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Abstract

This paper discusses some structural conditions under which Russellian propositions in the sense of J. Barwise and J. Etchemendy [2] are paradoxical, and the computational complexity of the problems whether or not Russellian proposition is paradoxical, intrinsically paradoxical, and classical.

1 Introduction

In situation theory, there are at least two kinds of the propositions to be considered ([2], [3]), *Austinian* propositions and *Russellian* propositions. An Austinian proposition is true if the situation about the proposition is of the type. By contrast, a Russellian proposition is true if there is a situation such that the proposition is of the type. In general, a Russellian proposition is simpler than an Austinian one, and uniquely determines its type. So in this paper we deal with Russellian propositions.

First, we consider the *Liar sentence* expressed by (λ):

(λ) This proposition is not true.

Intuitively, we can understand that (λ) is paradoxical in the following way:

1. Let f be the proposition expressed by (λ), i.e.,
   (a) f: the proposition that "f is not true",
   (b) claim of f: f is not true.
2. If f were true, what it claims would have to be the case, and hence f would not be true. So f can not be true.
3. If f were not true, what it claims to be the case is in fact the case, so f must be true, which is a contradiction.
4. Hence f is neither true nor false; f is paradoxical.

The above f, which is called the *Liar paradox*, is expressed by a Russellian proposition $f = [Fa\ f]$.

In this paper we deal with the facts expressed by Russellian propositions in [2]. We give the conditions that a proposition is paradoxical and a proposition connected to a given proposition is paradoxical. Then, we consider the computational complexity of the problem whether or not the proposition is paradoxical.

2 Basic Definitions

In this section and the next we prepare some basic definitions according to Barwise and Etchemendy [2]. In the definitions the term *class* means the class in the axiomatic set theory. Since we deal with the definitions which depend on *nonwellfounded* sets ([1] and [2]), we have adopted *coinductive* definitions rather than *inductive* ones which depend on the wellfoundedness of set inclusion.

Definition 1

1. The propositional closure $\Gamma(X)$ of X is the smallest class containing X and closed under the operations $\vee, \wedge$.

2. The class *AtPROP* of atomic propositions is the largest class such that if $p \in AtPROP$, then p is of one of the following forms:

   (a) $[a\ H\ c]$ or $\overline{[a\ H\ c]}(= [a\ NH\ c])$, where a is $Claire$ or Max, c is a card, i.e., $c \in \{A\clubsuit, 2\clubsuit, \cdots, Q\spadesuit, K\spadesuit\}$;

   (b) $[a\ Bel\ q]$ or $\overline{[a\ Bel\ q]}(= [a\ NBel\ q])$, where $q \in \Gamma(AtPROP)$;

   (c) $[Tr\ p]$ or $\overline{[Tr\ p]}(= [Fa\ p])$.

   Then we define that $\overline{\overline{p}} = p$, $\overline{[\vee X]} = [\wedge\{\overline{p} | p \in X\}]$ and $\overline{[\wedge X]} = [\vee\{\overline{p} | p \in X\}]$.

3. $PROP = \Gamma(AtPROP)$.

A member of $PROP$ is called a *Russellian proposition.* Now we introduce the *propositional indeterminates* $\mathbf{p}, \mathbf{q_1}, \mathbf{q_2}$, ... which correspond to the demonstratives $\mathbf{this}, \mathbf{that_1}, \mathbf{that_2}$, ... respectively. We define the class $ParPROP$ of parametric propositions, a generalized class of $PROP$, by allowing additional atomic propositions of the four forms $[a\ Bel\ \mathbf{z}]$, $\overline{[a\ Bel\ \mathbf{z}]}$, $[Tr\ \mathbf{z}]$ and $\overline{[Tr\ \mathbf{z}]}$, where $\mathbf{z}$ is one of the indeterminates.

We now turn to the definition of truth for Russellian propositions. Informally a Russellian proposition is true just in case there are facts which make it true, and not true just in case there are no such facts. To define the truth, first we define a state of affairs and a situation which is a set of states of affairs.

Definition 2

1. $\sigma \in SOA$ if and only if σ is of one of the following forms:

   (a) $\langle H, a, c; i\rangle$,

   (b) $\langle Bel, a, p; i\rangle$,

   (c) $\langle Tr, p; i\rangle$,

   where $i = 0, 1$; $p \in \mathcal{M}$.

