Relational Graph Rewritings

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Abstract

This note presents a new formalization of graph rewritings which generalizes Ehrig's graph derivations and Raoult's graph rewritings. The graph rewritings, based on a primitive pushout construction in the category of graphs and partial functions preserving graph structures, can be always applied without gluing conditions only if a graph has a matching to a given rewriting rule. A more general sufficient condition for two rewritings to commute is also proved. The simplicity of our discussion comes from the usage of relational calculus (theory of binary relations).

1 Introduction

There are many researches [4],[5],[9],[12],[14] on graph rewritings (or reduction) from the view-point of category theory [10]. An advantage of categorical graph rewritings is to produce a universal reduction despite how to execut algorithms for applying production rules.

Ehrig et al. [4],[5] studied graph grammars for a wide class of graphs and functions preserving edges. It is well-known that the category of graphs in [4],[5] is a topos [6] and so it has pushouts [10, page 65]. However, even if a graph has a matching (or occurrence) to a production rule, their derivation of graphs does not work unless an involved pushout-complement [3] exists. Gluing conditions for the existence of pushout-complements in the category of graphs were investigated by Ehrig and Kreowski [3] and Kawahara [8].

Raoult [14] proposed another formalization of graph rewritings by regarding produnction rules as partial functions preserving graph structures. Although his definition of graphs seems to be easier to implement graph structures on electric computers using pointers, the gluing condition appears as a complicate problem [14, Proposition 5] whether involved pushouts exist in a category of graphs and partial graph morphisms.

Thus the conventional categorical graph rewritings require strong gluing conditions for actual execution of graph derivations. In this note we treat of the category of (simple) graphs (with or without labelled edges) and partial functions preserving graph structures, and present a new formalization of graph rewritings by using a primitive pushout construction in the category. Our graph rewritings can be always executed without any gluing conditions, only if a graph has a matching to a given rewriting rule. Moreover our fomalization of graph rewritings generalizes Ehrig's graph derivations [3],[5] and Raoult's graph rewritings [14] in a reasonable sense. Therefore our formalization offers a neat foundation of categorical graph rewritings which releases us from tedious gluing conditions. The framework of the note is elementary and the simplicity of our discussion comes from relational calculus (theory of binary relations) due to Kawahara [8].

This note consists of the following sections. In the section 2 we present minimum fundamentals on relational calculus for the later calculations. The main subjects of the note are discussed in the section 3. We set up the framework of our thoery, that is, the notions of (simple) graphs and partial morphisms between them are defined. For a given pair of partial functions from a common set into graphs a primitive pushout square is constructed, which shows the category of graphs and partial morphisms has pushouts. An observation of Ehrig's graph reduction [5] suggests our formalization of graph rewritings without gluing conditions. Moreover we give a more general sufficient condition for two graph rewritings to commute. Some examples related to graph rewritings are listed in the section 4. In the section 5 we state how to develope our formalization of graph rewritings for graphs with labelled edges.

2 Fundamentals on Relational Calculus

A *relation* α of a set A into another set B is a subset of the cartesian product $A \times B$ and denoted by $\alpha : A \rightharpoonup B$. The *inverse relation* $\alpha^\sharp : B \rightharpoonup A$ of α is a relation such that $(b, a) \in \alpha^\sharp$ if and only if $(a, b) \in \alpha$. The *composite* $\alpha\beta : A \rightharpoonup C$ of $\alpha : A \rightharpoonup B$ followed by $\beta : B \rightharpoonup C$ is a relation such that $(a, c) \in \alpha\beta$ if and only if there exists $b \in B$ with $(a, b) \in \alpha$ and $(b, c) \in \beta$.

As a relation of a set A into a set B is a subset of $A \times B$, the inclusion relation, union, intersection and difference of them are avialable as usual and denoted by $\sqsubseteq$, $\sqcup$, $\sqcap$ and $-$, respectively. The *identity relation* $\mathrm{id}_A : A \rightharpoonup A$ is a relation with $\mathrm{id}_A = \{(a, a) \in A \times A \mid a \in A\}$ (the diagonal set of A).

The followings are the basic properties of relations and indicate that the totality of sets and relations forms a category **Rel** with involution (or shortly I-category).

