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Baba, Kensuke
Graduate School of Information Science and Electrical Engineering, Kyushu University

Hirokawa, Sachio
Computer Center, Kyushu University

Fujita, Ken-etsu
Department of Computer Science, Interdisciplinary Faculty of Science and Engineering Shimane University



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# Technical Report

## Parallel Reduction in Type Free $\lambda\mu$-Calculus

**by**

Kensuke Baba, Sachio Hirokawa, Ken-etsu Fujita

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Department of Informatics
Kyushu University
Fukuoka 812-81, Japan

Email: baba@i.kyushu-u.ac.jp Phone: +81-92-641-3131(8806)

# Parallel Reduction in Type Free $\lambda\mu$-Calculus

Kensuke Baba[1], Sachio Hirokawa[2], and Ken-etsu Fujita[3]

[1] Graduate School of Information Science and Electrical Engineering, Kyushu University
Fukuoka, 812-8581 Japan, E-mail: baba@i.kyushu-u.ac.jp
[2] Computer Center, Kyushu University
[3] Department of Computer Science, Interdisciplinary Faculty of Science and Engineering
Shimane University, Matsue, 690-8504 Japan

**Abstract.** Typed $\lambda\mu$-calculus is known to be strongly normalizing and weakly Church-Rosser, and hence confluent. In fact, Parigot formulated a parallel reduction to prove confluency of typed $\lambda\mu$-calculus by "Tait-and-Martin-Löf" method. However, the diamond property does not hold for his parallel reduction. The confluency for type-free $\lambda\mu$-calculus cannot be derived from that of typed $\lambda\mu$-calculus and is not known. We analyzed granualities of the reduction rules. We consider a renaming and consecutive structural reductions as one step parallel reduction, and show that the new formulation of parallel reduction has the diamond property, which yields the correct proof of confluency of type free $\lambda\mu$-calculus. The diamond property of new parallel reduction is also shown for the call-by-value version of $\lambda\mu$-calculus contains the symmetric structural reduction rule.

## 1 Introduction

Parigot's $\lambda\mu$-calculus[12] is a formal system for propositional classical logic and can at the same time be considered as a functional programming language with continuation. The $\lambda\mu$-terms $M$ is constructed as

$$M = x \mid \lambda x.M \mid MM \mid \mu\alpha.M \mid [\alpha]M.$$

The calculus has the following basic reduction rules.

| | |
|---|---|
| $\beta$-reduction: | $(\lambda x.M)N \to M[x := N]$ |
| Structural reduction: | $(\mu\alpha.M)N \to \mu\alpha.M[[\alpha]w := [\alpha](wN)]$ |
| Renaming: | $[\beta](\mu\alpha.M) \to M[\alpha := \beta]$ |

We assume some familiarity to $\lambda$-calculus $[2, 7, 8]$. In the structural reduction, the substitution is defined as follows:

**1.** $x[[\alpha]w := [\alpha](wN)] = x$
**2.** $(\lambda x.M)[[\alpha]w := [\alpha](wN)] = \lambda x.M[[\alpha]w := [\alpha](wN)]$
**3.** $(MM)[[\alpha]w := [\alpha](wN)] = M[[\alpha]w := [\alpha](wN)]M[[\alpha]w := [\alpha](wN)]$
**4.** $(\mu\beta.M)[[\alpha]w := [\alpha](wN)] = \mu\beta.M[[\alpha]w := [\alpha](wN)]$
**5-1.** $([\beta]M)[[\alpha]w := [\alpha](wN)] = [\beta](M[[\alpha]w := [\alpha](wN)]N)$ if $\alpha = \beta$
**5-2.** $([\beta]M)[[\alpha]w := [\alpha](wN)] = [\beta]M[[\alpha]w := [\alpha](wN)]$ if $\alpha \neq \beta$

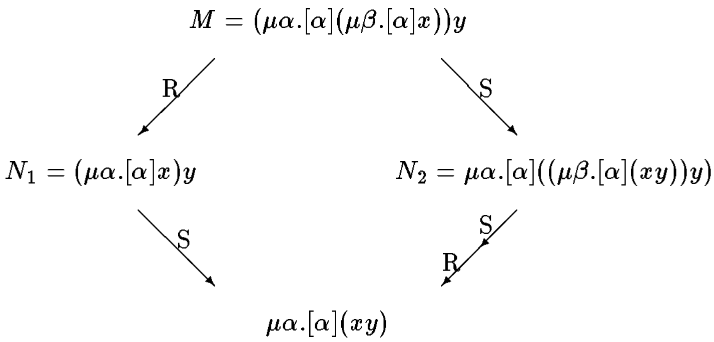
In [12], Parigot outlined the proof of confluency of $\lambda\mu$-calculus. He formulated the parallel reduction and claimed the diamond property for the parallel reduction:

$$\text{if } M \Rightarrow N \text{ then } N \Rightarrow M^*.$$

Here $M^*$ is a term obtained by reducing all the redexes in $M$. $M^*$ is usually referred as the "complete development" of $M$ [2]. The formulation of the parallel reduction is based on "Tait-and-Martin-Löf" method, which is explained clearly in [10]. The method is applicable to prove the confluence of many reduction systems. However, the method does not work for $\lambda\mu$-calculus. In fact, the diamond property does not hold for the formulation of parallel reduction in [12]. So the proof of confluence is not so trivial as it seems to be.

The $\lambda\mu$-calculus is known to be strongly normalizing[13] and weak Church-Rosser. For notions of deduction, these two properties yield confluency[2]. But type free $\lambda\mu$-calculus is not strongly normalizing. (For instance, the untypable term $(\lambda x.xx)(\lambda x.xx)$ dose not have normal form.) The correct proof of confluency of type free $\lambda\mu$-calculus is never published as far as we know.

