$Delta^p _2$Complete Lexicographically First

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Δ_2^p –Complete Lexicographically First Maximal Subgraph Problems

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Δ_2^p-Complete Lexicographically First Maximal Subgraph Problems
(Preliminary Report)

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Abstract

The lexicographically first maximal (lfm) induced path problem is shown Δ_2^p-complete. The lfm rooted tree problem is also analyzed. This problem is Δ_2^p-complete. But when restricted to topologically sorted directed acyclic graphs (dags), it allows a polynomial time algorithm. Moreover, the problem restricted to topologically sorted dags with degree 3 is shown in NC^2 while the problem for degree 4 is P-complete.

1. Introduction

Papadimitriou [Pa] is the first who gave a natural problem complete for Δ_2^p, which is the class of problems solvable in polynomial time using oracles in NP. He proved that the uniquely optimum traveling salesman problem is Δ_2^p-complete. Afterwards, Wagner [Wa] has found some Δ_2^p-complete problems related to optimization problems. This paper gives Δ_2^p-complete problems of a new kind. The importance of the Δ_2^p-completeness is not only due to its high complexity but also due to the observation that any Δ_2^p-complete problem is hard to efficiently parallelize even if NP-oracles are available.

For a given hereditary property π on graphs, we consider the problem of finding the lexicographically first maximal (abbreviated to lfm) subset U of vertices of a graph $G = (V, E)$ such that U induces a *connected* subgraph satisfying π, where we assume that V is linearly ordered as $V = \{1, ..., n\}$. Problems of this kind have been extensively studied in [AM], [Ma], [M1], [M2]. In particular, without the connectedness restriction, the P-completeness of the lfm subgraph problem for *any* nontrivial polynomial time testable hereditary property is proved in [M1] as an analogue of the results in [LY], [Y2]. However, since the connectedness is not necessarily inherited by subgraphs, a new analysis is required.

In this paper we are involved in the complexity analysis of problems of this kind. Our main result is that the lfm induced path problem is Δ_2^p-complete. We should here note that this problem is different from the lfm maximal path problem discussed in [AM] which was shown P-complete.

Some of the lfm connected subgraph problems for hereditary properties are polynomial time solvable. For example, the lfm clique problem is obviously in P. In Section 3, we prove a general theorem asserting that the lfm connected subgraph problem for a hereditary property is NP-hard if the property is satisfied by graphs with arbitrarily large diameters and is determined by blocks. Hence the connectedness makes the problem harder. The unfortunate point is that no general Δ_2^p-completeness result is known in this case.

In Section 4, we concentrate on a special problem, the lfm rooted tree problem. It is also possible to prove that this problem is Δ_2^p-complete even if the instances are directed acyclic graphs (abbreviated to dags). But if vertices of a dag are topologically sorted, it allows a polynomial time algorithm. Moreover, our analysis shows that the problem restricted to topologically sorted dags with degree 4 is P-complete and the degree bound 4 is proved to be optimal in the sense that the problem for degree 3 is, interestingly, solvable in NC^2. Finally, the complexity analysis of the lfm forest problem is given in comparison with the rooted tree problem. We show that for topologically sorted dags with degree 3 the problem is in NC^2 and the problem for not topologically sorted dags with degree 3 is P-complete although the lfm forest problem for undirected graphs with degree 3 is not known to be in NC^2 [M1].

2. The Lexicographically First Maximal Induced Path Problem is Δ_2^p-complete

For any graph property π, the lexicographically first maximal subgraph satisfying π is computed by the following greedy algorithm:

```
begin
    U ← ∅;
    for j = 1 to n do
      if U ∪ {j} can be extended to a subgraph of G satisfying π then U ← U ∪ {j}
end
```

From the algorithm it is clear that the lfm subgraph problem for π is in Δ_2^p if π is polynomial time testable.

A *path* is a connected graph of degree at most 2 with no cycle. The *lfm induced path problem* is to find the lfm subset of vertices that induces a path. We prove that this is Δ_2^p-complete.

Papadimitriou [Pa] defined the *deterministic satisfiability problem* and showed that it is Δ_2^p-complete. The problem is described as follows:

Let $x_1, \ldots, x_{k-1}$ be $k-1$ variables and $Y_1, \ldots, Y_k$ be k sets of variables. A formula $F(x_1, \ldots, x_{k-1}, Y_1, \ldots, Y_k)$ in conjunctive normal form is said to be *deterministic* if F consists of the following clauses:

(1) Either (y) or $(\bar{y})$ is a clause of F for each y in $Y_1 \cup Y_k$.

(2) For each $i = 1, \ldots, k-1$ and each y in Y_{i+1}, there are sets C_y^i and D_y^i of conjunctions of literals from $Y_i \cup \{x_i\}$ with the following properties:

(i) Exactly one of the conjunctions in $C_y^i \cup D_y^i$ is true for any truth assignment (this can be checked in polynomial time).

(ii) F contains clauses $(\alpha \to y)$ and $(\beta \to \bar{y})$ for each $\alpha \in C_y^i$ and each $\beta \in D_y^i$.

DSAT(Deterministic Satisfiability)

Instance: A deterministic formula $F_0(x_1, ..., x_{k-1}, Y_1, ..., Y_k)$ and $k-1$ formulas in 3-conjunctive normal form $F_1(Y_1, Z_1), \dots , F_{k-1}(Y_{k-1}, Z_{k-1})$, where $\{x_1, ..., x_{k-1}\}$, $Y_1, ..., Y_k$, $Z_1, ..., Z_{k-1}$ are mutually disjoint sets of variables.

Question: Is there a truth assignment $\hat{x}_1, ..., \hat{x}_{k-1}, \hat{Y}_1, ..., \hat{Y}_k$ satisfying (i) and (ii).

(i) $F_0(\hat{x}_1, ..., \hat{x}_{k-1}, \hat{Y}_1, ..., \hat{Y}_k) = t$.

(ii) $F_i(\hat{Y}_i, Z_i)$ is satisfiable $\Longleftrightarrow \hat{x}_i = t$ for $i = 1, ..., k-1$.

