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Abstract

This note presents a new formalization of graph rewritings which generalizes traditional graph rewritings. Relational notions of graphs and their rewritings are introduced and several properties about graph rewritings are discussed using relational calculus (theory of binary relations). Single pushout approaches to graph rewritings proposed by Raoult and Kennaway are compared with our rewritings of relational (labeled) graph. Moreover a more general sufficient condition for two rewritings to commute and a theorem concerning critical pairs useful to demonstrate the confluency of graph rewriting systems are also given.

1 Introduction

There are many researches [1-7,9,13,14,16-18,20-22] on graph grammars and graph rewritings which have a lot of applications including software specification, data bases, analysis of concurrent systems, developmental biology and many others. In these one of the advantages of categorical graph rewritings is to produce a universal reduction which eases theoretical investigation considerably.

Ehrig et al. [3, 4, 5] proposed algebraic graph grammars for a wide class of graphs and graph homomorphisms preserving graph structures. It is well-known that the category of graphs in [4, 7] is a topos ([8]) and so it has pushouts. In their formalization of graph grammars with double pushouts gluing conditions for existence of pushout-complements in the category of graphs provide an essential mean of controlling the semantics of rewriting rules. Gluing conditions are investigated by Ehrig and Kreowski [4] and Kawahara [9]. Using single pushouts and regarding production rules as partial functions preserving graph structures, another framework of graph rewritings were formalized by Raoult [22] and Kennaway [13]. Recently Ehrig and Löwe [3, 16, 18] studied rewritings based on single pushouts in Sig-algebras and proved pushout completeness for restricted signatures with monadic operator symbols only.

In this note we treat the category of (simple) graphs (with or without labeled edges) and partial functions preserving graph structures, and present a new formalization of graph rewritings by using a primitive pushout construction in the category. Graphs and morphisms introduced here are simple cases of relational structures and structure morphisms in the sense of [5, 16, 17]. However the notion of partial morphisms between graphs (as will be defined in Section 3) is briefly different from those of [5, 16, 17]. Thus graph rewritings in this note are always executed without any gluing conditions, only if a rewriting rule has a matching to a graph, and partially generalize graph derivations [4] and graph rewritings [22] in a reasonable sense. Moreover we state a more general sufficient condition for two rewritings to commute and

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critical pairs useful to demonstrate the confluency of graph rewriting systems. The framework of the note is elementary and the simplicity of discussions comes from the usage of relational calculus (theory of binary relations).

This note consists of the following sections. In Section 2 we present minimum fundamentals on relational calculus for the later calculations. The main subjects of this note are discussed in Section 3. We set up a new framework of graph rewritings, that is, the notions of (simple) graphs and partial morphisms between them are defined. For a pair of partial functions from a common set into graphs a primitive pushout square is constructed, which indicates that the category of graphs and partial morphisms has pushouts. At the end of the section we prove a more general sufficient condition for two graph rewritings to commute and a theorem on critical pairs useful to demonstrate the confluency of graph rewriting systems. In Section 4 we compare our approach with other approaches by Ehrig, Löwe, Kennaway and Okada [3, 14, 18, 21]. Some examples related to graph rewritings are listed in Section 5. In Section 6 we state how to develop our formalization of graph rewritings for graphs with labeled edges which contains graphs in the sense of Raoult [22].

2 Fundamentals on Relational Calculus

A *relation* α of a set A into another set B is a subset of the cartesian product $A \times B$ and denoted by $\alpha : A \rightarrow B$. The *inverse relation* $\alpha^\sharp : B \rightarrow A$ of α is a relation such that $(b, a) \in \alpha^\sharp$ if and only if $(a, b) \in \alpha$. The *composite* $\alpha\beta : A \rightarrow C$ of $\alpha : A \rightarrow B$ followed by $\beta : B \rightarrow C$ is a relation such that $(a, c) \in \alpha\beta$ if and only if there exists $b \in B$ with $(a, b) \in \alpha$ and $(b, c) \in \beta$.

As a relation of a set A into a set B is a subset of $A \times B$, the inclusion relation, union, intersection and difference of them are available as usual and denoted by \sqsubseteq , \sqcup , \sqcap and $-$, respectively. The *identity relation* $\text{id}_A : A \rightarrow A$ is a relation with $\text{id}_A = \{(a, a) \in A \times A \mid a \in A\}$ (the diagonal set of A).

The followings are the basic properties of relations and indicate that the totality of sets and relations forms a category **Rel** with involution (or shortly I-category).

2.1 Proposition (I-category) *Let $\alpha, \alpha' : A \rightarrow B$, $\beta, \beta' : B \rightarrow C$ and $\gamma : C \rightarrow D$ be relations. Then,*

- (a) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (associative),
- (b) $\text{id}_A\alpha = \alpha\text{id}_B = \alpha$ (identity),
- (c) $\alpha^\sharp^\sharp = \alpha$, $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$ (involutive),
- (d) *If $\alpha \sqsubseteq \alpha'$ and $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha'\beta'$ and $\alpha^\sharp \sqsubseteq \alpha'^\sharp$ (monotone).*

■

The distributive law for relations is trivial but indispensable in our relational calculus.

2.2 Proposition (Distributive Law) *The distributive law $\alpha(\bigsqcup_{\lambda \in \Lambda} \beta_\lambda)\gamma = \bigsqcup_{\lambda \in \Lambda} \alpha\beta_\lambda\gamma$ holds for relations $\alpha : A \rightarrow B$, $\beta_\lambda : B \rightarrow C$ ($\lambda \in \Lambda$) and $\gamma : C \rightarrow D$.*

■

A *partial function* f of a set A into a set B is a relation $f : A \rightarrow B$ with $f^\sharp f \sqsubseteq \text{id}_B$ and it is denoted by $f : A \rightarrow B$. A *(total) function* f of a set A into a set B is a relation $f : A \rightarrow B$ with $f^\sharp f \sqsubseteq \text{id}_B$ and $\text{id}_A \sqsubseteq ff^\sharp$, and it is also denoted by $f : A \rightarrow B$. Clearly a function is a partial function. Note that the identity relation id_A of a set A is a function. The definitions of partial functions and (total) functions here coincide with ordinary ones. A function $f : A \rightarrow B$ is injective if and only if $ff^\sharp = \text{id}_A$ and surjective if and only if $f^\sharp f = \text{id}_B$.

2.3 Proposition If $\alpha, \beta : A \rightarrow B$ are relations and $f : X \rightarrow A$, $g : Y \rightarrow B$ are partial functions, then $f(\alpha \sqcap \beta)g^\sharp = f\alpha g^\sharp \sqcap f\beta g^\sharp$. Moreover if $\alpha \sqsubseteq \beta$ then $f(\alpha - \beta)g^\sharp = f\alpha g^\sharp - f\beta g^\sharp$. ■

Given a relation $\alpha : A \rightarrow B$, the *domain* $d(\alpha) : A \rightarrow A$ of α is a relation defined by $d(\alpha) = \alpha\alpha^\sharp \sqcap \text{id}_A$. The domain $d(\alpha^\sharp) : B \rightarrow B$ of α^\sharp corresponds with the image of α . A partial function $f : A \rightarrow B$ is a function if and only if $d(f) = \text{id}_A$.