We think that the reason why the diamond property does not hold for the parallel reduction is in the sequential nature of the structural reduction rule. Consider a term $M = (\mu\alpha.[\alpha](\mu\beta.[\alpha]x))y$ which has a renaming redex and a structural redex. We have the terms $N_1 = (\mu\alpha.[\alpha]x)y$ and $N_2 = \mu\alpha.[\alpha]((\mu\beta.[\alpha](xy))y)$ by a renaming and a structural reduction respectively. Then we have $M \Rightarrow N_1$ and $M \Rightarrow N_2$. If the diamond property would hold, $N_1$ and $N_2$ were reducible to the same term $M^*$ in one step reduction. However, this is impossible. After the structural reduction, the "residual" of renaming redex in $M$ is no longer a renaming redex in $N_2$. To make the residual back to a renaming redex, we need another step of structural reduction. We consider such a successive sequence of structural reductions as a one step parallel reduction. With such a formulation, we prove the strong diamond property for the parallel reduction.




We consider the $\lambda\mu$-calculus as a programming language and reductions as computation. The reduction rules of $\lambda\mu$-calculus captures the mechanism of functional programming languages with control[3, 4, 6]. However we can not apply an arbitrary reduction for implementation of programming language. Usually we fix a reduction strategy. A call-by-value $\lambda\mu$-calculus $\lambda\mu_v$ was first considered by Ong and Stewart [11]. The $\lambda\mu_v$-calculus contains another reduction rule so called "symmetric structural reduction" such that:

$$N(\mu\alpha.M) \to \mu\alpha.M[[\alpha]w := [\alpha](Nw)].$$

Note that a subsystem is not always confluent even if the whole system is confluent. Therefore, the confluence of $\lambda\mu$ does not yield the confluence of $\lambda\mu_v$, even if we ignore the symmetric structural reduction rule. We shall formulate an appropriate parallel reduction for $\lambda\mu_v$ and prove the strong diamond property.

## 2 Parallel Reduction in $\lambda\mu$-Calculus

We define the parallel reduction as follows. The rules 1–8 are obtained by a straightforward application of Tait-and-Martin-Löf method to $\beta$-reduction, structural reduction and renaming. The last inference rule 9 is introduced in the present paper. It combines a renaming and a consecutive sequence of structural reduction. It is easy to see that the transitive and reflexive closure of "$\to$" is identical to the transitive closure of "$\Rightarrow$".

**Definition 1.**

**1.** $x \Rightarrow x$

**2.** $\dfrac{M \Rightarrow M'}{\lambda x.M \Rightarrow \lambda x.M'}$

**3.** $\dfrac{M \Rightarrow M' \quad N \Rightarrow N'}{MN \Rightarrow M'N'}$

**4.** $\dfrac{M \Rightarrow M'}{\mu\alpha.M \Rightarrow \mu\alpha.M'}$

**5.** $\dfrac{M \Rightarrow M'}{[\alpha]M \Rightarrow [\alpha]M'}$

**6.** $\dfrac{M \Rightarrow M' \quad N \Rightarrow N'}{(\lambda x.M)N \Rightarrow M'[x := N']}$

**7.** $\dfrac{M \Rightarrow M' \quad N \Rightarrow N'}{(\mu\alpha.M)N \Rightarrow \mu\alpha.M'[[\alpha]w := [\alpha](wN')]}$

**8.** $\dfrac{M \Rightarrow M'}{[\beta](\mu\alpha.M) \Rightarrow M'[\alpha := \beta]}$

**9.** $\dfrac{M \Rightarrow M' \quad N_1 \Rightarrow N_1' \quad \cdots \quad N_n \Rightarrow N_n'}{[\beta]((\mu\alpha.M)N_1 \cdots N_n) \Rightarrow M'[[\alpha]w := [\beta](wN_1' \cdots N_n')]}$

We define the complete development $M^*$ of a term $M$ as follows.

**Definition 2.**

**1.** $M = x$. *Then* $M^* = x$.

**2.** $M = \lambda x.M_1$. *Then* $M^* = \lambda x.M_1^*$.

**3.** $M = M_1 M_2$.

**3.1** $M_1 = \lambda x.M_3$. *Then* $M^* = M_3^*[x := M_2^*]$.

**3.2** $M_1 = \mu\alpha.M_3$. *Then* $M^* = \mu\alpha.M_3^*[[\alpha]w := [\alpha](wM_2^*)]$.

**3.3** $M^* = M_1^* M_2^*$ *o.w.*

**4.** $M = \mu\alpha.M_1$. *Then* $M^* = \mu\alpha.M_1^*$.

**5.** $M = [\alpha]M_1$.

**5.1** $M_1 = \mu\beta.M_2$. *Then* $M^* = M_2^*[\beta := \alpha]$.

**5.2** $M_1 = (\mu\beta.M_2)N_1 \cdots N_n$. *Then* $M^* = M_2^*[[\beta]w := [\alpha](wN_1^* \cdots N_n^*)]$.

**5.3** $M^* = [\alpha]M_1^*$ *o.w.*

## 3 Diamond Property of Parallel Reduction

A gap in the proof of confluence in [12] was *(2)* of the following lemma. Without the rule 9, *(2)* does not hold.

**Lemma 1.** *(1) If $M \Rightarrow M'$ and $N \Rightarrow N'$, then $M[x := N] \Rightarrow M'[x := N']$.*
*(2) If $M \Rightarrow M'$ and $N \Rightarrow N'$, then $M[[\alpha]w := [\alpha](wN)] \Rightarrow M'[[\alpha]w := [\alpha](wN')]$.*
*(3) If $M \Rightarrow M'$, then $M[\beta := \alpha] \Rightarrow M'[\beta := \alpha]$.*

**Proof.** *(1)* is easily shown by induction on the structure of $M \Rightarrow M'$.
*(3)* is trivial.
*(2)* is proved by induction on the structure of $M \Rightarrow M'$. Most cases are routine. Non-trivial cases are when the last inference of $M \Rightarrow M'$ is either 8 or 9. To save the space of the paper, we explain only the case 8.