Lemma 1 [Pa]. *DSAT is Δ_2^p-complete.*

Lemma 2. *For a formula F in conjunctive normal form, we can construct a graph G_F with specified vertices a, w_0, w_1 of degree 1 such that F is satisfieable (resp. not satisfiable) if and only if the lfm induced path of G_F is a path from a to w_1 (resp. w_0).*

Proof. For simplicity we assume that F is in 3-conjunctive normal form. Let $c_1, ..., c_m$ be the clauses of F and let $x_1, ..., x_n$ be the variables occurring in $c_1, ..., c_m$. For each variable x_i, we use the graph in Fig. 1(a) called the *variable graph*, where $k_i = \max\{|\{c_j : c_j \text{ contains } x_i\}|, |\{c_j : c_j \text{ contains } \bar{x}_i\}|\}$. We call the subgraph induced by $d_i, x_i, \bar{x}_i$ the *value assignment part* for x_i. For each clause $c_j = \alpha_j + \beta_j + \gamma_j$, we use the graph in Fig. 1(b) called the *clause graph* and c_j a *clause vertex*. We call vertices $\alpha_j[c_j]_p$, $\beta_j[c_j]_p$, $\gamma_j[c_j]_p$ $(p = 0, 1)$ *literal vertices*. An example of construction for a formula $F = (x_1 + x_2)(x_1 + \bar{x}_2)(\bar{x}_1 + x_2)(\bar{x}_1 + \bar{x}_2)$ in 2-conjunctive normal form is given in Fig. 2 together with the numbering of vertices, where some edges are not drawn since they make the figure ugly. As shown in Fig. 2, the variable graphs are concatenated in the order of $x_1, ..., x_n$ and the clause graphs are connected using square vertices $z_1, ..., z_m$. It also has special vertices a, b, h_0, h_1, h_2, w_0 and w_1 which are wired as shown. We put edges $\{h_1, z_j\}$ for $j = 1, ..., m$. Due to these edges, $z_1, ..., z_m$ are forbidden to be chosen when h_1 has been chosen before. We also add edges $\{h_0, u\}$ for all literal vertices u. These edges separate the clause vertices from the variable graphs when h_0 is chosen. We connect the vertex x_i (resp. $\bar{x}_i$) to vertices $\bar{x}_i[c_j]_0$, $\bar{x}_i[c_j]_1$ (resp. $x_i[c_j]_0$, $x_i[c_j]_1$) if clause c_j contains the literal $\bar{x}_i$ (resp. $\bar{x}_i$). The vertex x_i (resp. $\bar{x}_i$) is also connected to vertices $\bar{x}_i^{(1)}, \bar{x}_i^{(2)}, ...$ (resp. $x_i^{(1)}, x_i^{(2)}, ...$) as shown in Fig. 1(a). When x_i (resp. $\bar{x}_i$) has been chosen, these edges prevent the vertices named with the literal x_i (resp. $\bar{x}_i$) from being chosen. The vertices $x_i^{(2k-1)}$ and $x_i^{(2k+1)}$ (resp. $\bar{x}_i^{(2k-1)}$ and $\bar{x}_i^{(2k+1)}$) are connected to $x_i[c_j]_0$ and $x_i[c_j]_1$ (resp. $\bar{x}_i[c_j]_0$ and $\bar{x}_i[c_j]_1$), respectively, if the literal x_i in c_j is the kth occurrence of x_i in $c_1, ..., c_m$ counted from left to right. These four vertices are ordered as numbered in Fig. 2 and work to capture the vertex c_j. For the graph G_F with the specified vertex order, the lfm subset U of vertices which induces a path must contain all vertices in $B = \{a, b, c_1, ..., c_m, h_2, d_i, e_i, f_i, g_i \ : \ i = 1, ..., n\}$ since the set B is obviously extendable

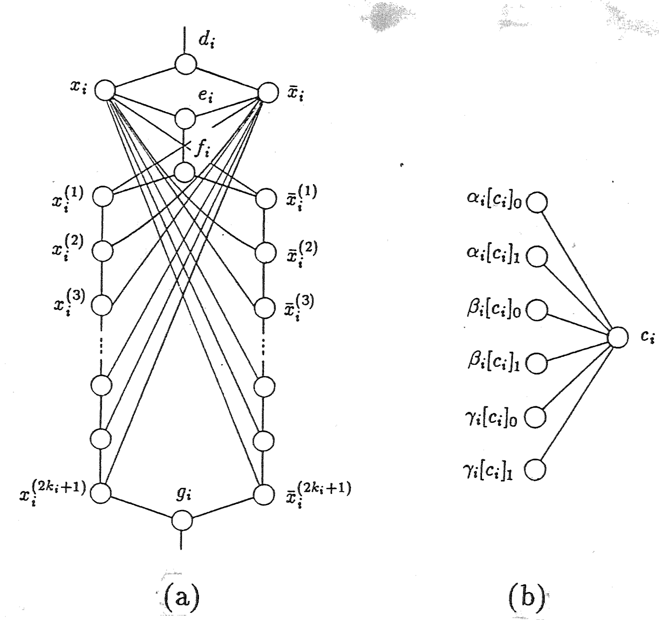


Fig. 1.

to a path. These vertices are colored black in Fig. 2. Moreover, either x_i or $\bar{x}_i$ must be chosen into U since deletion of both vertices raises two vertices d_i, e_i of degree 1 other than the vertex a. Furthermore, either h_0 or h_1 must be chosen. The choice of h_0 or h_1 depends on the satisfiability of the formula.