The following proposition is useful for manipulating domains of partial functions.

2.4 Proposition Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be relations and $f : A \rightarrow B$ a partial function. Then

- (a) $d(\alpha\beta)d(\alpha) = d(\alpha\beta)$ (or $d(\alpha\beta) \sqsubseteq d(\alpha)$),
- (b) $d(f\beta)f = fd(\beta)$.

■

2.5 Proposition Let $\alpha : A \rightarrow A$, $\theta : B \rightarrow B$ be relations and let $f : A \rightarrow B$ be a partial function. If $\theta \sqsubseteq f^\sharp\alpha f$, then $\theta = f^\sharp f\theta f^\sharp f$. ■

We denote the category of sets and functions by **Set** and the category of sets and partial functions by **Pfn**. Both of **Set** and **Pfn** have all small limits and colimits, so in particular, they have pushouts [19, 22, 13, 20]. Note that **Pfn** is equivalent to the category of sets with a base point (a selected element) and base point preserving functions. We assume that the readers are familiar with pushout constructions [22, 16, 20, 10] in **Pfn**. A singleton set $\{*\}$ is denoted by 1 and the maximum relation from a set A into 1 by $\Omega_A : A \rightarrow 1$, that is, $\Omega_A = \{(a, *) | a \in A\}$.

The following basic properties of pushouts in **Pfn** are indispensable for the later arguments in Section 3.

2.6 Proposition Let a square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

be a pushout in **Pfn**.

- (a) If g is an injective function, then so is h .
- (b) For a function $t : X \rightarrow C$ the composite $tk : X \rightarrow D$ is a function if and only if $d(t^\sharp) \sqcap d(g^\sharp) \sqsubseteq d(k)$.

■

3 Rewritings for Simple Graphs

3.1 Definition A (simple) graph $\langle A, \alpha \rangle$ is a pair of a set A and a relation $\alpha : A \rightarrow A$. A partial morphism f of a graph $\langle A, \alpha \rangle$ into a graph $\langle B, \beta \rangle$, denoted by $f : \langle A, \alpha \rangle \rightarrow \langle B, \beta \rangle$, is a partial function $f : A \rightarrow B$ satisfying $d(f)\alpha f \sqsubseteq f\beta$.

It is easily seen that a partial morphism among graphs is a partial function preserving edges on its domain of definitions.

Let $f : \langle A, \alpha \rangle \rightarrow \langle B, \beta \rangle$ and $g : \langle B, \beta \rangle \rightarrow \langle C, \gamma \rangle$ be partial morphisms of graphs. Since $d(f)\alpha f \sqsubseteq f\beta$ and $d(g)\beta g \sqsubseteq g\gamma$, we have $d(fg)\alpha fg = d(fg)d(f)\alpha fg$ (by 2.4(a)) $\sqsubseteq d(fg)f\beta g = fd(g)\beta g$ (by 2.4(b)) $\sqsubseteq fg\gamma$. Hence the composite of two partial morphisms of graphs is also a partial morphism of graphs. Thus we have the category $\mathbf{Pfn}(\mathbf{Graph})$ of graphs and partial morphisms between them.

The following theorem constructs a primitive pushout for a pair of partial functions from a common set into graphs.

3.2 Theorem *If $\langle B, \beta \rangle$ and $\langle C, \gamma \rangle$ are graphs and the square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & (1) & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

is a pushout in \mathbf{Pfn} , then $h : \langle B, \beta \rangle \rightarrow \langle D, \delta \rangle$ and $k : \langle C, \gamma \rangle \rightarrow \langle D, \delta \rangle$ are partial morphisms of graphs, where $\delta = h^\sharp \beta h \sqcup k^\sharp \gamma k$. Moreover, if $h' : \langle B, \beta \rangle \rightarrow \langle D', \delta' \rangle$ and $k' : \langle C, \gamma \rangle \rightarrow \langle D', \delta' \rangle$ are partial morphisms of graphs satisfying $fh' = gk'$, then there exists a unique partial morphism $t : \langle D, \delta \rangle \rightarrow \langle D', \delta' \rangle$ of graphs such that $h' = ht$ and $k' = kt$.

Proof. First we see that $h : \langle B, \beta \rangle \rightarrow \langle D, \delta \rangle$ and $k : \langle C, \gamma \rangle \rightarrow \langle D, \delta \rangle$ are partial morphisms of graphs. It simply follows from $d(h)\beta h \sqsubseteq hh^\sharp \beta h$ (by $d(h) = hh^\sharp \sqcap id_B$) $\sqsubseteq h\delta$ (by $\delta = h^\sharp \beta h \sqcup k^\sharp \gamma k$). Next assume that $h' : \langle B, \beta \rangle \rightarrow \langle D', \delta' \rangle$ and $k' : \langle C, \gamma \rangle \rightarrow \langle D', \delta' \rangle$ are partial morphisms of graphs satisfying $fh' = gk'$. Then we have $d(h')\beta h' \sqsubseteq h'\delta'$ and $d(k')\gamma k' \sqsubseteq k'\delta'$. As (1) is a pushout in \mathbf{Pfn} , there exists a unique partial function $t : D \rightarrow D'$ such that $h' = ht$ and $k' = kt$. It suffices to prove that $d(t)\delta t \sqsubseteq t\delta'$. But it follows from

$$\begin{aligned} d(t)\delta t &\sqsubseteq tt^\sharp(h^\sharp \beta h \sqcup k^\sharp \gamma k)t \quad (d(t) = tt^\sharp \sqcap id_D) \\ &= t(t^\sharp h^\sharp \beta ht \sqcup t^\sharp k^\sharp \gamma kt) \quad (\text{by (2.2)}) \\ &= t(h'^\sharp \beta h' \sqcup k'^\sharp \gamma k') \quad (h' = ht, k' = kt) \\ &= t(h'^\sharp d(h')\beta h' \sqcup k'^\sharp d(k')\gamma k') \quad (h' = d(h')h', k' = d(k')k') \\ &= t(h'^\sharp h'\delta' \sqcup k'^\sharp k'\delta') \quad (d(h')\beta h' \sqsubseteq h'\delta', d(k')\gamma k' \sqsubseteq k'\delta') \\ &\sqsubseteq t(\delta' \sqcup \delta') \quad (h'^\sharp h' \sqsubseteq id_{D'}, k'^\sharp k' \sqsubseteq id_{D'}) \\ &= t\delta'. \end{aligned}$$

This completes the proof. ■

Note that the graph $\langle D, \delta \rangle$ in the above proof is unique up to isomorphisms. The following is exactly a corollary of the last theorem.