**Case 8.** The last inference rule is 8.

By definition of $M \Rightarrow M'$, $M$ and $M'$ have the form $M = [\beta](\mu\gamma.M_1)$, $M' = M_1'[\gamma := \beta]$ and $M \Rightarrow M'$ has the following form.

$$\frac{M_1 \Rightarrow M_1'}{M = [\beta](\mu\gamma.M_1) \Rightarrow M_1'[\gamma := \beta] = M'}\ 8$$

Since $\gamma$ is a bound variable, we can assume $\gamma \neq \alpha$.

**Case 8.1** $\alpha = \beta$.

Then we have

$$\begin{aligned} M[[\alpha]w := [\alpha](wN)] &= ([\alpha](\mu\gamma.M_1))[[\alpha]w := [\alpha](wN)] \\ &= [\alpha]((\mu\gamma.M_1[[\alpha]w := [\alpha](wN)])N), \\ M'[[\alpha]w := [\alpha](wN')] &= M_1'[\gamma := \alpha][[\alpha]w := [\alpha](wN')] \\ &= M_1'[[\alpha]w := [\alpha](wN')][[\gamma]w := [\gamma](wN')][\gamma := \alpha] \end{aligned}$$

By induction hypothesis for $M_1 \Rightarrow M_1'$, we have $M_1[[\alpha]w := [\alpha](wN)] \Rightarrow M_1'[[\alpha]w := [\alpha](wN')]$. Thus we have $[\alpha]((\mu\gamma.M_1[[\alpha]w := [\alpha](wN)])N) \Rightarrow M_1'[[\alpha]w := [\alpha](wN')][[\gamma]w := [\alpha](wN')][\gamma := \alpha]$ by the rule 9. Hence Lemma holds.

**Case 8.2** $\alpha \neq \beta$.

Then we have

$$\begin{aligned} M[[\alpha]w := [\alpha](wN)] &= ([\beta](\mu\gamma.M_1))[[\alpha]w := [\alpha](wN)] \\ &= [\beta](\mu\gamma.M_1[[\alpha]w := [\alpha](wN)]), \\ M'[[\alpha]w := [\alpha](wN')] &= M_1'[\gamma := \beta][[\alpha]w := [\alpha](wN)] \\ &= M_1'[[\alpha]w := [\alpha](wN)][\gamma := \beta]. \end{aligned}$$

By induction hypothesis for $M_1$, we have $M_1[[\alpha]w := [\alpha](wN)] \Rightarrow M_1'[[\alpha]w := [\alpha](wN')]$. Therefore we have $[\beta](\mu\gamma.M_1[[\alpha]w := [\alpha](wN)]) \Rightarrow M_1'[[\alpha]w := [\alpha](wN')][\gamma := \beta]$ by the rule 8. Thus Lemma holds. QED

**Theorem 1.** *For any $\lambda\mu$-term $M$ and $M'$, if $M \Rightarrow M'$ then $M' \Rightarrow M^*$.*

The proof is by induction on the structure of $M \Rightarrow M'$ and is shown in Appendix.

**Theorem 2.** *If $M \Rightarrow M_1$ and $M \Rightarrow M_2$, then there exists some $M_3$ such that $M_1 \Rightarrow M_3$ and $M_2 \Rightarrow M_3$.*

**Proof.** Put $M_3 = M^*$. Then Theorem holds by Theorem 1. QED

Since the transitive and reflexive closure of "$\rightarrow$" is identical to the transitive closure of "$\Rightarrow$", we have the confluence of $\lambda\mu$-calculus.

**Theorem 3.** *$\lambda\mu$-calculus is confluent.*

## 4 Parallel Computation in Call-by-Value Calculus

A call-by-value version of $\lambda\mu$-calculus was first provided by Ong and Stewart [11]. As compared with the call-by-name system, one can adopt some reduction rules more in the call-by-value system; so-called symmetric structural reduction [12] such that $N(\mu\alpha.M) \rightarrow \mu\alpha.M[[\alpha]w := [\alpha](Nw)]$. It is known that adding such reduction rules breaks down the confluence unless the above term $N$ is in the form of a value. In this section, the notion of values as an extended form is introduced based on observation in [5].

$$V ::= x \mid \lambda x.M \mid [\alpha]M$$

This notion is closed under both a value-substitution and substitutions induced by structural reduction and symmetric structural reduction defined below.

A context $\mathcal{E}[\,]$ with a hole $[\,]$ is defined as usual, such that

$$\mathcal{E} ::= [\,] \mid \mathcal{E}M \mid V\mathcal{E}.$$

For $n \geq 0$ and a term $N$, we will write $\mathcal{E}_n[\mathcal{E}_{n-1}[\cdot\cdot\mathcal{E}_1[N]\cdot\cdot]]$ for $\mathcal{E}[N]$, where each $\mathcal{E}_i \not\equiv [\,]$ is either in the form of $V[\,]$ or $[\,]M$. For simplicity, such $\mathcal{E}_i$ also denotes the value $V$ or the term $M$.

The call-by-value $\lambda\mu$-calculus consists of the following reduction rules:

| | |
|---|---|
| $\beta_v$-reduction | $(\lambda x.M)V \to_v M[x := V]$ |
| Structural reduction | $(\mu\alpha.M_1)M_2 \to_v \mu\alpha.M_1[[\alpha]w := [\alpha](wM_2)]$ |
| Symmetric structural reduction | $V(\mu\alpha.M) \to_v \mu\alpha.M[[\alpha]w := [\alpha](Vw)]$ |
| Renaming reduction | $[\beta](\mu\alpha.V) \to_v V[\alpha := \beta]$ |

This renaming rule is different from that in [11]. The distinction is essential under the extended form of values, and this form of renaming would also be natural from the viewpoint of CPS-translation such as in [5].

