

## A New Series of $\Delta^p_{2}$ -Complete Problemse

Miyano, Satoru

Research Institute of Fundamental Information Science Kyushu University

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Satoru Miyano

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Research Institute of Fundamental Information Science  
Kyushu University 33  
Fukuoka 812, Japan

E-mail: [miyano@rifis.sci.kyushu-u.ac.jp](mailto:miyano@rifis.sci.kyushu-u.ac.jp)

Phone: 092(641)1101 Ext.4471

# A New Series of $\Delta_2^p$ -Complete Problems

Satoru Miyano

Research Institute of Fundamental Information Science

Faculty of Science, Kyushu University

## Abstract

We prove that the lexicographically first maximal connected subgraph problem for a graph property  $\pi$  is  $\Delta_2^p$ -complete if  $\pi$  is hereditary, determined by the blocks, and nontrivial on connected graphs.

## 1. Introduction

The class  $\Delta_2^p$  consists of problems solvable in polynomial time using oracles in NP. Recently some  $\Delta_2^p$ -complete problems have been reported [6, 13, 14, 17]. In [13] we have shown that the lexicographically first maximal induced path problem is  $\Delta_2^p$ -complete. This paper gives a very general theorem that derives a new series of  $\Delta_2^p$ -complete problems related to lexicographically first maximal subgraph problems.

For a given hereditary property  $\pi$  on graphs, we consider the problem of finding the lexicographically first maximal (abbreviated to lfm) subset  $U$  of vertices of a graph  $G = (V, E)$  such that  $U$  induces a *connected* subgraph satisfying  $\pi$ , where we assume that  $V$  is linearly ordered as  $V = \{1, \dots, n\}$ . Problems of this kind have been extensively studied in [1, 2, 5, 8, 9, 10, 11, 12, 13, 15, 16]. In particular, without the connectedness restriction, the P-completeness of the lfm subgraph problem for *any* nontrivial polynomial time testable hereditary property is proved in [11] as an analogue of the results in Lewis and Yannakakis [7], Yannakakis [19], Yannakakis [20], Asano and Hirata [3], Watanabe et al. [18]. However, since the connectedness is not necessarily inherited by subgraphs, a new analysis is required.

Some of the lfm connected subgraph problems for hereditary properties are polynomial time solvable. For example, the lfm clique problem is obviously in P. We prove a general theorem asserting that the lfm connected subgraph problem for a graph property  $\pi$  is  $\Delta_2^p$ -complete if  $\pi$  is hereditary, determined by blocks, and nontrivial on connected graphs. Hence the connectedness makes the problem drastically harder.

## 2. $\Delta_2^p$ -Completeness Theorem

For any graph property  $\pi$ , the lexicographically first maximal subgraph satisfying  $\pi$  is computed by the following greedy algorithm:

```
begin
   $U \leftarrow \emptyset$ ;
  for  $j = 1$  to  $n$  do
    if  $U \cup \{j\}$  can be extended to a sub-
      graph of  $G$  satisfying  $\pi$ 
      then  $U \leftarrow U \cup \{j\}$ 
end
```

It is clear from the algorithm that the lfm subgraph problem for  $\pi$  is in  $\Delta_2^p$  if  $\pi$  is polynomial time testable. We consider the following decision problem:

**Definition 2.1.**

**LFMCSP**( $\pi$ ) (the lfm connected subgraph problem for  $\pi$ )

**Instance:** A graph  $G = (V, E)$  and a vertex  $v \in V$ , where  $V = \{1, \dots, n\}$ .

**Question:** Let  $U$  be the lfm subset of  $V$  whose induced subgraph, denoted  $\langle U \rangle$ , is a connected subgraph satisfying  $\pi$ . Then  $v \in U$ ?

Papadimitriou [14] defined the *deterministic satisfiability problem* and showed that it is  $\Delta_2^P$ -complete. We use this problem as a basis of reduction. The problem is described as follows:

**Definition 2.2.** Let  $x_1, \dots, x_{k-1}$  be  $k-1$  variables and  $Y_1, \dots, Y_k$  be  $k$  sets of variables. A boolean formula  $F_0(x_1, \dots, x_{k-1}, Y_1, \dots, Y_k)$  in conjunctive normal form is said to be *deterministic* if  $F_0$  consists of the following clauses:

1. Either  $(y)$  or  $(\bar{y})$  is a clause of  $F_0$  for each  $y$  in  $Y_1 \cup Y_k$ .
2. For each  $i = 1, \dots, k-1$  and each  $y$  in  $Y_{i+1}$ , there are sets  $C_y^i$  and  $D_y^i$  of conjunctions of literals from  $Y_i \cup \{x_i\}$  with the following properties:
  - (a) Exactly one of the conjunctions in  $C_y^i \cup D_y^i$  is true for any truth assignment (this can be checked in polynomial time).
  - (b)  $F$  contains clauses  $(\alpha \rightarrow y)$  and  $(\beta \rightarrow \bar{y})$  for each  $\alpha \in C_y^i$  and each  $\beta \in D_y^i$ .