We show that h_1 (resp. h_0) is chosen into U if and only if F is satisfiable (resp. not satisfiable). Assume that all vertices in B have been chosen. Suppose that h_1 can be chosen. Then none of the square vertices $z_1, ..., z_m$ can be chosen. As mentioned before, either x_i or $\bar{x}_i$ must be chosen. If x_i (resp. $\bar{x}_i$) is chosen, then no vertices of the forms $\bar{x}_i^{(k)}$, $\bar{x}_i[c_j]_p$ (resp. $x_i^{(k)}$, $x_i[c_j]_p$) can be chosen. Therefore the choice of h_1 implies that the set $B \cup \{h_1\} \cup \{v_i \ : \ i = 1, ..., n\}$ is also extendable to a path for some choice of v_i for $i = 1, ..., n$, where v_i is either x_i or $\bar{x}_i$. In this situation, this means that F is satisfiable since each c_j must be connected to some chosen vertex in the variable graph via a literal vertex. Conversely, if F is satisfiable, let $(\hat{x}_1, ..., \hat{x}_n)$ be the lexicographically first bit vector which makes the formula true. Then by taking h_1 together with the vertices in the variable graphs corresponding to the bit vector $(\hat{x}_1, ..., \hat{x}_n)$ into U, we see that U induces a path from a to w_1. Hence if F is not satisfiable, h_1 cannot be chosen. Therefore h_0 must be chosen. This implies that no literal vertex can be chosen. Therefore z_1 must be chosen to capture $c_1, ..., c_m$. Hence w_1 cannot be chosen and all $z_1, ..., z_m$ must be chosen into U. In fact $U = B \cup \{x_1, x_1^{(1)}, ...\} \cup ... \cup \{x_n, x_n^{(1)}, ...\} \cup \{h_0\} \cup \{z_1, ..., z_m\} \cup \{w_0\}$.□

Theorem 3. *The lfm induced path problem is Δ_2^p-complete.*

Sketch of Proof. We give a reduction from DSAT. Let $F_0(x_1, ..., x_{k-1}, Y_1, ..., Y_k)$,

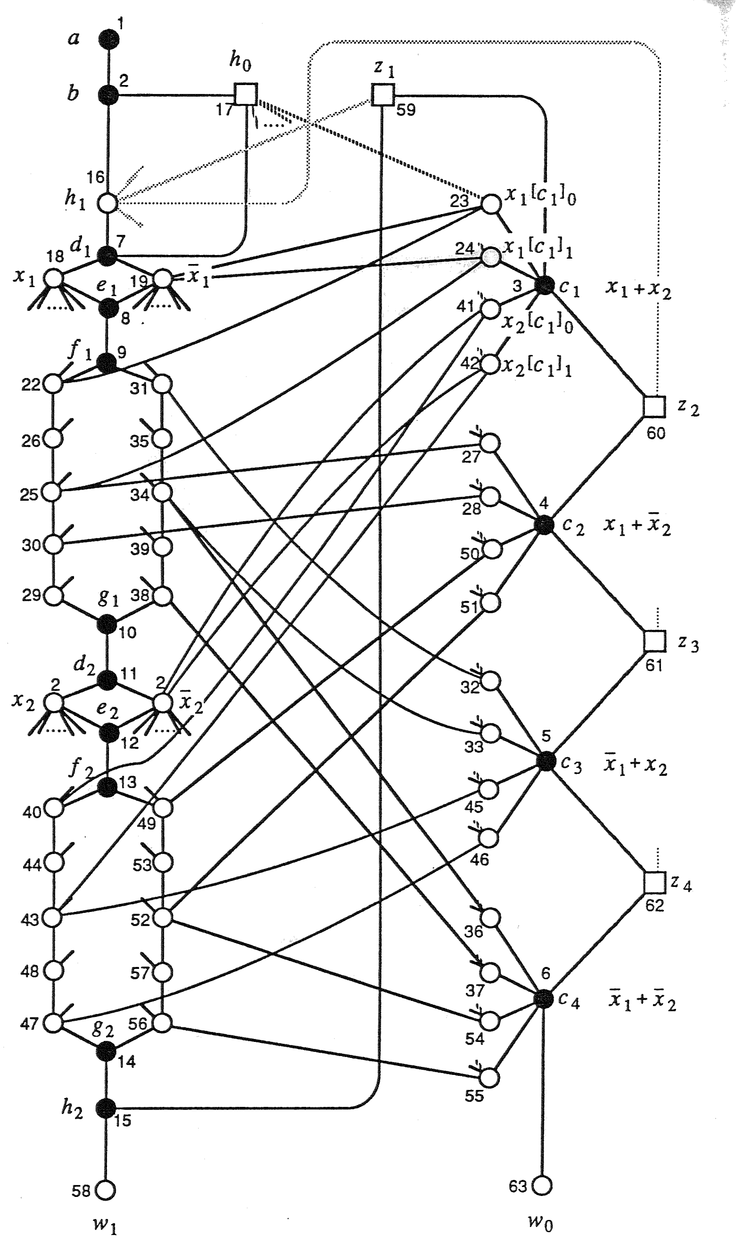

a
b
h0
z1
h1
d1
x1
e1
f1
g1
d2
x2
e2
f2
g2
h2
w1
x1[c1]0
x1[c1]1
c1
x1 + x2
x2[c1]0
x2[c1]1
z2
c2
x1 + x̄2
z3
c3
x̄1 + x2
z4
c4
x̄1 + x̄2
w0

Fig. 2.

$F_1(Y_1, Z_1)$, ..., $F_{k-1}(Y_{k-1}, Z_{k-1})$ constitute an instance of DSAT. We construct a graph $G(F_0, ..., F_{k-1})$ in the following way (see Fig. 3) and show that $(F_0, ..., F_{k-1}) \in$DSAT if and only if the lfm induced path reaches the vertex w_1, where variables are ordered as $Y_1, Z_1, ..., Y_{k-1}, Z_{k-1}, Y_k$.

The graph $G(F_0, ..., F_{k-1})$ is defined as follows: First we take the graph G_{F_0} that is constructed in Lemma 2 for F_0 but, for a moment, we ignore the variables in $Y_1, ..., Y_{k-1}$. We have to decide whether there is a truth assignment $\hat{x}_1, ..., \hat{x}_{k-1}, \hat{Y}_1, ..., \hat{Y}_k$ such that

(i) $F_0(\hat{x}_1, ..., \hat{x}_{k-1}, \hat{Y}_1, ..., \hat{Y}_k) = t$.

