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Abstract

We consider a numerical method to verify the existence and uniqueness of the solutions of nonlinear hyperbolic problems with guaranteed error bounds. Using a C^1 finite element solution and an inequality constituting a bound on the norm of the inverse operator of the linearized operator, we numerically construct a set of functions which satisfy the hypothesis of Banach's fixed point theorem for a continuous map on L^p-space in a computer. We present detailed verification procedures and give some numerical examples.

1 Introduction

In two preceding papers [5, 6], we discussed the numerical verification of the existence of solutions to nonlinear parabolic and hyperbolic equations in the one-dimensional case. This verification method is based on Plum's formulation [10] of verification methods and weak formulation for determining a bound on the inverse norm of the linearized operator. In this paper, we describe a numerical verification method that demonstrates existence and uniqueness of solutions to nonlinear hyperbolic equations. In order to ensure existence and uniqueness, we use the idea contained in Nakao's method [3, 12]. Another method used for hyperbolic equations [7] requires that the nonlinear map in question is retractive in a neighborhood of the solution. Our method is not subject to such a condition. Thus our method has the potential of being applicable to a more general class of hyperbolic equations.

In the following section, we introduce the problem considered and its fixed point formulation. In Section 3, a fundamental theorem which contains the verification conditions of our method is presented. In Section 4, using a weak formulation, we estimate the inverse norm of the linearized operator and give the algorithm for our method. Section 5 contains some examples that illustrate our method.

2 Problem and the Fixed Point Formulation

Consider the problem of finding a function u that satisfies the following relations:

$$\text{find } u \text{ satisfying } u \in L^2(J; H_0^1(\Omega)), u_t \in L^2(J; L^2(\Omega))$$

$$\begin{cases} u_{tt} - \triangle u &= -f(x,t,u) \quad (x,t) \in \Omega \times J, \\ u(x,0) &= 0 \qquad\qquad\quad x \in \Omega, \\ \frac{\partial u}{\partial t}(x,0) &= 0 \qquad\qquad\quad x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open interval on $\mathbf{R}$ or a bounded rectangular domain in $\mathbf{R}^2$. Let $J = (0,T)$ with $T > 0$, and let $Q = \Omega \times J$. Also, suppose $f(x,t,\cdot) : L^p(Q) \to L^2(Q)$ is a continuous map for fixed $(x,t) \in Q$ and some satisfying $2 \leq p \leq 6$.

Next, we define the time-dependent Sobolev spaces H by

$$H \equiv L^2(J; H_0^1(\Omega)) \cap H^1(J; L^2(\Omega))$$

with norm

$$\|u\|_H^2 = \int_J \|\nabla u(t)\|_{L^2(\Omega)}^2 dt + \int_J \|u_t(t)\|_{L^2(\Omega)}^2 dt,$$

where $\|\cdot\|_{L^p(\Omega)}$ is the usual $L^p(\Omega)$ norm. Then the problem (1) is equivalent to that of finding:

$$u \in \tilde{H} = \{\phi \in H | \phi(\cdot,0) = 0, \phi_t(\cdot,0) = 0\}$$

such that

$$(u_{tt}, v) + (\nabla u, \nabla v) = (-f(\cdot,u), v) \qquad t \in J, \quad v \in H_0^1(\Omega), \tag{2}$$

where $(\cdot,\cdot)$ is the usual $L^2(\Omega)$ inner product.

Let u_h be an approximate solution of (1). It is most common to think of such a solution as some finite element solution depending on h. Then suppose the following conditions hold for the nonlinear map f in (1):

(A1)$f : L^p(Q) \to L^2(Q)$ is continuous and maps bounded sets into bounded sets.

(A2)f is Fréchet differentiable in $L^p(Q)$.

We denote the Fréchet derivative of f at u_h by $f'(\cdot, u_h)$.