**Definition 2.3.**

**Deterministic Satisfiability (DSAT)**

**Instance:** A deterministic formula

$F_0(x_1, \dots, x_{k-1}, Y_1, \dots, Y_k)$  and  $k-1$  formulas in 3-conjunctive normal form  $F_1(Y_1, Z_1), \dots,$

$F_{k-1}(Y_{k-1}, Z_{k-1})$ , where  $\{x_1, \dots, x_{k-1}\}, Y_1, \dots, Y_k, Z_1, \dots, Z_{k-1}$  are mutually disjoint sets of variables.

**Question:** Decide whether there is a truth assignment  $\hat{x}_1, \dots, \hat{x}_{k-1}, \hat{Y}_1, \dots, \hat{Y}_k$  satisfying 1 and 2.

1.  $F_0(\hat{x}_1, \dots, \hat{x}_{k-1}, \hat{Y}_1, \dots, \hat{Y}_k) = 1$ .
2.  $F_i(\hat{Y}_i, Z_i)$  is satisfiable  $\iff \hat{x}_i = 1$  for  $i = 1, \dots, k-1$ .

**Lemma 2.1** [14]. *DSAT is  $\Delta_2^P$ -complete.*

**Remark 2.1.** For an instance  $(F_0, \dots, F_{k-1})$  of DSAT, we may assume that clauses in  $F_0$  are of conjunctive normal form. For example, clause  $(\alpha \rightarrow y)$  can be written in the form of disjunction of literals since  $\alpha$  is a conjunction of literals.

**Lemma 2.2.** *Let  $F_i(Y_i, Z_i)$  be a formula in 3-conjunctive normal form. Then there is a formula  $F'(Y_i, Z'_i)$  in 3-conjunctive normal form satisfying the following conditions:*

- (i) *Each clause in  $F'(Y_i, Z'_i)$  contains at most one literal from  $Y_i$ .*
- (ii) *For any truth assignment  $\hat{Y}_i$ ,  $F(\hat{Y}_i, Z_i)$  is satisfiable if and only if  $F'(\hat{Y}_i, Z'_i)$  is satisfiable.*

**Proof.** We just give an idea of construction. For a clause  $(y_1 + y_2 + y_3)$  with  $y_1, y_2, y_3 \in Y_i$ , we replace it by  $(y_1 + \bar{u})(y_2 + \bar{v})(y_3 + u + v)$  using new variables  $u, v$  which shall be put into  $Z'_i$ .  $\square$

A graph property  $\pi$  is said to be *hereditary* on induced subgraphs if, whenever a graph  $G$  satisfies  $\pi$ , all vertex-induced subgraphs of  $G$  also satisfy  $\pi$ . We say that  $\pi$  is *nontrivial* if  $\pi$  is satisfied by infinitely many graphs and there is a graph violating  $\pi$ . We say that  $\pi$  is *determined by the blocks* [18] if for any graphs  $G_1$  and  $G_2$  satisfying  $\pi$  the graph

formed by identifying a vertex of  $G_1$  and a vertex of  $G_2$  also satisfies  $\pi$ .

A *block* is a connected graph with at least two vertices which contains no cutpoint. We use the following result (see [4]).

**Lemma 2.3.** *Let  $G$  be a block with at least three vertices and let  $v$  be a vertex of  $G$ . Then there is an edge  $\{u, v\}$  such that the graph obtained by deleting vertices  $u$  and  $v$  together with adjacent edges is connected.*

Our main result is the following theorem.

**Theorem 2.4.** *Let  $\pi$  be a hereditary property satisfying the following conditions:*

- (i)  $\pi$  is determined by the blocks.
- (ii)  $\pi$  is nontrivial on connected graphs.

Then  $LFMCSP(\pi)$  is  $\Delta_2^p$ -complete.