We will show that the new parallel reduction can also be applicable to the confluence proof for the call-by-value system of $\lambda\mu$-calculus, contrary to the straightforward use of parallel reduction in [11]. To prove this, we define parallel reduction $\gg$ as follows:

**Definition 3.**

1. $x \gg x$
2. $\dfrac{M \gg M'}{\lambda x.M \gg \lambda x.M'}$
3. $\dfrac{M \gg M' \quad N \gg N'}{MN \gg M'N'}$
4. $\dfrac{M \gg M'}{\mu\alpha.M \gg \mu\alpha.M'}$
5. $\dfrac{M \gg M'}{[\alpha]M \gg [\alpha]M'}$
6. $\dfrac{M \gg M' \quad V \gg N'}{(\lambda x.M)V \gg M'[x := N']}$
7. $\dfrac{M \gg M' \quad N \gg N'}{(\mu\alpha.M)N \gg \mu\alpha.M'[[\alpha]w := [\alpha](wN')]}$
8. $\dfrac{M \gg M' \quad V \gg N'}{V(\mu\alpha.M) \gg \mu\alpha.M'[[\alpha]w := [\alpha](N'w)]}$
9. $\dfrac{V \gg M' \quad \mathcal{E}_1 \gg \mathcal{E}'_1 \quad \cdots \quad \mathcal{E}_n \gg \mathcal{E}'_n \quad \mathcal{E} = \mathcal{E}_n[\cdot\cdot\mathcal{E}_1[\,]\cdot\cdot] \quad \mathcal{E}' = \mathcal{E}'_n[\cdot\cdot\mathcal{E}'_1[\,]\cdot\cdot]}{[\alpha](\mathcal{E}[\mu\beta.V]) \gg M'[[\beta]w := [\alpha](\mathcal{E}'[w])]}$

It can now be seen that the transitive and reflexive closure of $\to_v$ is equivalent to the transitive closure of $\gg$.

**Lemma 2.** *(1) If $V \gg M$, then $M$ is also in the form of a value.*

*(2) If $M \gg N$ and $V \gg N'$, then $M[x := V] \gg N[x := N']$.*

*(3) If $M \gg M'$ and $N \gg N'$, then $M[[\alpha]w := [\alpha](wN)] \gg M'[[\alpha]w := [\alpha](wN')]$.*

*(4) If $M \gg M'$ and $V \gg N'$, then $M[[\alpha]w := [\alpha](Vw)] \gg M'[[\alpha]w := [\alpha](N'w)]$.*

*(5) If $M \gg M'$, then $M[\beta := \alpha] \gg M'[\beta := \alpha]$.*

*(6) Let $n \geq 0$. Let $\mathcal{E}[\,] = \mathcal{E}_n[\cdot\cdot\mathcal{E}_1[\,]\cdot\cdot]$ and $\mathcal{E}'[\,] = \mathcal{E}'_n[\cdot\cdot\mathcal{E}'_1[\,]\cdot\cdot]$. If $M \gg M'$ and $\mathcal{E}_i \gg \mathcal{E}'_i$ $(1 \leq i \leq n)$, then $M[[\alpha]w := [\alpha](\mathcal{E}[w])] \gg M'[[\alpha]w := [\alpha](\mathcal{E}'[w])]$.*

**Proposition 1.** *For any $\lambda\mu$-term $M$, there exists $M^*$ such that for any $N$, $N \gg M^*$ whenever $M \gg N$.*

**Proof.** By induction on the derivation of $\gg$. Here, the complete development $M^*$ can be given inductively as follows:

**Definition 4.**

**1.** *$M = x$. Then $M^* = x$.*
**2.** *$M = \lambda x.M$. Then $M^* = \lambda x.M^*$.*
**3.** *$M = M_1M_2$.*
**3.1** *$M_1 = \lambda x.M_3$ and $M_2 = V_2$. Then $M^* = M_3^*[x := V_2^*]$.*
**3.2** *$M_1 = \mu\alpha.M_3$. Then $M^* = \mu\alpha.M_3^*[[\alpha]w := [\alpha](wM_2^*)]$.*
**3.3** *$M_1 = V_1$ and $M_2 = \mu\alpha.M_4$. Then $M^* = \mu\alpha.M_4^*[[\alpha]w := [\alpha](V_1^*w)]$.*
**3.4** *$M^* = M_1^*M_2^*$ o.w.*
**4.** *$M = \mu\alpha.M_1$. Then $M^* = \mu\alpha.M_1^*$.*
**5.** *$M = [\alpha]M_1$.*
**5.1** *$M_1 = \mathcal{E}[\mu\beta.V_2]$. Then $M^* = V_2^*[[\beta]w := [\alpha](\mathcal{E}^*[w])]$, where $\mathcal{E}^*[\,]$ is defined as $\mathcal{E}_n^*[\cdot\cdot\mathcal{E}_1^*[\,]\cdot\cdot]$ for $\mathcal{E}[\,] = \mathcal{E}_n[\cdot\cdot\mathcal{E}_1[\,]\cdot\cdot]$ and $n \geq 0$.*

**5.2** $M^* = [\alpha]M_1^*$ *o.w.*

We show only the case $M$ of $[\alpha]M_1$. The remaining cases can also be justified following a similar pattern.
1. Case $[\alpha]M_1$ of $[\alpha](\mathcal{E}[\mu\beta.V])$:

1-1. $M \gg N = [\alpha]N_1$ is derived from $\mathcal{E}[\mu\beta.V] \gg N_1$ by 5:

1-1-1. $\mathcal{E}[\mu\beta.V] \equiv \mu\beta.V$:

In this case, $\mu\beta.V \gg N_1 = \mu\beta.N_2$ is derived from $V \gg N_2$ by 4, where $N_2$ is also a value. From the induction hypothesis, we have $N_2 \gg V^*$, and hence $N = [\alpha](\mu\beta.N_2) \gg V^*[\beta := \alpha] = M^*$ is obtained by 9.

