An Algebraic Formalization of Fuzzy Relations

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An Algebraic Formalization of Fuzzy Relations

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Abstract

This paper provides an algebraic formalization of mathematical structures formed by fuzzy relations with sup-min composition. A simple proof of a representation theorem for Boolean relation algebras satisfying Tarski rule and point axiom has been given by G. Schmidt and T. Ströhlein. Unlike Boolean relation algebras, fuzzy relation algebras are not Boolean but equipped with semi-scalar multiplication. First we present a set of axioms for fuzzy relation algebras and improve the definition of point relations. Then by using relational calculus a representation theorem for such relation algebras is deduced without Tarski rule.

Keywords : fuzzy relations, relation algebras, relational calculus, representation theorem.

1 Introduction

Since Zadeh's invention the concept of fuzzy sets has been extensively investigated in mathematics, science and engineering. The notion of fuzzy relations is also a basic one in processing fuzzy information in relational structures, see e.g. Pedrycz [9]. Goguen [2] generalized the concepts of fuzzy sets and relations taking values on partially ordered sets. Fuzzy relational equations were initiated by Sanchez [11] and applied to medical models of diagnosis.

On the other hand theory of relations, namely relational calculus, has a long history, see [8, 12, 13] for more details on the history. Almost modern formalizations of relation algebras are affected by the work of Tarski [14]. Mac Lane [7] and Puppe [10] exposed a categorical basis for calculus of additive relations. Freyd and Scedrov [1] developed and summarized categorical relational calculus, which they called allegories. Kawahara and Mizoguchi [3, 5, 6] developed relational methodology for assertion semantics of programs and theory of graph transformations (or graph grammars). A relational approach to set theory is studied by Kawahara [4]. However the authors should mention that some ideas and results in [4] have been already given by [1]. Concerning with applications to the relational theory of graphs and programs, Schmidt and Ströhlein [12] gave a simple proof of a representation theorem for Boolean relation algebras satisfying Tarski rule and point axiom. Also they wrote an excellent text book [13] on relations and graphs with many useful examples in computer science.

The aim of the paper is to provide an algebraic formalization for fuzzy relations. Fuzzy relations treated here are homogeneous ones on a set X with values in the unit interval $[0,1]$, that is, functions $R : X \times X \to [0,1]$. The set of all such fuzzy relations on X constitutes a fuzzy relation algebra. Unlike Boolean relation algebras, fuzzy relation algebras are not Boolean and

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so Schröder rule [12], an important axiom of Boolean relation algebras, does not work in our case. Instead of Schröder rule we prefer to adopt Dedekind formula, which will be stated as axiom A5 in 3.1. As fuzzy relations are naturally equipped with semi-scalar multiplication by scalars in the unit interval, additional axioms A6(a)–A6(h) are required. However regular (or crisp in [2]) relations are contained in fuzzy relations and form a Boolean subalgebra. The concept of regular relations is well formalized in terms of semi-scalar multiplication, and then we introduce axiom A7 for semi-Boolean algebras, which claims that regular relations have their complements (Cf. 2.1(c)). The final axiom A8, called point axiom, asserts that pair relations formed by point relations play a role of atoms. These are main different points from Boolean case.

In section 2 we first recall fundamental operations on fuzzy relations such as subset, union, intersection, sup-min composition, inverse and semi-scalar multiplication. Then some basic properties of concrete fuzzy relations accepted as axioms are proved. The section 3 provides a set of the axioms A1–A7 for fuzzy relation algebras and the basic properties of fuzzy relation algebras. In section 4 we first state the definition of point relations and a point axiom A8. Then some useful properties on point relations are given for the proof of main results. In particular, the point axiom A8 induces a function assigning a concrete fuzzy relation on the set of point relations to an abstract relation in a fuzzy relation algebra. In section 5 we show main theorems such as a representation, an insertion and an isomorphism theorems for fuzzy relation algebras without assumption of Tarski rule [12].

2 Fuzzy Relations

Let $[0,1]$ be the unit interval, that is, the set of all real numbers k with $0 \le k \le 1$. In the paper a real number $k \in [0,1]$ will be called a scalar and denoted by lower case Roman letters such as $k, r, s, \cdots$ and so on. (All real numbers k appearing in the paper are scalars, that is, $0 \le k \le 1$.) The supremum (the least upper bound) and the infimum (the greatest lower bound) of a family $\{k_\lambda\}$ of scalars will be denoted by $\vee_\lambda k_\lambda$ and $\wedge_\lambda k_\lambda$, respectively. In particular, $k \vee k' = \max\{k, k'\}$ and $k \wedge k' = \min\{k, k'\}$.

A fuzzy relation R on a set X is a function $R : X \times X \to [0,1]$. For $x, y \in X$ the value $R(x,y) \in [0,1]$ means the degree how far x and y are related under R. Throughout this section all fuzzy relations are those on a fixed set X. The set of all fuzzy relations on X will be denoted by $\mathbf{Rel}(X)$. A fuzzy relation R is contained in a fuzzy relation S, written $R \subseteq S$, if $R(x,y) \le S(x,y)$ for all $x, y \in X$. The zero relation O and the universal relation L are fuzzy relations with $O(x,y) = 0$ and $L(x,y) = 1$ for all $x, y \in X$, respectively. It is trivial that $\subseteq$ is a partial order, and $O \subseteq R \subseteq L$ for all fuzzy relations R. For a family $\{R_\lambda\}_\lambda$ of fuzzy relations we define fuzzy relations $\cup_\lambda R_\lambda$ and $\cap_\lambda R_\lambda$ as follows:

$$(\cup_\lambda R_\lambda)(x,y) = \vee_\lambda R_\lambda(x,y)$$

and

$$(\cap_\lambda R_\lambda)(x,y) = \wedge_\lambda R_\lambda(x,y)$$

for all $x, y \in X$. It is obvious that $\cup_\lambda R_\lambda$ and $\cap_\lambda R_\lambda$ are the least upper bound and the greatest lower bound of a family $\{R_\lambda\}_\lambda$, respectively, with respect to the order $\subseteq$. The composite $RS(= R;S)$ of a fuzzy relation R followed by a fuzzy relation S is defined by

$$(RS)(x,y) = \vee_{z \in X}[R(x,z) \wedge S(z,y)]$$

for all $x, y \in X$. This composition of fuzzy relations is called as sup-min composition. The associativity $(RS)T = R(ST)$ holds for all fuzzy relations R, S, and T. The identity relation