**Proof.** Let  $G_\pi$  be a connected graph with minimum number of vertices which violates  $\pi$ . Since  $\pi$  is nontrivial on connected graphs and hereditary, the complete graph  $K_2$  satisfies  $\pi$ . Therefore  $G_\pi$  is a block with at least three vertices since  $\pi$  is determined by the blocks. We denote  $G_\pi$  as Fig. 1(a), where bold lines represent edges adjacent to vertices  $u, v, w$ , respectively. We put labels  $a, b, c$  to specify the correspondence with  $u, v, w$ . By Lemma 2.3 we can assume that three vertices  $u, v, w$  are chosen so that the graph remains connected after deletion of  $v, w$ . Fig. 1(b) shows a graph obtained by adding a new vertex  $v'$  and edges in the same way as  $v$ . We use such abbreviation in the following construction.

Before getting into the reduction, we start with the following lemma which gives a basic construction in the reduction.

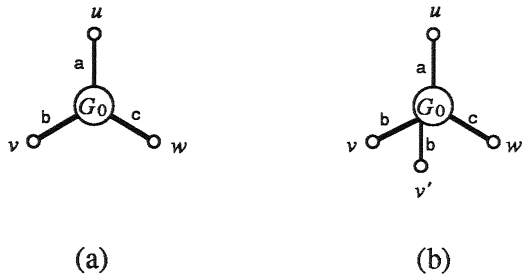


Fig. 1.

**Lemma 2.5.** *For a formula  $F(x_1, \dots, x_n) = c_1 c_2 \dots c_m$  in conjunctive normal form with variables  $x_1, \dots, x_n$ , we can construct a graph  $G_F$  with specified vertices  $h_1, h_0$  and an order on vertices such that  $F$  is satisfiable (resp. not satisfiable) if and only if  $h_1 \in U$  (resp.  $h_0 \in U$ ), where  $U$  is the lfm subset of vertices of  $G_F$  which induces a connected subgraph satisfying  $\pi$ .*

**Proof.** For variable  $x_i$ , we construct the variable graph  $G[x_i]$  in Fig. 2(a) using  $G_\pi$ , where  $s_i = |\{c_j \mid c_j \text{ contains } x_i\}|$  and  $t_i = |\{c_j \mid c_j \text{ contains } \bar{x}_i\}|$ . When  $s_i = 0$  (resp.  $t_i = 0$ ), we do not put edge  $\{f_i, d_{i+1}\}$ . We call vertices in gray circles which are copies of  $G_0$  *gray vertices*. Let  $V(x_i)$  (resp.  $V(\bar{x}_i)$ ) be the set of vertices  $x_i^k$ ,  $k = 1, \dots, t_i$  (resp.  $\bar{x}_i^k$ ,  $k = 1, \dots, s_i$ ). Let  $S$  be the set of black and gray vertices of  $G[x_i]$  and let  $\tilde{S}$  be any maximal set containing  $S$  whose induced subgraph is connected and satisfies  $\pi$ . Then it can be easily checked that  $\tilde{S}$  is either  $S \cup V(x_i) \cup \{x_i\}$  or  $S \cup V(\bar{x}_i) \cup \{\bar{x}_i\}$ .

For simplicity we deal with clauses with three literals but the argument below can be extended to the general case by a slight modification. The clause graph  $H[c_j]$  for  $c_j = (\alpha_j + \beta_j + \gamma_j)$  is shown in Fig. 2(b). Let  $V(c_j)$  be the set of three vertices labeled with literals  $\alpha_j, \beta_j, \gamma_j$ . These vertices shall be con-

nected to vertices in variable graphs corresponding to the literals. Again let  $C$  be the set of black and gray vertices of  $H[c_j]$  and  $\tilde{C}$  be any maximal set containing  $C$  whose induced subgraph is connected and satisfies  $\pi$ . Then exactly one of  $\alpha, \beta, \gamma$  can be put into  $\tilde{C}$ .

We also use the graph  $R$  in Fig. 2(c) called the *root graph*. We call vertex  $d_0$  the *root*.