Now, as well known [4], for each $g \in L^2(Q)$, if $a \in C^1([0,T]; L^\infty(\Omega))$, the following problem has a unique solution $\phi \in \tilde{H}$:

$$(\phi_{tt}, v) + (\nabla\phi, \nabla v) + (a\phi, v) = (g, v) \qquad v \in H_0^1(\Omega), \quad t \in J. \tag{3}$$

We denote the above correspondence by $Ag = \phi$. Moreover, assuming that we can set $a = f'(\cdot, u_h)$, we define the fixed-point operator T by

$$Tu \equiv A[f'(\cdot, u_h)u - f(\cdot, u)]. \tag{4}$$

Then, from (A1),(A2), and the fact that the injection $H \hookrightarrow L^p(Q)$ is continuous, the operator $T : L^p(Q) \to L^p(Q)$ is continuous.

The map $f(\cdot, u) = gu^m$ is an example that satisfies assumption (A1) and (A2), where $g \in L^\infty(Q)$, and m is an arbitrary nonnegative integer satisfying $1 \le m \le 3$. In this case, $f'(\cdot, u_h)\psi = mgu_h^{m-1}\psi$ for $\psi \in L^p(Q)$, and if $u_h \in C^1([0,T]; L^\infty(Q))$, (4) is well-posed.

Remark 1. In the one-dimensional case we can choose $2 \le p < \infty$, which implies $1 \le m < \infty$ in the above example. In any case, we assume that the nonlinearity of f has a polynomial form.

3 Verification Condition

In order to transform (4) into a "residual-form", setting $v = u - u_h$, we introduce the operator $\tilde{T} : L^p(Q) \to L^p(Q)$ defined by

$$\tilde{T}v \equiv T(u_h + v) - u_h. \tag{5}$$

Then if we wish to find some solution of the given problem that is close to u_h, we may look for the fixed-point of $\tilde{T}$ that is close to 0. To construct a set V which includes a solution to (1), taking some real number α, we set

$$V \equiv \{v \in L^p(Q) | \|v\|_{L^p(Q)} \le \alpha\}. \tag{6}$$

Next, we choose the positive real numbers β and γ such that

$$\|\tilde{T}(0)\|_{L^p(Q)} \le \beta, \tag{7}$$

$$\|\tilde{T}'(v_1)v_2\|_{L^p(Q)} \le \gamma \qquad \forall v_1, v_2 \in V, \tag{8}$$

and define the set $K \subset L^p(Q)$ by

$$K \equiv \{v \in L^p(Q) | \|v\|_{L^p(Q)} \le \beta + \gamma\}. \tag{9}$$

Our verification condition is described in the following theorem, which is similar to those in [3, 12].

Theorem 1. If $K \subset V$ holds for V(that is, if $\beta + \gamma \le \alpha$), then there exists a solution to

$$v = \tilde{T}(v)$$

in K, and this solution is unique within the set V.

Proof.

First, using the same number α used in (6), we define a scaling norm $\|\cdot\|_V$ by

$$\|x\|_V = \frac{\|x\|_{L^p(Q)}}{\alpha} \qquad \forall x \in L^p(Q).$$

Second, the following two relations follow from the definition of this norm:

$$\tilde{T}(V) \subset V \qquad \text{and} \qquad \|\tilde{T}(v_2) - \tilde{T}(v_1)\|_V \le k\|v_2 - v_1\|_V \qquad \forall v_1, v_2 \in V$$

for some k satisfying $0 < k < 1$. Banach's fixed point theorem then gives the desired result (see [12] for details).

□

4 Constants in the verification condition

In this section we describe how to estimate β and γ introduced in the previous section. We assume that the constants C_1 and C_2 are known and that they satisfy

$$\begin{aligned} \|Ar\|_H &\leq C_1\|r\|_{L^2(Q)} \qquad \forall r \in L^2(Q), & (10) \\ \|Ar\|_{L^p(Q)} &\leq C_2\|Ar\|_H. & (11) \end{aligned}$$

If we then compute an approximate solution u_h satisfying $d[u_h] \equiv u_{htt} - \triangle u_h + f(u_h) \in L^2(Q)$, the following relations holds:

$$\begin{aligned} \|\tilde{T}(0)\|_{L^p(Q)} &= \|Ad[u_h]\|_{L^p(Q)} \\ &\leq C_2\|Ad[u_h]\|_H \\ &\leq C_1C_2\|d[u_h]\|_{L^2(Q)}. \end{aligned} \tag{12}$$