2.1 I-category *Let $\alpha, \alpha' : A \rightharpoonup B$, $\beta, \beta' : B \rightharpoonup C$ and $\gamma : C \rightharpoonup D$ be relations. Then,*

(a) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (*associative*),
(b) $\mathrm{id}_A\alpha = \alpha\mathrm{id}_B = \alpha$ (*identity*),
(c) $\alpha^{\sharp\sharp} = \alpha$, $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$ (*involutive*),
(d) *If $\alpha \sqsubseteq \alpha'$ and $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha'\beta'$ and $\alpha^\sharp \sqsubseteq \alpha'^\sharp$* (*nomotone*).

The distributive law for relations is trivial but indespensable in our relational calculus.

2.2 Distributive Law *The distributive law $\alpha(\sqcup_{\lambda\in\Lambda}\beta_\lambda)\gamma = \sqcup_{\lambda\in\Lambda}\alpha\beta_\lambda\gamma$ holds for relations $\alpha : A \rightharpoonup B$, $\beta_\lambda : B \rightharpoonup C$ ($\lambda \in \Lambda$) and $\gamma : C \rightharpoonup D$.*

A *partial function* f of a set A into a set B is a relation $f : A \rightharpoonup B$ with $f^\sharp f \sqsubseteq \mathrm{id}_B$ and it is denoted by $f : A \to B$. A *(total) function* f of a set A into a set B is a relation $f : A \rightharpoonup B$ with $f^\sharp f \sqsubseteq \mathrm{id}_B$ and $\mathrm{id}_A \sqsubseteq ff^\sharp$, and it is also denoted by $f : A \to B$. Clearly a function is a partial function. Note that the identity relation id_A of a set A is a function. The readers easily understand our definitions of partial functions and (total) functions are coincide with ordinary ones.

2.3 Proposition *Let $\alpha, \beta : A \rightharpoonup B$ be relations. If $f : X \rightarrow A$ and $g : Y \rightarrow B$ are partial functions, then $f(\alpha \sqcap \beta)g^\sharp = f\alpha g^\sharp \sqcap f\beta g^\sharp$ and $f(\alpha - \beta)g^\sharp = f\alpha g^\sharp - f\beta g^\sharp$.*

Given a relation $\alpha : A \rightharpoonup B$, the *domain* $d(\alpha) : A \rightharpoonup A$ of α is a relation defined by $d(\alpha) = \alpha\alpha^\sharp \sqcap \mathrm{id}_A$. A partial function $f : A \rightarrow B$ is a function if and only if $d(f) = \mathrm{id}_A$.

The following proposition is useful for manipulating domains of partial functions.

2.4 Proposition *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$ be relations and $f : A \rightarrow B$ a partial function. Then*

(a) $d(\alpha\beta)d(\alpha) = d(\alpha\beta)$ (*or* $d(\alpha\beta) \sqsubseteq d(\alpha)$),

(b) $d(f\beta)f = fd(\beta)$.

2.5 Proposition *Let $\alpha : A \rightharpoonup A$, $\theta : B \rightharpoonup B$ be relations and let $f : A \rightarrow B$ be a partial function. If $\theta \sqsubseteq f^\sharp \alpha f$, then $\theta = f^\sharp f \theta f^\sharp f$.*

We denote the category of sets and functions by **Set** and the category of sets and partial functions by **Pfn**. Both of **Set** and **Pfn** have all small limits and colimits, so in particular, they have pushouts [10],[9],[14],[12]. Note that **Pfn** is equivalent to the category of sets with a base point (a selected element) and base point preserving functions. We assume that the readers are familiar with pushout constructions [14],[12] in **Pfn**.

A singleton set $\{*\}$ is denoted by 1 and the maximum relation from a set A into 1 by $\Omega_A : A \rightharpoonup 1$, that is, $\Omega_A = \{(a, *) | a \in A\}$. For a partial function $f : A \rightarrow B$ a relation $f^\sharp \Omega_A : B \rightarrow 1$ corresponds to the image of f.