1-1-2. $\mathcal{E}[\mu\beta.V] \equiv \mathcal{E}_n[\cdot\cdot\mathcal{E}_1[\mu\beta.V]\cdot\cdot]$ $(n \geq 1)$:
Since $\mathcal{E}_1[\mu\beta.V]$ is not a value, $\mathcal{E}_n[\cdot\cdot\mathcal{E}_1[\mu\beta.V]\cdot\cdot] \gg N_1$ must be derived from $\mathcal{E}_1[\mu\beta.V] \gg N_2'$ and $\mathcal{E}_j \gg \mathcal{E}_j'$ $(2 \leq j \leq n)$ by the successive use of 3, where $N_1 = \mathcal{E}_n'[\cdot\cdot\mathcal{E}_2'[N_2']\cdot\cdot]$. Here, we have two cases for $\mathcal{E}_1$ and two derivations for each of those.

1-1-2-1. $\mathcal{E}_1[\mu\beta.V] \equiv V_1(\mu\beta.V)$:

1-1-2-1-1. $V_1(\mu\beta.V) \gg N_2' = N_3'N_4'$ is derived from $\mu\beta.V \gg N_4'$ and $V_1 \gg N_3'$ by 3:
Since $\mu\beta.V \gg N_4' = \mu\beta.N_5'$ must be derived from $V \gg N_5'$ by 4, we have $N_5' \gg V^*$ by the induction hypothesis, where $N_5'$ is also a value. Let $\mathcal{E}_1'$ be $N_3'[\,]$, where $N_3'$ is a value. Then the induction hypothesis gives $N_3' \gg V_1^*$ abbreviated as $\mathcal{E}_1' \gg \mathcal{E}_1^*$. From the induction hypotheses for $\mathcal{E}_j$ $(2 \leq j \leq n)$, we also have $\mathcal{E}_j' \gg \mathcal{E}_j^*$, and then $[\alpha](\mathcal{E}_n'[\cdot\cdot\mathcal{E}_1'[\mu\beta.N_5']\cdot\cdot]) \gg V^*[[\beta]w := [\alpha](\mathcal{E}_n^*[\cdot\cdot\mathcal{E}_1^*[w]\cdot\cdot])]$ is obtained by 9.

1-1-2-1-2. $V_1(\mu\beta.V) \gg N_2' = N_3'[[\alpha]w := [\alpha](N_4'w)]$ is derived from $V_1 \gg N_3'$ and $V \gg N_4'$ by 8:

The induction hypotheses give $N_3' \gg V_1^*$ and $N_4' \gg V^*$. From the substitution lemma, we have $N_4'[[\beta]w := [\beta](N_3'w)] \gg V^*[[\beta]w := [\beta](V_1^*w)]$, where $N_4'$ is also a value and values are closed under substitutions. The induction hypotheses for $\mathcal{E}_j$ $(2 \leq j \leq n)$ also give $\mathcal{E}_j' \gg \mathcal{E}_j^*$. Hence, the use of 9 derives

$$[\alpha](\mathcal{E}_n'[\cdot\cdot\mathcal{E}_2'[\mu\beta.N_4'[[\beta]w := [\beta](N_3'w)]]\cdot\cdot])$$
$$\gg (V^*[[\beta]w := [\beta](V_1^*w)])[[\beta]w := [\alpha](\mathcal{E}_n^*[\cdot\cdot\mathcal{E}_2^*[w]\cdot\cdot])],$$

whose right-hand side is equivalent to $V^*[[\beta]w := [\alpha](\mathcal{E}_n^*[\cdot\cdot\mathcal{E}_1^*[w]\cdot\cdot])]$, where $\mathcal{E}_1^*$ is $V_1^*[\,]$.

1-1-2-2. $\mathcal{E}_1[\mu\beta.V] \equiv (\mu\beta.V)M_2$:

In this case, we have two derivations for $(\mu\beta.V)M_2 \gg N_2'$ by the use of 3 or 7. Each case can be verified following a similar pattern to the above two cases.

1-2. $M \gg N = N'[[\alpha]w := [\beta](\mathcal{E}'[w])]$ is derived from $V \gg N'$ and $\mathcal{E}_i \gg \mathcal{E}_i'$ $(1 \leq i \leq n)$, where $\mathcal{E}[\,] = \mathcal{E}_n[\cdot\cdot\mathcal{E}_1[\,]\cdot\cdot]$ and $\mathcal{E}'[\,] = \mathcal{E}_n'[\cdot\cdot\mathcal{E}_1'[\,]\cdot\cdot]$:
The successive application of the substitution lemma to the induction hypotheses.

2. Otherwise:
The straightforward use of the induction hypothesis. QED

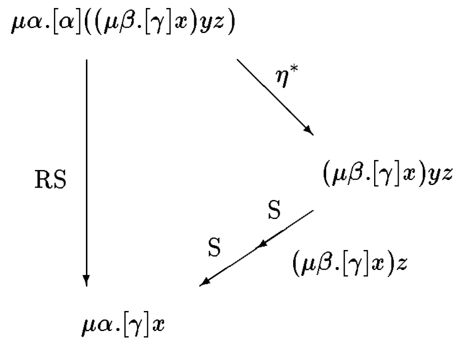
Finally, the confluence for the call-by-value $\lambda\mu$-calculus can be confirmed, since $\gg$ has the diamond property.