I is a fuzzy relation such that $I(x,y)=1$ if $x=y$ and $I(x,y)=0$ otherwise. The unitary law $RI=IR=R$ and the zero law $RO=OR=O$ hold for all R. The inverse (or transpose) $R^\sharp$ of a fuzzy relation R is defined by

$$R^\sharp(x,y)=R(y,x)$$

for all $x,y\in X$. The semi-scalar multiplication kR of a fuzzy relations R by a scalar k is a fuzzy relation such that

$$(kR)(x,y)=kR(x,y)$$

for all $x,y\in X$. A fuzzy relation R is called nonzero if $R\neq O$. A fuzzy relation R is regular (or crisp in [2]) if $R(x,y)=0$ or $R(x,y)=1$ for all $x,y\in X$. It is evident [2] that fuzzy relations and their operations defined above satisfy almost all axioms stated in the next section, except for A5(Dedekind formula), A6(f) and A7(semi-Boolean algebra), which will be proved in the following:

Proposition 2.1 *Let R,S,T be fuzzy relations on X. Then*

(a) $RS\cap T\subseteq R(S\cap R^\sharp T)$ *(Dedekind formula),*

(b) $(kR)S=(kR)(S\cap kL)$,

(c) *R is regular if and only if there is a fuzzy relation S such that $L=R\cup S$ and $R\cap S=O$,*

(d) *R is regular if and only if $R\cap kL=kR$ for all scalars k.*

Proof. (a) Remarking $R^\sharp(z,x)\wedge T(x,y)\le(R^\sharp T)(z,y)$ it follows from

$$\begin{aligned}(RS\cap T)(x,y) &= \vee_z[R(x,z)\wedge S(z,y)]\wedge T(x,y)\\ &= \vee_z[R(x,z)\wedge S(z,y)\wedge T(x,y)]\\ &= \vee_z[R(x,z)\wedge S(z,y)\wedge R^\sharp(z,x)\wedge T(x,y)]\\ &\le \vee_z[R(x,z)\wedge S(z,y)\wedge(R^\sharp T)(z,y)]\\ &= \vee_z[R(x,z)\wedge(S\cap R^\sharp T)(z,y)]\\ &= [R(S\cap R^\sharp T)](x,y)\end{aligned}$$

for all $x,y\in X$.
(b) It is immediate from

$$\begin{aligned}[(kR)(S\cap kL)](x,y) &= \vee_z[kR(x,z)\wedge S(z,y)\wedge k]\\ &= \vee_z[kR(x,z)\wedge S(z,y)]\\ &= [(kR)S](x,y)\end{aligned}$$

for all $x,y\in X$.
(c) Assume that R is regular and define $S(x,y)=1-R(x,y)$ for all $x,y\in X$. Then it is clear that $R\cup S=L$ and $R\cap S=O$. The converse is trivial (Cf. Proof of 3.4(c)).
(d) It is immediate from

$$\begin{aligned}R\cap kL=kR &\iff \forall k\in[0,1]:R(x,y)\wedge k=kR(x,y)\\ &\iff R(x,y)=0 \text{ or } R(x,y)=1.\quad\square\end{aligned}$$

In relational calculus ([1, 4, 13]) a function R on X is a relation satisfying the univalency $R^\sharp R\subseteq I$ and the totality $I\subseteq RR^\sharp$. Note that

$$\begin{aligned}R^\sharp R\subseteq I &\iff \forall y,y':R^\sharp R(y,y')=\vee_{x\in X}[R^\sharp(y,x)\wedge R(x,y')]\le I(y,y')\\ &\iff \forall x,y,y':y\neq y'\Rightarrow R(x,y)\wedge R(x,y')=0,\end{aligned}$$

and

$$I \subseteq RR^\sharp \iff \forall x : R^\sharp R(x,x) = \vee_{y\in X}[R(x,y) \wedge R^\sharp(y,x)] = 1$$
$$\iff \forall x : \vee_{y\in X} R(x,y) = 1.$$

Thus total relations are nonzero if X is not empty.

To capture the concept of points in relation algebras Schmidt and Ströhlein [12] defined a notion of vector relations. A vector relation R is a fuzzy relation such that $LR = R$, namely

$$LR = R \iff \forall x,y : R(x,y) = \vee_{z\in X}[L(x,z) \wedge R(z,y)]$$
$$\iff \forall x,y : R(x,y) = \vee_{z\in X} R(z,y)$$
$$\iff \forall x,x',y : R(x,y) = R(x',y).$$

The following lemma motivates the definition of point relations in fuzzy relation algebras.

Lemma 2.2 *Let X be a nonempty set. If R is a regular fuzzy relation with $R^\sharp R \subseteq I$, $I \subseteq RR^\sharp$ and $LR = R$, then there is an element $y_0 \in X$ such that for all $x, y \in X$, $R(x,y) = 1$ if $y = y_0$ and $R(x,y) = 0$ otherwise.*

Proof. Since X is not empty there is an element $x_0 \in X$. By the regularity and the totality of R there is an element $y_0 \in X$ such that $R(x_0, y_0) = 1$. But it follows from $LR = R$ that $R(x, y_0) = 1$ for all $x \in X$, and from $R^\sharp R \subseteq I$ that $R(x,y) = R(x,y) \wedge R(x,y_0) = 0$ for $y \neq y_0$. $\Box$

3 Axioms for Fuzzy Relations

The section 3 provides a set of the axioms A1–A7 for fuzzy relation algebras and some basic properties of fuzzy relation algebras are described. A fuzzy relation algebra $\mathcal{F}$, which will be defined below, is an algebraic structure on a nonempty set $\mathcal{F}$. Elements of $\mathcal{F}$ are called relations and denoted by Greek letters such as $\alpha, \beta, \cdots$ and so on. The composite of a relation α followed by a relation β will denoted by $\alpha\beta$, and the multiplication of α by a scalar $k \in [0,1]$ will be written as $k\alpha$, unless confusion occurs.