2. $s \in SIT$ if and only if s is a subset of SOA.

A member of SOA is called a *state of affairs* (or *soa*, for short), and a member of SIT a *situation.* We call $\langle H, a, c; 1\rangle$ and $\langle H, a, c; 0\rangle$ *duals* of one another (and similarly for soa's involving Bel and Tr).

Definition 3 We define the *makes true relation* to be the unique relation $\models \subseteq SIT \times PROP$ satisfying:

1. $s \models [a\ H\ c] \iff \langle H, a, c; 1\rangle \in s$
2. $s \models \overline{[a\ H\ c]} \iff \langle H, a, c; 0\rangle \in s$
3. $s \models [a\ Bel\ p] \iff \langle Bel, a, p; 1\rangle \in s$
4. $s \models \overline{[a\ Bel\ p]} \iff \langle Bel, a, p; 0\rangle \in s$
5. $s \models [Tr\ p] \iff \langle Tr, p; 1\rangle \in s$
6. $s \models \overline{[Tr\ p]} \iff \langle Tr, p; 0\rangle \in s$
7. $s \models [\wedge X] \iff s \models p$ for each $p \in X$
8. $s \models [\vee X] \iff s \models p$ for some $p \in X$.

Definition 4 Let $\mathcal{M}$ be a class of soa's.

1. A proposition p is *made true by* $\mathcal{M}$, denoted by $\mathcal{M} \models p$, if there is a set $s \subseteq \mathcal{M}$ such that $s \models p$; p is *made false by* $\mathcal{M}$, denoted by $\mathcal{M} \not\models p$, if there is no such s.
2. A proposition p is *true in* $\mathcal{M}$, denoted by $True_{\mathcal{M}}(p)$, if $\langle Tr, p; 1\rangle \in \mathcal{M}$; *false in* $\mathcal{M}$, denoted by $False_{\mathcal{M}}(p)$, if $\langle Tr, p; 0\rangle \in \mathcal{M}$.
3. $\mathcal{M}$ is *coherent* if no soa and its dual are in $\mathcal{M}$.
4. $\mathcal{M}$ is a *weak model* if it is a coherent class of soa's satisfying:

   (a) if $\langle Tr, p; 1\rangle \in \mathcal{M}$, then $\mathcal{M} \models p$

   (b) if $\langle Tr, p; 0\rangle \in \mathcal{M}$, then $\mathcal{M} \not\models p$

   for any $p \in PROP$.

Definition 5 Let $\mathcal{M}$ be a weak model.

1. $\mathcal{M}$ is *T-closed* if it satisfies the condition: $\langle Tr, p; 1\rangle \in \mathcal{M} \iff \mathcal{M} \models p$.
2. $\mathcal{M}$ is *N-closed* if it satisfies the condition: $\langle Tr, p; 0\rangle \in \mathcal{M} \iff \mathcal{M} \models \overline{p}$.
3. $\mathcal{M}$ is *almost semantically closed* (*asc*, for short) if it is both T- and N-closed. We call almost semantically closed models simply *models*.
4. $\mathcal{M}$ is a *maximal model* if $\mathcal{M}$ is not properly contained in any other model.

Intuitively, the maximal model is a model that *necessarily* involves each soa or its dual. For the (asc) model, the following lemma holds.

Lemma 1 (Barwise and Etchemendy)

1. $\mathcal{M} \models [Tr\ p] \iff \mathcal{M} \models p$
2. $\mathcal{M} \models [Fa\ p] \iff \mathcal{M} \models \overline{p}$

3. $\mathcal{M} \models [Fa[Fa\ p]] \iff \mathcal{M} \models p$

4. $\mathcal{M} \models [Tr\ [p \wedge p']] \iff \mathcal{M} \models [Tr\ p] \wedge [Tr\ p']$

5. $\mathcal{M} \models [Tr\ [p \vee p']] \iff \mathcal{M} \models [Tr\ p] \vee [Tr\ p']$

6. $\mathcal{M} \models [Fa\ [p \wedge p']] \iff \mathcal{M} \models [Fa\ p] \vee [Fa\ p']$

7. $\mathcal{M} \models [Fa\ [p \vee p']] \iff \mathcal{M} \models [Fa\ p] \wedge [Fa\ p']$

We will use the above proposition in Theorem 1 in the next section.