3.3 Corollary *The category $\mathbf{Pfn}(\mathbf{Graph})$ of graphs and partial morphisms has pushouts. ■*

A partial morphism $f : \langle A, \alpha \rangle \rightarrow \langle B, \beta \rangle$ is said to be a *morphism* of graphs if $f : A \rightarrow B$ is a function. In other words, $f : \langle A, \alpha \rangle \rightarrow \langle B, \beta \rangle$ is a morphism of graphs if and only if f is a function with $\alpha f \sqsubseteq f\beta$. It is trivial that the composition of two morphisms of graphs is also a morphism of graphs and so one can consider the category \mathbf{Graph} of graphs and morphisms between them.

3.4 Definition A *rewriting rule* p is a triple of two graphs $\langle A, \alpha \rangle$, $\langle B, \beta \rangle$ and a partial function $f : A \rightarrow B$. (Note that f need not to be a partial morphism of graphs.) A *matching* to p is a morphism $g : \langle A, \alpha \rangle \rightarrow \langle G, \xi \rangle$ of graphs. Construct a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ G & \xrightarrow{k} & H \end{array}$$

in \mathbf{Pfn} and define $\eta = h^\sharp \beta h \sqcup k^\sharp (\xi - g^\sharp \alpha g) k$. Then the graph $\langle G, \xi \rangle$ is said to be rewritten into a graph $\langle H, \eta \rangle$ by applying a rewriting rule p along a matching g , and denoted by $\langle G, \xi \rangle \Rightarrow_{p/g} \langle H, \eta \rangle$. More precisely $\langle G, \xi \rangle \Rightarrow_{p/g} \langle H, \eta \rangle$ is called a graph rewriting with a rewriting square

$$\begin{array}{ccc} \langle A, \alpha \rangle & \xrightarrow{f} & \langle B, \beta \rangle \\ g \downarrow & & \downarrow h \\ \langle G, \xi \rangle & \xrightarrow{k} & \langle H, \eta \rangle . \end{array}$$

(Note that rewriting squares are not necessarily pushouts in the category of graphs and partial morphisms.)

The next proposition states a sufficient condition that rewriting squares are pushouts.

3.5 Proposition *Let $g : \langle A, \alpha \rangle \rightarrow \langle G, \xi \rangle$ be a matching to a rewriting rule $p = (\langle A, \alpha \rangle, \langle B, \beta \rangle, f : A \rightarrow B)$. If $f : \langle A, \alpha \rangle \rightarrow \langle B, \beta \rangle$ is a partial morphism of graphs, then a rewriting square*

$$\begin{array}{ccc} \langle A, \alpha \rangle & \xrightarrow{f} & \langle B, \beta \rangle \\ g \downarrow & & \downarrow h \\ \langle G, \xi \rangle & \xrightarrow{k} & \langle H, \eta \rangle \end{array}$$

with $\eta = h^\sharp \beta h \sqcup k^\sharp (\xi - g^\sharp \alpha g) k$ is a pushout in $\mathbf{Pfn}(\mathbf{Graph})$.

Proof. By the virtue of **3.2** it suffices to show that $\eta = h^\sharp \beta h \sqcup k^\sharp \xi k$. First note that $f^\sharp \alpha f \sqsubseteq f^\sharp f \beta \sqsubseteq \beta$ since $d(f) \alpha f \sqsubseteq f \beta$. Thus we have

$$\begin{aligned} \eta &= h^\sharp \beta h \sqcup k^\sharp (\xi - g^\sharp \alpha g) k \\ &\sqsupseteq h^\sharp f^\sharp \alpha f h \sqcup k^\sharp (\xi - g^\sharp \alpha g) k \\ &= k^\sharp g^\sharp \alpha g k \sqcup k^\sharp (\xi - g^\sharp \alpha g) k \\ &= k^\sharp \{g^\sharp \alpha g \sqcup (\xi - g^\sharp \alpha g)\} k \\ &= k^\sharp \xi k. \end{aligned}$$

and

$$\begin{aligned} \eta &= h^\sharp \beta h \sqcup k^\sharp (\xi - g^\sharp \alpha g) k \sqcup k^\sharp \xi k \\ &= h^\sharp \beta h \sqcup k^\sharp \xi k. \end{aligned}$$

This completes the proof. ■

The last proposition suggests that our graph rewritings coincide with those of Raoult [22] if rewriting rules are partial morphisms of graphs. It is easy to understand that almost all results about the confluency and concurrency of graph rewritings in [22] are analogously valid in our case. The following is a general sufficient condition for two graph rewritings to commute (or to be strongly confluent).

3.6 Theorem Let $p_\lambda = (\langle A_\lambda, \alpha_\lambda \rangle, \langle B_\lambda, \beta_\lambda \rangle, f_\lambda : A_\lambda \rightarrow B_\lambda)$ be rewriting rules, $g_\lambda : \langle A_\lambda, \alpha_\lambda \rangle \rightarrow \langle G, \xi \rangle$ matchings to p_λ and $\langle G, \xi \rangle \Rightarrow_{p_\lambda/g_\lambda} \langle H_\lambda, \eta_\lambda \rangle$ a graph rewriting induced by a rewriting square

$$\begin{array}{ccc} \langle A_\lambda, \alpha_\lambda \rangle & \xrightarrow{g_\lambda} & \langle G, \xi \rangle \\ f_\lambda \downarrow & & \downarrow k_\lambda \\ \langle B_\lambda, \beta_\lambda \rangle & \xrightarrow{h_\lambda} & \langle H_\lambda, \eta_\lambda \rangle \end{array}$$

for $\lambda = 0, 1$. If $f_\lambda : \langle A_\lambda, \alpha_\lambda \rangle \rightarrow \langle B_\lambda, \beta_\lambda \rangle$ ($\lambda = 0, 1$) is partial morphisms of graphs and $d(g_0^\#) \sqcap d(g_1^\#) \sqsubseteq d(k_0) \sqcap d(k_1)$, then there exist a matching $g'_\lambda : \langle A, \alpha_\lambda \rangle \rightarrow \langle H_{1-\lambda}, \eta_{1-\lambda} \rangle$ ($\lambda = 0, 1$) and a graph $\langle H, \eta \rangle$ such that $\langle H_{1-\lambda}, \eta_{1-\lambda} \rangle \Rightarrow_{p_\lambda/g'_\lambda} \langle H, \eta \rangle$ ($\lambda = 0, 1$).

