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出版情報：DOI Technical Report. 159, 1999-02-17. Department of Informatics, Kyushu University

DOI-TR-159

DOI Technical Report

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February 17, 1999

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Abstract

In this paper we deal with 1-D finite cellular automata with a triplet local transition rule 14 and 142. And we investigate their behaviors and they are compared with each other.

1 Introduction

Cellular automata are discrete dynamical systems, which have discrete time and space, and have a local transition rule. Cellular automata were introduced by J. von Neumann as a simulator of a system having self-reproduction and universal computation. Since cellular automata have wide variety and their behavior are similar to those of complex systems as fractal and chaotic phenomenon for the last two decades many researchers investigated their behaviors and applied in mathematics, physics, biology, computer science and so on[1, 2, 3, 4]. Although Cellular automata have simple structure, their behaviors are very complicated. Behaviors of linear cellular automata were investigated by using algebraic methods[5, 8, 9, 10]. But those of many nonlinear cellular automata except some cellular automata[6, 7] are not analysed yet since we cannot investigate them by algebraic methods.

The transition rule of nonlinear cellular automata is difficult to deal with because of not to be able to use algebraic methods. But their behaviors are not always difficult. The numbers of limit cycles and transient length of many nonlinear cellular automata obey simple rules in particular.

In this paper behaviors of the nonlinear cellular automata CA-$14_{\alpha-\beta}(m)$ and CA-$142_{\alpha-\beta}(m)$ are compared while being investigated. We investigate their behaviors and show rules of the number of limit cycles, period length and transient length.

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2 Preliminaries

In this section, we define 1-dimensional finite cellular automata and necessary notations for after discussion.

X is the set $\{0,1\}^m$, which is called a configuration space, and an element in X is called a configuration. Usually a configuration $(x_1, x_2, \cdots, x_m)$ denotes $x_1 x_2 \cdots x_m$ for short.

The triplet local transition rule f is a function $\{0,1\}^3 \longrightarrow \{0,1\}$, and f is represented as follows:

111	110	101	100	011	010	001	000
r_7	r_6	r_5	r_4	r_3	r_2	r_1	r_0

Rule number R of f is defined as follows:

$$R = 2^7 r_7 + 2^6 r_6 + \cdots + 2^0 r_0.$$

For instance,

$$14 = 2^3 \times 0 + 2^2 \times 1 + 2^1 \times 1$$

and

$$142 = 2^7 \times 0 + 2^3 \times 1 + 2^2 \times 1 + 2^1 \times 1,$$

so the triplet local transition rule of rule number 14 and 142 are illustrated as follows:

	111	110	101	100	011	010	001	000
Rule 14	0	0	0	0	1	1	1	0
Rule 142	1	0	0	0	1	1	1	0

A cellular automaton which is dealed with in this paper is a pair (X, δ) where a global transition function $\delta : X \to X$ is defined by a triplet local transition rule f as follows;

$$\delta(x_1 x_2 \cdots x_m) = (f(\alpha x_1 x_2) f(x_1 x_2 x_3) \cdots f(x_{m-1} x_m \beta))$$

We call a pair (α, β) a boundary. The boundary condition is cyclic if and only if $\alpha = x_m$ and $\beta = x_1$, and the boundary condition is fixed if α and β are fixed in $\{0,1\}$. And in particular we called the boundary condition $a - b$ if $\alpha = a$ and $\beta = b$. The triplet local transition rule of rule number R (or the global transition function δ defined by f of rule number R) is denoted by "rule R" for short.

The cellular automaton (X, δ) such that $X = \{0,1\}^m$ and the rule number of a triplet local transition rule f which define δ is R and its boundary condition is $a - b$ is denoted by CA-$R_{a-b}(m)$.

The configuration x is on a limit cycle of period length T if there exists a positive integer s such that $\delta^s(x) = x$, and $T = min\{s \geq 1 | \delta^s(x) = x\}$. And the configurations $x(1), x(2), \cdots, x(T-1)$ form a limit cycle of period length T if $x(i+1) = \delta(x(i))$ and $x(T) = x(1)$, and $x(i)$ is on a limit cycle of period length T where $1 \leq i \leq T-1$. A limit cycle of period length T is denoted by a T-cycle, in particular 1-cycle is called a fixed point. And a number of a limit cycle of period length T is denoted by $\gamma_T(m)$. A configuration x is non-initial if there exists a configuration y such that $x = \delta(y)$. The height $h(x)$ of x from a limit cycle is defined as follows:

$$h(x) = min\{s \geq 0 | \delta^s(x) \text{ is on a limit cycle}\}$$

Then the transient length $H(m)$ of CA-$R_{a-b}(m)$ is defined as follows;

$$H(m) = max\{h(x) | x \in \{0,1\}^m\}$$

The symmetric transition rule f^T of a local transition rule f is defined as follows;

$$f^T(x, y, z) = f(z, y, x)$$

The reverse transition rule $\overline{f}$ of a local transition rule f is defined as follows;

$$\overline{f}(x, y, z) = 1 - f(1 - x, 1 - y, 1 - z)$$

For example, the symmetric rule, the reverse rule and the symmetric reverse rule of rule 14 are rule 84, rule 143 and rule 213 respectively. And the symmetric rule, the reverse rule and the symmetric reverse rule of rule 142 are rule 212, rule 142 and rule 212 respectively. They are illustrated as follows;

	111	110	101	100	011	010	001	000
Rule 84	0	1	0	1	0	1	0	0
Rule 143	1	0	0	0	1	1	1	1
Rule 212	1	1	0	1	0	1	0	0
Rule 213	1	1	0	1	0	1	0	1

So CA-$14_{\alpha-\beta}(m)$, CA-$84_{\beta-\alpha}(m)$, CA-$143_{\overline{\alpha}-\overline{\beta}}(m)$ and CA-$213_{\overline{\beta}-\overline{\alpha}}(m)$ are isomorphic each other, and CA-$142_{\alpha-\beta}(m)$, CA-$212_{\beta-\alpha}(m)$, CA-$142_{\overline{\alpha}-\overline{\beta}}(m)$ and CA-$212_{\overline{\beta}-\overline{\alpha}}(m)$ are isomorphic each other.

For the discussion in this paper the following notations are defined;
Let A be a subsequence. Then the sequence composed of k A's, the sequence composed of k bits taken from some A's and an arbitrary bit are denoted by A^k, A^*_k and $*$, respectively. For example, $(011)^2 = 011011$, $(010)^*_5 = 01001$ and $(010)^*_6 = (010)^2$.

3 Behaviors of CA-$14_{\alpha-\beta}(m)$ and CA-$142_{\alpha-\beta}(m)$

In this section we investigate behaviors of CA-$14_{\alpha-\beta}(m)$ and CA-$142_{\alpha-\beta}(m)$ and compare their behaviors with each other. And we investigate the differences between their behaviors caused by the difference between rule 14 and rule 142.

The number of limit cycles and transient length of CA-$14_{\alpha-\beta}(m)$ and CA-$142_{\alpha-\beta}(m)$ are as table 1 and table 2.

α-β	m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0-0	$1-cycle$	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9
	$tran.len.$	0	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0-1	$1-cycle$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	$tran.len.$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29
1-0	$1-cycle$	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8
	$tran.len.$	1	1	3	4	5	6	7	8	9	10	11	12	13	14	15
1-1	$1-cycle$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	$tran.len.$	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28

Table 1: CA-$14_{\alpha-\beta}(m)$

α-β	m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0-0	$1-cycle$	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9
	$tran.len.$	0	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0-1	$1-cycle$	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8
	$tran.len.$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1-0	$1-cycle$	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8
	$tran.len.$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1-1	$1-cycle$	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9
	$tran.len.$	0	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Table 2: CA-$142_{\alpha-\beta}(m)$

From table 1 and table 2 we can easily see that CA-$14_{\alpha-\beta}(m)$ and CA-$142_{\alpha-\beta}(m)$ behave regularly. And CA-$14_{\alpha-0}(m)$ and CA-$142_{\alpha-0}(m)$ behave the same but CA-$14_{\alpha-1}(m)$ and CA-$142_{\alpha-1}(m)$ do differently. So it is conjectured that at boundary condition $\alpha - 0$ there exist no effects on the number of limit cycles and transient length of CA-$14_{\alpha-0}(m)$ and CA-$142_{\alpha-0}(m)$ by the different between rule 14 and rule 142. In order to show it we will investigate their behaviors in detail.

First necessary and sufficient conditions for a configuration c to be a fixed point are as follows;

Lemma 1 *1. The configuration c is a fixed point of CA-$14_{\alpha-\beta}(m)$ if and only if c satisfies the following conditions.*

- *$\alpha c \beta$ contains no 001.*
- *$\alpha c \beta$ contains no 110.*
- *$\alpha c \beta$ contains no 111.*

2. The configuration c is a fixed point of CA-$142_{\alpha-\beta}(m)$ if and only if c satisfies the following conditions;

- *$\alpha c\beta$ contains no* 001.
- *$\alpha c\beta$ contains no* 110.

