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Abstract

This article presents a relational formalization of axiomatic set theory, including so-called ZFC and the anti-foundation axiom (AFA) due to P. Aczel. The relational framework of set theory provides a general methodology for the fundamental study on computer and information sciences such as theory of graph transformation, situation semantics and analysis of knowledge dynamics in distributed systems. To demonstrate the feasibility of relational set theory some fundamental theorems of set theory, for example, Cantor-Bernstein-Schröder theorem, Cantor's theorem, Rieger's theorem and Mostowski's collapsing lemma are proved.

1 Introduction

The study on (binary) relations on sets has been begun together with the pioneering works of set theory and since then theory of relations has been extensively investigated by many mathematicians from the view points of logic, algebra, topology and computer science. For more detailed history of studies on relations the reader refer to R.D. Muddux [14] and G. Schmidt and T. Ströhlein [16]. From a view of category theory S. Mac Lane [11, 12] initiated theory of additive relations and D. Puppe [15] established a notion of I-categories that was a start point of categorical theory of relations. Peter Freyd [3] investigated theory of allegories as a basis for theory of relations and Max Kelly [10] studied relations relative to factorization systems. Topos theory [4, 5] is well-known as a categorical model of higher-order intuitionistic set theory and has been extensively studied by categorists and logicians.

This paper presents relational set theory as categorical set theory slightly different from topoi or allegories to give another categorical perspective of axiomatic set theory. Relational set theory mainly consists of formalizing the traditional axioms of set theory in terms of relations. Thus the relationship between the traditional set theory and relational set theory is rather clear and hence the author expects that one, who is not an expert of set theory and logic, can easily understand relational set theory and apply it to various fields of mathematics and computer science.

The recent developement of computer science requires more fundamental studies on information analysis from mathematics and logic. For example, J. Barwise and P. Aczel urge to construct a new set theory, so-called non-well-founded set theory (or hyperset theory), as a basic language to analyze complicated linguistic phenomena such as circularity in semantics of natural languages and knowledge dynamics in distributed systems. This naturally requires more philosophical arguments and sensitive treatments of set theory for computer scientists.

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After all relational set theory would be helpfull for a simple intorduction of axiomatic set theory into computer science.

This article presents a relational formalization of axiomatic set theory, including so-called ZFC and the anti-foundation axiom (AFA) due to P. Aczel. The relational framework of set theory provides a general methodology for the fundamental study on computer and information sciences such as theory of graph transformation, situation semantics and analysis of knowledge dynamics in distributed systems. To demonstrate the feasibility of relational set theory some fundamental theorems of set theory, for example, Cantor-Bernstein-Schröder theorem, Cantor's theorem, Rieger's theorem and Mostowski's collasping lemma are proved.

2 Theory of Meroi

In the first section we introduce a notion of meroi, that is, relational categories on which a relational model of set theory will be discused. A morphism in a meros will be called a relation and a relation α from A into B will be denoted by $\alpha : A \rightharpoondown B$. The composite of a relation $\alpha : A \rightharpoondown B$ followed by a relation $\beta : B \rightharpoondown C$ is denoted as $\alpha\beta : A \rightharpoondown C$ and the identity relation of A as $\mathrm{id}_A : A \rightharpoondown A$.

Definition 2.1 *A category $\mathcal{C}$ is an I-category if it satisfies the following:*
[Lattice] *Let A and B be objects of $\mathcal{C}$. The collection $\mathcal{C}(A, B)$ of all relations of A into B is a lattice by an ordering $\sqsubseteq$. The least relation $0_{A,B}$ and the greatest relation $\Theta_{A,B}$ from A into B exist. The infimum (or greatest lower bound) and the supremum (or least upper bound) of two relations $\alpha, \alpha' : A \rightharpoondown B$ are denoted by $\alpha \sqcap \alpha'$ and $\alpha \sqcup \alpha'$, respectively.*
[Involution] *There is an involution operator $\sharp$ assigning to each relation $\alpha : A \rightharpoondown B$ its invesre relation $\alpha^\sharp : B \rightharpoondown A$ is defined so that for relations $\alpha, \alpha' : A \rightharpoondown B$, $\beta, \beta' : B \rightharpoondown C$ and $\gamma : C \rightharpoondown D$*

(a) $\alpha^{\sharp\sharp} = \alpha$, $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$ *(involutive),*

(b) *If $\alpha \sqsubseteq \alpha'$ and $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha'\beta'$ and $\alpha^\sharp \sqsubseteq \alpha'^\sharp$ (monotone).* □

In an I-category $\mathcal{C}$ a (total) function $f : A \to B$ is a relation $f : A \rightharpoondown B$ such that $f^\sharp f \sqsubseteq \mathrm{id}_B$ (univalent) and $\mathrm{id}_A \sqsubseteq ff^\sharp$ (total). Also a partial function $f : A \to B$ is a relation $f : A \rightharpoondown B$ satisfying $f^\sharp f \sqsubseteq \mathrm{id}_B$. A function $f : A \to B$ is called an injection if $ff^\sharp = \mathrm{id}_A$, a surjection if $f^\sharp f = \mathrm{id}_B$, and a bijection if $ff^\sharp = \mathrm{id}_A$ and $f^\sharp f = \mathrm{id}_B$.

Definition 2.2 *A meros [$\mu\varepsilon\rho o\varsigma$](relational category) $\mathcal{C}$ is an I-category satisfying the following:*
[Complete Heyting Algebra] *For all objects A and B of $\mathcal{C}$ the collection $\mathcal{C}(A, B)$ of all relations of A into B is a complete Heyting algebra.*
[Rationality] *For each relation $\alpha : A \rightharpoondown B$ there exists a pair of functions $f : X \to A$ and $g : X \to B$ such that $\alpha = f^\sharp g$ and $ff^\sharp \sqcap gg^\sharp = \mathrm{id}_X$.*
[Dedekind Formula] *If $\alpha : A \rightharpoondown B, \beta : B \rightharpoondown C$ and $\gamma : A \rightharpoondown C$ are relations, then $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$.*
[Terminability] *There is an object 1 such that $0_{1,1} \neq \mathrm{id}_1 = \Theta_{1,1}$ and $\Theta_{A,1}\Theta_{1,B} = \Theta_{A,B}$ for objects A and B.*
[Quotient Relation] *For relations $\beta : A \to C$ and $\gamma : B \to C$ there is a quotient relation $\beta \div \gamma : A \rightharpoondown B$ such that $\alpha\gamma \sqsubseteq \beta \Leftrightarrow \alpha \sqsubseteq \beta \div \gamma$ for any relation $\alpha : A \to B$.* □

The subsequent argements will be done in a (fixed) meros $\mathcal{C}$.

The following is the basic propeorties of meros deduced from the existence of quotient relations.

[Zero Relations] $0\alpha = 0$ and $\alpha 0 = 0$ for all relation $\alpha : A \rightharpoondown B$.

Proof. $0 \sqsubseteq 0 \div \alpha \Leftrightarrow 0\alpha \sqsubseteq 0 \Leftrightarrow 0\alpha = 0$. $\Box$

[Distributive Law] For relations $\alpha : A \rightharpoondown B$, $\beta_\lambda : B \rightharpoondown C$ $(\lambda \in \Lambda)$ and $\gamma : C \rightharpoondown D$ the distributive law $\alpha(\bigsqcup_{\lambda\in\Lambda} \beta_\lambda)\gamma = \bigsqcup_{\lambda\in\Lambda} \alpha\beta_\lambda\gamma$ holds.