3 The Conditions of the Paradox

At first, we give the definition that the proposition is paradoxical. Any Russellian proposition is the *unique* solution $p = r(p)$ of the equation $\mathbf{p} = r(\mathbf{p})$, and the *uniqueness* is guaranteed by our metatheory ZFC/AFA in [1] and [2]. Here $p = r(p)$ is in $PROP$ and $\mathbf{p} = r(\mathbf{p})$ is in $ParPROP$. For example, $\mathbf{p} = [Fa\ \mathbf{p}]$ has the unique solution $p = [Fa\ p]$, and $\mathbf{p} = [Max\ H\ A\clubsuit]$ has the unique solution $p = [Max\ H\ A\clubsuit]$. In this paper, for the simplicity, the Russellian proposition $p = r(p)$ does not include any q_i in the right-hand side of $p = r(p)$. That is to say, Russellian sentences do not include any $\mathbf{that_i}$. (See [1] and [2] for more details.)

Definition 6 A proposition p is *paradoxical in* $\mathcal{M}$ if for any maximal model $\mathcal{N} \supseteq \mathcal{M}$, neither $\langle Tr, p; 1\rangle \in \mathcal{N}$ nor $\langle Tr, p; 0\rangle \in \mathcal{N}$, i.e., $\langle Tr, p; 1\rangle \notin \mathcal{N}$ and $\langle Tr, p; 0\rangle \notin \mathcal{N}$.

It is clear that p is paradoxical in $\mathcal{M}$ if and only if $\langle Tr, p; 1\rangle \in \mathcal{N} \iff \langle Tr, p; 0\rangle \in \mathcal{N}$ holds for any maximal model $\mathcal{N} \supseteq \mathcal{M}$.

Example 1 Let $p = [Max\ H\ A\clubsuit]$. For any maximal model $\mathcal{M}$, if $\langle H, Max, A\clubsuit; 1\rangle \in \mathcal{M}$ then p is true in $\mathcal{M}$, and if $\langle H, Max, A\clubsuit; 0\rangle \in \mathcal{M}$ then p is false in $\mathcal{M}$, and there are only two cases.

Example 2 $p = [Fa\ p]$. As $\langle Tr, p; 1\rangle \in \mathcal{M} \iff \langle Tr, p; 0\rangle \in \mathcal{M}$ holds for any model $\mathcal{M}$, this proposition is paradoxical in any model.

Example 3 $p = [Max\ H\ A\clubsuit] \vee [Fa\ p]$. Then for any maximal model $\mathcal{M}$, if $\langle H, Max, A\clubsuit; 1\rangle \in \mathcal{M}$ then p is true in $\mathcal{M}$. If $\langle H, Max, A\clubsuit; 0\rangle \in \mathcal{M}$, by using that $\overline{p} = \overline{[Max\ H\ A\clubsuit]} \wedge [Tr\ p]$,

$$\begin{aligned} &\langle Tr, p; 0\rangle \in \mathcal{M} \\ &\iff \langle H, Max, A\clubsuit; 0\rangle \in \mathcal{M} \text{ and } \langle Tr.p; 1\rangle \in \mathcal{M} \\ &\implies \langle Tr, p; 1\rangle \in \mathcal{M}. \end{aligned}$$

Hence $\langle Tr, p; 0\rangle \notin \mathcal{M}$. On the other hand,

$$\begin{aligned} &\langle Tr, p; 1\rangle \in \mathcal{M} \\ &\iff \langle H, Max, A\clubsuit; 1\rangle \in \mathcal{M} \text{ or } \langle Tr, p; 0\rangle \in \mathcal{M} \\ &\implies \langle Tr, p; 0\rangle \in \mathcal{M}. \end{aligned}$$

So $\langle Tr, p; 1\rangle \notin \mathcal{M}$. Hence p is paradoxical in $\mathcal{M}$.

Definition 7

1. A proposition is *paradoxical* if it is paradoxical in some model.

2. A proposition is *intrinsically paradoxical* if it is paradoxical in any model.

3. A proposition is *classical* if it is not paradoxical.

Example 4 $p = [Fa\ p]$ is intrinsically paradoxical by Example 2, $p = [Max\ H\ A\clubsuit] \vee [Fa\ p]$ is paradoxical but not intrinsically by Example 3, and $p = [Max\ H\ A\clubsuit]$ is classical by Example 1.