Definition 3.1 A fuzzy relation algebra $\mathcal{F} = (\mathcal{F}, \sqsubseteq, \sqcup, \sqcap, ;, ^\sharp, \cdot, O, L, I)$ is an algebraic structure over a nonempty set $\mathcal{F}$ satisfying the following:
A1. [Complete distributive lattice] A hexad $(\mathcal{F}, \sqsubseteq, \sqcup, \sqcap, O, L)$ is a complete distributive lattice with the least element O and the greatest element L. That is,
(a) $\sqsubseteq$ is a partial order on $\mathcal{F}$, (b) $\forall \alpha \in \mathcal{F} : O \sqsubseteq \alpha \sqsubseteq L$,
(c) $\sqcup_\lambda \beta_\lambda \sqsubseteq \alpha \iff \forall \lambda : \beta_\lambda \sqsubseteq \alpha$, (d) $\alpha \sqsubseteq \sqcap_\lambda \beta_\lambda \iff \forall \lambda : \alpha \sqsubseteq \beta_\lambda$, (e) $\alpha \sqcap (\sqcup_\lambda \beta_\lambda) = \sqcup_\lambda (\alpha \sqcap \beta_\lambda)$.
A2. [Monoid] A quartet $(\mathcal{F}, ;, I, O)$ is a monoid with a unit element I and a zero element O. That is,
(a) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, (b) $\alpha I = I\alpha = \alpha$, (c) $\alpha O = O\alpha = O$.
A3. [Distributive Law] $\alpha(\sqcup_\lambda \beta_\lambda) = \sqcup_\lambda \alpha\beta_\lambda$.
A4. [Involution] An operation $^\sharp : \mathcal{F} \to \mathcal{F}$ is an involution on $\mathcal{F}$. That is,
(a) $(\alpha^\sharp)^\sharp = \alpha$, (b) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (c) If $\alpha \sqsubseteq \beta$, then $\alpha^\sharp \sqsubseteq \beta^\sharp$.
A5. [Dedekind formula] $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$.
A6. [Semi-scalar multiplication] An operation $\cdot : [0,1] \times \mathcal{F} \to \mathcal{F}$ is a semi-scalar multiplication on $\mathcal{F}$. That is,
(a) $0\alpha = O$ and $1\alpha = \alpha$, (b) $k(k'\alpha) = (kk')\alpha$, (c) $k(\sqcup_\lambda \alpha_\lambda) = \sqcup_\lambda k\alpha_\lambda$ and $k(\sqcap_\lambda \alpha_\lambda) = \sqcap_\lambda k\alpha_\lambda$,
(d) $(\wedge_\lambda k_\lambda)\alpha = \sqcap_\lambda k_\lambda \alpha$, (e) $k(\alpha\beta) = (k\alpha)(k\beta)$, (f) $(k\alpha)\beta = (k\alpha)(\beta \sqcap kL)$, (g) $(k\alpha)^\sharp = k\alpha^\sharp$,
(h) If $k\alpha \sqsubseteq k\beta$ and $k > 0$, then $\alpha \sqsubseteq \beta$.
A7. [Semi-Boolean algebra] If $\alpha \sqcap kL = k\alpha$ for all scalars k, then there is a relation β such that $\alpha \sqcup \beta = L$ and $\alpha \sqcap \beta = O$. $\Box$

Note that complete distributive lattices are equivalent to complete Brouwerian lattices or complete Heyting algebras.

Let $\mathcal{F} = (\mathcal{F}, \sqsubseteq, \sqcup, \sqcap, ;, ^\sharp, \cdot, O, L, I)$ be a fuzzy relation algebra. A fuzzy relation algebra $\mathcal{F}$ with $L = O$ is trivial and not worth mention. Throughout the rest of the paper all discussions will be done in a fixed fuzzy relation algebra $\mathcal{F}$ with $L \neq O$. All elements in the fuzzy relation algebra $\mathcal{F}$ are called relations for short.

Proposition 3.2 *Let α, β, β' be relations and k, k' scalars.*

(a) *If $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha\beta'$ and $\beta\alpha \sqsubseteq \beta'\alpha$.*

(b) *$O^\sharp = O$, $L^\sharp = L$ and $I^\sharp = I$.*

(c) *$(\alpha \sqcup \beta)^\sharp = \alpha^\sharp \sqcup \beta^\sharp$ and $(\alpha \sqcap \beta)^\sharp = \alpha^\sharp \sqcap \beta^\sharp$.*

(d) *If $\alpha \sqsubseteq \beta$, then $k\alpha \sqsubseteq k\beta$.*

(e) *If $k \leq k'$, then $k\alpha \sqsubseteq k'\alpha$. In particular, $k\alpha \sqsubseteq \alpha$ and $kO = O$.*

(f) *$k\alpha \sqcup k'\alpha = (k \vee k')\alpha$.*

(g) *$(k\alpha)\beta \sqsubseteq k(\alpha L)$ and $\alpha(k\beta) \sqsubseteq k(L\beta)$.*

(h) *If $kL = L$, then $k = 1$.*

Proof. (a) If $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha\beta \sqcup \alpha\beta' = \alpha(\beta \sqcup \beta') = \alpha\beta'$ by A3. (b) $O^\sharp \sqsubseteq O^{\sharp\sharp} = O$ since $O \sqsubseteq O^\sharp$ and $L = L^{\sharp\sharp} \sqsubseteq L^\sharp$ since $L^\sharp \sqsubseteq L$ and $I^\sharp = I^\sharp I = I^\sharp I^{\sharp\sharp} = (I^\sharp I)^\sharp = I^{\sharp\sharp} = I$. (c) First note that $\alpha^\sharp \sqcup \beta^\sharp \sqsubseteq (\alpha \sqcup \beta)^\sharp$. Hence $\alpha \sqcup \beta = \alpha^{\sharp\sharp} \sqcup \beta^{\sharp\sharp} \sqsubseteq (\alpha^\sharp \sqcup \beta^\sharp)^\sharp$ and $(\alpha \sqcup \beta)^\sharp \sqsubseteq (\alpha^\sharp \sqcup \beta^\sharp)^{\sharp\sharp} = \alpha^\sharp \sqcup \beta^\sharp$. (d) If $\alpha \sqsubseteq \beta$, then $k\alpha = k(\alpha \sqcap \beta) = k\alpha \sqcap k\beta \sqsubseteq k\beta$. (e) If $k \leq k'$, then $k\alpha = (k \wedge k')\alpha = k\alpha \sqcap k'\alpha \sqsubseteq k'\alpha$. (f) Assume $k \leq k'$. Then $k\alpha \sqcup k'\alpha = k'\alpha = (k \vee k')\alpha$ by (e). (g) $(k\alpha)\beta = (k\alpha)(\beta \sqcap kL) \sqsubseteq (k\alpha)(kL) = k(\alpha L)$ by A6(f) and A6(e). (h) Assume $kL = L$ and $0 \leq k < 1$. Then by A6(b) it is trivial that $k^n L = L$ for all natural numbers n. Hence $L = \sqcap_{n \geq 0} k^n L = (\wedge_{n \geq 0} k^n)L = 0L = O$ by A6(d), which contradicts the hypothesis $L \neq O$. □

In view of 2.1(d) the concept of regular relations in fuzzy relation algebras is defined by the following:

Definition 3.3 A relation α is regular if $\alpha \sqcap kL = k\alpha$ for all scalars k. □

Note that $L \sqcap kL = kL$ from $kL \sqsubseteq L$ by 2.1(c). This means that the universal relation L is regular. Also the zero relation O is clearly regular.