Proof. $(\sqcup_\lambda \delta_\lambda)\alpha \sqsubseteq \beta \Leftrightarrow \sqcup_\lambda \delta_\lambda \sqsubseteq \beta \div \alpha \Leftrightarrow \forall\lambda : \delta_\lambda \sqsubseteq \beta \div \alpha \Leftrightarrow \forall\lambda : \delta_\lambda\alpha \sqsubseteq \beta \Leftrightarrow \sqcup_\lambda \delta_\lambda\alpha \sqsubseteq \beta$ (Thus the existence of quotient relations is equivalent to the distributive law.) $\Box$

Let A be an object of a meros $\mathcal{C}$. An element a of A is a function $a : 1 \to A$ and will be denoted by $a \in A$. We will write $\Theta_{1,A}$ for ∇_A. Note that $!_A = \nabla_A^\sharp$ is a unique function of A into 1, and $\Theta_{A,B} = \nabla_A^\sharp \nabla_B$.

Proposition 2.3 *Let A be an object of a meros $\mathcal{C}$ and $G(A)$ the collection of all relations $u : A \rightharpoondown A$ with $u \sqsubseteq \mathrm{id}_A$. Then the function which assignins $\nabla_A u$ to each $u \in G(A)$ is an isomorphism of $G(A)$ onto $\mathcal{C}(1, A)$ as complete Heyting algebras.*

$$G(A) \cong \mathcal{C}(1, A)$$

Proof. Omitted. $\Box$

Proposition 2.4 *Let $\mathcal{C}$ be a meros and A an object of $\mathcal{C}$. Then for an element $x \in A$ and a relation $\rho : 1 \rightharpoondown A$*

(a) *$x \sqcap \rho = 0$ if and only if $\rho x^\sharp = 0$,*

(b) *$x \sqsubseteq \rho$ if and only if $\rho x^\sharp = \mathrm{id}_1$.*

Proof. (a) If $x \sqcap \rho = 0$, then $\rho x^\sharp = 0$ from $\rho x^\sharp = \rho x^\sharp \sqcap \mathrm{id}_1 \sqsubseteq (\rho \sqcap x)x^\sharp$. If $\rho x^\sharp = 0$, then $x \sqcap \rho = 0$ from $x \sqcap \rho \sqsubseteq (\rho x^\sharp \sqcap \mathrm{id}_1)x$. (b) If $x \sqsubseteq \rho$, then $\mathrm{id}_1 \sqsubseteq xx^\sharp \sqsubseteq \rho x^\sharp$. If $\rho x^\sharp = \mathrm{id}_1$, then $x = \mathrm{id}_1 x = \rho x^\sharp x \sqsubseteq \rho$. $\Box$

Theorem 2.5 *Let A be an object of a meros $\mathcal{C}$. Then the following statements are equivalent:*

(a) $\sqcup_{x\in A} x = \nabla_A$.

(b) $\sqcup_{x\in A} x^\sharp x = \mathrm{id}_A$.

(c) $[\forall x \in A(x\alpha \sqsubseteq x\beta) \Rightarrow \alpha \sqsubseteq \beta]$ *for all relations $\alpha, \beta : A \rightharpoondown B$.*

Proof. (a)$\Rightarrow$(b) It is clear that $\nabla_A(\sqcup_{x\in A} x^\sharp x) = \sqcup_{x\in A} \nabla_A x^\sharp x = \sqcup_{x\in A} x$ since $\nabla_A x^\sharp = \nabla_1 = \mathrm{id}_1$. Hence we have $\nabla_A(\sqcup_{x\in A} x^\sharp x) = \nabla_A$ by (a) and so $\sqcup_{x\in A} x^\sharp x = \mathrm{id}_A$. (b)$\Rightarrow$(c) If $\forall x \in A(x\alpha \sqsubseteq x\beta)$, then $\alpha = \mathrm{id}_A\alpha = (\sqcup_{x\in A} x^\sharp x)\alpha$ (by (b)) $= \sqcup_{x\in A} x^\sharp x\alpha \sqsubseteq \sqcup_{x\in A} x^\sharp x\beta = \beta$. (c)$\Rightarrow$(a) Note that $\nabla_A y^\sharp = \mathrm{id}_1 = yy^\sharp \sqsubseteq (\sqcup_{x\in A} x)y^\sharp$ for all $y \in A$. Hence $\nabla_A \sqsubseteq \sqcup_{x\in A} x$ by (c). $\Box$

Theorem 2.6 *Let $\mathcal{C}$ be a meros. Consider the following statements:*

(a) *A nonzero relation $\rho : 1 \rightharpoonup A$ is nonempty for all objects A,*

(b) $\mathcal{C}(1,1) = \{0, \mathrm{id}_1\}$.

(c) $\sqcup_{x \in A} x = \nabla_A$ *for all objects A,*

(d) $\mathcal{C}(1, A)$ *is a Boolean algebra for all objects A,*

Then (a)$\Rightarrow$(b) and (a)+(c)$\Leftrightarrow$(a)+(d) $\Leftrightarrow$(b)+(c) hold.

Proof. (a)$\Rightarrow$(b) Let $\xi : 1 \rightharpoonup 1$ be a nonzero relation ($\xi \neq 0$). Then ξ is nonempty by (a) and so $x \sqsubseteq \xi$ for some function $x : 1 \to 1$. But $x = \mathrm{id}_1$. Hence $\xi = \mathrm{id}_1$, which shows (b). (a)+(c)$\Rightarrow$(d) Let $\rho : 1 \rightharpoonup A$ be a relation. Then by (c) we have

$$\rho = \rho \sqcap \nabla_A = \rho \sqcap (\sqcup_{x \in A} x) = \sqcup_{x \in A} (\rho \sqcap x).$$

If $\rho \sqcap x$ is nonzero, then (a) claims that there exists $y \in A$ such that $y \sqsubseteq \rho \sqcap x (\sqsubseteq x)$ and so $\rho \sqcap x = x$. Hence $\rho = \sqcup\{x \in A | \rho \sqcap x \neq 0\}$. Set $\bar{\rho} = \sqcup\{x \in A | \rho \sqcap x = 0\}$. Then $\rho \sqcap \bar{\rho} = 0$ and $\rho \sqcup \bar{\rho} = \nabla_A$, which proves that $\mathcal{C}(1, A)$ is a Boolean algebra. (a)+(d)$\Rightarrow$(c) Set $\rho = \sqcup_{x \in A} x$. Then by (d) there is a relation $\bar{\rho} : 1 \rightharpoonup A$ such that $\rho \sqcap \bar{\rho} = 0$ and $\rho \sqcup \bar{\rho} = \nabla_A$. If $\bar{\rho} \neq 0$, then by (a) $x \sqsubseteq \bar{\rho}$ for some $x \in A$ and $x \sqsubseteq \rho \sqcap \bar{\rho} = 0$, which is a contradiction. Hence $\bar{\rho} = 0$ and $\rho = \nabla_A$. (b)+(c)$\Rightarrow$(a) Let $\rho : 1 \rightharpoonup A$ be a nonzero relation. Then by (c) we have $\rho = \sqcup_{x \in A} (\rho \sqcap x)$. Hence $\rho \sqcap x \neq 0$ for some $x \in A$. But $\rho \sqcap x \neq 0 \Leftrightarrow \rho x^\sharp \neq 0$ (by 2.4(a)) $\Leftrightarrow \rho x^\sharp = \mathrm{id}_1$ (by (b))$\Leftrightarrow x \sqsubseteq \rho$ (by 2.4(b)). This shows that ρ is nonempty. $\square$

Remark that the above (c)+(d) implies neither (a) nor (b). (Consider $\mathcal{C} = \mathbf{Rel} \times \mathbf{Rel}$.)