Similarly, we can obtain

$$\begin{aligned} \|\tilde{T}'(v_1)v_2\| &\leq C_1C_2\|f'(u_h+v_1)v_2 - f'(u_h)v_2\|_{L^2(Q)} \\ &\leq C_1C_2G(\|v_2\|_{L^p(Q)}). \end{aligned} \tag{13}$$

Here $G : [0,\infty) \to [0,\infty)$ is a monotonically increasing function that satisfies

$$G(\|v_2\|_{L^p(Q)}) \leq C_{(u_h,\alpha)}\|v_2\|_{L^p(Q)},$$

where $C_{(u_h,\alpha)}$ is a positive constant independent of $v_2 \in V$. Since $f'(v) : L^P(Q) \to L^2(Q)$ is a bounded and linear operator for any $v \in L^p(Q)$, we can construct a function G satisfying the above inequality.

In what follows, we consider the open intervals $I_{x_1} = (a_{x_1}, b_{x_1})$ and $I_{x_2} = (a_{x_2}, b_{x_2})$ for real numbers $a_{x_1} < b_{x_1}$ and $a_{x_2} < b_{x_2}$ and define $\Omega = I_{x_1} \times I_{x_2}$ and $d = max\{b_{x_1} - a_{x_1}, b_{x_2} - a_{x_2}\}$.

Lemma 1. Let $\underline{a}$ and $\overline{a}$ denote constants satisfying $\underline{a} \leq a(x,t) \leq \overline{a}$ for almost all $(x,t) \in Q$. Then C_1 in (10) is given by

$$C_1 = \sqrt{Te^{cT}},$$

where $c = max(1-\underline{a}, \dfrac{d^2}{\pi^2}\|a_t\|_{L^\infty(Q)})$ for $\underline{a} < 0$ and $c = max(1, \dfrac{d^2}{\pi^2}\|a_t\|_{L^\infty(Q)})$ for $\underline{a} \geq 0$.

Proof.

We first define

$$b(\phi, v) \equiv (\nabla\phi, \nabla v) + (a\phi, v) \qquad \phi, v \in H_0^1(\Omega), \tag{14}$$

and set $v = \phi_t$ in (3), yielding

$$(\phi_{tt}(t), \phi_t) + b(\phi(t), \phi_t) = (g(t), \phi_t). \tag{15}$$

Then using the following relation

$$\frac{d}{dt}b(\phi,\phi) = 2b(\phi,\phi_t) + (a_t\phi, \phi),$$

we obtain

$$\frac{d}{dt}(\|\phi_t\|^2_{L^2(\Omega)} + b(\phi,\phi)) - (a_t\phi,\phi) = 2(g,\phi_t).$$

Integrating both sides with respect to t gives

$$\|\phi_t\|^2_{L^2(\Omega)} + b(\phi,\phi) = \int_0^t (a_t\phi,\phi)ds + 2\int_0^t (g,\phi_t)ds.$$

Here we note that

$$\begin{aligned}
&\text{if } \underline{a} \geq 0, \text{ then } b(\phi,\phi) \geq \|\nabla\phi\|^2_{L^2(Q)},\\
&\text{if } \underline{a} < 0, \text{ then } b(\phi,\phi) \geq \|\nabla\phi\|^2_{L^2(Q)} + \underline{a}\|\phi\|^2_{L^2(Q)},
\end{aligned}$$

and

$$\|\phi(t)\|^2_{L^2(\Omega)} \leq \|\phi(0)\|^2_{L^2(\Omega)} + \int_0^t \|\phi_s(s)\|^2_{L^2(\Omega)}ds = \int_0^t \|\phi_s(s)\|^2_{L^2(\Omega)}ds.$$

We only prove the case for $\underline{a} < 0$, because we may put $\underline{a} = 0$ for $\underline{a} \geq 0$ in the following.