Notice that $\mathcal{M} \models \overline{p}$ is not equivalent to $\mathcal{M} \not\models p$ in Russellian propositions. In fact, $\mathcal{M} \models \overline{p}$ implies $\mathcal{M} \not\models p$, but the converse does not always hold. A Russellian proposition $p = r(p)$ is *connective* if it includes $\vee$ or $\wedge$, and it is *non-connective* if it includes neither $\vee$ nor $\wedge$.

Theorem 1 If $p = r(p)$ is paradoxical, then one of the following conditions holds:

1. $\mathcal{M} \models r(p) \iff \mathcal{M} \models [Fa\ p]$ for any model $\mathcal{M}$,
2. $\mathcal{M} \models r(p) \iff \mathcal{M} \models r_1(p) \vee r_2(p)$ for any model $\mathcal{M}$, and there is a model $\mathcal{M}_0$ such that
$$\mathcal{M}_0 \models r_1(p) \iff \mathcal{M}_0 \models [Fa\ p]$$
$$\mathcal{M}_0 \not\models r_2(p),$$
3. $\mathcal{M} \models r(p) \iff \mathcal{M} \models r_1(p) \wedge r_2(p)$ for any model $\mathcal{M}$, and there is a model $\mathcal{M}_0$ such that
$$\mathcal{M}_0 \models r_1(p) \iff \mathcal{M}_0 \models [Fa\ p]$$
$$\mathcal{M}_0 \not\models \overline{r_2(p)}.$$

Before we prove Theorem 1, we consider the following examples to explain the notation.

Example 5 For $p = [Fa\ p] \vee [Tr[Fa\ p]]$, $r(p)$ is $[Fa\ p] \vee [Tr[Fa\ p]]$, $r_1(p)$ is $[Fa\ p]$, and $r_2(p)$ is $[Tr[Fa\ p]]$, like the conditions in Theorem 1, where we can replace the subscript 1 with 2.

Example 6 For $p = [Fa[Tr[[Fa\ p] \vee [Tr\ p]]]]$, it holds that
$$\mathcal{M} \models [Fa[Tr[[Fa\ p] \vee [Tr\ p]]]] \iff \mathcal{M} \models [Fa[Fa\ p]] \wedge [Fa[Tr\ p]]$$
by Lemma 1. Then, the equivalence in Theorem 1,
$$\mathcal{M} \models r(p) \iff \mathcal{M} \models r_1(p) \wedge r_2(p)$$
means that $r(p)$ is $[Fa[Tr[[Fa\ p] \vee [Tr\ p]]]]$, $r_1(p)$ is $[Fa[Fa\ p]]$, and $r_2(p)$ is $[Fa[Tr\ p]]$, where we can replace the subscript 1 with 2.

Proof of Theorem 1. Suppose that $p = r(p)$ is paradoxical. Then one of the following holds:
$$(a)\quad \mathcal{M} \models r(p) \iff \mathcal{M} \models [Fa\ p],$$
$$(b)\quad \mathcal{M} \models r(p) \iff \mathcal{M} \models r_1(p) \vee r_2(p),$$
$$(c)\quad \mathcal{M} \models r(p) \iff \mathcal{M} \models r_1(p) \wedge r_2(p).$$
In fact, if not, one of the following holds:
$$\mathcal{M} \models r(p) \iff \mathcal{M} \models [a\ H(NH)\ c],$$
$$\mathcal{M} \models r(p) \iff \mathcal{M} \models [a\ Bel(NBel)\ r'(p)],$$
$$\mathcal{M} \models r(p) \iff \mathcal{M} \models [Tr\ p],$$
which contradicts the assumption that $p = r(p)$ is paradoxical.

If $p = r(p)$ is non-connective then $p = r(p)$ is intrinsically paradoxical or classical. The non-connective proposition is paradoxical in some model if and only if it is intrinsically paradoxical. Hence if the case (a) holds then $p = r(p)$ is paradoxical.