Proposition 2.7 *Suppose that a nonzero relation $\rho : 1 \rightharpoonup A$ is nonempty for an object A of a meros $\mathcal{C}$. Then $x \sqsubseteq \alpha \sqcup \beta$ if and only if $x \sqsubseteq \alpha$ or $x \sqsubseteq \beta$ for an element $x \in A$ and relations $\alpha, \beta : 1 \rightharpoonup A$.*

Proof.Assume that $x \sqsubseteq \alpha \sqcup \beta$. Then $x = x \sqcap (\alpha \sqcup \beta) = (x \sqcap \alpha) \sqcup (x \sqcap \beta)$ and so $x \sqcap \alpha \neq 0$ or $x \sqcap \beta \neq 0$. Assume that $x \sqcap \alpha \neq 0$. Then from the hypothesis there is $y \in A$ such that $y \sqsubseteq x \sqcap \alpha (\sqsubseteq x)$. Hence $x \sqcap \alpha = x (= y)$ and so $x \sqsubseteq \alpha$. $\square$

For a relation $\rho : 1 \rightharpoonup A$ of a meros $\mathcal{C}$ define $\rho^\circ = \sqcup\{x \in A | x \sqsubseteq \rho\}$.

Proposition 2.8 (a) $\rho^\circ \sqsubseteq \rho$,

(b) $x \sqsubseteq \rho \Leftrightarrow x \sqsubseteq \rho^\circ$,

(c) $\rho^{\circ\circ} = \rho^\circ$,

(d) *If ρ is empty, then $\rho^\circ = 0$, and if ρ is nonempty, then ρ° is nonempty and so $\rho^\circ \neq 0$,*

(e) *If $x \sqsubseteq \rho_1$ implies $x \sqsubseteq \rho_2$ for all $x \in A$, then ${\rho_1}^\circ \sqsubseteq {\rho_2}^\circ$,*

(f) *If $\rho_1 \sqsubseteq \rho_2$, then ${\rho_1}^\circ \sqsubseteq {\rho_2}^\circ$.*

(g) *If $f : A \to B$ is an injection, then $\rho^\circ f = (\rho f)^\circ$.*

Proof. (a) It is trivial. (b) If $x \sqsubseteq \rho^\circ$, then $x \sqsubseteq \rho$ by (a). If $x \sqsubseteq \rho$, then $x \sqsubseteq \rho^\circ$ by the definition of ρ°. (c) It easily follows from (b) that $\{x \in A | x \sqsubseteq \rho\} = \{x \in A | x \sqsubseteq \rho^\circ\}$ and so $\rho^{\circ\circ} = \rho^\circ$. (d) It is immediate from the definition of ρ°. (e) Since ρ_2 is a upper bound of $\{x \in A | x \sqsubseteq \rho_1\}$, ${\rho_1}^\circ \sqsubseteq \rho$ and so ${\rho_1}^\circ \sqsubseteq {\rho_2}^\circ$ by (c). (f) follows from (e). $\square$

Definition 2.9 (a) *A meros $\mathcal{C}$ is separable if $\neg\rho^\circ = (\neg\rho)^\circ$ holds for all relations $\rho : 1 \rightharpoonup A$ in $\mathcal{C}$.*

(b) *A meros $\mathcal{C}$ is Boolean if $\mathcal{C}(1, A)$ is a Boolean algebra for all objects A of $\mathcal{C}$.*

Proposition 2.10 *Let $\mathcal{C}$ be a separable Boolean meros. Then*

(a) $\neg\rho^\circ = (\neg\rho^\circ)^\circ$ *for all* $\rho : 1 \rightharpoonup A$*,*

(b) $(\neg\rho^\circ)^\circ = (\neg\rho)^\circ$ *for all* $\rho : 1 \rightharpoonup A$*,*

(c) $x \sqcap \rho^\circ = 0 \iff x \sqcap \rho = 0$ *for all* $x \in A$*,*

(d) $\sqcup_{x\in A} x = \nabla_A$ *for all objects* A*,*

(e) $\rho = \nabla_A$ *iff* $\neg\rho$ *is empty for* $\rho : 1 \rightharpoonup A$*.*

Proof. (a) $\neg\rho^\circ = \neg\rho^{\circ\circ} = (\neg\rho^\circ)^\circ$. (b) $(\neg\rho^\circ)^\circ = (\neg\rho)^{\circ\circ} = (\neg\rho)^\circ$. (c) If $x \sqcap \rho^\circ = 0$, then $x \sqsubseteq \neg\rho^\circ = (\neg\rho)^\circ \sqsubseteq \neg\rho$ and so $x \sqcap \rho = 0$. (d) Set $\rho = \nabla_A$. Then $\rho^\circ = \sqcup_{x\in A} x$ and $\neg\rho^\circ = (\neg\rho)^\circ = 0^\circ = 0$. Hence $\rho^\circ = \nabla_A$. (e) If $\rho = \nabla_A$, then $\neg\rho = 0$ and so $\neg\rho$ is empty. Conversely, if $\neg\rho$ is empty, then $\neg\rho^\circ = (\neg\rho)^\circ = 0$ and so $\rho^\circ = \neg\neg\rho^\circ = \neg 0 = \nabla_A$. $\square$

Example 2.11 *In $\mathcal{C} = \mathbf{Rel} \times \mathbf{Rel}$ consider $\rho = (a, b) \sqcup (a', 0)$, $a = (a, b)$ and $y = (a, b')$. Then $\rho^\circ = x$, $x^\circ = x$, $y^\circ = y$, $x^\circ \sqcap y^\circ = x \sqcap y = (a, 0)$, $(x \sqcap y)^\circ = 0$, $(a, 0)^\circ \sqcup (0, b)^\circ = 0$ and $[(a, 0) \sqcup (0, b)]^\circ = (a, b)$.*

As stated in [Complete Heyting Algebra] for relational categories the collection $\mathcal{C}(A, B)$ of A into B is a complete Heyting algebra. For relations $\alpha, \beta : A \rightharpoonup B$ the psuedo-complement of α relative to β will be denoted by $\alpha \Rightarrow \beta$ and the negation of α by $\alpha \Rightarrow 0$, respectively. The negation of $u \in G(A)$ will be represented as $\neg u$. (Note that $\neg u = (u \Rightarrow 0) \sqcap \mathrm{id}_A$.)

In the rest of the section we show Cantor-Bernstein-Schröder theorem in the framework of Boolean meroi.

Theorem 2.12 *Suppose that $\mathcal{C}$ is a Boolean meros. If $f : A \to B$ and $g : B \to A$ are injections, then there is a bijection $h : A \to B$.*

Proof. Since $\mathcal{C}(1, B)$ is a Boolean algebra, there is a relation $\xi : 1 \rightharpoonup B$ (the complement of $\nabla_A f$) such that $\xi \sqcup \nabla_A f = \nabla_B$ and $\xi \sqcap \nabla_A f = 0$. Set $\rho = \xi g\{\sqcup_{n=0}^{\infty} (fg)^n\}$. Then $\rho : 1 \rightharpoonup A$ satisfies $\rho = \xi g \sqcup \rho f g$. There is a unique relation $u : A \rightharpoonup A$ such that $u \sqsubseteq \mathrm{id}_A$ and $\nabla_A u = \rho$. Note that $\nabla_A u g^\sharp = \xi \sqcup \nabla_A u f$ (by $g g^\sharp = \mathrm{id}_B$) and $u g^\sharp g = u$ (since $u g^\sharp g \sqsubseteq \mathrm{id}_A$ and $\nabla_A u g^\sharp g = \nabla_A u$). Also we have $\xi f^\sharp = 0$ from

$$\xi f^\sharp = \xi f^\sharp \sqcap \nabla_A \sqsubseteq (\xi \sqcap \nabla_A f) f^\sharp = 0.$$

As $\mathcal{C}(1, A)$ is a Boolean algebra, there is a relation $\neg u : A \rightharpoonup A$ such that $\neg u \sqcup u = \mathrm{id}_A$ and $\neg u \sqcap u = 0$. An identity $u g^\sharp f^\sharp (\neg u) = 0$ follows from

$$\nabla_A u g^\sharp f^\sharp (\neg u) = \xi f^\sharp (\neg u) \sqcup \nabla_A u f f^\sharp (\neg u) = 0(\neg u) \sqcup \nabla_A u (\neg u) = 0 \quad (f f^\sharp = \mathrm{id}_A).$$

Set $h = (\neg u) f \sqcup u g^\sharp : A \rightharpoonup B$. Then the following shows that h is a bijection of A onto B.