2.6 Proposition *Let a square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

be a pushout in **Pfn** *and let $t : X \rightarrow C$ be a function. Then the composite $tk : X \rightarrow D$ is a function if and only if $t^\sharp \Omega_X \sqcap g^\sharp \Omega_A \sqsubseteq g^\sharp g k \Omega_D$.*

3 Rewritings for Simple Graphs

A *(simple) graph* $< A, \alpha >$ is a pair of a set A and a relation $\alpha : A \rightharpoonup A$. A *partial morphism* f of a graph $< A, \alpha >$ into a graph $< B, \beta >$, denoted by $f :< A, \alpha > \rightarrow < B, \beta >$, is a partial function $f : A \rightarrow B$ satisfying $d(f)\alpha f \sqsubseteq f\beta$. It is easily seen that a partial morphism among graphs is a partial function preserving edges on its domain of definitions.

Let $f :< A, \alpha > \rightarrow < B, \beta >$ and $g :< B, \beta > \rightarrow < C, \gamma >$ be partial morphisms of graphs. Since $d(f)\alpha f \sqsubseteq f\beta$ and $d(g)\beta g \sqsubseteq g\gamma$, we have $d(fg)\alpha fg = d(fg)d(f)\alpha fg$ (by 2.4(a)) $\sqsubseteq d(fg)f\beta g = fd(g)\beta g$ (by 2.4(b)) $\sqsubseteq fg\gamma$. Hence the composite of two partial morphisms of graphs is also a partial morphism of graphs. Thus we have the category of (simple) graphs and partial morphisms between them.

The following theorem constructs a primitive pushout for a pair of partial functions from a common set into graphs.

3.1 Theorem *If $< B, \beta >$ and $< C, \gamma >$ are graphs and if the square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & (1) & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

is a pushout in **Pfn**, *then $h :< B, \beta > \to < D, \delta >$ and $k :< C, \gamma > \to < D, \delta >$ are partial morphisms of graphs, where $\delta = h^\sharp \beta h \sqcup k^\sharp \gamma k$. Moreover, if $h' :< B, \beta > \to < D', \delta' >$ and $k' :< C, \gamma > \to < D', \delta' >$ are partial morphisms of graphs satisfying $fh' = gk'$, then there exists a unique partial morphism $t :< D, \delta > \to < D', \delta' >$ of graphs such that $h' = ht$ and $k' = kt$.*

Proof. First we see that $h :< B, \beta > \to < D, \delta >$ and $k :< B, \beta > \to < D, \delta >$ are partial morphisms of graphs. It simply follows from $d(h)\beta h \sqsubseteq hh^\sharp \beta h$ (by $d(h) = hh^\sharp \sqcap \mathrm{id}_B$) $\sqsubseteq h\delta$ (by $\delta = h^\sharp \beta h \sqcup k^\sharp \gamma k$). Next assume that $h' :< B, \beta > \to < D', \delta' >$ and $k' :< C, \gamma > \to < D', \delta' >$ are partial morphisms of graphs satisfying $fh' = gk'$. Then we have $d(h')\beta h' \sqsubseteq h'\delta'$ and $d(k')\gamma k' \sqsubseteq k'\delta'$. As (1) is a pushout in **Pfn**, there exists a unique partial function $t : D \to D'$ such that $h' = ht$ and $k' = kt$. It suffices to prove that $d(t)\delta t \sqsubseteq t\delta'$. But it follows from

$$\begin{array}{rcl} d(t)\delta t & \sqsubseteq & tt^\sharp(h^\sharp \beta h \sqcup k^\sharp \gamma k)t \quad (d(t) = tt^\sharp \sqcap \mathrm{id}_D) \\ & = & t(t^\sharp h^\sharp \beta h t \sqcup t^\sharp k^\sharp \gamma k t) \quad (\text{by } \mathbf{(2.2)}) \\ & = & t(h'^\sharp \beta h' \sqcup k'^\sharp \gamma k') \quad (h' = ht,\ k' = kt) \\ & = & t(h'^\sharp d(h')\beta h' \sqcup k'^\sharp d(k')\gamma k') \quad (h' = d(h')h',\ k' = d(k')k') \\ & = & t(h'^\sharp h'\delta' \sqcup k'^\sharp k'\delta') \quad (d(h')\beta h' \sqsubseteq h'\delta',\ d(k')\gamma k' \sqsubseteq k'\delta') \\ & \sqsubseteq & t(\delta' \sqcup \delta') \quad (h'^\sharp h' \sqsubseteq id_{D'},\ k'^\sharp k' \sqsubseteq id_{D'}) \\ & = & t\delta'. \end{array}$$

This completes the proof.