The graph  $G_F$  is constructed as follows: We connect graphs  $R, G[x_1], \dots, G[x_n]$  by identifying  $d_i$  for each  $i = 1, \dots, n - 1$ . We denote the resulting graph by  $T_F$  and call it the *trunk graph*. Consider clause  $c_j = (\alpha_j + \beta_j + \gamma_j)$ . Let the occurrence of literal  $\alpha_j$  (resp.  $\beta_j, \gamma_j$ ) in  $c_j$  be the  $k_1$ -th (resp.  $k_2$ -th,  $k_3$ -th) occurrence of  $\alpha_j$  (resp.  $\beta_j, \gamma_j$ ) counted from  $c_1$  to  $c_m$ . Then we put edges  $\{\alpha_j^{k_1}, \alpha_j\}, \{\beta_j^{k_2}, \beta_j\}, \{\gamma_j^{k_3}, \gamma_j\}$ , where  $\alpha_j^{k_1}, \beta_j^{k_2}, \gamma_j^{k_3}$  are vertices in variable graphs and  $\alpha_j, \beta_j, \gamma_j$  are vertices in  $V(c_j)$ . The clause graphs  $H[c_1], \dots, H[c_m]$  are connected to  $T_F$  in this way.

Finally we put edges  $\{h_0, v\}$  for all black vertices  $v$  except the root. Fig. 3 illustrates the whole construction of graph  $G_F$  focussed on  $G[x_p]$  and  $H[c_j]$ , where  $c_j = (x_p + x_q + \bar{x}_r)$ .

The vertices are ordered so that the following relations hold:

$$\begin{aligned} B &< h_1 < h_0 < x_1 < \bar{x}_1 < V(x_1) < V(\bar{x}_1) \\ &< \dots < x_n < \bar{x}_n < V(x_n) < V(\bar{x}_n) \\ &< V(c_1) < \dots < V(c_m), \end{aligned}$$

where  $B$  is the set of black and gray vertices.

Then it is clear from the definition of  $G_F$  that  $B \subset U$  since  $h_0$  is connected to all black vertices and  $\pi$  is determined by the blocks. It should be noticed that either  $h_1 \in U$  or  $h_0 \in U$  since  $G_\pi$  violates  $\pi$ . If  $h_1 \in U$ , then  $h_0 \notin U$ , and therefore  $\langle U \rangle$  can have no

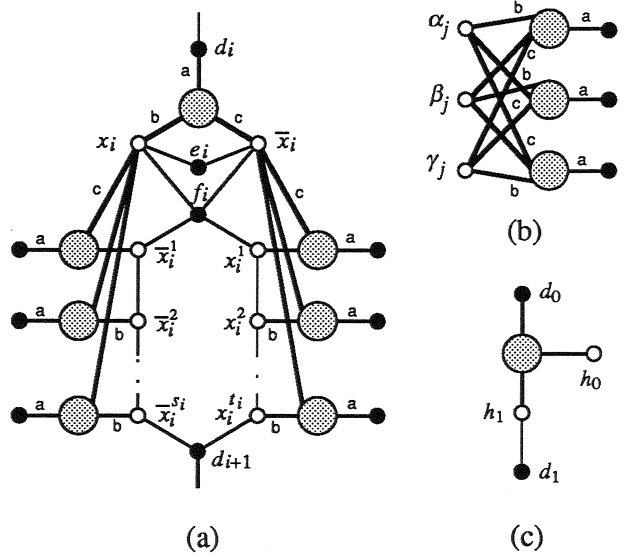


Fig. 2.

edge with  $h_0$  as an endpoint. For each variable  $x_i$ , either  $V(x_i) \cup \{x_i\} \subset U$  or  $V(\bar{x}_i) \cup \{\bar{x}_i\} \subset U$ . Since for each clause  $c_j$ ,  $U$  contains vertices in  $H[c_j]$ , one of the vertices in  $V(c_j)$  must be in  $U$  and joined to a vertex in  $U$  which belongs to a variable graph. It is now obvious that  $F$  is satisfied by the truth assignment defined by  $\hat{x}_i = 1$  (if  $x_i \in U$ ),  $\hat{x}_i = 0$  (if  $x_i \notin U$ ). Conversely, it can also be seen that  $h_1 \in U$  if  $F$  is satisfiable. Hence  $F$  is satisfiable (resp. not satisfiable) if and only if  $h_1 \in U$  (resp.  $h_0 \in U$ ).  $\square$

**Proof continued.** We shall give a reduction from DSAT. Let  $F_0(x_1, \dots, x_{k-1}, Y_1, \dots, Y_k)$  be a deterministic formula and let  $F_1(Y_1, Z_1), \dots, F_{k-1}(Y_{k-1}, Z_{k-1})$  be formulas in 3-conjunctive normal form. We construct a graph  $G(F_0, \dots, F_{k-1})$  and an order on it as follows.