(ii) $F_i(\hat{Y}_i, Z_i)$ is satisfiable $\Longleftrightarrow \hat{x}_i = t$ for $i = 1, ..., k-1$.

Since the condition (ii) must be kept, we replace the value assignment part for the variable x_i by $\tilde{G}_{F_i}$ which will be defined later and connect the variable graphs for Y_i of $\tilde{G}_{F_i}$ to the clause graphs of G_{F_0} as was done in Lemma 2. Then we assign numbers to vertices as in Fig. 2.

$\tilde{G}_{F_i}$ is defined for $F_i(Y_i, Z_i)$ using the construction in Lemma 2 but we need some changes. We just sketch a part of $\tilde{G}_{F_i}$ in Fig. 4, where it is assumed that literal $y_j^{(i)}$ from Y_i appears in clause c_{0l} of F_0 and literals $y_j^{(i)}$ and $\bar{y}_j^{(i)}$ appear in clauses $c_1^{(i)}$ and $c_2^{(i)}$ of $F_i(Y_i, Z_i)$, respectively.

The variable graph for $y_j^{(i)}$ in Y_i has a structure similar to the graph in Fig. 1(a). It consists of two parts as shown in Fig. 4. The upper part is connected to the clause graphs of F_0 as in Fig. 2. The lower part is connected to the clause graphs of F_i in a way that for each literal we use only two vertices which are directly wired to the corresponding clause vertex as shown in Fig. 4. Each vertex on the left (resp. right) side of the lower part is wired to the vertex $y_j^{(i)}$ (resp. $\bar{y}_j^{(i)}$). We assume that all vertices on both sides are connected to some clause vertices of F_i. Therefore the left and right sides may have different numbers of vertices. It may be implicitly understood how these vertices are ordered.

Assume that choices of the vertices in the variable graphs for Y_i have been already done and they define a truth assignment $\hat{Y}_i$. Then by the construction it can be checked that a path can reach the vertex x_i (resp. $\bar{x}_i$) if and only if $F_i(\hat{Y}_i, Z_i)$ is satisfiable.

We show that if the vertex h_1 in Fig. 3 is chosen then $(F_0, ..., F_{k-1})$ is in DSAT and the lfm induced path reaches the vertex w_1. As was seen in Lemma 2, if h_1 is chosen, then no triagle vertex can be chosen. Since F_0 is deterministic, truth values for Y_1 are uniquely determined. Therefore choices of the vertices of the variable graphs for Y_1 are also uniquely determined. By the fact mentioned in the former paragraph, $F_1(\hat{Y}_1, Z_1)$ is satisfiable (resp. not satisfiable) if and only if the vertex x_1 (resp. $\bar{x}_1$) is chosen. If x_1 (resp. $\bar{x}_1$) is chosen, then let $\hat{x}_1 = t$ (resp. f). Since F_0 is deterministic, the truth values for Y_2 are again uniquely determined by $\hat{Y}_1$ and $\hat{x}_1$. By induction, the choice of h_1 determines $\hat{Y}_1$, $\hat{x}_1$,...,$\hat{Y}_{k-1}$, $\hat{x}_{k-1}$ and $\hat{Y}_k$ uniquely and the condition (ii) is kept. Therefore $(F_0, ..., F_{k-1})$ is in DSAT. It can be also shown that the chosen vertices induces a path from a_0 to w_1 via h_1. Conversely, if the conditions (i) and (ii) are satisfied, then the lfm induced path contains h_1 and reaches w_1.□