(a) $h^\sharp h = [f^\sharp(\neg u) \sqcup gu][(\neg u)f \sqcup ug^\sharp] = f^\sharp(\neg u)f \sqcup gug^\sharp \sqsubseteq f^\sharp f \sqcup gg^\sharp = \mathrm{id}_B$,

(b) $hh^\sharp = [(\neg u)f \sqcup ug^\sharp][f^\sharp(\neg u) \sqcup gu] = (\neg u)ff^\sharp(\neg u) \sqcup (\neg u)fgu \sqcup ug^\sharp f^\sharp(\neg u) \sqcup ug^\sharp gu = (\neg u) \sqcup u = \mathrm{id}_A$,

(c) $\nabla_A h = \nabla_A(\neg u)f \sqcup \nabla_A ug^\sharp = \nabla_A(\neg u)f \sqcup \xi \sqcup \nabla_A uf = \nabla_A[(\neg u) \sqcup u]f \sqcup \xi = \nabla_A f \sqcup \xi = \nabla_B$. $\Box$

3 Axioms of Relational Set Theory

In this section we state axioms of relational set theory within separable Boolean meroi. The axioms are formalized for a relation $\Pi : V \rightharpoonup V$ which corresponds to the (universal) membership predicate $\ni$ in the traditional axiomatic set theory. Let $\mathcal{C}$ be a separable Boolean meros and $\Pi : V \rightharpoonup V$ a relation in $\mathcal{C}$. The axioms of relational set theory are as follows:

A.1 [axiom of extensionality] $\forall a, b \in V(a\Pi = b\Pi \Rightarrow a = b)$
A.2 [axiom of empty(null) set] $\exists x(=\emptyset) \in V(0 = x\Pi)$
A.3 [axiom of pairing] $\forall x, y \in V \exists z \in V(x \sqcup y = z\Pi)$
A.4 [axiom of union] $\forall x \in V \exists y \in V(x\Pi\Pi = y\Pi)$

When $\Pi : V \rightharpoonup V$ satisfies the axioms of union and pairing, for each element $x \in V$ there is an element $x \cup \{x\} \in V$ such that $(x \cup \{x\})\Pi = x\Pi \sqcup x$. That is, by the axiom of pairing there is $\{x\} \in V$ such that $\{x\}\Pi = x$, again by the same axiom there is $\{x, \{x\}\} \in V$ such that $\{x, \{x\}\}\Pi = x \sqcup \{x\}$, and finally by the axiom of union there is $x \cup \{x\} \in V$ such that $(x \cup \{x\})\Pi = \{x, \{x\}\}\Pi\Pi$. Then we have

$$(x \cup \{x\})\Pi = \{x, \{x\}\}\Pi\Pi = (x \sqcup \{x\})\Pi = x\Pi \sqcup x.$$

The next proposition shows that if a relation $\Pi : V \rightharpoonup V$ satisfies axioms (A.1) – (A.4), then V has infintary many elements.

Proposition 3.1 *Suppose that $\Pi : V \rightharpoonup V$ satisfies axioms (A.1) – (A.4) and define elements $x_n \in V$ $(n \geq 0)$ by $x_0\Pi = 0$ and $x_{n+1}\Pi = x_n\Pi \sqcup x_n$. Then*

(a) *$x_n\Pi^{n+1} = 0$ for $n \geq 0$.*

(b) *$x_n\Pi^n = x_0$ for $n \geq 0$.*

(c) *If $m \neq n$, then $x_m \neq x_n$.*

(d) *$x_{n+1}\Pi = \sqcup_{k=0}^n x_k$ for $n \geq 0$.*

(e) *$x_{n+m}\Pi^m = \sqcup_{k=0}^n x_k$ for $n \geq 0$ and $m > 0$.*

Proof. (a) For $n = 0$ we have $x_0\Pi = 0$ by the definition. Assume that $x_n\Pi^{n+1} = 0$. Then $x_{n+1}\Pi^{n+2} = (x_n\Pi \sqcup x_n)\Pi^{n+1} = x_n\Pi^{n+2} \sqcup x_n\Pi^{n+1} = 0 \sqcup 0 = 0$. (b) For $n = 0$ it is trivial that $x_0\Pi^0 = x_0$. Assume that $x_n\Pi^n = x_0$. Then $x_{n+1}\Pi^{n+1} = (x_n\Pi \sqcup x_n)\Pi^n = 0 \sqcup x_0 = x_0$. (c) Assume that $0 \leq m < n$. Then $x_m\Pi^n = x_m\Pi^{m+1}\Pi^{n-m-1} = 0$ by (a) and $x_n\Pi^n = x_0$ by (b). Hence $x_m\Pi^n \neq x_n\Pi^n$. (d) For $n = 0$ we have $x_1\Pi = x_0\Pi \sqcup x_0 = x_0$. Assume that $x_{n+1}\Pi = \sqcup_{k=0}^n x_k$ for $n \geq 0$. Then $x_{n+2}\Pi = x_{n+1}\Pi \sqcup x_{n+1} = \sqcup_{k=0}^{n+1} x_k$. (e) $x_{n+m+1}\Pi^{m+1} = x_{n+1+m}\Pi^m\Pi = (\sqcup_{k=0}^{n+1} x_k)\Pi = \sqcup_{k=0}^n x_k$. $\Box$

A.5 [axiom of infinity] $\exists a \in V(\emptyset \sqsubseteq a\Pi \wedge \forall x(x \sqsubseteq a\Pi \Rightarrow x \cup \{x\} \sqsubseteq a\Pi))$
A.6 [axiom of power set] $\forall x \in V \exists y \in V(x\Pi \div \Pi = y\Pi)$
A.7 [axiom of replacement] $\forall x \in V \forall$ pfn $f : V \to V \exists y \in V(x\Pi f = y\Pi)$
A.7' [axiom of subset (comprehension)] $\forall a \in V \forall \rho : 1 \rightharpoonup V[\rho \sqsubseteq a\Pi \Rightarrow \exists b \in V(\rho = b\Pi)]$

As is well-known, the axiom (A.7) of replacement includes the axiom (A.7') of subset. (Assume that $\rho \sqsubseteq a\Pi$ and choose a relation $u : V \rightharpoonup V$ such that $\rho = \nabla_V u$ and $u \sqsubseteq \mathrm{id}_V$. Then $\rho = a\Pi u$ since $\rho = \rho u \sqsubseteq a\Pi u \sqsubseteq \nabla_V u = \rho$.)

A.8 [axiom of foundation] $\forall a \in V[\exists x \in V(x \sqsubseteq a\Pi) \Rightarrow \exists x \in V(x \sqsubseteq a\Pi \wedge a\Pi \sqcap x\Pi = 0)]$
A.9 [axiom of choice]
$\forall a \in V \forall \alpha : V \rightharpoonup V[\nabla_V \alpha = a\Pi \Rightarrow \exists$ pfn $h : V \to V(h \sqsubseteq \alpha \wedge \nabla_V h^\sharp = \nabla_V \alpha^\sharp)]$

A Π-system $< j : M \to V, \mu : M \rightharpoonup M >$ is a pair of an injection $j : M \to V$ and a relation $\mu : M \rightharpoonup M$ such that for every relation $\rho : 1 \rightharpoonup M$ there is an element $x \in V$ with $\rho\mu j = x\Pi$.