By using the above inequalities, we have

$$\begin{aligned}
\|\phi_t\|^2_{L^2(\Omega)} + \|\nabla\phi\|^2_{L^2(\Omega)} \;&\leq\; -\underline{a}\int_0^t \|\phi_s\|^2_{L^2(\Omega)}ds\\
&+\; \|a_t\|_{L^\infty(Q)}\int_0^t \|\phi\|^2_{L^2(\Omega)}ds + \int_0^t \|g\|^2_{L^2(\Omega)}ds + \int_0^t \|\phi_s\|^2_{L^2(\Omega)}ds.
\end{aligned}$$

Furthermore, using Poincaré's inequality, we have

$$\|\phi\|_{L^2(\Omega)} \leq \frac{d}{\pi}\|\nabla\phi\|_{L^2(\Omega)}, \tag{16}$$

and setting

$$c = max(1-\underline{a}, \frac{d^2}{\pi^2}\|a_t\|_{L^\infty(Q)}),$$

we obtain

$$\|\phi_t\|^2_{L^2(\Omega)} + \|\nabla\phi\|^2_{L^2(\Omega)} \leq c(\int_0^t \|\phi_s\|^2_{L^2(\Omega)}ds + \int_0^t \|\nabla\phi\|^2_{L^2(\Omega)}ds) + \int_0^t \|g\|^2_{L^2(\Omega)}ds.$$

Finally, Gronwall's lemma provides the relation

$$\|\phi_t\|^2_{L^2(Q)} + \|\nabla\phi\|^2_{L^2(Q)} \leq Te^{cT}\|g\|^2_{L^2(Q)}.$$

□

To calculate C_2 in (11), we slightly modify the proof of the Sobolev Embedding theorem.

Proposition 1. (e.g.[1]) Let $\tilde{x_1} = (x_2, x_3), \tilde{x_2} = (x_1, x_3), \tilde{x_3} = (x_1, x_2)$, and let $I_k (k = 1, 2, 3)$ be bounded open intervals. The function f is defined by

$$f(x) = f(x_1, x_2, x_3) = f_1(\tilde{x_1}) f_2(\tilde{x_2}) f_3(\tilde{x_3}),$$

where $f_k \in L^2(\Omega_k)(k = 1, 2, 3)$, and $\Omega_1 = I_2 \times I_3$, $\Omega_2 = I_1 \times I_3, \Omega_3 = I_1 \times I_2$. Then $f \in L^1(\Omega)$, and the following inequality holds:

$$\|f\|_{L^1(\Omega)} \le \|f_1\|_{L^2(\Omega_1)} \|f_2\|_{L^2(\Omega_2)} \|f_3\|_{L^2(\Omega_3)},$$

where $\Omega = I_1 \times I_2 \times I_3$.

Lemma 2. C_2 in (11) is given by

$$C_2 = \begin{cases} \frac{p}{4}|Q|^{\frac{1}{p}} & (\Omega \text{ is one dimensional}) \\ (\frac{1}{2})^{\frac{2}{3}} \frac{2\sqrt{3}}{9} p |Q|^{\frac{6-p}{6p}} & (\Omega \text{ is two dimensional}). \end{cases}$$

Proof.

The proof for the one-dimensional case is found in [2, 6].

We consider the two dimensional case. For any $w \in C^1([0, T]; C_0^\infty(\Omega))$ and arbitrary $(x, t) \in Q$, we have

$$|w(x_1, x_2, t)| \le \frac{1}{2} \int_{I_{x_1}} |w_{x_1'}(x_1', x_2, t)| dx_1',$$

$$|w(x_1, x_2, t)| \le \frac{1}{2} \int_{I_{x_2}} |w_{x_2'}(x_1, x_2', t)| dx_2'$$

and

$$|w(x_1, x_2, t)| \le \int_J |w_{t'}(x_1, x_2, t'))| dt'.$$

Then using Proposition 1 for $f(x_1, x_2, t) = |w(x_1, x_2, t)|^{\frac{3}{2}}$, we obtain

$$\int_Q |w(x_1, x_2, t)|^{\frac{3}{2}} dxdydt \le \frac{1}{2} \|w_{x_1}\|_{L^1(Q)}^{\frac{1}{2}} \|w_{x_2}\|_{L^1(Q)}^{\frac{1}{2}} \|w_t\|_{L^1(Q)}^{\frac{1}{2}}.$$

This implies

$$\|w\|_{L^{\frac{3}{2}}(Q)} \le (\frac{1}{2})^{\frac{2}{3}} \|w_{x_1}\|_{L^1(Q)}^{\frac{1}{3}} \|w_{x_2}\|_{L^1(Q)}^{\frac{1}{3}} \|w_t\|_{L^1(Q)}^{\frac{1}{3}}. \tag{17}$$