Proof. By virtue of **3.3** we can construct the following three pushouts in the category of graphs and partial morphisms :

$$\begin{array}{ccccc} & & \langle A_0, \alpha_0 \rangle & \xrightarrow{f_0} & \langle B_0, \beta_0 \rangle \\ & & g_0 \downarrow & (0) & \downarrow h_0 \\ \langle A_1, \alpha_1 \rangle & \xrightarrow{g_1} & \langle G, \xi \rangle & \xrightarrow{k_0} & \langle H_0, \eta_0 \rangle \\ f_1 \downarrow & (1) & \downarrow k_1 & (2) & \downarrow h'_0 \\ \langle B_1, \beta_1 \rangle & \xrightarrow{h_1} & \langle H_1, \eta_1 \rangle & \xrightarrow{h'_1} & \langle H, \eta \rangle \end{array}$$

Set $g'_\lambda = g_\lambda k_{1-\lambda}$ ($\lambda = 0, 1$). Then g'_λ ($\lambda = 0, 1$) is a function from the assumption and **2.6**, and so $g'_\lambda : \langle A_\lambda, \alpha_\lambda \rangle \rightarrow \langle H_{1-\lambda}, \eta_{1-\lambda} \rangle$ is a matching to p_λ ($\lambda = 0, 1$). Since two squares (0)+(2) and (1)+(2) are pushouts in the category of graphs and partial morphisms, we have a graph rewriting $\langle H_{1-\lambda}, \eta_{1-\lambda} \rangle \Rightarrow_{p_\lambda/g'_\lambda} \langle H, \eta \rangle$ ($\lambda = 0, 1$) by the definition **3.4**, which proves the theorem. ■

The rest of this section is concerned with critical pairs [15, 22, 21] that are useful to demonstrate the confluency of actual graph rewriting systems. A basic idea on critical pairs in graph rewriting systems was initiated by Raoult [22]. Our approach is an extension of his method.

In what follows we assume that rewriting rules are morphisms of graphs and matchings are injective morphisms of graphs. Therefore rewriting squares hereafter are pushouts from **3.5** and so they will be called rewriting pushouts. An essential point of the dicussion below is due to **2.6** stating that pushouts in $\mathbf{Pfn}(\mathbf{Graph})$ preserve injective morphisms of graphs.

3.7 Definition A *rewriting system* P is simply a family of rewriting rules (morphisms of graphs). Let $\langle G, \xi \rangle \Rightarrow_{f_\lambda/g_\lambda} \langle H_\lambda, \eta_\lambda \rangle$ be a graph rewriting induced by a rewriting pushout

$$\begin{array}{ccc} \langle A_\lambda, \alpha_\lambda \rangle & \xrightarrow{g_\lambda} & \langle G, \xi \rangle \\ f_\lambda \downarrow & & \downarrow k_\lambda \\ \langle B_\lambda, \beta_\lambda \rangle & \xrightarrow{h_\lambda} & \langle H_\lambda, \eta_\lambda \rangle \end{array}$$

with $f_\lambda \in P$ for $\lambda = 0, 1$. The pair of graph rewritings $\langle G, \xi \rangle \Rightarrow_{f_\lambda/g_\lambda} \langle H_\lambda, \eta_\lambda \rangle$ is called *confluent* on P if there exist rewriting rules $f'_\lambda \in P$ and graph rewritings $\langle H_\lambda, \eta_\lambda \rangle \Rightarrow_{f'_\lambda/g'_\lambda} \langle H, \eta \rangle$ ($\lambda = 0, 1$) induced by rewriting pushouts

$$\begin{array}{ccc} \langle A'_\lambda, \alpha'_\lambda \rangle & \xrightarrow{g'_\lambda} & \langle H_\lambda, \eta_\lambda \rangle \\ f'_\lambda \downarrow & & \downarrow k'_\lambda \\ \langle B_\lambda, \beta_\lambda \rangle & \xrightarrow{h'_\lambda} & \langle H, \eta \rangle \end{array}$$

satisfying $k_0 k'_0 = k_1 k'_1$.

Let I be a set and $0 : I \rightarrow I$ the empty relation. Then $\langle I, 0 \rangle$ is a discrete graph over I , that is, a graph without edges. When $\langle A, \alpha \rangle$ is a graph, every function $f : I \rightarrow A$ always induces a morphism $f : \langle I, 0 \rangle \rightarrow \langle A, \alpha \rangle$ of graphs.

3.8 Definition Let f_λ be a rewriting rule in a rewriting system P ($\lambda = 0, 1$). A critical pair formed from f_0 and f_1 in P is a pair of morphisms $t_\lambda : \langle S, \sigma \rangle \rightarrow \langle T_\lambda, \tau_\lambda \rangle$ ($\lambda = 0, 1$) of graphs such that all squares in the following diagram are pushouts in **Graph** for some pair of injective functions $i_\lambda : I \rightarrow A_\lambda$ ($\lambda = 0, 1$).

$$\begin{array}{ccccc} \langle I, 0 \rangle & \xrightarrow{i_0} & \langle A_0, \alpha_0 \rangle & \xrightarrow{f_0} & \langle B_0, \beta_0 \rangle \\ i_1 \downarrow & & \downarrow s_0 & & \downarrow u_0 \\ \langle A_1, \alpha_1 \rangle & \xrightarrow{s_1} & \langle S, \sigma \rangle & \xrightarrow{t_0} & \langle T_0, \tau_0 \rangle \\ f_1 \downarrow & & \downarrow t_1 & & \\ \langle B_1, \beta_1 \rangle & \xrightarrow{u_1} & \langle T_1, \tau_1 \rangle & & \end{array}$$

Note that if A_0 and A_1 are finite sets then the set of critical pairs formed from f_0 and f_1 is finite.

3.9 Lemma *If a graph rewriting $\langle G, \xi \rangle \Rightarrow_{f_\lambda} \langle H_\lambda, \eta_\lambda \rangle$ is induced by a rewriting pushout*

$$\begin{array}{ccc} \langle A_\lambda, \alpha_\lambda \rangle & \xrightarrow{g_\lambda} & \langle G, \xi \rangle \\ f_\lambda \downarrow & (1) & \downarrow k_\lambda \\ \langle B_\lambda, \beta_\lambda \rangle & \xrightarrow{h_\lambda} & \langle H_\lambda, \eta_\lambda \rangle \end{array}$$

for $\lambda = 0, 1$, then there exist a critical pair $t_\lambda : \langle S, \sigma \rangle \rightarrow \langle T_\lambda, \tau_\lambda \rangle$ ($\lambda = 0, 1$) and matchings $s_\lambda : \langle A_\lambda, \alpha_\lambda \rangle \rightarrow \langle S, \sigma \rangle$ ($\lambda = 0, 1$) and $s : \langle S, \sigma \rangle \rightarrow \langle G, \xi \rangle$ such that the rewriting pushout (1) is decomposed into two pushouts in **Graph** through $\langle S, \sigma \rangle$ as follows:

$$\begin{array}{ccccc} \langle A_\lambda, \alpha_\lambda \rangle & \xrightarrow{s_\lambda} & \langle S, \sigma \rangle & \xrightarrow{s} & \langle G, \xi \rangle \\ f_\lambda \downarrow & & \downarrow t_\lambda & & \downarrow k_\lambda \\ \langle B_\lambda, \beta_\lambda \rangle & \xrightarrow{u_\lambda} & \langle T_\lambda, \tau_\lambda \rangle & \xrightarrow{v_\lambda} & \langle H_\lambda, \eta_\lambda \rangle \end{array}$$