Proof.

1. Let $d = \delta(c)$.
(sufficient condition) Since $f(111) = f(110) = 0$ and $f(001) = 1$ if $\alpha c\beta$ contains 111, 110 or 001, then c is not a fixed point. So if c is a fixed point then $\alpha c\beta$ contains no 111, no 110 and no 001.
(necessary condition) Note that $f(xyz) = y$ for $xyz \neq 111, 110$ and 001. So if $\alpha c\beta$ contains no 111, no 110 and no 001, then trivially c is a fixed point.

2. (sufficient condition) Assume that $\alpha c\beta$ contains 001 or 110. Since $f(110) = 0$ and $f(001) = 1$ c is not a fixed point.
(necessary condition) Assume that $\alpha c\beta$ contains no 001 and no 110. Then c is a fixed point since $f(xyz) = y$ for $xyz \neq 001$ and 110.

□

By the above lemma we can easily get the configurations which are fixed points of CA-$14_{\alpha-\beta}(m)$ and CA-$142_{\alpha-\beta}(m)$ as the following corollaries.

Corollary 1 *1. All fixed points of CA-$14_{0-0}(m)$ are $(10)^i 0^{m-2i}$ and $(10)^*_m$ where $0 \leq i \leq [\frac{m-1}{2}]$.*

2. All fixed points of CA-$14_{1-0}(m)$ are $(01)^i 0^{m-2i}$ where $0 \leq i \leq [\frac{m}{2}]$.

3. The configuration $(10)^*_m$ is the unique fixed point of CA-$14_{0-1}(m)$.

4. The configuration $(01)^*_m$ is the unique fixed point of CA-$14_{1-1}(m)$.

Corollary 2 *1. All fixed points of CA-$142_{0-0}(m)$ are $(10)^i 0^{m-2i}$ and $(10)^*_m$ where $0 \leq i \leq [\frac{m}{2}]$.*

2. All fixed points of CA-$142_{1-0}(m)$ are $(01)^i 0^{m-2i}$ where $0 \leq i \leq [\frac{m}{2}]$.

3. All fixed points of CA-$142_{0-1}(m)$ are $(10)^i 1^{m-2i}$ where $0 \leq i \leq [\frac{m}{2}]$.

4. All fixed points of CA-$142_{1-1}(m)$ are $(01)^i 1^{m-2i}$ and $(01)^*_m$ where $0 \leq i \leq [\frac{m}{2}]$.

Lemma 2 *The followings are common behaviors of CA-$14_{\alpha-0}(m)$ and CA-$142_{\alpha-0}(m)$.*

1. *Let c and c' be configurations such that $c_{i+1} = 0$ and $c' = c_1 c_2 \cdots c_i 0^{m-i}$, $d = \delta(c)$ and $d' = \delta(c')$. Then*
$$d_1 d_2 \cdots d_i = d'_1 d'_2 \cdots d'_i.$$

2. *Let c be a configuration and $d = \delta(c)$. If $c_{m-i}c_{m-i+1}\cdots c_m = 0^{i+1}$ then $d_{m-i}d_{m-i+1}\cdots d_m = 0^{i+1}$ where $0 \le i \le m-1$.*

3. *Let c be an arbitrary configuration and $d = \delta(c)$. Then the number of the subsequences 01 of $\alpha d0$ is equal to that of $\alpha c0$.*

4. *Let c be an arbitrary configuration and $d = \delta(c)$. Then the number of the subsequences 10 of $\alpha d0$ is equal to that of $\alpha c0$.*

Proof.

1. It is trivial.

2. It is trivial.

3. First we assume that $c_ic_{i+1} = 01$. Then if $c_{i-1} = 0$ then $d_{i-1}d_i = 01$, and if $c_{i-1} = 1$ then $d_id_{i+1} = 01$. Next we assume that $d_id_{i+1} = 01$.

   - Case of CA-$14_{\alpha-0}(m)$
     From $d_{i+1} = 1$ we have $c_ic_{i+1}c_{i+2} = 011, 010$ or 001 and from $d_i = 0$ $c_{i-1}c_ic_{i+1}c_{i+2} = 1011, 1010$ or $*001$.
   - Case of CA-$142_{\alpha-0}(m)$
     From $d_{i+1} = 1$ we have $c_ic_{i+1}c_{i+2} = 111, 011, 010$ or 001. $c_ic_{i+1}c_{i+2} = 111$ contradicts with $d_i = 0$. If $d_id_{i+1} = 01$ then $c_{i-1}c_ic_{i+1}c_{i+2} = 1011, 1010$ or $*001$.