Note that the graph $< D, \delta >$ is unique up to isomorphisms. The following is exactly a corollary of the last theorem.

3.2 Corollary *The category of graphs and partial morphisms has pushouts.*

A partial morphism $f :< A, \alpha > \to < B, \beta >$ is said to be a *morphism* of graphs if $f : A \to B$ is a function. It is trivial that the composition of two morphisms of graphs is also a morphism of graphs and so one can consider the category of graphs and morphisms betweem them.

3.3 Observation Now consider a direct derivation of Ehrig's fast productions [5], that is, assume that the following two squares are pushouts in the category of graphs and morphisms and that m is an injective function.

$$\begin{array}{ccccc} < A, \alpha > & \xleftarrow{m} & < D, \delta > & \xrightarrow{f} & < B, \beta > \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle h} \\ < G, \xi > & \xleftarrow[n]{} & < E, \varepsilon > & \xrightarrow[k]{} & < H, \eta > \end{array}$$

Then $\delta m \sqsubseteq m\alpha$, $\delta s \sqsubseteq s\varepsilon$, $\delta f \sqsubseteq f\beta$, $\xi = g^\sharp \alpha g \sqcup n^\sharp \varepsilon n$ and $\eta = h^\sharp \beta h \sqcup k^\sharp \varepsilon k$ by **3.1**. Since $nn^\sharp = \mathrm{id}_E$ by the pushout property it is easy to see that $s^\sharp \delta s \sqsubseteq \varepsilon$ and

$$\begin{array}{rcl} n(\xi - g^\sharp \alpha g)n^\sharp & = & (ng^\sharp \alpha g n^\sharp \sqcup nn^\sharp \varepsilon nn^\sharp) - ng^\sharp \alpha g n^\sharp \quad (\text{by } \mathbf{2.3}) \\ & = & (ng^\sharp \alpha g n^\sharp \sqcup \varepsilon) - ng^\sharp \alpha g n^\sharp \quad (nn^\sharp = \mathrm{id}_E) \\ & = & \varepsilon - ng^\sharp \alpha g n^\sharp \\ & \sqsubseteq & \varepsilon. \end{array}$$

Hence $n(\xi - g^\sharp \alpha g)n^\sharp \sqcup s^\sharp \delta s \sqsubseteq \varepsilon$. Now put $\hat{\varepsilon} = n(\xi - g^\sharp \alpha g)n^\sharp \sqcup s^\sharp \delta s$. From $n^\sharp \varepsilon n - g^\sharp \alpha g = n^\sharp n(n^\sharp \varepsilon n - g^\sharp \alpha g)n^\sharp n$ (by **2.5**) $= n^\sharp(\varepsilon - ng^\sharp \alpha g n^\sharp)n$ (by **2.2**) and $nn^\sharp = \mathrm{id}_E$, we have

$$\begin{aligned} g^\sharp \alpha g \sqcup n^\sharp \hat{\varepsilon} n &= g^\sharp \alpha g \sqcup n^\sharp n(\xi - g^\sharp \alpha g)n^\sharp n \sqcup n^\sharp s^\sharp \delta s n \quad \text{(by } \mathbf{2.3}) \\ &= g^\sharp \alpha g \sqcup n^\sharp(\varepsilon - ng^\sharp \alpha g n^\sharp)n \sqcup n^\sharp s^\sharp \delta s n \\ &= g^\sharp \alpha g \sqcup (n^\sharp \varepsilon n - g^\sharp \alpha g) \sqcup n^\sharp s^\sharp \delta s n \\ &= g^\sharp \alpha g \sqcup n^\sharp \varepsilon n \sqcup n^\sharp s^\sharp \delta s n \\ &= g^\sharp \alpha g \sqcup n^\sharp \varepsilon n \quad (s^\sharp \delta s \sqsubseteq \varepsilon) \\ &= \xi. \end{aligned}$$