Substituting $w = |v|^t (t \ge 1)$ into (17) and using some density arguments gives

$$\|u\|_{L^{\frac{3}{2}t}(Q)}^t \le (\frac{1}{2})^{\frac{2}{3}} t \|u^{t-1}\|_{L^2(Q)} \|u_{x_1}\|_{L^2(Q)}^{\frac{1}{3}} \|u_{x_2}\|_{L^2(Q)}^{\frac{1}{3}} \|u_t\|_{L^2(Q)}^{\frac{1}{3}}. \tag{18}$$

Then Hölder's inequality, we obtain

$$\int_Q |u|^{2(t-1)} dxdydt \le \|u\|_{L^{\frac{3}{2}t}(Q)} |Q|^{\frac{4-t}{3t}}. \tag{19}$$

We conclude from (18) and (19) that

$$\begin{aligned}
\|u\|_{L^{\frac{3}{2}t}(Q)} &\leq (\frac{1}{2})^{\frac{2}{3}}t|Q|^{\frac{4-t}{6t}}\|u_{x_1}\|_{L^2(Q)}^{\frac{1}{3}}\|u_{x_2}\|_{L^2(Q)}^{\frac{1}{3}}\|u_t\|_{L^2(Q)}^{\frac{1}{3}} \\
&\leq (\frac{1}{2})^{\frac{2}{3}}t|Q|^{\frac{4-t}{6t}}\frac{1}{3}(\|u_{x_1}\|_{L^2(Q)}+\|u_{x_2}\|_{L^2(Q)}+\|u_t\|_{L^2(Q)}) \\
&\leq (\frac{1}{2})^{\frac{2}{3}}t|Q|^{\frac{4-t}{6t}}\frac{\sqrt{3}}{3}\|u\|_H.
\end{aligned}$$

Setting $t = \frac{2}{3}p$, we obtain the desired conclusion.

□

Finally, we describe the algorithm for finding a real number α that satisfies the verification condition in Theorem 1,

$$\beta + \gamma \leq \alpha. \tag{20}$$

What we present here is the most basic such algorithm [3]. Since γ depends on α, we write $\gamma = \gamma(\alpha)$. The algorithm is as follows:

1. Compute a constant β satisfying (8).
2. Set $\alpha = \beta$.
3. Compute $\gamma(\alpha)$ satisfying (8).
4. Check the conditions (20), $\beta + \gamma(\alpha) \leq \alpha$. If this condition is satisfied, then stop. This means that verification is completed.
5. Otherwise, make the replacement

$$\alpha \leftarrow (1+\delta)\alpha$$

   for a certain positive number δ and return to 3.

If the maximum iteration number exceeds some maximal value that we decide in advance without satisfying (20), the verification fails.

5 Verification procedures and numerical examples

Let S_h be a finite-dimensional subspace of $H_0^1(\Omega) \cap H^2(\Omega)$ depending on h and let N be the dimension of S_h. Then we can represent u_h by

$$u_h(x,t) = \sum_{i=1}^{N} u_i(t)\hat{\phi}_i(x),$$

where the $\hat{\phi}_i$ are base functions in S_h.

The function $u_i(t)$ constitutes the time-dependent coefficient of the base function $\hat{\phi}_i(x)$. For the discretization of time, we take equal time steps of length Δt and define

$$t_k = k\Delta t, \qquad k = 0, 1, 2, \cdots.$$

We used the Newmark method [11], which generates the following relation:

$$u_i^{(n)}(t+\Delta t) = u_i^{(n)}(t) + \Delta t \dot{u}_i^{(n)}(t) + \Delta t^2[\beta \ddot{u}_i^{(n)}(t+\Delta t) + (\frac{1}{2}-\beta)\ddot{u}_i^{(n)}(t)] \tag{21}$$

$$\dot{u}_i^{(n)}(t+\Delta t) = \dot{u}_i^{(n)}(t) + \Delta t[\theta \ddot{u}_i^{(n)}(t+\Delta t) + (1-\theta)\ddot{u}_i^{(n)}(t)], \tag{22}$$

where $\dot{u}_i = \dfrac{du_i}{dt}$ and $\ddot{u}_i = \dfrac{d^2u_i}{dt^2}$, and θ and β are some non-negative parameters.