Assume the case (b) holds, and that $p = r(p)$ is paradoxical in some $\mathcal{M}_0$. If p is paradoxical in $\mathcal{M}_0$ then $\langle Tr, p; 1\rangle \in \mathcal{M}_0 \iff \langle Tr, p; 0\rangle \in \mathcal{M}_0$ holds. Hence
$$\langle Tr, p; 1\rangle \in \mathcal{M}_0 \iff \mathcal{M}_0 \models r_1(p) \text{ or } \mathcal{M}_0 \models r_2(p),$$
and

$$\langle Tr, p; 0\rangle \in \mathcal{M}_0 \iff \mathcal{M}_0 \models \overline{r_1(p)} \text{ and } \mathcal{M}_0 \models \overline{r_2(p)}.$$

holds. By the supposition, since

$$(\mathcal{M}_0 \models r_1(p) \text{ or } \mathcal{M}_0 \models r_2(p)) \iff (\mathcal{M}_0 \models \overline{r_1(p)} \text{ and } \mathcal{M}_0 \models \overline{r_2(p)})$$

either 1 or 2 below holds:

1. $\mathcal{M}_0 \models r_1(p) \iff \mathcal{M}_0 \models \overline{r_1(p)}$ and $\mathcal{M}_0 \models \overline{r_2(p)}$,
2. $\mathcal{M}_0 \models r_2(p) \iff \mathcal{M}_0 \models \overline{r_1(p)}$ and $\mathcal{M}_0 \models \overline{r_2(p)}$.

In case of 1 holds,

$$\begin{aligned} &\mathcal{M}_0 \models r_1(p) \\ &\iff \mathcal{M}_0 \models \overline{r_1(p)} \text{ and } \mathcal{M}_0 \models \overline{r_2(p)} \\ &\iff \langle Tr, p; 0\rangle \in \mathcal{M}_0 \iff \mathcal{M}_0 \models [Fa\ p]. \end{aligned}$$

Hence $\mathcal{M}_0 \models r_1(p) \iff \mathcal{M}_0 \models [Fa\ p]$. Similarly in case of 2 holds, $\mathcal{M}_0 \models r_2(p) \iff \mathcal{M}_0 \models [Fa\ p]$. Consider the following two cases:

(i) Assume that $\mathcal{M}_0 \models r_1(p) \iff \mathcal{M}_0 \models [Fa\ p]$ holds. This means that $\mathcal{M}_0 \not\models r_2(p)$ does not hold, i.e., $\mathcal{M}_0 \models r_2(p)$. Then

$$\begin{aligned} &\langle Tr, p; 0\rangle \in \mathcal{M}_0 \\ &\iff \langle Tr, p; 1\rangle \in \mathcal{M}_0 \text{ and } \mathcal{M}_0 \models \overline{r_2(p)} \\ &\implies \langle Tr, p; 1\rangle \in \mathcal{M}_0 \text{ and } \mathcal{M}_0 \not\models r_2(p). \end{aligned}$$

Hence $\langle Tr, p; 0\rangle \notin \mathcal{M}_0$. If $\langle Tr, p; 1\rangle \notin \mathcal{M}_0$, then $\langle Tr, p; 1\rangle \notin \mathcal{M}_0 \iff \langle Tr, p; 0\rangle \notin \mathcal{M}_0$ and $\mathcal{M}_0 \not\models r_2(p)$. As $\mathcal{M}_0 \not\models r_2(p)$ does not hold, $\langle Tr, p; 1\rangle \in \mathcal{M}_0$. Hence $p = r(p)$ is true in $\mathcal{M}_0$, which contradicts the assumption that $p = r(p)$ is paradoxical in $\mathcal{M}_0$. Hence $\mathcal{M}_0 \not\models r_2(p)$.

(ii) Assume that $\mathcal{M}_0 \models r_2(p) \iff \mathcal{M}_0 \models [Fa\ p]$ holds. Then we can similarly prove $\mathcal{M}_0 \not\models r_1(p)$ by replacing $r_1(p)$ with $r_2(p)$.

Since $(\mathcal{M}_0 \models r_1(p) \iff \mathcal{M}_0 \models [Fa\ p]) \iff (\mathcal{M}_0 \models \overline{r_1(p)} \iff \mathcal{M}_0 \models [Tr\ p])$, if the case (c) holds, then we may have a similar proof to (b). □

The converse of Theorem 1 does not hold, but the following corollary holds:

Corollary 1 If one of the following conditions holds, then $p = r(p)$ is paradoxical.