Proposition 3.4 *Let α and β be relations and k a scalar.*

(a) *If β is regular, then $(k\alpha)\beta = k(\alpha\beta)$. (If α is regular, then $\alpha(k\beta) = k(\alpha\beta)$).*

(b) *If α and β are regular, then so are $\alpha \sqcup \beta$, $\alpha \sqcap \beta$, $\alpha^\sharp$, and $\alpha\beta$.*

(c) *If $L = \alpha \sqcup \beta$ and $\alpha \sqcap \beta = O$, then both of α and β are regular.*

(d) *If $L = \alpha \sqcup \beta$, $\alpha \sqcap \beta = O$ and $L\alpha = \alpha$, then $L\beta = \beta$.*

(e) *If β is regular and $k\alpha \sqsubseteq \beta$ for $k > 0$, then $\alpha \sqsubseteq \beta$.*

Proof. (a) Assume that β is regular. Then $(k\alpha)\beta = (k\alpha)(\beta \sqcap kL) = (k\alpha)(k\beta) = k(\alpha\beta)$ by A6(f) and A6(e). (b) $(\alpha \sqcup \beta) \sqcap kL = (\alpha \sqcap kL) \sqcup (\beta \sqcap kL) = k\alpha \sqcup k\beta = k(\alpha \sqcup \beta)$ by A1(e) and A6(c). $\alpha^\sharp \sqcap kL = (\alpha \sqcap kL)^\sharp = (k\alpha)^\sharp = k\alpha^\sharp$ by A4, A6(g), 3.2(c) and 3.2(b). $(\alpha \sqcap \beta) \sqcap kL = (\alpha \sqcap kL) \sqcap (\beta \sqcap kL) = k\alpha \sqcap k\beta = k(\alpha \sqcap \beta)$ by A6(c). $\alpha\beta \sqcap kL \sqsubseteq \alpha[\beta \sqcap \alpha^\sharp(kL)] \sqsubseteq \alpha[\beta \sqcap k(LL)] \sqsubseteq \alpha(\beta \sqcap kL) = \alpha(k\beta) = k(\alpha\beta)$ by A5, 3.2(g), 3.2(b) and (a). (c) First by 3.2(e) $k\alpha \sqsubseteq \alpha$ and $\alpha \sqcap k\beta = O$ for a scalar k. Then $\alpha \sqcap kL = \alpha \sqcap k(\alpha \sqcup \beta) = (\alpha \sqcap k\alpha) \sqcup (\alpha \sqcap k\beta) = k\alpha$ by A6(e) and A1(e). (d) Note that $\beta = I\beta \sqsubseteq L\beta$ by A2(b) and A1(b), and $L\beta \sqcap \alpha \sqsubseteq L(\beta \sqcap L^\sharp\alpha) \sqsubseteq L(\beta \sqcap L\alpha) = L(\beta \sqcap \alpha) = LO = O$ by A5, A1(b) and A2(c). Hence $L\beta = L\beta \sqcap L = L\beta \sqcap (\alpha \sqcup \beta) = (L\beta \sqcap \alpha) \sqcup (L\beta \sqcap \beta) = \beta$. (e) Note that $k\alpha \sqsubseteq kL$ by A1(b) and 3.1(d). As β is regular $k\alpha \sqsubseteq \beta \sqcap kL = k\beta$. Hence $\alpha \sqsubseteq \beta$ by the axiom A6(h). □

¿From the last proposition 3.4(b) and 3.4(c) it is immediate that the set of all regular relations in a fuzzy relation algebra $\mathcal{F}$ forms a (Boolean) relation algebra in the sense of [12].

4 Point Axiom

In this section we state a concept of point relations and a point axiom in fuzzy relation algebras. The concept of point relations defined here contains the so-called Tarski rule [14] and so is stronger than that of [13]. Consequently the Tarski rule is not necessary in proving the representability theorem of relation algebras.

Proposition 4.1 *Let α be a regular relation such that $L\alpha = \alpha$. Then the following four conditions are equivalent : (a) $I \sqsubseteq \alpha\alpha^\sharp$, (b) $L = \alpha\alpha^\sharp$, (c) $L = \alpha L$, (d) $\alpha L = kL$ for some $k > 0$.*

Proof. (a)⇒(b) If $I \sqsubseteq \alpha\alpha^\sharp$, then $L = LI \sqsubseteq L\alpha\alpha^\sharp = \alpha\alpha^\sharp$. (b)⇒(c) If $L = \alpha\alpha^\sharp$, then $L = \alpha\alpha^\sharp \sqsubseteq \alpha L$. (c)⇒(a) If $L = \alpha L$, then $I = I \sqcap L = I \sqcap \alpha L \sqsubseteq \alpha(\alpha^\sharp I \sqcap L) = \alpha\alpha^\sharp$. (c)⇒(d) It is trivial. (d)⇒(c) If $kL \sqsubseteq \alpha L$ for $k > 0$, then $L \sqsubseteq \alpha L$ by 3.4(e) since αL is regular. □

In view of lemma 2.2 the concept of point relations in fuzzy relation algebras is defined as follows:

Definition 4.2 A point relation x is a regular relation such that $x^\sharp x \sqsubseteq I$, $I \sqsubseteq xx^\sharp$ and $Lx = x$. (Point relations will be denoted by lower case Roman letters such as $x, y, z, \cdots$.) □

Note that a point relation x is nonzero from its totality $I \sqsubseteq xx^\sharp$. For point relations x, y a relation $x^\sharp y$ is called a pair relation. Every pair relation $x^\sharp y$ is nonzero since $y \sqsubseteq x(x^\sharp y)$ by the totality $I \sqsubseteq xx^\sharp$ of x. The point axiom, which will be stated in 4.4, asserts that pair relations play a role of atomic relations. The last proposition 4.1 indicates that Tarski rule [14, 13] for point relations is equivalent to the totality [6]. This enables us to deduce main theorems in section 5 for fuzzy relation algebras without Tarski rule.