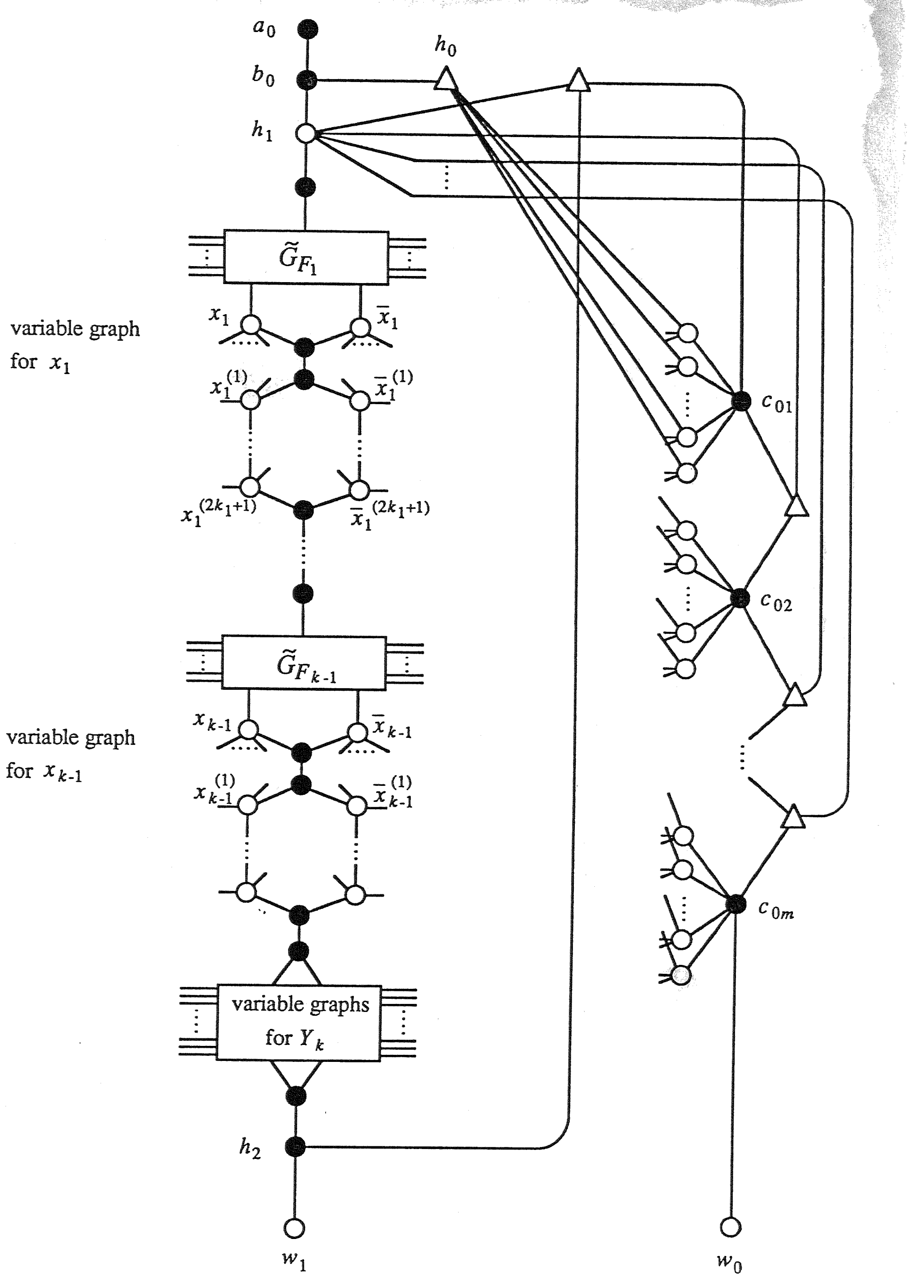

a_0
b_0
h_0
h_1
$\tilde{G}_{F_1}$
variable graph
for x_1
x_1
$\overline{x}_1$
$x_1^{(1)}$
$\overline{x}_1^{(1)}$
$x_1^{(2k_1+1)}$
$\overline{x}_1^{(2k_1+1)}$
$\tilde{G}_{F_{k-1}}$
variable graph
for x_{k-1}
x_{k-1}
$\overline{x}_{k-1}$
$x_{k-1}^{(1)}$
$\overline{x}_{k-1}^{(1)}$
variable graphs
for Y_k
h_2
w_1
c_{01}
c_{02}
c_{0m}
w_0

Fig. 3.

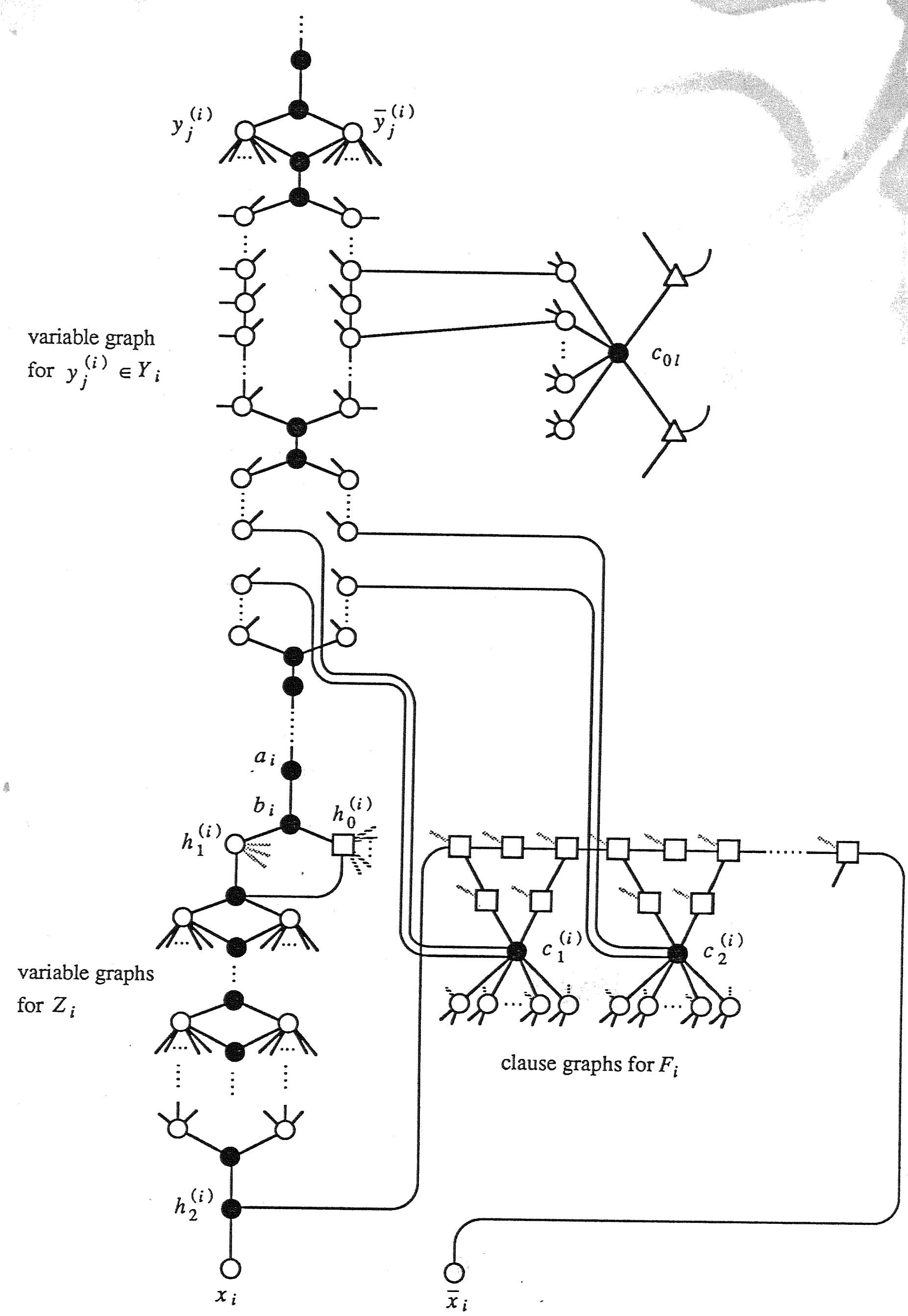

variable graph
for $y_j^{(i)} \in Y_i$
c_{0l}
a_i
b_i
$h_0^{(i)}$
$h_1^{(i)}$
$c_1^{(i)}$
$c_2^{(i)}$
variable graphs
for Z_i
clause graphs for F_i
$h_2^{(i)}$
x_i
$\bar{x}_i$

Fig. 4.