A.10 [anti-foundation axiom (AFA)] For each Π-system $< j : M \to V, \mu : M \rightharpoonup M >$ there is a unique function (decoration) $d : M \to V$ such that $d\Pi = \mu d$.

$$\begin{array}{ccc} M & \xrightarrow{\mu} & M \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ V & \xrightarrow[\Pi]{} & V \end{array}$$

Theorem 3.2 *Suppose that a relation $\Pi : V \rightharpoonup V$ satisfies the axiom (A.7') of subset. Let $i : A \to V$ and $j : P(A) \to V$ be injections such that $\nabla_A i = a\Pi$ and $\nabla_{P(A)} j = a\Pi \div \Pi$ for $a \in V$.*

(a) *For every relation $\rho : 1 \rightharpoonup A$ there is an element $r \in P(A)$ such that $\rho = r\Pi_A$, where $\Pi_A = j\Pi i^\sharp : P(A) \rightharpoonup A$.*

(b) *There is no surjection $f : A \to P(A)$.*

Proof. (a) As $\rho i \sqsubseteq \nabla_A i = a\Pi$ by the axiom of subset there is $b \in V$ such that $\rho i = b\Pi$ and so $b \sqsubseteq a\Pi \div \Pi = \nabla_{P(A)} j$. Hence $r = bj^\sharp \in P(A)$ and $\rho = \rho i i^\sharp = b\Pi i^\sharp = bj^\sharp j \Pi i^\sharp = r\Pi_A$. (b) Let $f : A \to P(A)$ be a function and set $u = f\Pi_A \sqcap \mathrm{id}_A : A \rightharpoonup A$. The computation

$$\begin{array}{rcl} f^\sharp u & = & f^\sharp(f\Pi_A \sqcap \mathrm{id}_A) \\ & \sqsubseteq & f^\sharp f \Pi_A \sqcap f^\sharp \\ & \sqsubseteq & \Pi_A \sqcap f^\sharp \\ & \sqsubseteq & f^\sharp(f\Pi_A \sqcap \mathrm{id}_A) \\ & = & f^\sharp u \end{array}$$

indicates that $f^\sharp u = \Pi_A \sqcap f^\sharp$. By (a) there is a function $r : 1 \to P(A)$ such that $r\Pi_A = \nabla_A(\neg u)$. Then we have

$$\begin{array}{rcl} rf^\sharp u & = & r(\Pi_A \sqcap f^\sharp) \\ & = & r\Pi_A \sqcap rf^\sharp \\ & = & \nabla_A(\neg u) \sqcap rf^\sharp \\ & = & rf^\sharp(\neg u). \end{array}$$

(Note that $\nabla_A(\neg u) \sqcap \rho \sqsubseteq [\nabla_A \sqcap \rho(\neg u)](\neg u) = \rho(\neg u) \sqsubseteq \nabla_A(\neg u) \sqcap \rho$.) Hence $rf^\sharp u = rf^\sharp(\neg u) = 0$ and by the rationality of relations there is a relation $v : A \rightharpoonup A$ such that $\nabla_A v = rf^\sharp$ and $v \sqsubseteq \mathrm{id}_A$. By $\nabla_A vu = 0$ we have $vu = 0$ and so $v \sqsubseteq \neg u$. Analogously $v(\neg u) = 0$ from

$\nabla_A v(\neg u) = 0$. Therefore $v = vv \sqsubseteq v(\neg u) = 0$ and $rf^\sharp = 0$. Now assume that $f : A \to P(A)$ is a surjection, that is, $f^\sharp f = \mathrm{id}_{P(A)}$. Then $r = rf^\sharp f = 0f = 0$, which is a contradiction. $\square$

4 Rieger's Theorem

In this section Rieger's theorem is proved in the framework of relational set theory. We begin with introduction of a full decoration which is a key notion for Rieger's theorem.

Definition 4.1 *A function $f : M \to V$ is a full decoration of a relation $\mu : M \rightharpoondown M$ into a relation $\Pi : V \rightharpoondown V$ iff $f\Pi = \mu f$, $ff^\sharp = \mathrm{id}_M$ and $\nabla_M f = \nabla_M f \div \Pi$.* $\square$

$$\begin{array}{ccc} M & \xrightarrow{\mu} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[\Pi]{} & V \end{array}$$

Lemma 4.2 *Let $f : M \to V$ be a full decoration of $\mu : M \to M$ into $\Pi : V \rightharpoondown V$. If $\rho f = v\Pi$ for a relation $\rho : 1 \rightharpoondown M$ and $v \in V$, then there is an element $a \in M$ such that $\rho = a\mu$.*

Proof. First $v\Pi = \rho f \sqsubseteq \nabla_M f$ and $v \sqsubseteq \nabla_M f \div \Pi = \nabla_M f$. Hence $v = af$ for some $a : 1 \to M$. Thus $\rho f = v\Pi = af\Pi = a\mu f$ and $\rho = a\mu$ by $ff^\sharp = \mathrm{id}_M$. $\square$

A relation $\Pi : V \rightharpoondown V$ is a model of ZFC$^-$ if it satisfies axioms (A1.) – (A.7) and (A.9).

Theorem 4.3 *If $\Pi : V \rightharpoondown V$ is a model of ZFC$^-$ and $f : M \to V$ is a full decoration of $\mu : M \to M$ into $\Pi : V \rightharpoondown V$, then $\mu : M \rightharpoondown M$ is a model of ZFC$^-$.*

Proof. (Extensionality) $[\forall a, b \in M(a\mu = b\mu \Rightarrow a = b)]$
Let $a, b : 1 \to M$ be functions such taht $a\mu = b\mu$. Then $af\Pi = a\mu f = b\mu f = bf\Pi$. Hence $af = bf$ by the extensionality of Π and $a = b$ by $ff^\sharp = \mathrm{id}_M$.
(Null Set) $[\exists a \in M(0 = a\mu)]$
Let $0 : 1 \rightharpoondown M$ be a null relation. Then $0f = 0 : 1 \rightharpoondown V$ and by the Axiom of Null Set $0f = 0 = v\Pi$ for some function $v : 1 \to V$. Hence by the lsat lemma $0 = a\mu$ for some function $a : 1 \to M$.
(Pairing) $[\forall a, b \in M \exists d \in M(a \sqcup b = d\mu)]$
Let $a, b : 1 \to M$ be functions. By Axiom of Pairing for Π there is a function $v : 1 \to V$ such that $(a \sqcup b)f = af \sqcup bf = v\Pi$. Hence by the last lemma there is a function $d : 1 \to M$ such that $a \sqcup b = d\mu$.
(Union) $[\forall a \in M \exists b \in M(a\mu\mu = b\mu)]$
Let $a : 1 \to M$ be a function. Then $a\mu\mu f = a\mu f\Pi = af\Pi\Pi$ and by the Axiom of Union $a\mu\mu f = v\Pi$ for some function $v : 1 \to V$. Hence by the lsat lemma $a\mu\mu = b\mu$ for some function $b : 1 \to M$.
(Powerset) $[\forall a \in M \exists b \in M(a\mu \div \mu = b\mu)]$
Let $a : 1 \to M$ be a function. Then we have $(a\mu \div \mu)f \sqsubseteq af\Pi \div \Pi$ from $(a\mu \div \mu)f\Pi = (a\mu \div \mu)\mu f \sqsubseteq a\mu f = af\Pi$. By the axiom of powerset $(a\mu \div \mu)f \sqsubseteq u\Pi$ for some $u : 1 \to V$ and by the axiom of subset $(a\mu \div \mu)f = v\Pi$ for some $v : 1 \to V$. Hence by the last lemma $a\mu \div \mu = b\mu$ for some $b : 1 \to M$.
(Infinity) $[\exists m \in M(\emptyset_M \sqsubseteq m\mu \wedge \forall x \in M(x \sqsubseteq m\mu \Rightarrow x \cup_M \{x\}_M \sqsubseteq m\mu))]$ Where $\emptyset_M$ is an element of M such that $\emptyset_M \mu = 0$, and $x \cup_M \{x\}_M$ such that $(x \cup_M \{x\}_M)\mu = x\mu \sqcup x$.
Let $a \in V$ be an element of V with the infinity property and set a relation $\rho = a\Pi f^\sharp : 1 \rightharpoondown M$.