Proof. Construct a pullback

$$\begin{array}{ccc} I & \xrightarrow{i_0} & A_0 \\ i_1 \downarrow & & \downarrow g_0 \\ A_1 & \xrightarrow{g_1} & G \end{array}$$

in **Set** and a pushout

$$\begin{array}{ccc} \langle I, 0 \rangle & \xrightarrow{i_0} & \langle A_0, \alpha_0 \rangle \\ i_1 \downarrow & (2) & \downarrow s_0 \\ \langle A_1, \alpha_1 \rangle & \xrightarrow{s_1} & \langle S, \sigma \rangle \end{array}$$

in **Graph**. Since (1) is a pushout there exists a unique morphism $s : \langle S, \sigma \rangle \rightarrow \langle G, \xi \rangle$ such that $g_\lambda = s_\lambda s$ ($\lambda = 0, 1$). Remark that s_0, s_1 and s are injective. Also construct a rewriting pushout

$$\begin{array}{ccc} \langle A_\lambda, \alpha_\lambda \rangle & \xrightarrow{s_\lambda} & \langle S, \sigma \rangle \\ f_\lambda \downarrow & & \downarrow t_\lambda \\ \langle B_\lambda, \beta_\lambda \rangle & \xrightarrow{u_\lambda} & \langle T_\lambda, \tau_\lambda \rangle \end{array}$$

for $\lambda = 0, 1$. Thus we have a critical pair (t_0, t_1) formed from f_0 and f_1 and there exists a unique morphism v_λ of graphs such that $sk_\lambda = t_\lambda v_\lambda$ and $h_\lambda = u_\lambda v_\lambda$ ($\lambda = 0, 1$).

$$\begin{array}{ccccc} \langle A_\lambda, \alpha_\lambda \rangle & \xrightarrow{s_\lambda} & \langle S, \sigma \rangle & \xrightarrow{s} & \langle G, \xi \rangle \\ f_\lambda \downarrow & & \downarrow t_\lambda & (3) & \downarrow k_\lambda \\ \langle B_\lambda, \beta_\lambda \rangle & \xrightarrow{u_\lambda} & \langle T_\lambda, \tau_\lambda \rangle & \xrightarrow{v_\lambda} & \langle H_\lambda, \eta_\lambda \rangle \end{array}$$

By the basic property of pushouts the square (3) is a pushout. This completes the proof. \blacksquare

A rewriting system P is *confluent* if every pair of rewritings $\langle G, \xi \rangle \Rightarrow_{f_\lambda} \langle H_\lambda, \eta_\lambda \rangle$ ($\lambda = 0, 1$) in P is confluent on P . The following is a main theorem of the note, which asserts that the confluency of rewriting systems are reduced to that of critical pairs.

3.10 Theorem *A graph rewriting system P is confluent if and only if every critical pair in P is confluent.*

Proof. The only-if part is trivial. So we will show the if part. Assume that $f_\lambda : \langle A_\lambda, \alpha_\lambda \rangle \rightarrow \langle B_\lambda, \beta_\lambda \rangle$ is a rewriting rule in P and $\langle G, \xi \rangle \Rightarrow_{f_\lambda/g_\lambda} \langle H_\lambda, \eta_\lambda \rangle$ is a graph rewriting induced by a rewriting pushout

$$\begin{array}{ccc} \langle A_\lambda, \alpha_\lambda \rangle & \xrightarrow{g_\lambda} & \langle G, \xi \rangle \\ f_\lambda \downarrow & (1) & \downarrow k_\lambda \\ \langle B_\lambda, \beta_\lambda \rangle & \xrightarrow{h_\lambda} & \langle H_\lambda, \eta_\lambda \rangle \end{array}$$

for $\lambda = 0, 1$. By **3.9** there exists a critical pair $t_\lambda : \langle S, \sigma \rangle \rightarrow \langle T_\lambda, \tau_\lambda \rangle$ ($\lambda = 0, 1$) such that the rewriting pushout (1) is decomposed into two pushouts in **Graph** as follows:

$$\begin{array}{ccccc} \langle A_\lambda, \alpha_\lambda \rangle & \xrightarrow{s_\lambda} & \langle S, \sigma \rangle & \xrightarrow{s} & \langle G, \xi \rangle \\ f_\lambda \downarrow & & \downarrow t_\lambda & & \downarrow k_\lambda \\ \langle B_\lambda, \beta_\lambda \rangle & \xrightarrow{u_\lambda} & \langle T_\lambda, \tau_\lambda \rangle & \xrightarrow{v_\lambda} & \langle H_\lambda, \eta_\lambda \rangle \end{array}$$

From the assumption that every critical pair in P is confluent there exists a pair of rewriting pushouts

$$\begin{array}{ccc} \langle A'_\lambda, \alpha'_\lambda \rangle & \xrightarrow{g'_\lambda} & \langle T_\lambda, \tau_\lambda \rangle \\ f'_\lambda \downarrow & & \downarrow k'_\lambda \\ \langle B'_\lambda, \beta'_\lambda \rangle & \xrightarrow{h'_\lambda} & \langle T, \tau \rangle \end{array}$$

($\lambda = 0, 1$) such that $t_0 k'_0 = t_1 k'_1$. Construct a pushout

$$\begin{array}{ccc} \langle S, \sigma \rangle & \xrightarrow{t_0 k'_0 = t_1 k'_1} & \langle T, \tau \rangle \\ s \downarrow & & \downarrow k' \\ \langle G, \xi \rangle & \xrightarrow{h'} & \langle H', \eta' \rangle. \end{array}$$

Then there exists a unique morphism $w_\lambda : \langle H_\lambda, \eta_\lambda \rangle \rightarrow \langle H', \eta' \rangle$ ($\lambda = 0, 1$) of graphs making the square (*) below a pushout.

$$\begin{array}{ccccc} \langle A'_\lambda, \alpha'_\lambda \rangle & \xrightarrow{g'_\lambda} & \langle T_\lambda, \tau_\lambda \rangle & \xrightarrow{v_\lambda} & \langle H_\lambda, \eta_\lambda \rangle \\ f'_\lambda \downarrow & & \downarrow k'_\lambda & (*) & \downarrow w_\lambda \\ \langle B'_\lambda, \beta'_\lambda \rangle & \xrightarrow{h'_\lambda} & \langle T, \tau \rangle & \xrightarrow{k'} & \langle H', \eta' \rangle \end{array}$$