Proposition 4.3 *Let x, x_0, y, y_0 be point relations and k a scalar. Then the following holds:*

(a) *If $x \sqsubseteq y$, then $x = y$.*

(b) *If $x^\sharp y \sqsubseteq x_0^\sharp y_0$, then $x = x_0$ and $y = y_0$.*

(c) *$\alpha \sqcap x^\sharp y = k(x^\sharp y)$ if and only if $x\alpha \sqcap y = ky$.*

Proof. (a) Using $I \sqsubseteq xx^\sharp$, $x^\sharp \sqsubseteq y^\sharp$ and $yy^\sharp \sqsubseteq I$ we have $y \sqsubseteq xx^\sharp y \sqsubseteq xy^\sharp y \sqsubseteq x$. (b) Assume $x^\sharp y \sqsubseteq x_0^\sharp y_0$. Then $y = Ly = Lx^\sharp y \sqsubseteq Lx_0^\sharp y_0 = y_0$ by 4.1 and so $y = y_0$ by (a). Similarly $x = x_0$. (c) First note that $x\alpha \sqcap y = x(\alpha \sqcap x^\sharp y)$ and $\alpha \sqcap x^\sharp y = x^\sharp(x\alpha \sqcap y)$, because $x\alpha \sqcap y \sqsubseteq x(\alpha \sqcap x^\sharp y) \sqsubseteq xx^\sharp(x\alpha \sqcap y) \sqsubseteq L(x\alpha \sqcap y) \sqsubseteq x\alpha \sqcap y$, and $\alpha \sqcap x^\sharp y \sqsubseteq x^\sharp(x\alpha \sqcap y) \sqsubseteq x^\sharp x(\alpha \sqcap x^\sharp y) \sqsubseteq \alpha \sqcap x^\sharp y$ by $x^\sharp x \sqsubseteq I$. Now assume $\alpha \sqcap x^\sharp y = k(x^\sharp y)$. Then $x\alpha \sqcap y = x(\alpha \sqcap x^\sharp y) = x[k(x^\sharp y)] = k[x(x^\sharp y)] = k(Ly) = ky$. Conversely assume $x\alpha \sqcap y = ky$. Then $\alpha \sqcap x^\sharp y = x^\sharp(x\alpha \sqcap y) = x^\sharp(ky) = k(x^\sharp y)$ (since $x^\sharp$ is regular). □

By making use of the last definition of point relations in fuzzy relation algebras we add the following axiom:

Definition 4.4 A fuzzy relation algebra $\mathcal{F}$ satisfies the point axiom if :
A8. [Point axiom] For each nonzero relation α there is a scalar $k > 0$ and point relations x, y such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$. □

In what follows we assume that a fixed fuzzy relation algebra $\mathcal{F}$ satisfies the point axiom A8.

Proposition 4.5 *Let α and β be relations, x and y point relations, and k and k' scalars. Then the following holds:*

(a) *If α is a nonzero relation with $L\alpha = \alpha$, then there exist a scalar $k > 0$ and a point relation y such that $ky \sqsubseteq \alpha$.*

(b) *If $k\alpha \sqsubseteq k'\beta$ for a nonzero regular relation α, then $k \le k'$.*

(c) *If $\alpha \sqsubseteq x^\sharp y$, then there is a unique scalar k such that $\alpha = k(x^\sharp y)$.*

(d) *If $\alpha \sqcap x^\sharp y = k(x^\sharp y)$, then $k = \max\{r \in [0,1] \mid r(x^\sharp y) \sqsubseteq \alpha\}$.*

(e) *If $x \neq y$, then $x \sqcap y = O$ and $xy^\sharp = O$.*

Proof. (a) From the point axiom A8 there exist a scalar $k > 0$ and point relations x, y such that $k(x^\sharp y) \sqsubseteq \alpha$. Hence $ky = k(Ly) = k(xx^\sharp y) = x[k(x^\sharp y)] \sqsubseteq L\alpha = \alpha$ by 4.1 and 3.3(a). (b) First we will show that $kL \sqsubseteq k'L$. By the point axiom A8 there are a scalar $r > 0$ and point relations x, y such that $r(x^\sharp y) \sqsubseteq \alpha$. But $x^\sharp y \sqsubseteq \alpha$ by 3.4(e) since α is regular. Hence $L = L(x^\sharp y)L \sqsubseteq L\alpha L$ by 4.1 and $kL \sqsubseteq k(L\alpha L) = L(k\alpha)L \sqsubseteq L(k'\beta)L = k'(L\beta L) \sqsubseteq k'L$. Now assume $k' < k$. Then $k'/k \in [0,1]$ and $kL \sqsubseteq k[(k'/k)L]$ and so $L \sqsubseteq (k'/k)L$ by A6(h). Thus $k'/k = 1$ follows from by 3.2(h), which is a contradiction. Therefore $k \le k'$. (c) It is trivial that if $\alpha \neq O$ then $\alpha = 0(x^\sharp y)$ by A6(a). Next assume $\alpha \neq O$. Then by the point axiom there are a scalar $k > 0$ and point relations x_0, y_0 such that $\alpha \sqcap x_0^\sharp y_0 = k(x_0^\sharp y_0)$. Hence $k(x_0^\sharp y_0) \sqsubseteq \alpha \sqsubseteq x^\sharp y$, and so $x = x_0$ and $y = y_0$ by 3.4(e) and 4.3(c), which implies $\alpha = k(x^\sharp y)$. The uniqueness of k follows from (b). (d) Define a set $R = \{r \in [0,1] \mid r(x^\sharp y) \sqsubseteq \alpha\}$ of scalars. If $r \in R$, then $r(x^\sharp y) \sqsubseteq \alpha \sqcap x^\sharp y = k(x^\sharp y)$ and so $r \le k$ by (b). On the other hand, $k \in R$ follows from $k(x^\sharp y) = \alpha \sqcap x^\sharp y \sqsubseteq \alpha$. Hence k is the maximum element of R. (e) Assume that $x \neq y$ and $x \sqcap y \neq O$. Note that $L(x \sqcap y) = x \sqcap y$ from $x \sqcap y \sqsubseteq L(x \sqcap y) \sqsubseteq Lx \sqcap Ly = x \sqcap y$. Then by (a) there are a scalar $r > 0$ and a point relation z such that $rz \sqsubseteq x \sqcap y$. But $x \sqcap y$ is regular by 3.4(b). Hence we have $z \sqsubseteq x \sqcap y$ by 3.4(e) and so $x = z = y$ by 4.3(a). Finally if $x \sqcap y = O$, then $xy^\sharp = xy^\sharp \sqcap L \sqsubseteq (x \sqcap Ly)y^\sharp = (x \sqcap y)y^\sharp = O$. □

Proposition 4.6 *Let X be the set of all point relations in $\mathcal{F}$. Then the following holds:*

(a) $L = \sqcup_{z \in X} z$,

(b) $I = \sqcup_{z \in X} z^\sharp z$.