Then $\rho f = a\Pi f^\sharp f \sqsubseteq a\Pi$ and so by the axiom of subset there is $a' \in V$ such that $\rho f = a'\Pi$. Also by the last lemma there is $m \in M$ such that $\rho = m\mu$. Note that $\emptyset_M f = \emptyset$ from the axiom of extensionality and $\emptyset_M f\Pi = \emptyset_M \mu f = 0f = 0 = \emptyset\Pi$. Hence

$$\emptyset_M = \emptyset_M f f^\sharp = \emptyset f^\sharp \sqsubseteq a\Pi f^\sharp = \rho = m\mu.$$

Next note that $(x \cup_M \{x\}_M)f = xf \cup \{xf\}$ for each element $x \in M$. It follows at once from

$$(x \cup_M \{x\}_M)f\Pi = (x \cup_M \{x\}_M)\mu f = (x\mu \sqcup x)f = x\mu f \sqcup xf = xf\Pi \sqcup xf = (xf \cup \{xf\})\Pi.$$

At last assume that $x \sqsubseteq m\mu$ for $x \in M$. Then $xf \sqsubseteq m\mu f = a\Pi f^\sharp f \sqsubseteq a\Pi$ and by the axiom of infinity $xf \cup \{xf\} \sqsubseteq a\Pi$. Therefore

$$x \cup_M \{x\}_M = (x \cup_M \{x\}_M)ff^\sharp = (xf \cup \{xf\})f^\sharp \sqsubseteq a\Pi f^\sharp = m\mu.$$

(Replacement) $[\forall a \in M \forall \text{ pfn } k : M \to M \exists b \in M(a\mu k = b\mu)]$
First note that $a\mu k f = a\mu f f^\sharp k f = af\Pi f^\sharp k f$ and $f^\sharp k f : V \rightharpoondown V$ is a pfn. Hence by the axiom of replacement there is a function $v : 1 \to V$ such that $a\mu k f = v\Pi$ and so from the last lemma $a\mu k = b\mu$ for some function $b : 1 \to M$.
(Choice) $\forall a \in M \forall \alpha : M \rightharpoondown M[\nabla_M \alpha = a\mu \Rightarrow \exists \text{ pfn } k : M \to M(k \sqsubseteq \alpha \wedge \nabla_M k^\sharp = \nabla_M \alpha^\sharp)]$
Assume that $\nabla_M \beta = b\mu$ for a relation $\beta : M \rightharpoondown M$ and $b \in M$. Then $\nabla_V(f^\sharp \beta f) = \nabla_M \beta f = b\mu f = bf\Pi$. Hence by the axiom of choice for Π there is a pfn $h : V \to V$ such that $h \sqsubseteq f^\sharp \beta f$ and $\nabla_V h^\sharp = \nabla_V (f^\sharp \beta f)^\sharp = \nabla_M \beta^\sharp f$. Note that $h = hf^\sharp f$ from $\nabla_V h \sqsubseteq \nabla_M f$. Define a pfn $k = fhf^\sharp : M \to M$. Then $k = fhf^\sharp \sqsubseteq \beta$ and $\nabla_M k^\sharp = \nabla_M f h^\sharp f^\sharp = \nabla_V f^\sharp f h^\sharp f^\sharp = \nabla_V h^\sharp f^\sharp = \nabla_M \beta^\sharp f f^\sharp = \nabla_M \beta^\sharp$. $\Box$

5 Well-Founded Relations

In this section we assume that a meros $\mathcal{C}$ is Boolean, that is, for every object A and B the collection $\mathcal{C}(A, B)$ of all relations from A into B is a complete Boolean algebra.

Definition 5.1 *Let $\alpha : A \rightharpoondown A$ be a relation in a meros $\mathcal{C}$.*

(a) *A relation $\rho : 1 \rightharpoondown A$ is nonempty if there exists an element $a \in A$ such that $a \sqsubseteq \rho$.*

(b) *A relation $\alpha : A \rightharpoondown A$ is extensional if $a\alpha = b\alpha$ implies $a = b$ for $a, b \in A$.*

(c) *A relation $\alpha : A \rightharpoondown A$ is well-founded (wf) if for each nonempty relation $\rho : 1 \rightharpoondown A$ there exists an element $a \in A$ such that $a \sqsubseteq \rho$ and $\rho \sqcap a\alpha = 0$.* $\Box$

Proposition 5.2 *If $\alpha : A \rightharpoondown A$ is a wf relation, then*

(a) *$\nabla_A(\alpha \sqcap \mathrm{id}_A)$ is empty,*

(b) *$x\alpha x^\sharp = 0$ for every $x \in A$,*

(c) *There is no nonempty relation $\rho : 1 \rightharpoondown A$ such that $\forall x \in A\, [x \sqsubseteq \rho \Rightarrow \exists y \in A\, (y \sqsubseteq \rho \sqcap x\alpha)]$.*

Proof. (a) Assume that $\nabla_A(\alpha \sqcap \mathrm{id}_A)$ is nonempty. As α is wf there is some $a \in A$ such that $a \sqsubseteq \nabla_A(\alpha \sqcap \mathrm{id}_A)$ and $\nabla_A(\alpha \sqcap \mathrm{id}_A) \sqcap a\alpha = 0$. Then

$$a \sqsubseteq a(\alpha \sqcap \mathrm{id}_A)^\sharp(\alpha \sqcap \mathrm{id}_A) = a(\alpha \sqcap \mathrm{id}_A) \sqsubseteq a\alpha$$

and so $a \sqsubseteq \nabla_A(\alpha \sqcap \mathrm{id}_A) \sqcap a\alpha$. This is a contradiction. Therefore $\nabla_A(\alpha \sqcap \mathrm{id}_A)$ is empty. (Assume that empty relations are zero. Then $\nabla_A(\alpha \sqcap \mathrm{id}_A) = 0$ and so $\alpha \sqcap \mathrm{id}_A = 0$.) (b) Set $\rho = x : 1 \rightharpoonup A$. By wf property of α there is $y \in A$ such that $y \sqsubseteq x$ and $x \sqcap y\alpha = 0$. From $y \sqsubseteq x$ it is trivial that $y = x$. Hence applying Dedekind Formula we have $x\alpha x^\sharp = \mathrm{id}_1 \sqcap x\alpha x^\sharp \sqsubseteq (x \sqcap x\alpha)x^\sharp = 0x^\sharp = 0$ by $x \sqcap x\alpha = 0$. (c) Assume that such nonempty relation ρ exists. By wf property we have $x \in A$ such that $x \sqsubseteq \rho$ and $\rho \sqcap x\alpha = 0$. From the assumption there is $y \in A$ such that $y \sqsubseteq \rho \sqcap x\alpha$. Hence $y \sqsubseteq \rho \sqcap x\alpha = 0$, which is a contradiction. $\square$

Proposition 5.3 *Let $\alpha : A \rightharpoonup A$ and $\beta : B \rightharpoonup B$ be relations and $i : A \to B$ an injection with $i\beta = \alpha i$.*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow[i]{} & B \end{array}$$