A *rooted tree* is a directed acyclic graph with a special vertex with indegree 0 called the *root* such that every vertex except the root is of indegree 1 and reachable from the root. It is also possible to prove the following theorem in a similar way. The reduction is simpler than that for Theorem 3 and we omit it.

Theorem 4. *The lfm rooted tree problem restricted to directed acyclic graphs is* Δ_2^p*-complete.*

The basic idea in the proofs of Lemma 2 and Theorem 3 can be used to show the Δ_2^p-completeness for another properties, for instance, maximum degree k, directed path, etc. Unfortunately, we do not have a general result like [M1] for P-complete problems.

3. The Connectedness Condition Makes the Problems Hard

A graph property π is said to be *hereditary* on induced subgraphs if, whenever a graph G satisfies π, all vertex-induced subgraphs of G also satisfy π. We say that π is *nontrivial* if π is satisfied by infinitely many graphs and there is a graph violating π. We say that π is *determined by the blocks* [Y1] if for any graphs G_1 and G_2 satisfying π the graph formed by identifying a vertex of G_1 and a vertex of G_2 also satisfies π. We define the *diameter* $\delta^*(\pi)$ by $\sup\{\delta^*(G) : G \text{ satisfies } \pi\}$, where $\delta^*(G)$ is the diameter of G.

We consider the following problem:

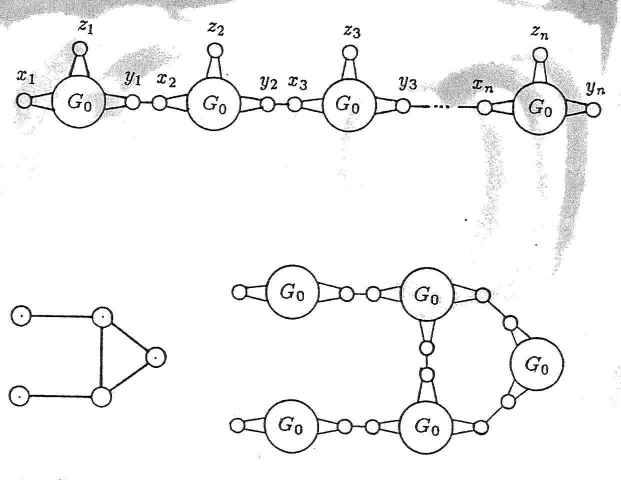
LFMCSP(π)(the lfm connected subgraph problem for π)
Instance: A graph $G = (V, E)$ and a subset $S \subseteq V$, where $V = \{1, ..., n\}$.
Question: Let U be the lfm subset of V which induces a connected subgraph satisfying π. Then $S \subseteq U$?

The following lemma is well known [Be].

Lemma 5. *Let G be a connected graph with at least two vertices. Then G has at least two vertices which are not cutpoints. Moreover, G has exactly two such vertices if and only if G is a path.*

Theorem 6. *Let π be a property hereditary on induced subgraphs satisfying the following conditions:*
(i) π is nontrivial on connected graphs.
(ii) $\delta^*(\pi) = \infty$.
(iii) π is determined by the blocks.
Then LFMCSP(π) is NP-hard.

Proof. Since $\delta^*(\pi) = \infty$ and π is hereditary, it can be proved by considering the shortest paths of the graphs satisfying π that all paths satisfy π. Let G_3 be a connected graph with minimum number of vetices which violates π. Since all paths satisfy π, G_3



(b) (c)

Fig. 5.

is not a path and therefore contains at least three vertices. Therefore by Lemma 5 G_3 contains at least three vertices a, b, c that are not cutpoints. Let G_0 be the subgraph of G_3, not necessarily connected, obtained by deleting a, b, c from G_3.

It is known that the Hamiltonian path problem restricted to planar graphs with degree at most 3 is NP-complete [GJT]. We shall give a reduction from this problem. Before getting into the detail, we consider the graph in Fig. 5(a), where the graph formed by x_i, y_i, z_i and G_0 is G_3 and x_i, y_i, z_i correspond a, b, c but we do not care the way how x_i, y_i, z_i are identified with a, b, c. By the choice of G_3, the graph obtained by deleting any vertex of a, b, c satisfies π. Therefore, by (iii) the graph obtained by deleting all $z_1, ..., z_m$ from the graph in Fig. 5(a) satisfies π. Moreover, it is connected since a, b, c are not cutpoints. However, adding any of $z_1, ..., z_m$ violates π since G_3 violates π.

Let $G = (V, E)$ be a graph with degree at most 3. Let G_2 be the graph formed by deleting vertex c from G_3. For each vertex v of G with degree 3 (resp. degree 2 or 1), we replace it by G_3 (resp. G_2) (see Fig. 5(b)-(c)). Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be the resulting graph. Let $\tilde{W} \subset \tilde{V}$ be the set of vertices on G_0's. We can give an order on $\tilde{V}$ so that $\tilde{W} < \tilde{V} - \tilde{W}$. Now let $\tilde{U}$ be the lfm subset of vertices of $\tilde{G}$ which induces a connected subgraph satisfying π. Then it is not hard to see that the following statements are equivalent.

(1) G has a Hamiltonian path.

(2) $\tilde{W} \subset \tilde{U}$.

Thus LFMCSP(π) is at least NP-hard.□

Examples of the properties that satisfy the conditions of Theorem 5 are planar, bipartite, cycle-free, etc [Y1]. The reduction in Theorem 6 also shows that if G_3 is chosen to be planar then the problem restricted to planar graphs is also NP-hard.

Conjecture. The above result can be strengthened to Δ_2^p-completeness.

4. The LFM Rooted Tree Problem for Topologically Sorted Dags

We say that a dag $G = (V, A)$ with $V = \{1, ..., n\}$ is *topologically sorted* if each arrow (i, j) in A satisfies $i < j$.