**Theorem 4.** *The call-by-value $\lambda\mu$-calculus has the confluence.*

## 5 Related Works and Further Problems

Parallel reduction is very clear and intuitive idea which means to reduce a number of redexes (existing in the term) simultaneously. It is often applied to prove the confluence of reduction system. However, a naive formulation parallel reduction does not always work. The $\lambda\mu$-calculus is one of such reduction systems. We showed that the difficulty is in the sequentiality of the structural reduction. So we think that consecutive sequence of structural reduction should be considered as one step of parallel reduction. As pointed out in Takahashi [10], the idea does not work for $\lambda\eta^{-1}$, i.e., $\lambda$-calculus with $\eta$-expansion:$M \to \lambda x.Mx$. The confluence of $\lambda\eta^{-1}$ is proved in [1, 9]. Jay and Ghani [9] proved the confluence by introducing "parallel expansion" which includes, roughly speaking, a consecutive application of $\eta^{-1}$: $M \to \lambda x_1.Mx_1 \to \lambda x_1x_2.Mx_1x_2 \to \lambda x_1x_2x_3.Mx_1x_2x_3 \to \cdots$. Van Raamsdonk[14] introduced a notion of "superdevelopment" to prove confluence of the orthogonal combinatory reduction systems. A superdevelopment is a reduction sequence in which besides redexes that descend from the initial term some redexes that are created during reduction may be contracted. A key of these works is to overcome some sequentiality of reduction. We cannot tell ,at the moment, what kind of reduction contains such sequentiality. To find some criterion to tell such sequentiality is a further work.

Parigot's $\lambda\mu$-calculus has another reduction rule $\eta^*$: $\mu\alpha.[\alpha]M \to M$ if $\alpha$ has no free occurrence in $M$. Consider a term $M = \mu\alpha.[\alpha]((\mu\beta.[\gamma]x)yz)$. Then $M$ has $\eta^*$-redex and the redex with respect to the rule 9 of Definition1. The reduction of each redex is represented by "$\eta^*$" and "RS" in the following figure.




By "RS", we reach $\mu\alpha.[\gamma]x$ with one step parallel reduction. If we apply $\eta^*$ first, we have $(\mu\beta.[\gamma]x)yz$ from which we cannot reach $\mu\alpha.[\gamma]x$ with one step parallel reduction. It seems that we can overcome this situation by counting a series of structural reductions as one step. We have a formulation of parallel reduction with $\eta^*$ with this idea. However, the definition of the complete development $M^*$ becomes very complex and will be discussed elsewhere.

## References


1. Y. Akama: "On Mints's Reduction for CCC-Calculus", *Lecture Notes in Computer Science* **664**, pp. 1–15, 1993.
2. H. P. Barendregt: "The Lambda Calculus," 2nd ed., North-Holland, Amsterdam, 1984.
3. Ph. de Groote: "On the Relation between the $\lambda\mu$-Calculus and the Syntactic Theory of Sequential Control", *Lecture Notes in Computer Science* **822**, pp. 31–43, 1994.
4. M. Felleisen, D. P. Friedman, E. Kohlbecker and B. Duba: "A Syntactic Theory of Sequential Control", *Theoretical Computer Science* **52**, pp. 205–237, 1987.
5. K. Fujita: "Calculus of Classical Proofs I", *Lecture Notes in Computer Science* **1345**, pp. 321–335, 1997.
6. T. Griffin: "A Formulae-as-Types Notion of Control", Conference Record of 17th ACM Symposium on Principles of Programming Languages, pp. 47–58, 1990.
7. J. R. Hindley: "Basic Simple Type Theory", Cambridge University Press, 1997.
8. J. R. Hindley, J. P. Seldin: "Introduction to Combinators and $\lambda$-Calculus", Cambridge University Press, 1986.
9. C. B. Jay and N. Ghani: "The Virtues of Eta-expansion ", LFCS report ECS-LFCS-92-243, October 1992.
10. M. Takahashi: "Parallel Reductions in $\lambda$-Calculus", *Information and Computation***118**, pp. 120–7, 1995.
11. C. -H. L. Ong and C. A. Stewart: "A Curry-Howard Foundation for Functional Computation with Control", Proc. 24th Annual ACM SIGPLAN-SIGACT Symposium of Principles of Programming Languages, 1997.
12. M. Parigot: "$\lambda\mu$-calculus: An Algorithmic Interpretation of Classical Natural Deduction", *Lecture Notes in Artificial Intelligence* **624**, pp. 190–201, Springer, 1992.
13. M. Parigot: "Strong Normalization for Second Order Classical Natural Deduction", Proc. 8th Annual IEEE Symposium on Logic in Computer Science, pp. 39–46, 1993.
14. F. van Raamsdonk: "Confluence and Superdevelopment", *Lecture Notes in Computer Science* **690**, pp. 168–182, 1993.


## Appendix:Proof of Theorem 1

**Theorem 4** *For any $\lambda\mu$-term $M$ and $M'$, if $M \Rightarrow M'$ then $M' \Rightarrow M^*$.*

**Proof.** By induction on the structure of $M \Rightarrow M'$.

**1.** $M = x$.

Then we have $M^* = M' = x$. Thus we have $M' \Rightarrow M^*$.

**2.** $M = \lambda x.M$.

Then $M \Rightarrow M'$ has the following form.

$$\frac{M_1 \Rightarrow M_1'}{M = \lambda x.M_1 \Rightarrow \lambda x.M_1' = M'}\ 2$$

By induction hypothesis we have $M_1' \Rightarrow M_1^*$. Thus we have $M' = \lambda x.M_1' \Rightarrow \lambda x.M_1^* = M^*$.

**3.** $M = M_1 M_2$.

**3.1.** $M = (\lambda x.M_3)M_2$.

Then we have $M^* = M_3^*[x := M_2^*]$ and the last inference rule of $M \Rightarrow M'$ is either 3 or 6.