For each  $i = 1, \dots, k - 1$ , we first construct a graph  $\tilde{G}_{F_i}$  in the following way: We denote by  $\tilde{F}_i(Z_i)$  be a formula obtained from  $F_i(Y_i, Z_i)$  by deleting all occurrences of literals from  $Y_i$ . Let  $Y_i = \{y_{i1}, \dots, y_{in_i}\}$ . For each  $y_{ip}$  in  $Y_i$  we use the

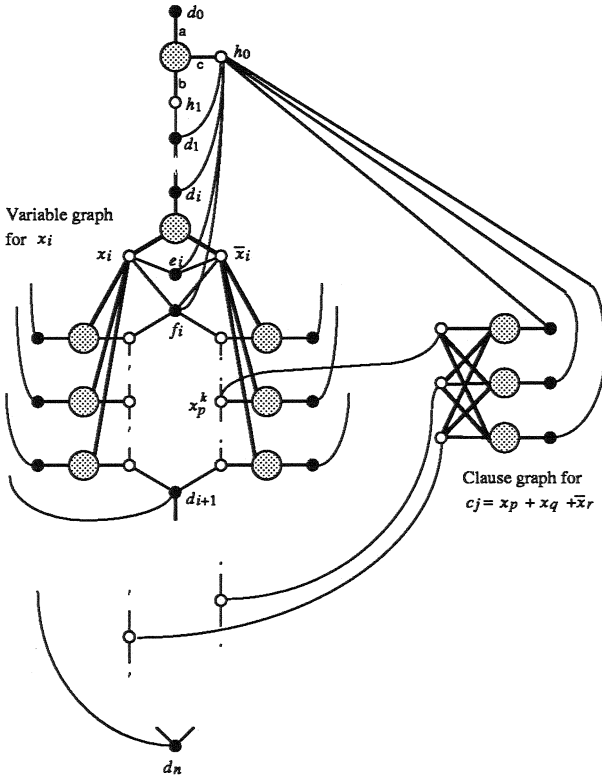


Fig. 3.

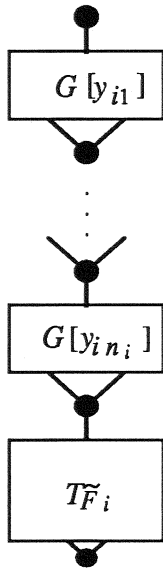


Fig. 4.

variable graph  $G[y_{ip}]$ , where the occurrences of literals  $y_{ip}$  and  $\bar{y}_{ip}$  are counted for  $F_0$  and  $F_i$ . We connect these variable graphs  $G[y_{i1}], \dots, G[y_{in_i}]$  and the trunk graph  $T_{\tilde{F}_i}$  consecutively as shown in Fig. 4. We denote by  $h_0^i, h_1^i$  the vertices corresponding to  $h_0, h_1$  in the construction in Lemma 2.5, respectively. We put an edge between  $h_0^i$  and each black vertex in the trunk graph  $T_{\tilde{F}_i}$  except the root. By Lemma 2.2 we can assume that each clause in  $F_i(Y_i, Z_i)$  contains at most one literal from  $Y_i$ . Let  $c_j^i$  be a clause in  $F_i(Y_i, Z_i)$ . If  $c_j^i$  contains only literals from  $Z_i$ , the clause graph  $H[c_j^i]$  is connected to the trunk graph  $T_{\tilde{F}_i(Z_i)}$  and we put edges between  $h_0^i$  and black vertices in  $H[c_j^i]$  in the same way as Lemma 2.5. If  $c_j^i$  contains a literal from  $Y_i$ , let  $c_j^i = (\alpha_j^i + \beta_j^i + \gamma_j^i)$ , where  $\alpha_j^i$  is a literal from  $Y_i$  and  $\beta_j^i, \gamma_j^i$  are literals from  $Z_i$ . For such clause, we use the graph  $\tilde{H}[c_j^i]$  shown in Fig. 5 instead of  $H[c_j^i]$ . Vertices  $\beta_j^i$  and  $\gamma_j^i$  are connected to the trunk graph in the same way and we put edges  $\{h_0^i, \beta_j^i\}, \{h_0^i, \gamma_j^i\}$ . For literal  $\alpha_j^i$ , let  $\alpha_j^i = y_{ip}$  for simplicity. Then we connect  $\tilde{H}[c_j^i]$  to some vertex in  $V(y_{ip})$  of the variable graph  $G[y_{ip}]$  as shown in Fig. 5. We denote by  $\tilde{G}_{\tilde{F}_i(Z_i)}$  the part consisting of the trunk graph and the clause graphs for  $\tilde{F}_i(Z_i)$ . Finally we add two vertices  $x_i$  and  $\bar{x}_i$  which are connected to  $h_1^i$  and  $h_0^i$ , respectively. This is the end of the construction of  $\tilde{G}_{F_i}$ .