(a) *If β is extensional, then so is α.*

(b) *If β is well-founded, then so is α.*

Proof. (a) Assume that $u\alpha = v\alpha$ for $u, v \in A$. Then $ui\beta = u$
$alphai = v\alpha i = vi\beta$ and so $ui = vi$ since β is extensional. Hence $u = uii^\sharp = vii^\sharp = v$ by $ii^\sharp = \mathrm{id}_A$. (b) Assume that $\rho : 1 \rightharpoonup A$ is nonempty. Then $\rho i : 1 \rightharpoonup B$ is also nonempty. As β is well-founded, there is $b \in B$ such that $b \sqsubseteq \rho i$ and $\rho i \sqcap b\beta = 0$. Note that $bi^\sharp : 1 \rightharpoonup B$ is a function by $b \sqsubseteq \rho i$ and $bi^\sharp \sqsubseteq \rho ii^\sharp = \rho$. Since $\alpha = \alpha ii^\sharp = i\beta i^\sharp$ by $i\beta = \alpha i$ we have

$$\rho \sqcap bi^\sharp\alpha = \rho \sqcap bi^\sharp i\beta i^\sharp \sqsubseteq \rho \sqcap b\beta i^\sharp \sqsubseteq (\rho i \sqcap b\beta)i^\sharp = 0i^\sharp = 0. \quad \square$$

Proposition 5.4 *(Assuming that nonzero relations are nonempty.) Let $\alpha : A \rightharpoonup A$ and $\beta : B \rightharpoonup B$ be relations and $f : A \to B$ a surjection with $\alpha f = f\beta$. If α is well-founded, then so is β.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow[f]{} & B \end{array}$$

Proof. Let $\rho : 1 \rightharpoonup B$ be a nonempty relation. Then $\rho f^\sharp$ is also nonzero (since $\rho = \rho f^\sharp f$ by $f^\sharp f = \mathrm{id}_B$) and so there is some $a \in A$ such that $a \sqsubseteq \rho f^\sharp$ and $\rho f^\sharp \sqcap \alpha = 0$. Finally $af \sqsubseteq \rho \sqsubseteq (\rho f^\sharp \sqcap \alpha)f = 0f^\sharp = 0$. $\square$

The following lemma shows the uniqueness of decorations in Mostowski's Collapsing Lemma.

Lemma 5.5 *Suppose that $\mathcal{C}$ is a separable Boolean meros. If $\alpha : A \rightharpoonup A$ is a wf relation and $\mu : M \rightharpoonup M$ is an extensional relation, then there is at most one function $d : A \to M$ such that $d\mu = \alpha d$.*

$$\begin{array}{ccc} A & \xrightarrow{d} & M \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\mu} \\ A & \xrightarrow[d]{} & M \end{array}$$

Proof. Let $d_i : A \to M$ be a function with $d_i\mu = \alpha d_i$ for $i = 0, 1$. Set $u = d_0{d_1}^\sharp \sqcap \mathrm{id}_A : A \rightharpoonup A$ and $\rho = \nabla_A u : 1 \to A$. (Note that $u^\sharp = u$.) It is clear that $d_0 = d_1$ iff $u = \mathrm{id}_A$ iff $\rho = \nabla_A$ iff $\neg\rho$ is empty. Hence it suffices to show that $\neg\rho$ is empty. Assume that $\neg\rho$ is nonempty. As α is a wf relation there is an element $a \in A$ such that $a \sqsubseteq \neg\rho$ and $\neg\rho \sqcap a\alpha = 0$. Then $a\alpha \sqsubseteq \neg\neg\rho = \rho$ and so

$$a\alpha = a\alpha \sqcap \rho = a\alpha \sqcap \nabla_A u = a\alpha u \sqsubseteq a\alpha d_0 {d_1}^\sharp.$$

Hence $a\alpha d_1 \sqsubseteq a\alpha d_0 {d_1}^\sharp d_1 \sqsubseteq a\alpha d_0$ and similarly $a\alpha d_0 \sqsubseteq a\alpha d_1$, which deduces $a\alpha d_0 = a\alpha d_1$. Moreover $ad_0\mu = a\alpha d_0 = a\alpha d_1 = ad_1\mu$ and $ad_0 = ad_1$ from the extensionality of μ. Hence we have $a = au$ from $au = ad_0{d_1}^\sharp \sqcap a = ad_1{d_1}^\sharp \sqcap a = a$. On the other hand $a \sqcap \rho = 0$ by $a \sqsubseteq \neg\rho$ and so $au = a \sqcap \nabla_A u = a \sqcap \rho = 0$ by

$$a \sqcap \nabla_A u \sqsubseteq (au^\sharp \sqcap \nabla_A)u = au^\sharp u = au \sqsubseteq a \sqcap \nabla_A u.$$

Therefore $a = au = 0$, which is a contradiction. $\Box$

Definition 5.6 *Let $\alpha : A \rightharpoonup A$ and $\mu : M \rightharpoonup M$ be relations. A partial function $t : A \to M$ is called a partial decoration of α into μ if $t\mu = \mathrm{d}(t)\alpha t$. (Where $\mathrm{d}(t) = tt^\sharp \sqcap \mathrm{id}_A$ is the domain relation of t.)*

$$\begin{array}{ccc} A & \xrightarrow{t} & M \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\mu} \\ A & \xrightarrow[t]{} & M \end{array}$$

Corollary 5.7 *Suppose that $\alpha : A \rightharpoonup A$ be a wf relation and $\mu : M \rightharpoonup M$ an extensional relation. Let $t_i : A \to M$ be a partial decoration of α into $\mu : M \rightharpoonup M$ with $\mathrm{d}(t_i)\alpha = \mathrm{d}(t_i)\alpha\mathrm{d}(t_i)$ $(i = 0, 1)$.*

(a) *If $\mathrm{d}(t_0) = \mathrm{d}(t_1)$, then $t_0 = t_1$.*

(b) *$\mathrm{d}(t_1)t_0 = \mathrm{d}(t_0)t_1$, that is, ${t_0}^\sharp t_1 \sqsubseteq \mathrm{id}_M$.*

Proof. (a) There is an injection $m : D \to A$ such that $\mathrm{d}(t_0) = m^\sharp m$. Define $\alpha' = m\alpha m^\sharp : D \rightharpoonup D$. Then we have $m\alpha = \alpha' m$ from $m\alpha = m\mathrm{d}(t_0)\alpha = m\mathrm{d}(t_0)\alpha\mathrm{d}(t_0) = m\alpha m^\sharp m = \alpha' m$. Thus α' is wf by 5.5(b). On the other hand mt_i is a function and $mt_i\mu = m\mathrm{d}(t_i)\alpha t_i = m\alpha t_i = \alpha' m t_i$. Therefore by 6.6 we have $mt_0 = mt_1$ and so $t_0 = m^\sharp m t_0 = m^\sharp m t_1 = t_1$. (b) Set $s_i = \mathrm{d}(t_{1-i})t_i (= \mathrm{d}(t_0)\mathrm{d}(t_1)t_i)$. First note that $\mathrm{d}(s_0) = \mathrm{d}(s_1) (= \mathrm{d}(t_0)\mathrm{d}(t_1))$. Then $\mathrm{d}(s_i)\alpha = \mathrm{d}(t_0)\mathrm{d}(t_1)\alpha = \mathrm{d}(t_0)\mathrm{d}(t_1)\alpha\mathrm{d}(t_0)\mathrm{d}(t_1) = \mathrm{d}(s_i)\alpha\mathrm{d}(s_i)$ and $s_i\mu = \mathrm{d}(t_{1-i})t_i\mu = \mathrm{d}(t_{1-i})\mathrm{d}(t_i)\alpha t_i = \mathrm{d}(t_{1-i})\mathrm{d}(t_i)\alpha\mathrm{d}(t_{1-i})t_i = \mathrm{d}(s_i)\alpha s_i$. Hence $s_0 = s_1$ by (a) and so ${t_0}^\sharp t_1 = (\mathrm{d}(t_0)t_0)^\sharp \mathrm{d}(t_1)t_1 = {t_0}^\sharp \mathrm{d}(t_0)\mathrm{d}(t_1)t_1 = {t_0}^\sharp \mathrm{d}(t_1)\mathrm{d}(t_0)t_1 = {s_0}^\sharp s_1 \sqsubseteq \mathrm{id}_M$. $\Box$