The above diagram induces a graph rewriting $\langle H_\lambda, \eta_\lambda \rangle \Rightarrow_{f'_\lambda/g'_\lambda v_\lambda} \langle H', \eta' \rangle$ ($\lambda = 0, 1$) in P . Hence the given pair of graph rewritings is confluent on P . \blacksquare

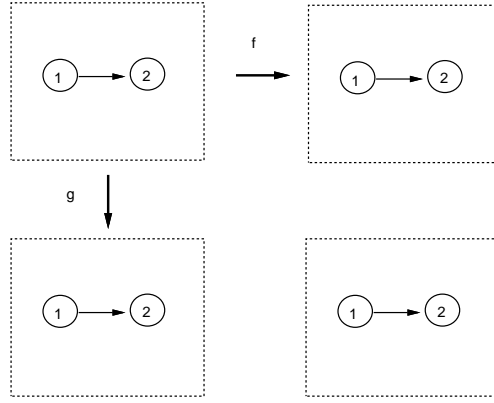


Figure 1:

4 Observation

We first compare our category of graphs with that of Löwe and Ehrig [18]. Let $\langle A, \alpha \rangle$ be a graph in our sense. We have two functions $i_{\alpha}p : \alpha \rightarrow A$ and $i_{\alpha}q : \alpha \rightarrow A$, where $i_{\alpha} : \alpha \rightarrow A \times A$ is an inclusion function and $p : A \times A \rightarrow A$ and $q : A \times A \rightarrow A$ are projections. Then $(\alpha, A, i_{\alpha}p, i_{\alpha}q)$ is naturally considered as a Sig-algebra with $Sig = \{s, t : E \rightarrow V\}$ in the sense of [18]. Thus a graph in our sense exactly corresponds to a Sig-algebra (G_E, G_V, s^G, t^G) such that a function $(s^G, t^G) : G_E \rightarrow G_V \times G_V$ is injective. A partial Sig-algebra morphism from $\langle A, \alpha \rangle$ to $\langle B, \beta \rangle$ is a tuple $(\langle A', \alpha' \rangle, i_f, t_f)$ of a subgraph $\langle A', \alpha' \rangle$ of $\langle A, \alpha \rangle$, an inclusion function $i_f : \langle A', \alpha' \rangle \rightarrow \langle A, \alpha \rangle$, and a (total) graph morphism $t_f : \langle A', \alpha' \rangle \rightarrow \langle B, \beta \rangle$. It corresponds to a notion of partial morphisms [23, 14] over **Graph**. But **Graph** has pushouts which are not hereditary in the sense of Kennaway [14], so the category of partial morphisms constructed from **Graph** is not pushout complete [14]. **Figure 1** illustrates a pushout which is not hereditary. Let $f : \langle A, \alpha \rangle \rightarrow \langle B, \beta \rangle$ be a partial morphism of graphs in our sense. We have the domain A' of partial function $f : A \rightarrow B$, an inclusion function $i_f : A' \rightarrow A$ and a function $t_f : A' \rightarrow B$ such that $f = i_f^{\sharp} t_f$. Define α' by constructing a pullback

$$\begin{array}{ccc}
 \alpha' & \xrightarrow{i_{\alpha'}} & A' \times A' \\
 \downarrow & \text{PB} & \downarrow i_f \times i_f \\
 \alpha & \xrightarrow{i_{\alpha}} & A \times A
 \end{array}$$

in **Set**. Since $d(f)\alpha f \sqsubseteq f\beta$ it follows by assumptions that $i_f : \langle A', \alpha' \rangle \rightarrow \langle A, \alpha \rangle$ and $t_f : \langle A', \alpha' \rangle \rightarrow \langle B, \beta \rangle$ are morphisms of graphs. But there may be many subgraphs $\langle A', \alpha^* \rangle$ of $\langle A, \alpha \rangle$ such that $t_f : \langle A', \alpha^* \rangle \rightarrow \langle B, \beta \rangle$ is a morphism of graphs. This is a difference between our partial morphisms of graphs and those in [18]. **Figure 1** indicates that Löwe's pushout construction is not closed under the subclass of our graphs and so it is meaningful to prove the pushout completeness of the category **Pfn(Graph)** in our sense(cf. **3.3**).

Ehrig and Löwe [16, 18] proved the pushout completeness of the category of Sig-algebras whose signature contains monadic operator symbols only. In this case the category of Sig-algebras is equivalent to a functor category over **Set** which is a topos [8].

Relations and partial functions can be similarly considered in topoi [9, 10]. We present a new pushout completeness [10] in the following

4.1 Theorem *If a topos **E** has the following properties:*

- (a) the set $\mathbf{Sub}(A)$ of subobjects of an object A is a complete lattice by inclusion,
- (b) the distributive law (cf. **2.2**) of relations holds,

then the category of partial functions in \mathbf{E} is finitely cocomplete. ■

Kennaway [14] introduced the notion of hereditary pushouts and showed that if \mathbf{E} satisfying the condition (a) of **4.1** has hereditary pushouts, then $P(\mathbf{E})$ has pushouts. When **4.1** holds, every pushout square in \mathbf{E} is also a pushout in the category of partial functions in \mathbf{E} , that is, it is hereditary [14].

Next we consider Ehrig's double pushout approach [3] in our category \mathbf{Graph} , that is, assume that the following two squares are pushouts in \mathbf{Graph} and that m is an injective function.

$$\begin{array}{ccccc} \langle A, \alpha \rangle & \xleftarrow{m} & \langle D, \delta \rangle & \xrightarrow{f} & \langle B, \beta \rangle \\ g \downarrow & & \downarrow s & & \downarrow h \\ \langle G, \xi \rangle & \xleftarrow{n} & \langle E, \varepsilon \rangle & \xrightarrow{k} & \langle H, \eta \rangle \end{array}$$

Then $\delta m \sqsubseteq m\alpha$, $\delta s \sqsubseteq s\varepsilon$, $\delta f \sqsubseteq f\beta$, $\xi = g^\sharp\alpha g \sqcup n^\sharp\varepsilon n$ and $\eta = h^\sharp\beta h \sqcup k^\sharp\varepsilon k$ by **3.2**. Since $nn^\sharp = \text{id}_E$ by the pushout property it is easy to see that $s^\sharp\delta s \sqsubseteq \varepsilon$ and

$$\begin{aligned} n(\xi - g^\sharp\alpha g)n^\sharp &= (ng^\sharp\alpha gn^\sharp \sqcup nn^\sharp\varepsilon nn^\sharp) - ng^\sharp\alpha gn^\sharp \quad (\text{by } \mathbf{2.3}) \\ &= (ng^\sharp\alpha gn^\sharp \sqcup \varepsilon) - ng^\sharp\alpha gn^\sharp \quad (nn^\sharp = \text{id}_E) \\ &= \varepsilon - ng^\sharp\alpha gn^\sharp \\ &\sqsubseteq \varepsilon. \end{aligned}$$