In this section we first show that the lfm rooted tree problem for topologically sorted dags can be solved in polynomial time. Then we consider the cases of degree constraints 3 and 4. Our results on the lfm rooted tree problem is summarized in Table I.

	rooted tree	**degree**	**forest**	**degree**
Dags	Δ_2^p-complete		P-complete	3
Topologically sorted dags	P-complete NC^2	4 3	P-complete NC^2	4 3

Table I.

Lemma 7. *The lfm rooted tree problem for topologically sorted dags is in P.*

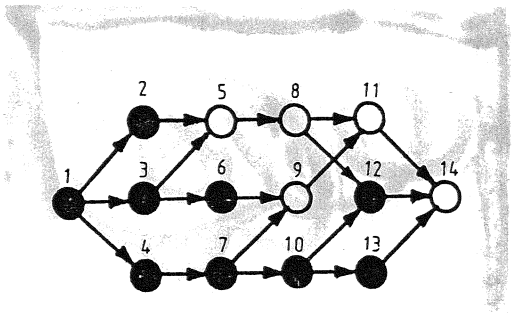
Proof. Let $G = (V, A)$ be a dag, where $V = \{1, ..., n\}$. The problem can be solved in polynomial time by the following greedy algorithm:

begin
  $U \leftarrow \{1\}$;
  for $j \leftarrow 2$ **to** n **do**
    if there is a unique $i \in U$ with $(i, j) \in A$ **then** $U \leftarrow U \cup \{j\}$
end

Before the execution of the j-th step, assume that U is the lfm subset of $\{1, ..., j-1\}$ that can be extended to a rooted tree. Moreover, assume that the subgraph induced by U forms a rooted tree. If j is adjacent to more than two vertices in U, then j has two incoming arrows from U since $i < j$ for any i in U. Therefore j cannot be addend into U. If j is not adjacent to any vertex in U, then there is no rooted tree containing $U \cup \{j\}$ since any i with (i, j) in A must satisfy $i < j$. If j is adjacent to exactly one vertex in U, then the graph induced by $U \cup \{j\}$ is a rooted tree. Hence j can be chosen if and only if it has exactly one incoming arrow from U.□

Theorem 8. *The lfm rooted tree problem for topologically sorted dags with degree at most 3 is in NC^2.*

Proof. Let $G = (V, A)$ be an instance of the problem with degree at most 3, where $V = \{1, ..., n\}$. Without loss of generality, we may assume that every vertex is reachable



The black (resp. white) vertices are chosen (resp. unchosen) vertices.

Fig. 6.

from the vertex 1 and the vertex 1 is the unique vertex with indegree 0. Let $V_d = \{i \in V : \text{indeg}(i) = d\}$ for $d = 0, ..., 3$. Let i be in V. As the greedy algorithm states, i can be chosen if and only if i has exactly one incoming arrow from a chosen vertex. Since vertices in V_3 have no outgoing arrows, they have no effect on the choices of vertices afterward. Therefore we concentrate on the subgraph G' induced by $V_0 \cup V_1 \cup V_2$. If i is in V_1, the choice of i depends on the unique vertex j with $(j, i) \in A$. Namely, i can be chosen if and only if j is chosen. If i is in V_2, there are exactly two predecessors $j, k < i$. When both j and k are chosen or neither j nor k is chosen, i cannot be chosen. On the other hand, when either j or k is chosen, i can be chosen. From this observation, it is not hard to see that i in $V_1 \cup V_2$ can be chosen if and only if there are odd number of paths from the vertex 1 to i (see Fig. 6.). By a method similar to the parallel transitive closure algorithm on adjacency matrices, we can compute in NC^2 the numbers of paths (modulo 2) between vertices in G'. After deciding the choices of the vertices in $V_1 \cup V_2$, the vertices in V_3 are examined. This can be also done in NC^2.□

Theorem 9. *The lfm rooted tree problem for topologically sorted planar dags with degree 4 is P-complete.*

Proof. We give a reduction from planar circuits described in [Go]. We may assume that each gate executes one of the operations $t, \neg x, \neg(x \vee y)$ and the fanout of a gate for $\neg(x \vee y)$ (resp. $t, \neg x$) is at most one (resp. two). We describe the reduction by using an example of a circuit in Fig. 7(a) and we omit the formal details. The circuit $B = (B_1, ..., B_6)$ is converted to the graph in Fig. 7(b). The vertices are ordered as $u_1 < v_1 < \cdots < u_6 < v_6$. This graph is not necessarily planar. Let U be the lfm vertex set to be computed. Then it can be easily checked that all u_j are chosen into U and that v_i is chosen into U if and only if value$(B_i) = t$.

The planarity is violated by arrows from u_j crossing the lines of the circuit. This difficulty can be resolved by replacing each such crossing like Fig. 7(c) with the graph in Fig. 7(d), where the vertices are suitably ordered. Then the choice of vertex v_i is the same as that of v_{ij} and there is a path from u_j to u_{ij} via either w_{ij} or v_{ij}. Since the palanr circuits in [Go] are very regular, it is easy to compute how the arrows from the square vertices cross the lines of the circuit. The degree of the resulting graph is still 4.□

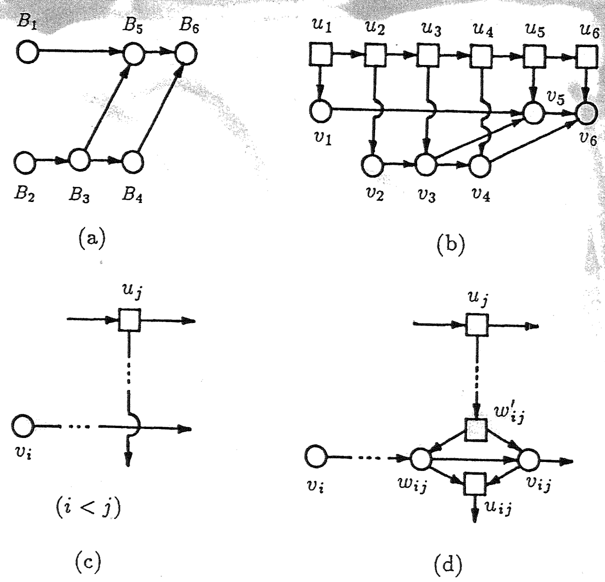


Fig. 7.