Thus $\hat{\varepsilon} : E \rightharpoonup E$ is the least relation such that $s^\sharp \delta s \sqsubseteq \hat{\varepsilon}$ and $\xi = g^\sharp \alpha g \sqcup n^\sharp \hat{\varepsilon} n$. Hence it is reasonable to assume that $\varepsilon = \hat{\varepsilon}$ (Cf. **4.1**). In this case we have

$$\begin{aligned} \eta &= h^\sharp \beta h \sqcup k^\sharp n(\xi - g^\sharp \alpha g)n^\sharp k \sqcup k^\sharp s^\sharp \delta s k \\ &= h^\sharp \beta h \sqcup k^\sharp n(\xi - g^\sharp \alpha g)n^\sharp k \sqcup h^\sharp f^\sharp \delta f h \quad (fh = sk) \\ &= h^\sharp \beta h \sqcup k^\sharp n(\xi - g^\sharp \alpha g)n^\sharp k \quad (f^\sharp \delta f \sqsubseteq \beta). \end{aligned}$$

Therefore we define our graph rewritings in order to include Ehrig's graph derivations [5].

3.4 Definition A *rewriting rule* p is a triple of two graphs $< A, \alpha >$, $< B, \beta >$ and a partial function $f : A \rightarrow B$. (Note that f need not to be a partial morphism of graphs.) A *matching* to p is a morphism $g :< A, \alpha > \rightarrow < G, \xi >$ of graphs. Then construct a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ G & \xrightarrow[k]{} & H \end{array}$$

in **Pfn** and define $\eta = h^\sharp \beta h \sqcup k^\sharp(\xi - g^\sharp \alpha g)k$. The graph $< H, \eta >$ is the resultant graph after applying a production rule p along a matching g, and denoted by $< G, \xi > \Rightarrow_{p/g} < H, \eta >$. A square

$$\begin{array}{ccc} < A, \alpha > & \xrightarrow{f} & < B, \beta > \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ < G, \xi > & \xrightarrow[k]{} & < H, \eta > \end{array}$$

is called the rewriting square for a rewriting rule p along a matching g. (Note that the rewriting square is not neccessarily a pushout in the category of graphs and partial morphisms.)

3.5 Proposition *Let $g :< A, \alpha > \rightarrow < G, \xi >$ be a matching to a rewriting rule $p = (< A, \alpha >, < B, \beta >, f : A \rightarrow B)$. If $f :< A, \alpha > \rightarrow < B, \beta >$ is a partial morphism of graphs, then the rewriting square*

$$\begin{array}{ccc} < A, \alpha > & \xrightarrow{f} & < B, \beta > \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ < G, \xi > & \xrightarrow[k]{} & < H, \eta > \end{array}$$

for p along g is a pushout in the category of graphs and partial morphisms.

Proof. By the virtue of **3.1** it suffices to show that $\eta = h^\sharp \beta h \sqcup k^\sharp \xi k$. First note that $f^\sharp \alpha f \sqsubseteq f^\sharp f \beta \sqsubseteq \beta$ since $d(f)\alpha f \sqsubseteq f\beta$. Thus we have

$$\begin{aligned} \eta &= h^\sharp \beta h \sqcup k^\sharp(\xi - g^\sharp \alpha g)k \\ &\sqsupseteq h^\sharp f^\sharp \alpha f h \sqcup k^\sharp(\xi - g^\sharp \alpha g)k \\ &= k^\sharp g^\sharp \alpha g k \sqcup k^\sharp(\xi - g^\sharp \alpha g)k \\ &= k^\sharp\{g^\sharp \alpha g \sqcup (\xi - g^\sharp \alpha g)\}k \\ &= k^\sharp \xi k. \end{aligned}$$

and

$$\begin{aligned}\eta &= h^\sharp \beta h \sqcup k^\sharp(\xi - g^\sharp \alpha g)k \sqcup k^\sharp \xi k \\ &= h^\sharp \beta h \sqcup k^\sharp \xi k.\end{aligned}$$

This complets the proof.