Proof. (a) Set $\alpha = \sqcup_{z \in X} z$. It is clear that $\alpha \sqsubseteq L$. As point relations are regular by the definition, α is also regular by 3.4(b) and by the axiom A7 there is a relation β such that $L = \alpha \sqcup \beta$ and $\alpha \sqcap \beta = O$. Then $L\beta = \beta$ by 3.4(d). Assume $\beta \neq O$. By 4.5(a) there are a scalar $k > 0$ and a point relation y such that $ky \sqsubseteq \beta$. But β is regular by 3.4(c) and so $y \sqsubseteq \beta$ by 3.4(e). Therefore $y \sqsubseteq \alpha \sqcap \beta = O$, which contradicts to $y \neq O$. This proves $\beta = O$. (b) It is clear that $\sqcup_{z \in X} z^\sharp z \sqsubseteq I$. Using $L = \sqcup_{z \in X} z$ by (a) we have $I = I \sqcap L = \sqcup_{z \in X}(I \sqcap z) \sqsubseteq \sqcup_{z \in X}(Iz^\sharp \sqcap I)z \sqsubseteq \sqcup_{z \in X} z^\sharp z$. □

¿From the point axiom A8 there exists a scalar k such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$. By 4.5(c) such scalar k is unique. For a relation α and point relations x, y a scalar $\chi(\alpha)(x, y)$ is defined to be the unique scalar k with $\alpha \sqcap x^\sharp y = k(x^\sharp y)$. Thus by 4.3(c) $\chi(\alpha)(x, y)$ is a unique scalar such that

$$x\alpha \sqcap y = \chi(\alpha)(x, y)y.$$

Thus $\chi(\alpha)$ defines a fuzzy relation on the set of all point relations in $\mathcal{F}$. In the next section we will prove that the function $\chi : \mathcal{F} \to \mathbf{Rel}(X)$ is an isomorphism of fuzzy relation algebras.

Proposition 4.7 *If $\alpha \sqcap x^\sharp y = k(x^\sharp y)$, then $r\alpha \sqcap k(x^\sharp y) = rk(x^\sharp y)$ for all scalars r.*

Proof. If $r = 0$, then the assertion is trivial. So we assume $r > 0$. As $r\alpha \sqcap k(x^\sharp y) \sqsubseteq k(x^\sharp y) \sqsubseteq x^\sharp y$, by 4.5(c) there is a scalar r' such that $r\alpha \sqcap k(x^\sharp y) = r'(x^\sharp y)$. ¿From $k(x^\sharp y) \sqsubseteq \alpha$ it follows that $rk(x^\sharp y) \sqsubseteq r\alpha \sqcap k(x^\sharp y) = r'(x^\sharp y)$ and so $rk \leq r'$ by 4.5(b). Also $r' \leq r$ follows from $r'(x^\sharp y) \sqsubseteq r\alpha$. Since $0 \leq r'/r \leq 1$ we have $r(r'/r)(x^\sharp y) = r'(x^\sharp y) \sqsubseteq r\alpha$ and $(r'/r)(x^\sharp y) \sqsubseteq \alpha$ by the axiom A6(h). Hence $(r'/r)(x^\sharp y) \sqsubseteq k(x^\sharp y)$ and $r'/r \leq k$. This proves $r' = rk$. □

5 Main Theorems

First we prove the representability theorem for fuzzy relation algebras asserting that every relation in a fuzzy relation algebra satisfying the point axiom A8, can be represented as a union of pairs of point relations with semi-scalar weights.

Theorem 5.1 *(Representation Theorem) Let X be the set of all point relations in $\mathcal{F}$. Then every relation α has a unique representation*

$$\alpha = \sqcup_{x,y \in X} \chi(\alpha)(x, y)(x^\sharp y).$$

Proof. Since $L = \sqcup_{y \in X} y$ by 4.6(a) we have

$$\begin{aligned} x\alpha &= x\alpha \sqcap L \\ &= x\alpha \sqcap (\sqcup_{y \in X} y) \\ &= \sqcup_{y \in X}(x\alpha \sqcap y) \\ &= \sqcup_{y \in X} \chi(\alpha)(x, y)y \end{aligned}$$

and so

$$\begin{aligned} \alpha &= I\alpha \\ &= \sqcup_{x \in X} x^\sharp x\alpha \\ &= \sqcup_{x \in X} x^\sharp[\sqcup_{y \in X} \chi(\alpha)(x, y)y] \\ &= \sqcup_{x \in X} \sqcup_{y \in X} x^\sharp[\chi(\alpha)(x, y)y] \\ &= \sqcup_{x,y \in X} \chi(\alpha)(x, y)(x^\sharp y) \end{aligned}$$

using $I = \sqcup_{x \in X} x^\sharp x$ by 4.6(b). Finally we show the uniqueness of the representation. Assume $\alpha = \sqcup_{x,y \in X} k_{x,y}(x^\sharp y)$. Then for all $x_0, y_0 \in X$ we have $x_0\alpha = \sqcup_{x,y \in X} k_{x,y}(x_0 x^\sharp y) = \sqcup_{y \in X} k_{x_0,y} y$ and $x_0\alpha \sqcap y_0 = \sqcup_{y \in X}[k_{x_0,y} y \sqcap y_0] = k_{x_0,y_0} y_0$. Hence $k_{x,y} = \chi(\alpha)(x, y)$ by 4.5(c). □

Corollary 5.2 *For every fuzzy relation algebra $\mathcal{F}$ the function $\chi : \mathcal{F} \to \mathbf{Rel}(X)$ is a bijection.*

Proof. If $\chi(\alpha) = \chi(\beta)$, then by the last theorem we have $\alpha = \sqcup_{x,y \in X} \chi(\alpha)(x,y)(x^\sharp y) = \sqcup_{x,y \in X} \chi(\beta)(x,y)(x^\sharp y) = \beta$, which shows that χ is injective. Given a fuzzy relation $R \in \mathbf{Rel}(X)$ we set $\alpha_R = \sqcup_{x,y \in X} R(x,y)(x^\sharp y)$. Then by the uniqueness of representation in the last theorem we have $R(x,y) = \chi(\alpha_R)(x,y)$, which means that χ is surjective. $\square$

Theorem 5.3 *For a relation α and a set $\{k_\lambda\}_\lambda$ of scalars the identity $\sqcup_\lambda k_\lambda \alpha = (\vee_\lambda k_\lambda)\alpha$ holds.*