Hence $n(\xi - g^\sharp\alpha g)n^\sharp \sqcup s^\sharp\delta s \sqsubseteq \varepsilon$. Now put $\hat{\varepsilon} = n(\xi - g^\sharp\alpha g)n^\sharp \sqcup s^\sharp\delta s$. From $n^\sharp\varepsilon n - g^\sharp\alpha g = n^\sharp n(n^\sharp\varepsilon n - g^\sharp\alpha g)n^\sharp n$ (by **2.5**) $= n^\sharp(\varepsilon - ng^\sharp\alpha gn^\sharp)n$ (by **2.2**) and $nn^\sharp = \text{id}_E$, we have

$$\begin{aligned} g^\sharp\alpha g \sqcup n^\sharp\hat{\varepsilon}n &= g^\sharp\alpha g \sqcup n^\sharp n(\xi - g^\sharp\alpha g)n^\sharp n \sqcup n^\sharp s^\sharp\delta s n \quad (\text{by } \mathbf{2.3}) \\ &= g^\sharp\alpha g \sqcup n^\sharp(\varepsilon - ng^\sharp\alpha gn^\sharp)n \sqcup n^\sharp s^\sharp\delta s n \\ &= g^\sharp\alpha g \sqcup (n^\sharp\varepsilon n - g^\sharp\alpha g) \sqcup n^\sharp s^\sharp\delta s n \\ &= g^\sharp\alpha g \sqcup n^\sharp\varepsilon n \sqcup n^\sharp s^\sharp\delta s n \\ &= g^\sharp\alpha g \sqcup n^\sharp\varepsilon n \quad (s^\sharp\delta s \sqsubseteq \varepsilon) \\ &= \xi. \end{aligned}$$

Thus $\hat{\varepsilon} : E \rightarrow E$ is the least relation such that $s^\sharp\delta s \sqsubseteq \hat{\varepsilon}$ and $\xi = g^\sharp\alpha g \sqcup n^\sharp\hat{\varepsilon}n$. Hence it is reasonable to assume that $\varepsilon = \hat{\varepsilon}$. In this case we have

$$\begin{aligned} \eta &= h^\sharp\beta h \sqcup k^\sharp n(\xi - g^\sharp\alpha g)n^\sharp k \sqcup k^\sharp s^\sharp\delta s k \\ &= h^\sharp\beta h \sqcup k^\sharp n(\xi - g^\sharp\alpha g)n^\sharp k \sqcup h^\sharp f^\sharp\delta f h \quad (fh = sk) \\ &= h^\sharp\beta h \sqcup k^\sharp n(\xi - g^\sharp\alpha g)n^\sharp k \quad (f^\sharp\delta f \sqsubseteq \beta). \end{aligned}$$

This shows that $\hat{\varepsilon} : E \rightarrow E$ is the least relation ε which makes the above squares pushouts. In our category of graphs \mathbf{Graph} the pushout complement is not always exist and not unique (cf. **5.1**). If there exists a pushout complement, our rewriting using single pushout coincides with the double pushout rewriting which uses the least pushout complement.

Finally we consider the boundary graphs (or B-graphs) due to Okada and Hayashi[21] in $\mathbf{Pfn}(\mathbf{Graph})$. If a matching $g : \langle A, \alpha \rangle \rightarrow \langle G, \xi \rangle$ to $p = (\langle A, \alpha \rangle, \langle B, \beta \rangle, f : A \rightarrow B)$ is an injective morphism of graphs such that $\text{deg}(g(a)) = \text{deg}(a)$ for each $a \in A$ on which f is undefined, then the rewritings coincides with those of B-graphs.

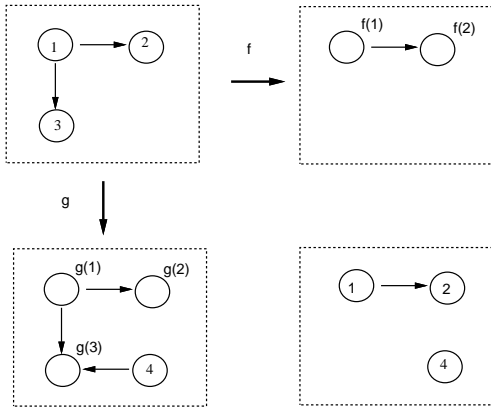


Figure 2:

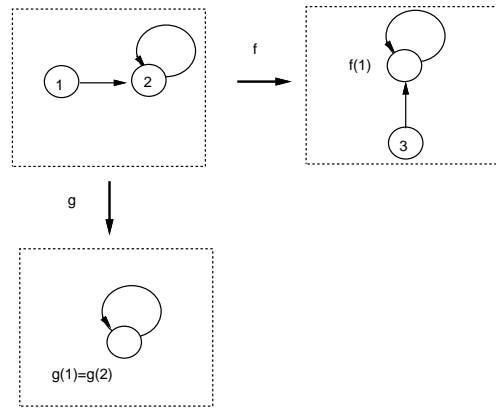


Figure 3:

5 Examples of Graph Rewritings

In this section a few examples related to graph rewritings are listed. The first example shows that pushout-complements are not unique in **Graph**.

5.1 Let $\alpha, \beta, \gamma : A \rightarrow A$ be relations with $\alpha \sqsubseteq \gamma \sqsubseteq \beta$. Then because of **3.2** the square

$$\begin{array}{ccc} \langle A, \alpha \rangle & \xrightarrow{\text{id}_A} & \langle A, \beta \rangle \\ \text{id}_A \downarrow & & \downarrow \text{id}_A \\ \langle A, \gamma \rangle & \xrightarrow{\text{id}_A} & \langle A, \beta \rangle \end{array}$$

is a pushout in the category of graphs and morphisms between them. Therefore the square is a pushout for any choice of γ satisfying $\alpha \sqsubseteq \gamma \sqsubseteq \beta$. The choice of $\hat{\varepsilon}$ in **4** means the most economical way to have pushout-complements.

Next we present two simple examples of graph rewritings to which conventional graph rewritings cannot be applied.

5.2 In **Figure 2** g is a neat morphism of graphs with respect to theories of Ehrig [3], Raoult [22] and ours. But f is not a morphism of graphs and it is not worth to be a rewriting rule in the sense of Raoult [22]. On the other hand f means a fast production in the double pushout approach of [3] but unfortunately the necessary pushout-complement does not exist since the gluing condition is not satisfied. However we have the bottom right resultant graph by applying our formalization.