1. $\mathcal{M} \models r(p) \iff \mathcal{M} \models [Fa\ p]$ for any model $\mathcal{M}$,
2. $\mathcal{M} \models r(p) \iff \mathcal{M} \models r_1(p) \vee r_2(p)$ for any model, and there is a model $\mathcal{M}_0$ such that for *any maximal model* $\mathcal{N} \supseteq \mathcal{M}_0$,

   $\mathcal{N} \models r_1(p) \iff \mathcal{N} \models [Fa\ p]$

   $\mathcal{N} \not\models r_2(p)$,
3. $\mathcal{M} \models r(p) \iff \mathcal{M} \models r_1(p) \wedge r_2(p)$ for any model, and there is a model $\mathcal{M}_0$ such that for *any maximal model* $\mathcal{N} \supseteq \mathcal{M}_0$,

   $\mathcal{N} \models r_1(p) \iff \mathcal{N} \models [Fa\ p]$

   $\mathcal{N} \not\models \overline{r_2(p)}$.

Let p be a connective proposition. Then by Theorem 1 there are the following two cases where p is paradoxical.

Definition 8

1. $p = r(p)$ is *or-paradoxical* if $p = r(p)$ is paradoxical such that $\mathcal{M} \models r(p) \iff \mathcal{M} \models r_1(p) \vee r_2(p)$ for any model $\mathcal{M}$.
2. $p = r(p)$ is *and-paradoxical* if $p = r(p)$ is paradoxical such that $\mathcal{M} \models r(p) \iff \mathcal{M} \models r_1(p) \wedge r_2(p)$ for any model $\mathcal{M}$.

According to Definition 8 above, we call the paradoxical proposition satisfying the second condition of Theorem 1 to be or-paradoxical, and the third condition to be and-paradoxical. On the other hand, the model $\mathcal{M}_0$ in Theorem 1 is not generally arbitrary, but for example if $r_1(p) = [Fa\ p]$, then $\mathcal{M}_0 \models r_1(p) \iff \mathcal{M}_0 \models [Fa\ p]$ holds for any model $\mathcal{M}_0$. We define this case as the special case of or-/and-paradoxical:

Definition 9

1. $p = r(p)$ is *strongly or-paradoxical* if $p = r(p)$ is or-paradoxical and $\mathcal{M} \models r_1(p) \iff \mathcal{M} \models [Fa\ p]$ for any model $\mathcal{M}$.
2. $p = r(p)$ is *strongly and-paradoxical* if $p = r(p)$ is and-paradoxical and $\mathcal{M} \models r_1(p) \iff \mathcal{M} \models [Fa\ p]$ for any model $\mathcal{M}$.

By Theorem 1, we can decide whether or not the proposition is paradoxical. Since $(\mathcal{M}_0 \models r_1(p) \iff \mathcal{M}_0 \models [Fa\ p]) \iff (\mathcal{M}_0 \models \overline{r_1(p)} \iff \mathcal{M} \models [Tr\ p])$, we can use one of the following conditions on $r_1(p)$

- $\mathcal{M}_0 \models r_1(p) \iff \mathcal{M}_0 \models [Fa\ p]$,
- $\mathcal{M}_0 \models \overline{r_1(p)} \iff \mathcal{M}_0 \models [Tr\ p]$.

Example 7
$p = [Fa\ p] \vee [Max\ H\ A\clubsuit]$ is strongly or-paradoxical.
$p = [Fa\ p] \vee \overline{[Max\ H\ A\clubsuit]}$ is strongly or-paradoxical.
$p = [Fa\ p] \wedge [Max\ H\ A\clubsuit]$ is strongly and-paradoxical.
$p = [Fa\ p] \wedge \overline{[Max\ H\ A\clubsuit]}$ is strongly and-paradoxical.

Example 8
$p = [[Fa\ p] \vee [Max\ H\ A\clubsuit]] \vee [Max\ H\ A\clubsuit]$ is or-paradoxical.
$p = [[Fa\ p] \wedge [Max\ H\ A\clubsuit]] \wedge [Max\ H\ A\clubsuit]$ is and-paradoxical.
$p = [[Fa\ p] \vee [Max\ H\ A\clubsuit]] \vee \overline{[Max\ H\ A\clubsuit]}$ is not paradoxical.
$p = [[Fa\ p] \wedge [Max\ H\ A\clubsuit]] \wedge \overline{[Max\ H\ A\clubsuit]}$ is not paradoxical.
$p = [[Fa\ p] \wedge [Max\ H\ A\clubsuit]] \vee [Max\ H\ A\clubsuit]$ is not paradoxical.
$p = [[Fa\ p] \vee [Max\ H\ A\clubsuit]] \wedge [Max\ H\ A\clubsuit]$ is not paradoxical.
$p = [[Fa\ p] \wedge [Max\ H\ A\clubsuit]] \vee \overline{[Max\ H\ A\clubsuit]}$ is or-paradoxical.
$p = [[Fa\ p] \vee [Max\ H\ A\clubsuit]] \wedge \overline{[Max\ H\ A\clubsuit]}$ is and-paradoxical.