Proof. First we show that $\sqcup_\lambda k_\lambda (x^\sharp y) = (\vee_\lambda k_\lambda)(x^\sharp y)$ for point relations x, y. It is obvious that $\sqcup_\lambda k_\lambda (x^\sharp y) \sqsubseteq (\vee_\lambda k_\lambda)(x^\sharp y)$ and so there is a scalar k such that $\sqcup_\lambda k_\lambda (x^\sharp y) = k(x^\sharp y)$ by the point axiom A8. Then $k \le \vee_\lambda k_\lambda$ by $k(x^\sharp y) \sqsubseteq (\vee_\lambda k_\lambda)(x^\sharp y)$ and 4.3(b). On the other hand $k_\lambda \le k$ from $k_\lambda(x^\sharp y) \sqsubseteq k(x^\sharp y)$ and so $\vee_\lambda k_\lambda \le k$. Therefore this proves $k = \vee_\lambda k_\lambda$. We are ready to prove the general case. Since $\alpha = \sqcup_{x,y \in X} \chi(\alpha)(x,y)(x^\sharp y)$ by the representability theorem 5.1 we have

$$\begin{aligned}
\sqcup_\lambda k_\lambda \alpha &= \sqcup_\lambda k_\lambda [\sqcup_{x,y \in X} \chi(\alpha)(x,y)(x^\sharp y)] \\
&= \sqcup_\lambda \sqcup_{x,y \in X} [k_\lambda \chi(\alpha)(x,y)(x^\sharp y)] \\
&= \sqcup_{x,y \in X} \sqcup_\lambda [k_\lambda \chi(\alpha)(x,y)(x^\sharp y)] \\
&= \sqcup_{x,y \in X} [(\vee_\lambda k_\lambda)\chi(\alpha)(x,y)(x^\sharp y)] \\
&= (\vee_\lambda k_\lambda)[\sqcup_{x,y \in X} \chi(\alpha)(x,y)(x^\sharp y)] \\
&= (\vee_\lambda k_\lambda)\alpha. \quad \square
\end{aligned}$$

The following theorem is known as Tarski rule for Boolean relation algebras [12, 14].

Theorem 5.4 *For a nonzero relation α in $\mathcal{F}$ there is a scalar $k > 0$ such that $L\alpha L = kL$.*

Proof. By means of the representation theorem 5.1 we have

$$\begin{aligned}
L\alpha L &= \sqcup_{x,y \in X} L[\chi(\alpha)(x,y)(x^\sharp y)]L \\
&= \sqcup_{x,y \in X} \chi(\alpha)(x,y)[L(x^\sharp y)L] \\
&= \sqcup_{x,y \in X} \chi(\alpha)(x,y)LL \\
&= \sqcup_{x,y \in X} \chi(\alpha)(x,y)L \\
&= [\vee_{x,y \in X} \chi(\alpha)(x,y)]L.
\end{aligned}$$

Set $k = \vee_{x,y \in X} \chi(\alpha)(x,y)$. Then $L\alpha L = kL$. By the way there exist point relations x, y such that $\chi(\alpha)(x,y) > 0$ by the point axiom A8. Hence $k > 0$, which completes the proof. $\square$

The following proposition shows that $\chi : \mathcal{F} \to \mathbf{Rel}(X)$ preserves all operations of fuzzy relations, that is, χ is a homomorphism of fuzzy relation algebras.

Proposition 5.5 *Let α, β be relations and k a scalar. Then the following holds:*

(a) $\chi(O) = O$, $\chi(L) = L$ *and* $\chi(I) = I$.

(b) *If* $\alpha \sqsubseteq \beta$, *then* $\chi(\alpha) \subseteq \chi(\beta)$.

(c) $\chi(\alpha \sqcup \beta) = \chi(\alpha) \cup \chi(\beta)$.

(d) $\chi(\alpha \sqcap \beta) = \chi(\alpha) \cap \chi(\beta)$.

(e) $\chi(\alpha^\sharp) = \chi(\alpha)^\sharp$.

(f) $\chi(k\alpha) = k\chi(\alpha)$.

(g) $\chi(\alpha\beta) = \chi(\alpha)\chi(\beta)$.

Proof. (a) The fist follows from $xO \sqcap y = O = 0y$, the second follows from $xL \sqcap y = L \sqcap y = y$ by $xL = L$, and the last follows from $xI \sqcap y = x \sqcap y$ and 4.5(e).
(b) $\chi(\alpha)(x,y)y = x\alpha \sqcap y \sqsubseteq x\beta \sqcap y = \chi(\beta)(x,y)y$ and so $\chi(\alpha)(x,y) \leq \chi(\beta)(x,y)$ by 4.5(b).
(c) It follows from

$$\begin{aligned}
\chi(\alpha \sqcup \beta)(x,y)y &= x(\alpha \sqcup \beta) \sqcap y \\
&= (x\alpha \sqcup x\beta) \sqcap y \\
&= (x\alpha \sqcap y) \sqcup (x\beta \sqcap y) \\
&= \chi(\alpha)(x,y)y \sqcup \chi(\beta)(x,y)y \\
&= [\chi(\alpha)(x,y) \vee \chi(\beta)(x,y)]y \\
&= [\chi(\alpha) \cup \chi(\beta)](x,y)y.
\end{aligned}$$

(d) First note that $x(\alpha \sqcap \beta) = x\alpha \sqcap x\beta$. It follows from $x^\sharp x \sqsubseteq I$ that $x(\alpha \sqcap \beta) \sqsubseteq x\alpha \sqcap x\beta \sqsubseteq x(\alpha \sqcap x^\sharp x\beta) \sqsubseteq x(\alpha \sqcap \beta)$. Hence we have

$$\begin{aligned}
\chi(\alpha \sqcap \beta)(x,y)y &= x(\alpha \sqcap \beta) \sqcap y \\
&= (x\alpha \sqcap x\beta) \sqcap y \\
&= (x\alpha \sqcap y) \sqcap (x\beta \sqcap y) \\
&= \chi(\alpha)(x,y)y \sqcap \chi(\beta)(x,y)y \\
&= [\chi(\alpha)(x,y) \wedge \chi(\beta)(x,y)]y \\
&= [\chi(\alpha) \cap \chi(\beta)](x,y)y.
\end{aligned}$$

(e) An identity $\chi(\alpha^\sharp) = \chi(\alpha)^\sharp$ follows from

$$\begin{aligned}
x\alpha \sqcap y = ky &\iff \alpha \sqcap x^\sharp y = k(x^\sharp y) \\
&\iff \alpha^\sharp \sqcap y^\sharp x = k(y^\sharp x) \\
&\iff y\alpha^\sharp \sqcap x = kx.
\end{aligned}$$