The last proposition suggests that our graph rewritings coincide with those of Raoult [14] if a production rule is a partial morphism of graphs. Hence our formalization of graph rewritings includes Ehrig's graph derivations [5] and Raoult's graph rewritings [14] in this sense.

It is easy to understand that analoguos results to Raoult's work [14] about the confluency and concurrency of graph rewritings are valid for our case. At the end of the section we state a general sufficient condition for two graph rewritings to commute (or to be strongly confluent).

3.6 Theorem *Let* $p_i = (< A_i, \alpha_i >, < B_i, \beta_i >, f_i : A_i \to B_i)$ *be rewriting rules,* $g_i :< A_i, \alpha_i > \to < G, \xi >$ *matchings to* p_i *and* $< G, \xi > \Rightarrow_{p_i/g_i} < H_i, \eta_i >$ *for* $i = 0, 1$. *If* $f_i :< A_i, \alpha_i > \to < B_i, \beta_i >$ *is partial morphisms of graphs* $(i = 0, 1)$ *and* $g_0^\sharp \Omega_{A_0} \sqcap g_1^\sharp \Omega_{A_1} \sqsubseteq g_1^\sharp g_0 k_0 \Omega_{D_0} \sqcap g_0^\sharp g_1 k_1 \Omega_{D_1}$, *then there exist matchings* $g_i' :< A, \alpha_i > \to < H_{1-i}, \eta_{1-i} >$ $(i = 0, 1)$ *and a graph* $< H, \eta >$ *such that* $< H_{1-i}, \eta_{1-i} > \Rightarrow_{p_i/g_i'} < H, \eta >$ $(i = 0, 1)$.

Proof. As f_0, g_0, f_1 and g_1 are partial morphisms of graphs, we can construct the following three pushouts in the category of graphs and partial morphisms between them by **3.2**:

$$\begin{array}{ccccc} & & < A_0, \alpha_0 > & \xrightarrow{f_0} & < B_0, \beta_0 > \\ & & {\scriptstyle g_0}\downarrow & (0) & \downarrow{\scriptstyle h_0} \\ < A_1, \alpha_1 > & \xrightarrow{g_1} & < G, \xi > & \xrightarrow{k_0} & < H_0, \eta_0 > \\ {\scriptstyle f_1}\downarrow & (1) & \downarrow{\scriptstyle k_1} & (2) & \downarrow{\scriptstyle h_0'} \\ < B_1, \beta_1 > & \xrightarrow[h_1]{} & < H_1, \eta_1 > & \xrightarrow[h_1']{} & < H, \eta > \end{array}$$

Set $g_i' = g_i k_{1-i}$ $(i = 0, 1)$. Then we can deduces that g_i' $(i = 0, 1)$ is a function because of **2.6**, and so $g_i' :< A_i, \alpha_i > \to < H_{1-i}, \eta_{1-i} >$ is a matching to p_i $(i = 0, 1)$. Since two squares (0)+(2) and (1)+(2) are pushouts in the category of graphs and partial morphisms between them, we have $< H_{1-i}, \eta_{1-i} > \Rightarrow_{p_i/g_i'} < H, \eta >$ $(i = 0, 1)$ by means of **3.4**, which proves the theorem.

Remark. Rewriting rules can be freely selected according to aims. Matchings to rewriting rules might be also restricted sometime. For example, if a matching $g :< A, \alpha > \to < G, \xi >$ to $p = (< A, \alpha >, < B, \beta >, f : A \to B)$ is an injective morphism of graphs such that $deg(g(a)) = deg(a)$ for each $a \in A$ on which f is undefined, then the rewritings coincides with those of boundary graphs (or B-graphs) due to Okada and Hayashi [13].

4 Examples of Graph Rewritings

In this section a few particular examples related to graph rewritings are listed. The first example shows that pushout-complements are not unique in the category of graphs.