   So the subsequence 01 is derived from itself and results to itself. Hence the number of 01 of $\alpha d0$ is equal to that of $\alpha c0$.

4. First we assume that $c_ic_{i+1} = 10$

   - Case of CA-$14_{\alpha-0}(m)$.
     If $c_{i-l}c_{i-l+1}\cdots c_{i-1} = 01^{l-1}$ then $d_{i-l}d_{i-l+1}\cdots d_{i+1} = 110^l$ or 010^l where $1 \le l \le i$. If $\alpha c_1c_2\cdots c_{i-1} = 1^i$ then $\alpha d_1d_2\cdots d_{i+1} = 10^{i+1}$.
   - Case of CA-$142_{\alpha-0}(m)$.
     If $c_{i-1} = 1$ then $d_{i-1}d_id_{i+1} = 100$. p And if $c_{i-1} = 0$ then $d_{i-1}d_id_{i+1} = *10$.

   Next we assume that $d_id_{i+1} = 10$.

   - Case of CA-$14_{\alpha-0}(m)$.
     From $d_i = 1$ we have $c_{i-1}c_ic_{i+1} = 011, 010$ or 001 and from $d_{i+1} = 0$ $c_{i-1}c_i\cdots c_{i+j} = 01^j0$ where $1 \le j \le l$ if $d_{i+2}d_{i+3}\cdots d_{i+l+1} = 0^l1$.
   - Case of CA-$142_{\alpha-0}(m)$.
     From $d_i = 1$ we have $c_{i-1}c_ic_{i+1} = 111, 011, 010$ or 001 and from $d_{i+1} = 0$ $c_{i-1}c_ic_{i+1}c_{i+2} = 010*, 0110$ or 1110.

So the subsequence 10 is derived from itself and results to itself. Hence the number of 10 of $\alpha d0$ is equal to that of $\alpha c0$.

□

We let R be 14 or 142 then the following corollary holds;

Corollary 3 *Let δ and δ' be the transition functions of CA-$R_{\alpha-0}(m)$ and CA-$R_{\alpha-0}(i)$, respectively, c and d be configurations of CA-$R_{\alpha-0}(m)$ such that $c_{i+1}c_{i+2}\cdots c_m = 0^{m-i}$ and $d = \delta(c)$, and c' and d' be configurations of CA-$R_{\alpha-0}(i)$ such that $c' = c_1c_2\cdots c_i$ and $d' = \delta'(c')$. Then*

$$d_1d_2\cdots d_i = d'.$$

And in the following lemma we investigate behaviors of the subsequence 1^l in CA-$14_{\alpha-\beta}(m)$ and CA-$142_{\alpha-0}(m)$.

Lemma 3 1. *For CA-$14_{\alpha-\beta}(m)$ the following hold; Let c be an arbitrary configuration. $\delta(c)$ contains no subsequence 111.*

2. *Let c be a configuration, $d = \delta(c)$ and $c_{i+1}c_{i+2}\cdots c_{i+l}$ and $d_{i'+1}d_{i'+2}\cdots d_{i'+l'}$ be nth subsequences of $\alpha c0$ and $\alpha d0$ such that $c_ic_{i+1}\cdots c_{i+l+1} = 01^l0$ and $d_{i'}d_{i'+1}\cdots d_{i'+l'+1} = 01^{l'}0$, respectively. Then for CA-$142_{\alpha-0}(m)$ if $l = 1$ then $l' = 1$ or 2, and if $l \leq 2$ then $l' = l-1$ or l.*

Proof.

1. Let $d = \delta(c)$. From the local transition rule 14 $d_i = 1$ is derived from only $c_{i-1} = 0$. So if $d_{i-1}d_id_{i+1} = 111$ then $c_{i-2}c_{i-1}c_i = 000$. But since $f(000) = 0$ $c_{i-2}c_{i-1}c_i = 000$ contradicts with $d_{i-1}d_id_{i+1} = 111$. Hence d contains no 111.