Let  $B_i$  be the set of all black and gray vertices of  $\tilde{G}_{F_i}$ . Let  $\hat{Y}_i$  be a truth assignment for variables in  $Y_i$ . Then if  $\hat{y}_{ip} = 1$ , then let  $\hat{V}(y_{ip}) = V(y_{ip})$  else  $\hat{V}(y_{ip}) = V(\bar{y}_{ip})$ . Let  $B_i(\hat{Y}_i) = B_i \cup \bigcup_{p=1}^{n_i} \hat{V}(y_{ip})$ . Assume that the order on white vertices on variable graphs for  $Z_i$  and clause graphs follows Lemma 2.5. In Fig. 5, it should be noticed that if  $y_{ip}^k \in B_i(\hat{Y}_i)$  then the black and gray vertices in  $\tilde{H}[c_j^i]$  are connected to  $G[y_{ip}]$  and none of  $\beta_j^i, \gamma_j^i$  can be selected.

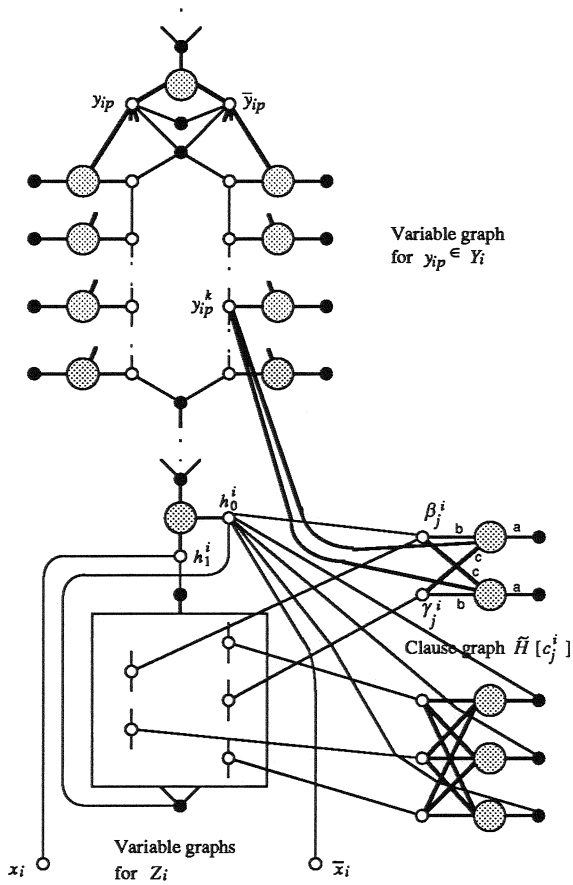


Fig. 5.

Then it can be seen that

**Fact.**  $F(\hat{Y}_i, Z_i)$  is satisfiable if and only if the lfm set containing  $B_i(\hat{Y}_i)$  whose induced subgraph is connected and satisfies  $\pi$  contains  $h_1^i$ .

The graph  $G(F_0, \dots, F_{k-1})$  shown in Fig. 6 illustrates the whole graph for  $(F_0, \dots, F_{k-1})$ . It is obtained by modifying the construction in Lemma 2.5. First we construct a trunk graph using the root graph  $R$  and the variable graphs  $G[x_1], \dots, G[x_{k-1}]$ . Then for each  $x_i$ , the part consisting of  $d_i, x_i, \bar{x}_i$  together with a copy of  $G_0$  is replaced by  $\tilde{G}_{F_i}$ . Then the graphs for clauses in  $F_0(x_1, \dots, x_{k-1}, Y_1, \dots, Y_k)$