We compute an approximate solution by combining the Newton iteration and the Newmark method (see [6] for details). Since $u_{htt} - \triangle u_h + f(\cdot, u_h) \in L^2(Q)$ is required, we use the piecewise cubic Hermite function as the base function in space and the piecewise cubic Hermite interpolation in time. Moreover, since f has a polynomial restriction and piecewise polynomials are used in space and time, we can compute $\|d[u_h]\|_{L^2(Q)}$ directly, stepwise in time.

One-dimensional case

In (1), we set

$f(x,t,u) = -Au^2 - k\sin\pi x(2+\pi^2t^2 - Akt^4\sin\pi x)$,

where A and k are constants, and we let $R = 1$ and $T = 1$. The exact solution is $u(x,t) = kt^2\sin\pi x$.

If we take $p = 4$, then $(A1)$ and $(A2)$ are satisfied. Then, since we have

$$\begin{aligned} \|f'(u_h+v_1)v_2 - f'(u_h)v_2\|^2_{L^2(Q)} &= 4A^2\int\int |v_1v_2|^2 dxdt \\ &\le 4A^2\|v_1\|^2_{L^4(Q)}\|v_2\|^2_{L^4(Q)} \le 4A^2\alpha^4, \end{aligned}$$

(8) is satisfied for

$$G(\alpha) = 2A\alpha^2.$$

We illustrate our algorithm with the numerical results of several examples, where NS and NT are the partition numbers of space and time, respectively, and $M = NS\times NT$. Since the cubic Hermite interpolation procedure is fourth-order accurate for all sufficiently smooth functions and the Newmark method is second-order accurate, we choose NS and NT to satisfy the relation $NT = NS \times NS$ when adjusting the accuracy. Generally speaking, it is difficult to describe the stability of the Newmark method for nonlinear problems, but according to [11], if $\theta = \frac{1}{2}$ and $\beta = \frac{1}{4}$, the Newmark method is unconditionally stable for linear hyperbolic equations. Thus we choose $\theta = \frac{1}{2}$ and $\beta = \frac{1}{4}$ in these examples.

In the following, α represents the verified error bound in Theorem 1.

We obtain sharper bounds than those in [6] thanks to the better constant C used here. We note that Theorem 1 ensures not only the existence of solutions but also their uniqueness.

Case 1: $A = 1.5, k = 1, NS = 60, M = 216000$		
Parameters	The result in [6]	Our new result
C	7.3890561	1.6487213
δ	0.0013745	0.0013745
α	0.0785593	0.0022922

Case 2: $A = 0.8, k = 2, NS = 100, M = 1000000$		
Parameters	The result in [6]	Our new result
C	8.16616992	1.6487213
δ	0.00075867	0.00075867
α	0.146602	0.001255

Table 1: Results in the one dimensional-case.

Parameters	$A = 0.1, k = 0.5$
NS	4
Δt	$\frac{1}{16}$
C	0.489561
δ	0.69574
α	0.252794

Table 2: Results in two dimensional case.

Two dimensional case

In (1), we set

$$f(x,t,u) = f(x_1,x_2,t,u) = -Au^2 - k\sin\pi x_1 \sin\pi x_2(2 + 2t^2\pi^2 - Akt^4 \sin\pi x_1 \sin\pi x_2),$$

and let $\Omega = (0,1) \times (0,1)$ and $T = 0.25$. The exact solution is $u(x_1,x_2,t) = kt^2 \sin\pi x_1 \sin\pi x_2$. The other conditions are the same as in the one-dimensional case.

Remark 2. In these computations, we used the usual floating-point number system with double precision. Therefore, the above results may include some unknown rounding errors. From the author's experiences, however, the order of magnitude of the effect of round-off errors is smaller than 10^{-10}. With this observation, we can assume that the numerical results are sufficiently reliable to at least six digits or so. Of course, we need to use arithmetic system with guaranteed accuracy for more rigorous verification.

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