4.1 Let $\alpha, \beta, \gamma : A \to A$ be relations with $\alpha \sqsubseteq \gamma \sqsubseteq \beta$. Then because of **3.1** the square

$$\begin{array}{ccc} < A, \alpha > & \xrightarrow{\mathrm{id}_A} & < A, \beta > \\ {\scriptstyle \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A} \\ < A, \gamma > & \xrightarrow[\mathrm{id}_A]{} & < A, \beta > \end{array}$$

is a pushout in the category of graphs and morphisms between them. Therefore the square is a pushout for any choice of γ satisfying $\alpha \sqsubseteq \gamma \sqsubseteq \beta$. The choice of $\hat{\varepsilon}$ in **3.3** means the most ecomomical way to have pushout-complements.

Next we present two simple examples of graph rewritings to which conventional graph rewritings cannot be applied.

4.2 In **Fig.1** g is a neat morphism of graphs in theories of Ehrig [5], Raoult [14] and ours. But f is not a morphism of graphs and it is not worth to be a rewriting rule in the sense of Raoult [14].

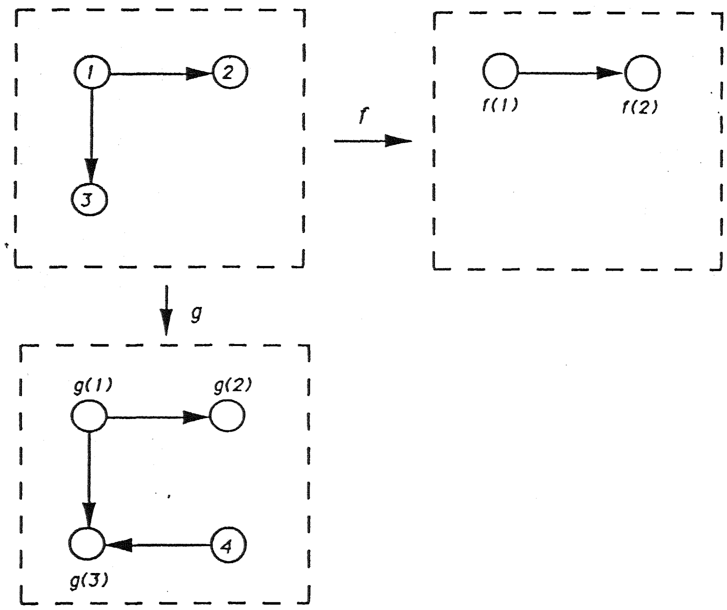
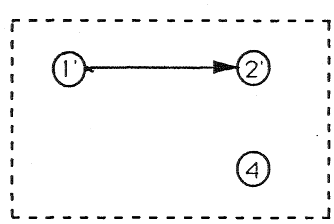


Figure 1:

On the other hand f means a fast production in [5] but unfornately the necesary pushout-complement does not exist since the gluing condition is not satisfied. However we have the following resultant graph applying our formalization:



4.3 In **Fig.2** g is a morphism of graphs and f is a partial morphism of graphs in all theories of Ehrig [5], Raoult [14] and ours. However graph rewritings of Ehrig [5] and Raoult [14] does not work again because the gluing conditions are not valid. In this case The resultant graph given by our graph rewritings is one point graph without edges.

The final example indicates a reason why matchings must be morphisms of graphs in the definition **3.4** of graph rewritings.

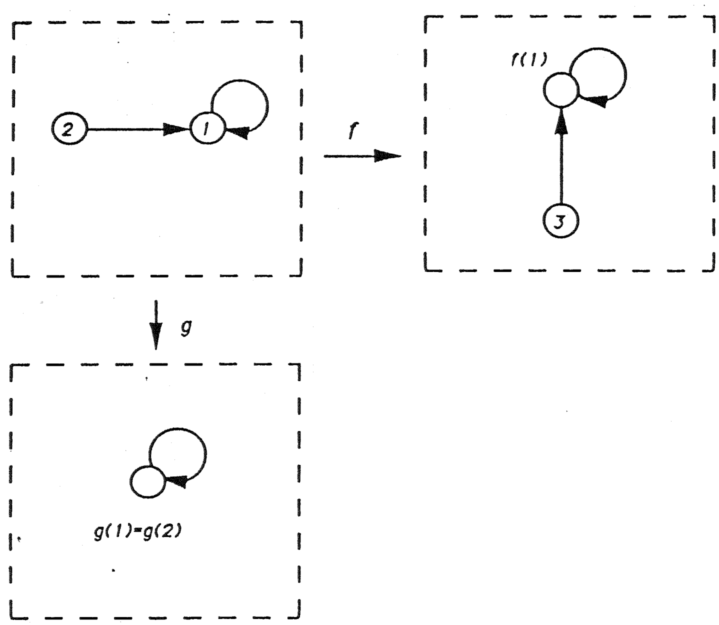


Figure 2:

4.4 Recall that matchings to rewriting rules are defined to be morphisms of graphs but not partial morphisms (Cf. **3.4**). We now observe what happens when matchings are allowed to be partial morphisms of graphs. First we note that any couple of rewriting rules being partial morphisms of graphs commute, because rewriting squares are pushouts in the category of graphs and partial morphisms by **3.5**. Hence every set of rewriting rules consisting of partial morphisms of graphs is strongly confluent, which seems to exceed.

Let $p = (< A, \alpha >, < B, \beta >, f : A \rightarrow B)$ and assume that $f(A) = B$ and there exists $a \in A$ such that f is undefined on a and a has no loops. (This rewriting rule p is not so special.) For any vertex v of an arbitrary graph $< G, \xi >$, define a matching $g :< A, \alpha > \rightarrow < G, \xi >$ such that $g(a) = v$ and undefined otherwise. Then g is in fact a partial morphism of graphs. The resultant graph H after applying p along g is a graph obtained by subtracting from G the vertex v and all edges connected with v. Thus this claims that any finite graph is reduced into the empty graph by iterating applications of p. Therefore these graph rewritings are nonsense.

5 Rewritings for Graphs with Labelled Edges

In this section we first define graphs with labelled edges and partial morphisms between them, and a primitive pushout construction similar to **3.1** is stated for graphs with labelled edges. The readers may easily understand anologies with results in the section **3** are also valid in this case.

Let Σ be a set of labels. *A graph* $< A, \alpha >$ *with* Σ*-labelled edges* is a pair of a set A and a collection $\alpha = \{\alpha_\sigma : A \rightharpoonup A \mid \sigma \in \Sigma\}$ of relations indexed by Σ. A *partial morphism* f of a graph $< A, \alpha >$ with Σ -labelled edges into a graph $< B, \beta >$ with Σ-labelled edges, denoted by $f :< A, \alpha > \rightarrow < B, \beta >$, is a partial function $f : A \rightarrow B$ satisfying $d(f)\alpha_\sigma f \sqsubseteq f\beta_\sigma$ for all $\sigma \in \Sigma$.

Similarly we have the category of graphs with Σ-labelled edges and partial morphisms between them. The following theorem constructs a primitive pushout for a pair of partial functions from a common set into graphs with labelled edges.

5.1 Theorem *If* $< B, \beta >$ *and* $< C, \gamma >$ *are graphs with* Σ*-labelled edges and if the square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & (1) & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

is a pushout in **Pfn**, *then* $h :< B, \beta > \rightarrow < D, \delta >$ *and* $k :< C, \gamma > \rightarrow < D, \delta >$ *are partial morphisms of graphs with* Σ*-labelled edges, where* $\delta_\sigma = h^\sharp \beta_\sigma h \sqcup k^\sharp \gamma_\sigma k$ *for each* $\sigma \in \Sigma$. *Moreover, if* $h' :< B, \beta > \rightarrow < D', \delta' >$ *and* $k' :< C, \gamma > \rightarrow < D', \delta' >$ *are partial morphisms of graphs with* Σ*-labelled edges satisfying* $fh' = gk'$, *then there exists a unique partial morphism* $t :< D, \delta > \rightarrow < D', \delta' >$ *of graphs with* Σ*-labelled edges such that* $h' = ht$ *and* $k' = kt$.

Similarly we have the following corollary from the last theorem.

5.2 Corollary *The category of graphs with* Σ*-labelled edges and partial morphisms between them has pushouts.*

Remark. A graph $< A, \alpha >$ with $\alpha^\sharp = \alpha$ is just an undirected graph. Hence almost all results in this note are also valid for undirected graphs.

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