**3.1.1.** The last inference rule is 3.

Then $M \Rightarrow M'$ has the following form.

$$\frac{\dfrac{M_3 \Rightarrow M_3'}{\lambda x.M_3 \Rightarrow \lambda x.M_3'}\ 2 \quad M_2 \Rightarrow M_2'}{M = (\lambda x.M_1)M_2 \Rightarrow (\lambda x.M_3')M_2' = M'}\ 3$$

By induction hypothesis, we have $M_2' \Rightarrow M_2^*$ and $M_3' \Rightarrow M_3^*$. Applying the rule 6, we have $(\lambda x.M_3')M_2' \Rightarrow M_3^*[x := M_2^*]$. Thus $M' \Rightarrow M^*$ holds.

**3.1.2.** The last inference rule is 6.

$M \Rightarrow M'$ has the following form.

$$\frac{M_3 \Rightarrow M_3' \quad M_2 \Rightarrow M_2'}{M = (\lambda x.M_3)M_2 \Rightarrow M_3'[x := M_2']}\ 6$$

By induction hypothesis, we have $M_2' \Rightarrow M_2^*$ and $M_3' \Rightarrow M_3^*$. By Lemma 1*(1)*, it follows $M_3'[x := M_2'] \Rightarrow M_3^*[x := M_2^*]$. Thus $M' \Rightarrow M^*$ and Theorem holds.

**3.2.** $M = (\mu\alpha.M_3)M_2$.

Then the last inference of $M \Rightarrow M'$ is either 3 or 7.

**3.2.1.** The last inference rule of $M \Rightarrow M'$ is 3.

Then $M \Rightarrow M'$ has the following form.

$$\frac{\dfrac{M_3 \Rightarrow M_3'}{\mu\alpha.M_3 \Rightarrow \mu\alpha.M_3'}\ 4 \quad M_2 \Rightarrow M_2'}{M = (\mu\alpha.M_3)M_2 \Rightarrow (\mu\alpha.M_3')M_2'}\ 3$$

By induction hypothesis we have $M_2' \Rightarrow M_2^*$ and $M_3' \Rightarrow M_3^*$. Applying the rule 7, we have $(\mu\alpha M_3')M_2' \Rightarrow \mu\alpha.M_3^*[[\alpha]w := [\alpha](wM_2^*)]$. Hence $M' \Rightarrow M^*$ and Theorem holds.

**3.2.2.** The last inference rule of $M \Rightarrow M'$ is 7.

Then $M \Rightarrow M'$ has the following form.

$$\frac{M_3 \Rightarrow M_3' \quad M_2 \Rightarrow M_2'}{(\mu\alpha.M_3)M_2 \Rightarrow \mu\alpha.M_3'[[\alpha]w := [\alpha](wM_2')]}\ 7$$

By induction hypothesis we have $M_2' \Rightarrow M_2^*$ and $M_3' \Rightarrow M_3^*$. Applying Lemma 1*(2)*, we have $\mu\alpha.M_3'[[\alpha]w := [\alpha](wM_2')] \Rightarrow \mu\alpha.M_3^*[[\alpha]w := [\alpha](wM_2^*)]$. Hence $M' \Rightarrow M^*$ and Theorem holds.

**3.3.** $M = M_1M_2$ and $M_1$ is not a $\lambda$-abstraction or a $\mu$-abstraction.

Then $M^* = M_1^*M_2^*$ and $M \Rightarrow M'$ has the following form.

$$\frac{M_1 \Rightarrow M_1' \quad M_2 \Rightarrow M_2'}{M = M_1M_2 \Rightarrow M_1'M_2' = M'}\ 3$$

By induction hypothesis, we have $M_1' \Rightarrow M_1^*$ and $M_2' \Rightarrow M_2^*$. Thus we have $M_1'M_2' \Rightarrow M_1^*M_2^*$. Therefore Theorem holds.

**4.** $M = \mu\alpha.M_1$.

Then we have $M^* = \mu\alpha.M_1^*$ and $M \Rightarrow M'$ has the following form.

$$\frac{M_1 \Rightarrow M_1'}{\mu\alpha.M_1 \Rightarrow \mu\alpha.M_1'}\ 4$$

By induction hypothesis we have $M_1' \Rightarrow M_1^*$. Applying the rule 4, we have $\mu\alpha.M_1' \Rightarrow \mu\alpha.M_1^*$. Hence Theorem holds.

**5.** $M = [\alpha]M_1$.

**5.1.** $M = [\alpha](\mu\beta.M_2)$.

Then we have $M^* = M_2^*[\beta := \alpha]$. The last inference rule of $M \Rightarrow M'$ is either 5 or 8.

**5.1.1.** The last inference rule of $M \Rightarrow M'$ is 5.

Then $M \Rightarrow M'$ has the following form.

$$\frac{\dfrac{M_2 \Rightarrow M_2'}{\mu\beta.M_2 \Rightarrow \mu\beta.M_2'}\ 4}{M = [\alpha](\mu\beta.M_2) \Rightarrow [\alpha](\mu\beta.M_2') = M'}\ 5$$

By induction hypothesis we have $M_2' \Rightarrow M_2^*$. Apply the rule 8. Then we have $[\alpha](\mu\beta.M_2') \Rightarrow M_2^*[\beta := \alpha]$. Hence $M \Rightarrow M'$ and Theorem holds.

**5.1.2.** The last inference rule of $M \Rightarrow M'$ is 8.

$$\frac{M_2 \Rightarrow M_2'}{M = [\alpha](\mu\beta.M_2) \Rightarrow [\alpha]M_2'[\beta := \alpha]) = M'}\ 8$$

By induction hypothesis, we have $M_2' \Rightarrow M_2^*$. Apply Lemma 1*(3)*. Then we have $M_2'[\beta := \alpha] \Rightarrow M_2^*[\beta := \alpha]$, hence $M' \Rightarrow M^*$.

**5.2.** $M = [\alpha]((\mu\beta.M_2)N_1 \cdots N_n)$.

Then we have $M^* = M_2^*[[\beta]w := [\alpha](wN_1^* \cdots N_n^*)]$. The last inference rule of $M \Rightarrow M'$ is either 5 or 9.