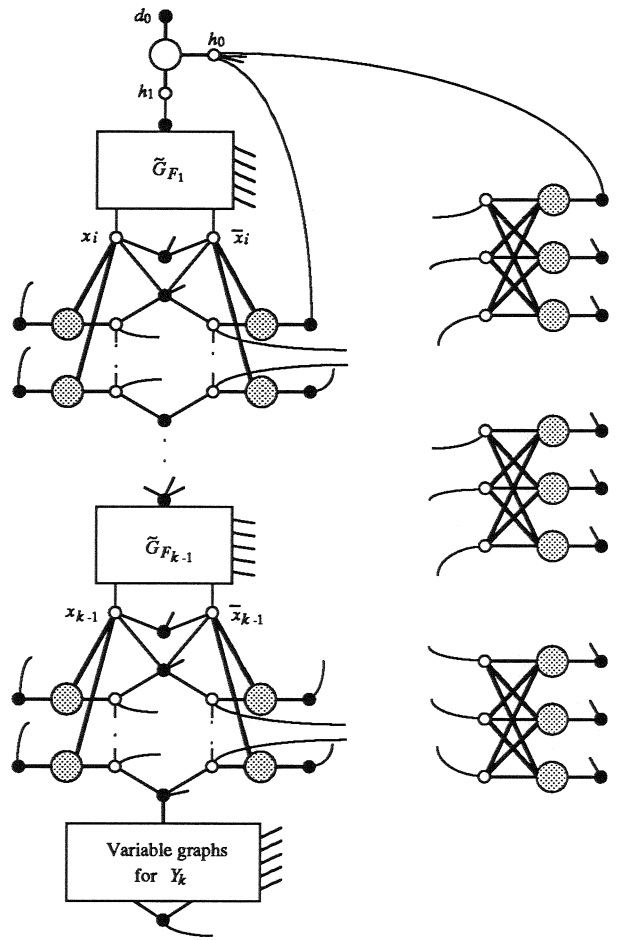


Fig. 6.

are connected to the trunk in the same way as Lemma 2.5 using the variable graphs for  $Y_1, \dots, Y_k$  and modified variable graphs for  $x_1, \dots, x_{k-1}$ . Finally we put edges connecting  $h_0$  and all black vertices on the variable graphs for  $Y_1, \dots, Y_k$  and the clause graphs for  $F_0$ .

Let  $\tilde{B}$  be the set of all black and gray vertices of  $G(F_0, \dots, F_{k-1})$ . We denote by  $W(Y_i)$  (resp.  $W(x_i)$ ,  $W(F_i)$ ,  $C(F_0)$ ) the set of all white vertices in the variable graphs for  $Y_i$  (resp. the variable graph for  $x_i$ , the graph  $\hat{G}_{\tilde{F}_i(Z_i)}$ , the clause graphs for  $F_0$ ). Then the vertices of  $G(F_0, \dots, F_{k-1})$  are ordered as follows:



$$\begin{aligned}
& \tilde{B} < h_1 < h_0 < W(Y_1) < W(Y_k) \\
& < W(x_1) < W(F_1) < \dots < W(Y_{k-1}) \\
& < W(x_{k-1}) < W(F_{k-1}) < C(F_0),
\end{aligned}$$

where the orders inside  $W(Y_i)$ ,  $W(x_i)$ ,  $W(F_i)$  and  $C(F_0)$  follow Lemma 2.5.

We shall show that  $(F_0, \dots, F_{k-1})$  is in DSAT if and only if  $h_1$  is in  $\tilde{U}$ , where  $\tilde{U}$  is the lfm subset of vertices such that  $\langle \tilde{U} \rangle$  is connected and satisfies  $\pi$ . It can be seen from the construction that  $\tilde{B} \subset \tilde{U}$ .

If  $h_1 \in \tilde{U}$ , then  $h_0 \notin \tilde{U}$ . Hence there is no edge connecting  $h_0$  and a black vertex in  $\langle \tilde{U} \rangle$ . First consider the variable graphs for  $Y_1$  and  $Y_k$ . For each  $y \in Y_1 \cup Y_k$ , either  $V(y) \subset \tilde{U}$  or  $V(\bar{y}) \subset \tilde{U}$ . Since for each  $y \in Y_1 \cup Y_k$  either  $(y)$  or  $(\bar{y})$  is a clause in  $F_0$  and the corresponding clause graph contains black vertices, it follows that  $V(y) \subset \tilde{U}$  (resp.  $V(\bar{y}) \subset \tilde{U}$ ) if and only if  $(y)$  (resp.  $(\bar{y})$ ) is a clause in  $F_0$ . Let  $\hat{Y}_1$  and  $\hat{Y}_k$  be a truth assignment defined by  $\hat{y} = 1$  (resp.  $\hat{y} = 0$ ) if  $(y)$  (resp.  $(\bar{y})$ ) is in  $F_0$  for  $y \in Y_1 \cup Y_k$ . From Fact, we see that  $F_1(\hat{Y}_1, Z_1)$  is satisfiable if and only if vertex  $x_1$  is in  $\tilde{U}$ . Therefore either  $V(x_1) \subset \tilde{U}$  or  $V(\bar{x}_1) \subset \tilde{U}$  holds according to the satisfiability of  $F_1(\hat{Y}_1, Z_1)$ . Let  $\hat{x}_1 = 1$  if  $x_1 \in \tilde{U}$  else  $\hat{x}_1 = 0$ .