5.3 In **Figure 3** g is a morphism of graphs and f is a partial morphism of graphs in all theories of Ehrig [3], Raoult [22] and ours. However graph rewritings of Ehrig [3] and Raoult [22] does not work again because the gluing conditions are not valid. In this case the resultant graph given by our graph rewritings is one point graph without edges.

The final two examples indicate reasons why rewriting rules are not restricted to morphisms of graphs and why matchings must be morphisms of graphs in the definition **3.4** of graph rewritings.

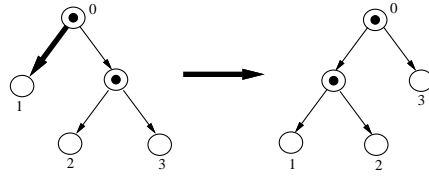


Figure 4:

5.4 In order to treat with more general graph rewritings we do not restrict rewriting rules to (partial) morphisms of graphs (Cf. **3.4**). **Figure 4** illustrates an important example of graph rewriting rules being not morphisms of graphs, because the edge denoted by a bold arrow is not preserved. The rule expresses usual associative laws.

5.5 Recall that matchings to rewriting rules are defined to be morphisms of graphs but not partial morphisms (Cf. **3.4**). We now observe what happens when matchings are allowed to be partial morphisms of graphs. First we note that any couple of rewriting rules being partial morphisms of graphs commute, because rewriting squares are pushouts in the category of graphs and partial morphisms by **3.5**. Hence every set of rewriting rules consisting of partial morphisms of graphs is strongly confluent, which seems to exceed. Let $p = (\langle A, \alpha \rangle, \langle B, \beta \rangle, f : A \rightarrow B)$ and assume that $f(A) = B$ and there exists $a \in A$ such that f is undefined on a and a has no loops. (This rewriting rule p is not so special.) For any vertex v of an arbitrary graph $\langle G, \xi \rangle$, define a matching $g : \langle A, \alpha \rangle \rightarrow \langle G, \xi \rangle$ such that $g(a) = v$ and undefined otherwise. Then g is in fact a partial morphism of graphs. The resultant graph H after applying p along g is a graph obtained by subtracting from G the vertex v and all edges connected with v . Thus this claims that any finite graph is reduced into the empty graph by iterating applications of p .

6 Rewritings for Graphs with Labeled Edges

In this section we first define graphs with labeled edges and partial morphisms between them, and a primitive pushout construction similar to **3.2** is stated for graphs with labeled edges. The readers may easily understand analogies with results in the section **3** are also valid in this case.

Let Σ be a set of labels. A graph $\langle A, \alpha \rangle$ with Σ -labeled edges is a pair of a set A and a collection $\alpha = \{\alpha_\sigma : A \rightarrow A \mid \sigma \in \Sigma\}$ of relations indexed by Σ . A *partial morphism* f of a graph $\langle A, \alpha \rangle$ with Σ -labeled edges into a graph $\langle B, \beta \rangle$ with Σ -labeled edges, denoted by $f : \langle A, \alpha \rangle \rightarrow \langle B, \beta \rangle$, is a partial function $f : A \rightarrow B$ satisfying $d(f)\alpha_\sigma f \sqsubseteq f\beta_\sigma$ for all $\sigma \in \Sigma$.

Similarly we have the category of graphs with Σ -labeled edges and partial morphisms between them. The following theorem constructs a primitive pushout for a pair of partial functions from a common set into graphs with labeled edges.

6.1 Theorem *If $\langle B, \beta \rangle$ and $\langle C, \gamma \rangle$ are graphs with Σ -labeled edges and if the square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & (1) & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

is a pushout in \mathbf{Pfn} , then $h : \langle B, \beta \rangle \rightarrow \langle D, \delta \rangle$ and $k : \langle C, \gamma \rangle \rightarrow \langle D, \delta \rangle$ are partial morphisms of graphs with Σ -labeled edges, where $\delta_\sigma = h^\# \beta_\sigma h \sqcup k^\# \gamma_\sigma k$ for each $\sigma \in \Sigma$. Moreover, if $h' : \langle B, \beta \rangle \rightarrow \langle D', \delta' \rangle$ and $k' : \langle C, \gamma \rangle \rightarrow \langle D', \delta' \rangle$ are partial morphisms of graphs with Σ -labeled edges satisfying $fh' = gk'$, then there exists a unique partial morphism $t : \langle D, \delta \rangle \rightarrow \langle D', \delta' \rangle$ of graphs with Σ -labeled edges such that $h' = ht$ and $k' = kt$.

Similarly we have the following corollary from the last theorem.

6.2 Corollary *The category of graphs $\mathbf{Pfn}(\Sigma\text{-Graph})$ with Σ -labeled edges and partial morphisms between them has pushouts.*

Remark. A graph $\langle A, \alpha \rangle$ with $\alpha^\# = \alpha$ is just an undirected graph. Hence almost all results in this note are also valid for undirected graphs.

6.3 Example Let N be the set of natural numbers. A graph $\langle A, \alpha \rangle$ with N -labeled edges satisfying the following conditions:

- (a) α_i is a partial function for any $i \in N$, i.e. $\alpha_i^\# \alpha_i \sqsubseteq \text{id}_A$,
- (b) $d(\alpha_j) \sqsubseteq d(\alpha_i)$ for any $i \leq j$ ($i, j \in N$),

is equivalent to a graph $\langle A, \delta_A : A \rightarrow A^* \rangle$ in the sense of Raoult [22]. Since graph morphisms in [22] are identical with morphisms of graphs with Σ -labeled edges, the category of graphs in [22] is a subcategory of $\mathbf{Pfn}(N\text{-Graph})$. Though the category of graphs in [22] does not have pushouts Raoult [22] showed a sufficient condition for existence of pushouts.

7 Concluding Remark

Ehrig and Löwe [16, 17, 18] have extensively developed the theory of graph rewritings using partial functions and single pushouts from an algebraic viewpoint. They reexamined that several properties of graph grammars can be simply proved within single pushout approaches and demonstrated the efficiency of the single pushout formalization. Kennaway [14] investigated the pushout completeness of abstract categories of partial morphisms. But their categories of graphs are different from $\mathbf{Pfn}(\mathbf{Graph})$.

We proved the pushout completeness of the category of simple graphs and partial morphisms using the relational calculus. We claim two points. First our notions and proofs are simple and clear. The relational calculus is convenient to deal with partial functions. Second our framework can be extended to more general relational categories which have many applications. For example, a relational structure $\langle A, \alpha \rangle$, a pair of a set A and a relation $\alpha : SA \rightarrow TA$, is considered as a generalization of graphs, where $S, T : \mathbf{Set} \rightarrow \mathbf{Set}$ are two functors. We can construct a category of relational structures in which similar properties to the case of simple graphs also hold [11].

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