A *forest* is a collection of disjoint rooted trees. This problem is easily seen to be P-complete since the property *forest* is hereditary [M1]. Therefore we pay attention to the consistency and the vertex degree.

It is straightforward to check that the reduction in the proof of Theorem 9 is also valid for the lfm forest problem. Thus we have the following corollary.

Corollary 10. *The lfm forest problem for topologically sorted planar dags with degree 4 is P-complete.*

The following result asserts that the degree bound 4 in Corollary 8 is optimal.

Theorem 11. *The lfm forest problem restricted to topologically sorted dags with degree at most 3 is in NC^2.*

Proof. We give a sketch of the algorithm. Given a topologically sorted dag $G = (V, A)$, let V_d be the set of the vertices with indegree d for $d = 0, ..., 3$. Let U denote the lfm subset that forms a forest. The following three facts can be easily observed.

(1) All vertices in $V_0 \cup V_1$ can be chosen into U.

(2) The graph obtained by reversing the arrows of the induced subgraph of V_2 is a forest.

(3) A vertex in V_3 can be chosen into U if and only if it is adjacent to at most one chosen vertex.

For each rooted tree T in the forest obtained from V_2, we associate it with a circuit C_T with an operation $\neg(x \wedge y)$ as shown in Fig. 8. Then we can see that a vertex in T can

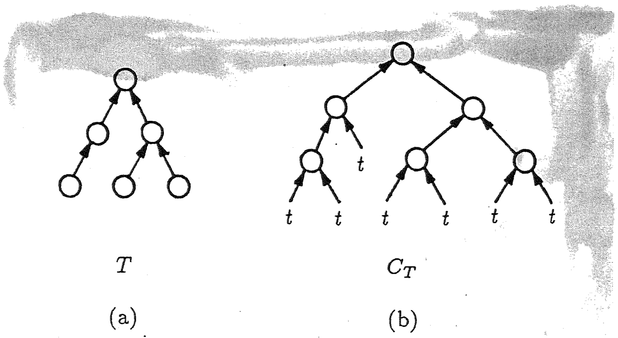


Fig. 8.

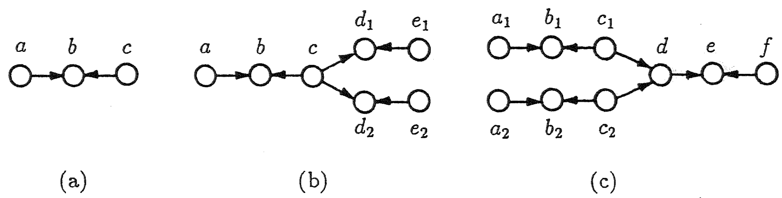


Fig. 9.

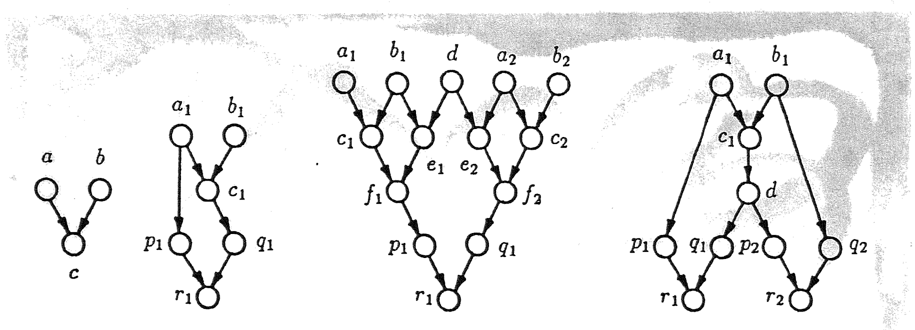
be chosen into U if and only if the value of the corresponding gate is t. Fortunately, the evaluation of such tree-like circuits can be done in NC^2. It is not hard to see that all other necessary computaitons can be also done in NC^2.□

For dags with degree 3 which are not topologically sorted, the problem turns to be P-complete.

Theorem 12. *The lfm forest problem for planar dags with degree 3 is P-complete.*

Proof. We give a reduciton from the planar circuit value problem. We describe how the gates for $f, \neg x$ and $x \wedge y$ can be simulated. We can assume that the fanout of the gates for f and $\neg x$ (resp. $x \wedge y$) is at most two (resp. one). The truth values are simulated by the graph in Fig. 9(a). The order on vertices follows the alphabetical order. When the vertex c is not chosen (resp. chosen), it represents f (resp. t). A gate for $\neg x$ with fanout 2 is simulated by the graph in Fig. 9(b), where (a, b, c) is the input and (c, d_i, e_i) $(i = 1, 2)$ are the outputs. A gate for $x \wedge y$ is simulated by the graph in Fig. 9(c) with the inputs (a_i, b_i, c_i) $(i = 1, 2)$ and the output (d, e, f).□

A dag is said to be *uniconnected* if for any pair (u, v) of vertices there is at most one path from u to v. Concerning the vertex degree, the following P-completeness result on uniconnected graphs might be interesting since uniconnected graphs have a property similar to forests.



(a) (b) (c) (d)

Fig. 10.

Theorem 13. *The lfm uniconnected subgraph problem for topologically sorted planar dags with degree 3 is P-complete.*

Proof. Again we give a reduction form the planar circuit value problem. We assume that vertices are ordered alphabetically. The truth values are represented by using the graph in Fig. 10(a). If the vertex c (resp. is not) chosen, it represents t (resp. f) while the vertices a and b are always chosen. The operations $\neg x, x \wedge y$ and the distribution of a value are simulated, respectively, by the graphs in Fig. 10(b)-(d), where (a_i, b_i, c_i) $(i = 1, 2)$ are the inputs and the resulting values are presented on (p_i, q_i, r_i) $(i = 1, 2)$.□

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