Lemma 5.8 *Let $\alpha : A \rightharpoonup A$ be a wf relation and $\mu : M \rightharpoonup M$ an extensional relation. If T is a collection of partial decorations $t : A \to M$ of α into μ with $\mathrm{d}(t)\alpha = \mathrm{d}(t)\alpha\mathrm{d}(t)$, then the least upper bound $s = \sqcup_{t\in T} t : A \rightharpoonup M$ of T is also a partial decoration with $\mathrm{d}(s)\alpha = \mathrm{d}(s)\alpha\mathrm{d}(s)$.*

Proof. It simply follows from the following computations:

(a) $\mathrm{d}(s)\alpha\mathrm{d}(s) = \sqcup_{t\in T}\mathrm{d}(t)\alpha\mathrm{d}(s) = \sqcup_{t\in T}\mathrm{d}(t)\alpha\mathrm{d}(t)\mathrm{d}(s) = \sqcup_{t\in T}\mathrm{d}(t)\alpha\mathrm{d}(t) = \sqcup_{t\in T}\mathrm{d}(t)\alpha = \mathrm{d}(s)\alpha$,

(b) For $t \in T$, $\mathrm{d}(t)s = \sqcup_{t'\in T}\mathrm{d}(t)t' = \sqcup_{t'\in T}\mathrm{d}(t')t($ 6.7(b) $) = \mathrm{d}(s)t = t$

(c) $\mathrm{d}(s)\alpha s = \sqcup_{t\in T}\mathrm{d}(t)\alpha s = \sqcup_{t\in T}\mathrm{d}(t)\alpha\mathrm{d}(t)s = \sqcup_{t\in T}\mathrm{d}(t)\alpha t = \sqcup_{t\in T} t\mu = s\mu$. $\Box$

Proposition 5.9 *Assume that $\Pi : V \rightharpoonup V$ is a model of ZFC and $\mathcal{C}(1,1) = \{0, \mathrm{id}_1\}$. If $a\Pi = \nabla_A i$ for $a \in V$ and an injection $i : A \to V$ and if a relation $\rho : 1 \rightharpoonup A$ is nonzero $(\rho \neq 0)$, then there is $r \in A$ with $r \sqsubseteq \rho$, that is, ρ is nonempty.*

Proof. Let $\rho \neq 0 : 1 \rightharpoonup A$ and set $\alpha = x^\sharp \rho i : V \rightharpoonup V$ for an arbitrary $x \in V$. First note that $\nabla_V \alpha \sqsubseteq \nabla_A i = a\Pi$ and by the axiom (A.7') of subset $\nabla_V \alpha = b\Pi$ for some $b \in V$. Applying the axiom (A.9) of choice to α there is a partial function $h : V \to V$ with $h \sqsubseteq \alpha$ and $\nabla_V h^\sharp = \nabla_V \alpha^\sharp$. On the other hand $\nabla_A \rho^\sharp = \mathrm{id}_1$ by the assumption $\mathcal{C}(1,1) = \{0, \mathrm{id}_1\}$. Hence we have $\nabla_V (xh)^\sharp = \nabla_V i^\sharp \rho^\sharp x x^\sharp = x x^\sharp = \mathrm{id}_1$ and so xh is a function from 1 into V. Finally from $xh \sqsubseteq x\alpha = \rho i$ it follows that $xhi^\sharp \sqsubseteq \rho$ and $xhi^\sharp \in A$. $\square$

Theorem 5.10 (Mostowski's Collapsing Lemma) *Assume that $\Pi : V \rightharpoonup V$ is a model of ZFC and $\mathcal{C}(1,1) = \{0, \mathrm{id}_1\}$. If $< i : A \to V, \alpha : A \rightharpoonup A >$ is a wf Π-system, then there is a unique decoration $d : A \to V$ such that $d\Pi = \alpha d$.*

Proof. By ?? it suffices to see the existence of such decorations. Note that the zero relation $0 : A \rightharpoonup V$ is a partial decoration into Π. Consider the collection T of all partial decoration into Π. Then by the last lemma $s = \sqcup_{t \in T} t$ is the greatest partial decoration into Π. Set $\rho = \nabla_V s^\sharp$. Note that $\rho = \nabla_A$ iff $\neg\rho$ is empty. So it suffices to see that $\neg\rho$ is empty. Assume that $\neg\rho$ is nonempty. By the well foundedness of α there is an element $a : 1 \to A$ such that $a \sqsubseteq \neg\rho$ and $\neg\rho \sqcap a\alpha = 0$. Then $a \sqcap \rho = 0$ and $a\alpha \sqsubseteq \neg\neg\rho = \rho$. By the way we have $a\alpha s = a\alpha i i^\sharp s = v\Pi i^\sharp s = w\Pi$ for some $v, w \in V$ because of the axiom (A.7). It is easy to see that $a\mathrm{d}(s) = 0$, $a\alpha a^\sharp = 0$ and $\mathrm{d}(s)\alpha a^\sharp = \mathrm{d}(s)\alpha\mathrm{d}(s)a^\sharp = 0$. Therefore we have $\mathrm{d}(s \sqcup a^\sharp w)\alpha\mathrm{d}(s \sqcup a^\sharp w) = \mathrm{d}(s \sqcup a^\sharp w)\alpha$, and $\mathrm{d}(s \sqcup a^\sharp w)\alpha[s \sqcup a^\sharp w] = [s \sqcup a^\sharp w]\Pi$, which contradicts to the maximality of $s : A \to V$. $\square$

Proposition 5.11 *Assume that $\Pi : V \rightharpoonup V$ is a model of ZFC, $\mathcal{C}(1,1) = \{0, \mathrm{id}_1\}$, $< i : A \to V, \alpha : A \rightharpoonup A >$ a Π-system, and $< j : M \to V, \mu : M \rightharpoonup M >$ a wf Π-system. If there is a decoration $d : A \to M$ of α into μ, then α is a wf relation.*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ M & \xrightarrow[\mu]{} & M \end{array}$$

Proof. Let $\rho : 1 \rightharpoonup A$ be a nonempty relation. Then ρd is nonempty. As $\alpha : A \rightharpoonup A$ is a wf relation there is an element $x \in V$ such that $x \sqsubseteq \rho d$ and $\rho d \sqcap x\mu = 0$. !!By 2.8 there is an element $a \in A$ with $a \sqsubseteq \rho$ and $x = ad$.!! Then $\rho \sqcap a\alpha \sqsubseteq (\rho \sqcap a\alpha)dd^\sharp \sqsubseteq (\rho d \sqcap a\alpha d)d^\sharp = (\rho d \sqcap ad\mu)d^\sharp = (\rho d \sqcap x\mu)d^\sharp = 0d^\sharp = 0$. Hence $\rho \sqcap a\alpha = 0$. $\square$

Proposition 5.12 *Let $\mu : M \rightharpoonup M$ be a wf relation. If $A \neq 0$ and $f : A \to A$ is an iso, then there is no decoration of f into μ.*

Proof. Assume that there is a decoration $d : A \to M$ of f.

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ M & \xrightarrow[\mu]{} & M \end{array}$$

It is immediate that $\rho = \nabla_A d \neq 0$ (since $\nabla_A = \nabla_M d^\sharp = \nabla_A d d^\sharp = 0$ if $\rho = \nabla_A d = 0$). As μ is a wf relation there is $x \in M$ such that $x \sqsubseteq \nabla_A d$ and $\nabla_A d \sqcap x\mu = 0$. Note that $x = xd^\sharp d$ and $x\mu = xd^\sharp d\mu = xd^\sharp f d \sqsubseteq \nabla_A d$. Hence $x\mu = \nabla_A d \sqcap x\mu = 0$ and $x = xd^\sharp d \sqsubseteq xd^\sharp f d d^\sharp f^\sharp d = x\mu d^\sharp f^\sharp d = 0$, which is a contradiction. $\square$

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