B_n|^-\}$ if Γ is a multiset $\{B_1, \cdots, B_n\}$.*

Proof. We will show this lemma by induction on the length of a cut-free proof of IL whose end sequent is $\Gamma \to A$. Here we will show the proof of the following two cases.

(1) Suppose that $\Gamma \to A$ is a lower sequent of weakening. Recall that the weakening rule is not a structural rule of $\mathrm{FL_{ec}}$.

(i) If Γ is B, Δ and $\Gamma \to A$ is obtained by

$$\frac{\Delta \to A}{B, \Delta \to A}(w \to),$$

then by induction hypothesis, $\mathrm{FL_{ec}} \vdash |\Delta|^- \to |A|^+$. On the other hand, $\mathrm{FL_{ec}} \vdash |B|^- \to 1$ by Lemma 3.2. Thus,

$$\frac{\begin{matrix}\vdots \\ |B|^- \to 1\end{matrix} \qquad \dfrac{|\Delta|^- \to |A|^+}{1, |\Delta|^- \to |A|^+}(1w)}{|B|^-, |\Delta|^- \to |A|^+}(cut).$$

(ii) If $\Gamma \to A$ is obtained by

$$\frac{\Gamma \to}{\Gamma \to A}(\to w),$$

then by induction hypothesis, $\mathrm{FL_{ec}} \vdash |\Gamma|^- \to$. By Lemma 3.2, $\mathrm{FL_{ec}} \vdash \perp\to |A|^+$. Therefore,

$$\frac{\dfrac{|\Gamma|^- \to}{|\Gamma|^- \to \perp}(\perp w) \qquad \begin{array}{c}\vdots\\ \perp\to |A|^+\end{array}}{|\Gamma|^- \to |A|^+}(cut).$$

(2) If $\Gamma \to A$ is a lower sequent of $(\supset\to)$, then it is obtained by

$$\frac{\Delta \to B \quad C, \Pi \to A}{B \supset C, \Delta, \Pi \to A}(\supset\to),$$

where Γ is $B \supset C, \Delta, \Pi$. In this case, $\mathrm{FL_{ec}} \vdash |\Delta|^- \to |B|^+$ and $\mathrm{FL_{ec}} \vdash |C|^-, |\Pi|^- \to |A|^+$ by induction hypothesis. What we want to show is $\mathrm{FL_{ec}} \vdash 1 \wedge (|B|^+ \supset |C|^-), |\Delta|^-, |\Pi|^- \to |A|^+$. But, this can be obtained by

$$\frac{\dfrac{|\Delta|^- \to |B|^+ \qquad |C|^-, |\Pi|^- \to |A|^+}{|B|^+ \supset |C|^-, |\Delta|^-, |\Pi|^- \to |A|^+}(\supset\to)}{1 \wedge (|B|^+ \supset |C|^-), |\Delta|^-, |\Pi|^- \to |A|^+}(\wedge 2 \to).$$

□

For any formula B containing no &'s, $\tilde{B}$ denotes the formula obtained from B by replacing 1 and $\perp$ by $p \supset p$ and $\neg(p \supset p)$, respectively, where p is a fixed propositional variable. Then, the following lemma can be easily verified.

Lemma 3.4 *For any formula A of IL, both $A \equiv \widetilde{|A|^-}$ and $A \equiv \widetilde{|A|^+}$ are provable in IL, where $B \equiv C$ is an abbreviation of $(B \supset C) \wedge (C \supset B)$.*

Lemma 3.5 *For any sequent $\Pi \to B$ (of FL_{ec}) $FL_{ec} \vdash \Pi \to B$ implies $IL \vdash \tilde{\Pi} \to \tilde{B}$, where $\tilde{\Pi}$ means $\{\tilde{C_1}, \cdots, \tilde{C_2}\}$ if Π is a multiset $\{C_1, \cdots, C_n\}$.*

Proof. Our lemma can be proved by induction of the length of the proof of $\Pi \to B$. Clearly, it is enough to check it only for initial sequents and rules of inference of $\mathrm{FL_{ec}}$ which are not of IL.

(i) Suppose that $\Pi \to B$ is either $\to 1$ or $\perp\to$. Then we must show that both $\to p \supset p$ and $\neg(p \supset p) \to$ are provable in IL. But this is trivial.

(ii) Suppose that $\Pi \to B$ is the lower sequent of $(1w)$. Then, the proof is of the form

$$\frac{\Delta \to B}{1, \Delta \to B}(1w).$$

By induction hypothesis, $\tilde{\Delta} \to \tilde{B}$ is provable in IL. Therefore, $p \supset p, \tilde{\Delta} \to \tilde{B}$ is also provable by using $(w \to)$. Similarly, we can treat the case $(\perp w)$. □

Combining Lemma 3.5 with Lemma 3.4, we have the following.

Lemma 3.6 *For any sequent* $\Gamma \to A$ *of IL,* $FL_{ec} \vdash |\Gamma|^- \to |A|^+$ *implies* $IL \vdash \Gamma \to A$.

By Lemmas 3.3 and 3.6, we have the following.

Theorem 3.7 *For any sequent* $\Gamma \to A$ *of IL,* $IL \vdash \Gamma \to A$ *if and only if* $FL_{ec} \vdash |\Gamma|^- \to |A|^+$.

Theorem 3.8 *The predicate logic* FL_{ec} *is undecidable.*

Proof. Suppose otherwise. Then, there exists an algorithm of deciding whether $|\Gamma|^- \to |A|^+$ is provable in $\mathrm{FL_{ec}}$ or not, for any given sequent $\Gamma \to A$ of IL. By Theorem 3.7, this brings about a decision algorithm for the intuitionistic predicate logic. But this contradicts Proposition 3.1. So the predicate logic $\mathrm{FL_{ec}}$ must be undecidable. □

Notice here that the proof of the above theorem depends highly on the existence of constants 1 and $\perp$.

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