Example 9
$p = [Fa\ p] \vee [[Max\ H\ A\clubsuit] \wedge \overline{[Max\ H\ A\clubsuit]}]$ is strongly or-paradoxical.
$p = [Fa\ p] \wedge [[Max\ H\ A\clubsuit] \vee \overline{[Max\ H\ A\clubsuit]}]$ is strongly and-paradoxical.
$p = [Fa\ p] \vee [[Max\ H\ A\clubsuit] \vee \overline{[Max\ H\ A\clubsuit]}]$ is not paradoxical.
$p = [Fa\ p] \wedge [[Max\ H\ A\clubsuit] \wedge \overline{[Max\ H\ A\clubsuit]}]$ is not paradoxical.

By Theorem 1, it is clear that the following corollary holds.

Corollary 2

1. Suppose that $p = r(p)$ is strongly or-paradoxical. If $\mathcal{M} \not\models r_2(p)$ holds for any model $\mathcal{M}$, then $p = r(p)$ is intrinsically paradoxical.

2. Suppose that $p = r(p)$ is strongly and-paradoxical. If $\mathcal{M} \not\models \overline{r_2(p)}$ holds for any model $\mathcal{M}$, then $p = r(p)$ is intrinsically paradoxical.

By the above corollary, the first two propositions of Example 8 are intrinsically paradoxical.

We consider whether there is an and-paradoxical proposition which is intrinsically paradoxical but not strong. If $p = r(p)$ is and-paradoxical but not strong, then there is a model $\mathcal{M}_0$ such that

$$\mathcal{M}_0 \models r_1(p) \iff \mathcal{M}_0 \models [Fa\ p]$$

does not hold, where $r_1(p)$ is the one in Theorem 1. In the model $\mathcal{M}_0$, $p = r(p)$ is not paradoxical, and hence $p = r(p)$ is not intrinsically paradoxical. The above fact holds also when we replace and- with or- in the above sentence. Therefore the following lemma holds:

Lemma 2 If the or-paradoxical proposition is not strong, then it is not intrinsically paradoxical. If the and-paradoxical proposition is not strong, then it is not intrinsically paradoxical.

We have obtained the conditions from the *outside* of the proposition given so far. Now we give the conditions from the *inside*. That is, given paradoxical proposition, we deal with the conditions that the *connected* propositions are paradoxical.

Lemma 3

1. Suppose $p = r_1(p)$ is paradoxical in $\mathcal{M}_0$. If for any maximal model $\mathcal{N} \supseteq \mathcal{M}_0$,

   $\mathcal{N} \models \overline{r_2(p)}$, or

   $\mathcal{N} \models r_2(p) \iff \mathcal{N} \models [Fa\ p]$,

   then $p = r_1(p) \vee r_2(p)$ is paradoxical in $\mathcal{M}_0$.

2. Suppose $p = r_1(p)$ is paradoxical in $\mathcal{M}_0$. If for any maximal model $\mathcal{N} \supseteq \mathcal{M}_0$,

   $\mathcal{N} \models r_2(p)$, or

   $\mathcal{N} \models r_2(p) \iff \mathcal{N} \models [Fa\ p]$,

   then $p = r_1(p) \wedge r_2(p)$ is paradoxical in $\mathcal{M}_0$.