2. From lemma 2 and their proof we see that the subsequence 01 and 10 are derived from themselves and nth subsequence $d_{i'+1}d_{i'+2}\cdots d_{i'+l'}$ of $\alpha d0$ is derived from nth subsequence $c_{i+1}c_{i+2}\cdots c_{i+l}$ of $\alpha c0$.

   - First we assume that $l = 1$, that is, $c_ic_{i+1}c_{i+2} = 010$. If $c_{i-1} = 1$ or $i = 0$ then $d_id_{i+1}d_{i+2} = 010$, and if $c_{i-1} = 0$ then $d_{i-1}d_id_{i+1}d_{i+2} = 0110$.
   - Next we assume that $l \geq 2$, that is, $c_ic_{i+1}\cdots c_{i+l+1} = 01^l0$. If $c_{i-1} = 1$ or $i = 0$ then $d_id_{i+1}\cdots d_{i+l} = 01^{l-1}0$, and if $c_{i-1} = 0$ then $d_{i-1}d_i\cdots d_{i+l} = 01^l0$.

   Hence if $l = 1$ then $l' = 1$ or 2, and if $l \geq 2$ then $l' = l$ or $l-1$.

□

The following lemma states how the subsequence 0^l behaves in CA-$14_{\alpha-0}(m)$ and CA-$142_{\alpha-0}(m)$.

Lemma 4 *1. Let c be a non-initial configuration, $d = \delta(c)$ and $c_{i+1}c_{i+2}\cdots c_{i+l}$ and $d_{i'+1}d_{i'+2}\cdots d_{i'+l'}$ be nth subsequences of $\alpha c0$ and $\alpha d0$ such that $c_ic_{i+1}\cdots c_{i+l+1} = 10^l1$ and $d_{i'}d_{i'+1}\cdots d_{i'+l'+1} = 10^{l'}1$, respectively. Then for CA-$14_{\alpha-0}(m)$ if $l = 1$ then $l' = 1$ or 2, and if $l \le 2$ then $l' = l-1$ or l.*

2. Let c be a configuration, $d = \delta(c)$ and $c_{i+1}c_{i+2}\cdots c_{i+l}$ and $d_{i'+1}d_{i'+2}\cdots d_{i'+l'}$ be nth subsequences of $\alpha c0$ and $\alpha d0$ such that $c_ic_{i+1}\cdots c_{i+l+1} = 10^l1$ and $d_{i'}d_{i'+1}\cdots d_{i'+l'+1} = 10^{l'}1$, respectively. Then for CA-$142_{\alpha-0}(m)$ if $l = 1$ then $l' = 1$ or 2, and if $l \le 2$ then $l' = l-1$ or l.

Proof.

1. From lemma 2 and their proof we see that the subsequence 01 and 10 are derived from themselves and nth subsequence $d_{i'+1}d_{i'+2}\cdots d_{i'+l'}$ of $\alpha d0$ is derived from nth subsequence $c_{i+1}c_{i+2}\cdots c_{i+l}$ of $\alpha c0$.

   - First we assume that $l = 1$, that is, $c_ic_{i+1}c_{i+2} = 101$. If $c_{i-1} = 0$ or $i = 0$ then $d_id_{i+1}d_{i+2} = 101$, and if $c_{i-1} = 1$ then $d_{i-1}d_id_{i+1}d_{i+2} = 1001$.
   - Next we assume that $l \ge 2$, that is, $c_ic_{i+1}\cdots c_{i+l+1} = 10^l1$. If $c_{i-1} = 0$ or $i = 0$ then $d_id_{i+1}\cdots d_{i+l} = 10^{l-1}1$, and if $c_{i-1} = 1$ then $d_{i-1}d_i\cdots d_{i+l} = 10^l1$.

   Hence if $l = 1$ then $l' = 1$ or 2, and if $l \ge 2$ then $l' = l$ or $l-1$.

2. By the same way as 1 we can show it.

□

For only CA-$14_{\alpha-\beta}(m)$ the following lemma holds. The behaviors stated in the following lemma are caused by $f(111) = 0$. So in CA-$142_{\alpha-\beta}(m)$ it does not hold.

Lemma 5 *For CA-$14_{\alpha-\beta}(m)$ the following hold;*

1. *Let c be a configuration and $d = \delta(c)$. If $d_id_{i+1} = 11$ then $c_{i-1}c_ic_{i+1} = 001$.*
2. *Let c be a configuration and $e = \delta^2(c)$. If $e_ie_{i+1}e_{i+2}e_{i+3} = 1101$ then $c_{i-2}c_{i-1}c_ic_{i+1} = 1101$ where $2 \le i \le m-3$.*
3. *Let c be a configurations such that $c_1c_2\cdots c_{i+1} = 0^i1$ $(1 \le i \le m-1)$ and $e = \delta^k(c)$ $(i+t \ge 3)$. Then $e_1e_2\cdots e_{i+k+1}$ contains no subsequence 1101.*

Proof.