(f) It follows from

$$\begin{aligned}
\chi(k\alpha)(x,y)y &= x(k\alpha) \sqcap y \\
&= k(x\alpha) \sqcap kL \sqcap y \quad (\ k(x\alpha) \sqsubseteq kL\ ) \\
&= k(x\alpha) \sqcap ky \\
&= k(x\alpha \sqcap y) \\
&= [k\chi(\alpha)(x,y)]y \\
&= [k\chi(\alpha)(x,y)]y.
\end{aligned}$$

(g) First we have $\chi(\alpha\beta)(x,y) = \vee_z \chi(\alpha z^\sharp z\beta)(x,y)$ from

$$\begin{aligned}
\chi(\alpha\beta)(x,y)y &= x\alpha\beta \sqcap y \\
&= x\alpha I\beta \sqcap y \\
&= x\alpha(\sqcup_z z^\sharp z)\beta \sqcap y \\
&= \sqcup_z(x\alpha z^\sharp z\beta \sqcap y) \\
&= \sqcup_z \chi(\alpha z^\sharp z\beta)(x,y)y \\
&= [\vee_z \chi(\alpha z^\sharp z\beta)(x,y)]y. \quad (\text{ by } 5.3\ )
\end{aligned}$$

To complete the proof it suffices to show $\chi(\alpha z^\sharp z\beta)(x,y) = \chi(\alpha)(x,z) \wedge \chi(\beta)(z,y)$. Note that $x\alpha \sqcap z = x\alpha z^\sharp z$ (by $x\alpha z^\sharp z \sqsubseteq x\alpha \sqcap Lz \sqsubseteq (x\alpha z^\sharp \sqcap L)z = x\alpha z^\sharp z$) and so

$$\begin{aligned}
[\chi(\alpha)(x,z) \wedge \chi(\beta)(z,y)]z &= \chi(\alpha)(x,z)z \sqcap \chi(\beta)(z,y)z \quad (\text{ by A6(d) }) \\
&= \chi(\alpha)(x,z)z \sqcap \chi(\beta^\sharp)(y,z)z \\
&= x\alpha \sqcap z \sqcap y\beta^\sharp \\
&= x\alpha z^\sharp z \sqcap y\beta^\sharp.
\end{aligned}$$

Set $r = \chi(\alpha z^\sharp z\beta)(x,y)$ and $s = \chi(\alpha)(x,z) \wedge \chi(\beta)(z,y)$. Then $x\alpha z^\sharp z\beta \sqcap y = ry$ and $x\alpha z^\sharp z \sqcap y\beta^\sharp = sz$. We have $r \leq s$ from

$$ry = x\alpha z^\sharp z\beta \sqcap y \sqsubseteq (x\alpha z^\sharp z \sqcap y\beta^\sharp)\beta \sqsubseteq (sz)\beta \sqsubseteq s(zL)$$

and $s \leq r$ from

$$sz = x\alpha z^\sharp z \sqcap y\beta^\sharp \sqsubseteq (x\alpha z^\sharp z\beta \sqcap y)\beta^\sharp \sqsubseteq (ry)\beta^\sharp \sqsubseteq r(yL).$$

Therefore $r = s$. □

The following is immediately deduced from the last proposition.

Corollary 5.6 *(Insertion Theorem) Let α and β be relations and let x and y be point relations. If $\chi(\alpha\beta)(x,y) > 0$, then there is a point relation z such that $\chi(\alpha)(x,z) > 0$ and $\chi(\beta)(z,y) > 0$.* □

Corollary 5.7 *(Isomorphism Theorem) Every fuzzy relation algebra $\mathcal{F}$ satisfying the point axiom is isomorphic to the fuzzy relation algebra* **Rel**(X) *on the set X of all point relations of $\mathcal{F}$.* □

The reader can easily understand that all results of the paper remain valid, even if the unit interval $[0,1]$ is replaced by a subset $\mathcal{L}$ of $[0,1]$ such that (0) $0, 1 \in \mathcal{L}$, (i) If $\{k_\lambda\}_\lambda \subseteq \mathcal{L}$, then $\vee_\lambda k_\lambda \in \mathcal{L}$ and $\wedge_\lambda k_\lambda \in \mathcal{L}$, (ii) If $k, k' \in \mathcal{L}$, then $kk' \in \mathcal{L}$, (iii) If $k, k' \in \mathcal{L}$ and $k < k'$, then $k = k'k''$ for some $k'' \in \mathcal{L}$. For example, $\mathcal{L} = \{0\} \cup \{k^n \mid n = 0, 1, 2, \cdots\}$ $(0 < k \leq 1)$.

References

[1] P. Freyd and A. Scedrov, Categories, allegories (North-Holland, Amsterdam, 1990).

[2] J.A. Goguen, L-fuzzy set, J. Math. Anal. Appl. **18** (1967) 145–174.

[3] Y. Kawahara, Pushout-complements and basic concepts of grammars in topoi, Theoretical Computer Science **77** (1990), 267–289.

[4] Y. Kawahara, Relational set theory, RIFIS Technical Report TR-CS-97, Kyushu University (1995).

[5] Y. Kawahara and Y. Mizoguchi, Categorical assertion semantics in topoi, Advances in Software Science and Technology **4**(1992), 137 – 150.

[6] Y. Kawahara and Y. Mizoguchi, Relational structures and their partial morphisms in the view of single pushout rewriting, Lecture Notes in Computer Science **776** (1994), 218 – 233.

[7] S. Mac Lane, An algebra of additive relations, Proc. Nat. Acad. Sci. U.S.A. **47**(1961), 1043–1051.

[8] R.D. Maddux, The origin of relation algebras in the development and axiomatization of the calculus of relations, Studia Logica, **50** (1991), 423–455.

[9] W. Pedrycz, Processing in relational structures : Fuzzy relational equations, Fuzzy Sets and Systems **40** (1991) 77–106.

[10] D. Puppe, Korrespondenzen in Abelschen Kategorien, Math. Ann. **148** (1962), 1-30.

[11] E. Sanchez, Resolution of composite fuzzy relation equations, Information and Control **30** (1976) 38–48.

[12] G. Schmidt and T. Ströhlein, Relation algebras : Concept of points and representability, Discrete Mathematics **54** (1985) 83–92.

[13] G. Schmidt and T. Ströhlein, Relations and graphs – Discrete Mathematics for Computer Science – (Springer-Verlag, Berlin, 1993).

[14] A. Tarski, On the calculus of relations, J. Symbolic Logic **6** (1941) 73–89.