Since  $F_0$  is deterministic, for each  $y_{2p} \in Y_2$  there are sets  $C_{y_{2p}}^1$  and  $D_{y_{2p}}^1$  of conjunctions of literals from  $Y_1 \cup \{x_1\}$  satisfying the conditions (a), (b) of Definition 2.2. For the truth assignment  $\hat{Y}_1$ ,  $\hat{x}_1$ , there is exactly one conjunction  $\gamma \in C_{y_{2p}}^1 \cup D_{y_{2p}}^1$  which is true under this truth assignment. If  $\gamma \in C_{y_{2p}}^1$ , then  $(\gamma \rightarrow y_{2p})$  is in  $F_0$ . By considering the clause graphs corresponding to  $(\gamma \rightarrow y_{2p})$ , we can see that  $V(y_{ip}) \subset \tilde{U}$  must hold since otherwise the connectedness of  $\langle \tilde{U} \rangle$  is violated. If  $\gamma \in D_{y_{2p}}^1$ , then  $V(\bar{y}_{ip}) \subset \tilde{U}$  must hold. Let  $\hat{y}_{2p} = 1$  (resp.  $\hat{y}_{2p} = 0$ ) if  $V(y_{ip}) \subset \tilde{U}$  (resp.  $V(\bar{y}_{ip}) \subset \tilde{U}$ ). With this truth assignment we can see that clauses  $(\alpha \rightarrow y_{2p})$  and

$(\beta \rightarrow \bar{y}_{2p})$  are satisfied for each  $\alpha \in C_{y_{2p}}^1$  and each  $\beta \in D_{y_{2p}}^1$ . In this way, we define  $\hat{Y}_2$ . Inductively we define  $\hat{x}_2, \hat{Y}_3, \dots, \hat{x}_{k-1}$ . Finally we can see that the truth assignment given to  $Y_k$  together with  $Y_1$  must coincide with the one determined from  $\hat{Y}_{k-1}$  and  $\hat{x}_{k-1}$  since the graph  $\langle \tilde{U} \rangle$  is connected. Thus we have shown the conditions 1 and 2 of Definition 2.3 hold. Hence  $(F_0, \dots, F_{k-1})$  is in DSAT.

The converse can also be shown by repeating a similar argument.  $\square$

Examples of the properties on undirected graphs that satisfy the conditions of Theorem 2.4 are “planar”, “outerplanar”, “bipartite”, “acyclic”, etc. But the property “clique” is hereditary but not determined by blocks. In this case LFMCSPP(“clique”) is P-complete [5].

### 3. Conclusion

We have shown a rather general  $\Delta_2^P$ -completeness theorem for the lfm connected subgraph problems. This result does not cover the lfm induced path problem [13]. We believe that we could expect a more general result which include the results in [13]. As a candidate, we give the following conjecture.

We define the *diameter*  $\delta(\pi)$  by  $\sup\{\delta(G) \mid G \text{ is a connected graph satisfying } \pi\}$ , where  $\delta(G)$  is the diameter of a graph  $G$ . For example,  $\delta(\text{“clique”})=1$  but  $\delta(\text{“planar”}) = \infty$ . For the former property, LFMCSPP( $\pi$ ) becomes P-complete [5] but from Theorem 2.4 LFMCSPP(“planar”) is  $\Delta_2^P$ -complete.

**Conjecture.** If a hereditary property  $\pi$  is non-trivial on connected graphs and satisfies  $\delta(\pi) = \infty$ , then LFMCSPP( $\pi$ ) is  $\Delta_2^P$ -complete.

It should be noticed that if a hereditary prop-

erty  $\pi$  is nontrivial on connected graphs and satisfies  $\delta(\pi) = \infty$  then all paths satisfy  $\pi$ .

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