Proof . By the supposition, $\mathcal{N} \models r_1(p) \iff \mathcal{N} \models \overline{r_1(p)}$. For $p = r_1(p) \vee r_2(p)$, suppose that $\mathcal{N} \models \overline{r_2(p)}$. Then

$$\begin{aligned}
&\langle Tr, p; 1\rangle \in \mathcal{N} \\
&\iff \mathcal{N} \models r_1(p) \vee r_2(p) \iff \mathcal{N} \models r_1(p) \text{ or } \mathcal{N} \models r_2(p) \\
&\iff \mathcal{N} \models r_1(p) \iff \mathcal{N} \models \overline{r_1(p)} \iff \mathcal{N} \models \overline{r_1(p)} \text{ and } \mathcal{N} \models \overline{r_2(p)} \\
&\iff \mathcal{N} \models \overline{r_1(p)} \wedge \overline{r_2(p)} \iff \mathcal{N} \models \overline{r_1(p) \vee r_2(p)} \\
&\iff \langle Tr, p; 0\rangle \in \mathcal{N}
\end{aligned}$$

If $\mathcal{N} \models r_2(p) \iff \mathcal{N} \models [Fa\ p]$, then trivially $p = r_1(p) \vee r_2(p)$ is paradoxical. $\square$

It is obvious that if $p = r_1(p)$ is intrinsically paradoxical, then $p = r_1(p) \vee r_2(p)$ is strongly or-paradoxical, and $p = r_1(p) \wedge r_2(p)$ is strongly and-paradoxical. By this lemma we can also check Examples 7 - 9.

4 The Computational Complexity

We consider the problem whether or not a Russellian proposition is paradoxical (classical, intrinsically paradoxical). We deal with this problem by the conditions in Theorem 1, while Rounds [5] solved it by the AFA graphs.

Let $p = r(p)$ be a Russellian proposition and *labels* be $H, NH, Bel, NBel, Tr, Fa, \vee$, and $\wedge$. We say that a proposition p is *satisfiable* if there is a model $\mathcal{M}$ such that p is true in $\mathcal{M}$, and that a proposition p is *valid* if p is true in any *maximal* model. If p is satisfiable, then there is a model $\mathcal{M}$ such that p is true in any maximal model $\mathcal{N} \supseteq \mathcal{M}$. The following computational complexity of the problem on a number of labels have been given by Rounds [5]:

Theorem 2 (Rounds)

1. The satisfiability problem for Russellian propositions is in **NP**.

2. The validity problem for Russellian propositions is in **co-NP**.

Note that the Russellian proposition $p = r(p)$ is satisfiable if and only if there is a model $\mathcal{M}$ such that $\mathcal{M} \models r(p)$, and by Theorem 1 and Corollary 2 the problem whether or not a Russellian proposition is paradoxical is in **NP**, and the problem whether or not a Russellian proposition is intrinsically paradoxical is in **co-NP**. This has also been given by Rounds [5] using the AFA graphs.

Theorem 3 (Rounds)

1. The problem whether or not a Russellian proposition is paradoxical is in **NP**.

2. The problem whether or not a Russellian proposition is intrinsically paradoxical is in **co-NP**.

3. The problem whether or not a Russellian proposition is classical is in **co-NP**.

We can verify the above results by using the Theorem 1. We show have an outline of the proof. Since a proposition $p = r(p)$ is satisfiable if and only if there is some model $\mathcal{M}$ such that $\mathcal{M} \models r(p)$, by Theorem 1 and Corollary 1, we can conclude that the computational complexity of the problem whether or not a Russellian proposition is paradoxical is the same as that of Russellian satisfiability problem. Hence this problem is in **NP**. Since a proposition $p = r(p)$ is valid if and only if $\mathcal{N} \models r(p)$ for any maximal model $\mathcal{N}$, we can conclude that the computational complexity of the problem whether or not a Russellian proposition is intrinsically paradoxical is also the same as that Russellian validity problem. Hence this problem is in **co-NP**. Finally, since a proposition is classical if and only if it is not paradoxical, the computational complexity of the problem whether or not a Russellian proposition is classical is in **co-NP**.

5 Conclusion

We have seen that a Russellian proposition is paradoxical depending only on the proposition of $[Fa\ p]$. We have presented conditions under which a Russellian proposition and a connective proposition are paradoxical. By using the conditions we have considered the various problems and their computational complexity. However, we have just dealt with Russellian propositions only including p which corresponds to the parametric proposition $\mathbf{p}$, i.e., to the demonstrative **this**, and have not dealt with the propositions including q_i, and neither the system of the AFA equations. We will discuss this problem elsewhere in a future.

Russellian propositions cannot express all of the paradoxes. For example, Nait Abdallah [4] discussed three kinds of paradoxes, Protagoras paradox, Newcomb's paradox, and the Hangman paradox. Needless to say, our present study cannot deal with these three paradoxes because these are the dilemmas rather than the paradoxes. We have left the problem whether or not we can have the framework to define paradox.

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