**5.2.1.** The last inference rule of $M \Rightarrow M'$ is 5.

Then $M \Rightarrow M'$ has the followoing form.

$$\frac{\dfrac{(\mu\beta.M_2)N_1 \Rightarrow Q' \quad N_2 \Rightarrow N_2' \quad \cdots \quad N_n \Rightarrow N_n'}{(\mu\beta.M_2)N_1 \cdots N_n \Rightarrow Q'N_1' \cdots N_n'}\ 3,\cdots,3}{M = [\alpha]((\mu\beta.M_2)N_1 \cdots N_n) \Rightarrow [\alpha](Q'N_1' \cdots N_n') = M'}\ 5$$

Then the last inference rule of $(\mu\beta.M_2)N_1 \Rightarrow Q'$ is either 3 or 7.

**5.2.1.1.** The last inference rule of $(\mu\beta.M_2)N_1 \Rightarrow Q'$ is 3.

Then $Q' = \mu\beta.M_2'$ for some $M_2'$ and $M \Rightarrow M'$ has the following form.

$$\frac{\dfrac{\dfrac{M_2 \Rightarrow M_2'}{\mu\beta.M_2 \Rightarrow \mu\beta.M_2'}\ 4 \quad N_1 \Rightarrow N_1' \quad N_2 \Rightarrow N_2' \quad \cdots \quad N_n \Rightarrow N_n'}{(\mu\beta.M_2)N_1 \cdots N_n \Rightarrow (\mu\beta.M_2')N_1' \cdots N_n'}\ 3,\cdots,3}{M = [\alpha]((\mu\beta.M_2)N_1 \cdots N_n) \Rightarrow [\alpha]((\mu\beta.M_2')N_1' \cdots N_n') = M'}\ 5$$

By induction hypothesis, we have $M_2' \Rightarrow M_2^*, N_1' \Rightarrow N_1^* \cdots, N_n' \Rightarrow N_n^*$. Apply the rule 9. Then we have $[\alpha]((\mu\beta.M_2')N_1' \cdots N_n') \Rightarrow M_2^*[[\beta]w := [\alpha](wN_1^* \cdots N_n^*)]$. Hence $M' \Rightarrow M^*$ holds.

**5.2.1.2.** The last inference rule of $(\mu\beta.M_2)N_1 \Rightarrow Q'$ is 7.

Then $M \Rightarrow M'$ has the following form.

$$\frac{\dfrac{\dfrac{M_2 \Rightarrow M_2' \quad N_1 \Rightarrow N_1'}{(\mu\beta.M_2)N_1 \Rightarrow \mu\beta.M_2'[[\beta]w := [\beta](wN_1')]}\ 7 \quad N_2 \Rightarrow N_2' \quad \cdots \quad N_n \Rightarrow N_n'}{(\mu\beta.M_2)N_1 \cdots N_n \Rightarrow (\mu\beta.M_2'[[\beta]w := [\beta](wN_1')])N_1' \cdots N_n'}\ 3,\cdots,3}{M = [\alpha]((\mu\beta.M_2)N_1 \cdots N_n) \Rightarrow [\alpha]((\mu\beta.M_2'[[\beta]w := [\beta](wN_1')])N_1' \cdots N_n') = M'}\ 5$$

By induction hypothesis, we have $M_2' \Rightarrow M_2^*$ and $N_1' \Rightarrow N_1^*$. By Lemma 1*(2)*, we have $M_2'[[\beta]w := [\beta](wN_1')] \Rightarrow M_2^*[[\beta]w := [\beta](wN_1^*)]$. On the other hand, we have $N_2' \Rightarrow N_2^*, \cdots N_n' \Rightarrow N_n^*$ by induction hypothesis. Apply the rule 9.

Then we have
$[\alpha]((\mu\beta.M_2'[[\beta]w := [\beta](wN_1')])N_2' \cdots N_n')$
$\Rightarrow M_2^*[[\beta]w := [\beta](wN_1^*)][[\beta]w := [\alpha](wN_2^* \cdots N_n^*)]$
$= M_2[[\beta]w := [\alpha](wN_1^*N_2^* \cdots N_n^*)]$.
Hence $M \Rightarrow M'$ and Theorem holds.

**5.2.2.** The last inference rule of $M \Rightarrow M'$ is 9.

Then $M \Rightarrow M'$ has the following form.

$$\frac{M_2 \Rightarrow M_2' \quad N_1 \Rightarrow N_1' \quad \cdots N_n \Rightarrow N_n'}{[\alpha]((\mu\beta.M_2)N_1 \cdots N_n) \Rightarrow M_2'[[\beta]w := [\alpha](wN_1' \cdots N_n')]}\ 9$$

By induction hypothesis we have $M_2' \Rightarrow M_2^*, N_1' \Rightarrow N_1^*, \cdots, N_n' \Rightarrow N_n^*$. Apply Lemma 1*(2)* and *(3)*, we have $M_2^*[[\beta]w := [\alpha](wN_1' \cdots N_n')] \Rightarrow M_2 * [[\beta]w := [\alpha](wN_1^* \cdots N_n^*)]$. Hence $M' \Rightarrow M^*$ and Theorem holds.

**5.3.** $M = [\alpha]M_1$ and $M_1$ is not of the form $M_1 = \mu\beta.M_2$ or $M_1 = (\mu\beta.M_2)N_1 \cdots N_n$.

Then we have $M^* = [\alpha]M_1^*$ and $M \Rightarrow M'$ has the following form.

$$\frac{M_1 \Rightarrow M_1'}{[\alpha]M_1 \Rightarrow [\alpha]M_1'}\ 5$$

By induction hypothesis we have $M_1' \Rightarrow M_1^*$. Applying the rule 5, we have $[\alpha]M_1' \Rightarrow [\alpha]M_1^*$. Hence Theorem holds.

QED