1. Let $d_id_{i+1} = 11$ where $1 \le i \le m-1$. Then $c_{i-1}c_i = 00$ from the local transitions rule 14. So we have $c_{i-1}c_ic_{i+1} = 001$ since $f(000) = 0$ and $f(001) = 1$.

2. Let $d = \delta(c)$. From the local transition rule 14 if $d_i = 1$ then $c_{i-1} = 0$. Now we assume that $e_i e_{i+1} e_{i+2} e_{i+3} = 1101$, then $d_{i-1} = d_i = d_{i+2} 0$. And since $f(000) = 0$ and $f(010) = 1$ we have $d_{i-1} d_i d_{i+1} d_{i+2} = 0010$. And from $d_{i+1} = 1$ we have $c_i = 0$, and since $f(000) \neq 1$ $c_{i+1} c_{i+2} = 11, 10$ or 01. But $c_{i+1} c_{i+2} = 01$ contradicts with $d_{i+2} = 0$. So $c_{i+1} c_{i+2} = 11$ or 10, which don't contradict with $d_{i+2} = 0$. Since $c_{i-1} c_i c_{i+1} \neq 000$ $c_{i-1} = 1$, and similarly $c_{iu-2} = 1$. Hence we have $c_{i-2} c_{i-1} c_i c_{i+1} = 1101$ if $e_i e_{i+1} e_{i+2} e_{i+3} = 1101$.

3. We prove it by induction on k. Let c be a configuration such that $c_1 c_2 \cdots c_{i+1} = 0^i 1$, $d = \delta(c)$, $b = \delta^2(c)$, $a = \delta^3(c)$, $e = \delta^k(c)$ and $g = \delta^{k+2}(c)$. First if $i \geq 2$ then $d_1 d_2 \cdots d_{i+2} = 0^{i-1} 110$, so $c_1 c_2 \cdots c_{i+1}$ and $d_1 d_2 \cdots d_{i+2}$ contain no 1101. And if $i = 1$ then $d_1 d_2 d_3 = 110$ and $b_1 b_2 b_3 = 100$. So we have $a_2 = 0$. Hence $b_! b_2 b_3 b_4$ and $a_1 a_2 \cdots a_5$ contain no 1101. Next we assume that $e_1 e_2 \cdots e_{i+k+1}$ contains no 1101. If $g_1 g_2 \cdots g_{i+k+3}$ contains any 1101 then by 2 $e_1 e_2 \cdots e_{i+k+1}$ contains some 1101. So $g_1 g_2 \cdots g_{i+k+3}$ does not contains any 1101.

$\square$

Lemma 6 *1. Let c be an arbitrary configuration and $e = \delta^k(c)$. Then for CA-$14_{0-1}(m)$ $e_{m-k} e_{m-k+1} \cdots e_m$ contains no subsequence 000 where $2 \leq k \leq m-1$.*

2. Let c be an arbitrary configuration and $d = \delta(c)$. Then for CA-$14_{0-0}(m)$

$$d_{m-1} d_m \neq 01.$$

3. Let c be a configuration and $d = \delta(c)$. Then for CA-$14_{1-\beta}(m)$

$$d_1 = 0.$$

Proof.

1. Let $d = \delta(c)$, $b = \delta^2(c)$, $e = \delta^k(c)$ and $g = \delta^{k+1}(c)$. First we assume that $b_{m-2} b_{m-1} b_m = 000$. Then since $d_{m-1} d_m 1 \neq 000$ $d_{m-1} = 1$. Similarly $d_{m-3} d_{m-2} = 11$. But by lemma 3 1 d contains no 111. So we have $b_{m-2} b_{m-1} b_m \neq 000$. Next we assume that $e_{m-k} e_{m-k+1} \cdots e_m$ contains no 000. By the assumption $g_{m-k} g_{m-k+1} g_m$ contains no 000. So we let $g_{m-k-1} g_{m-k} g_{m-k+1} = 000$. If $e_{m-k} e_{m-k+1} e_{m-k+2} \neq 000$ then $e_{m-k-2} e_{m-k-1} e_{m-k} = 111$. But by lemma 3 1 e contains no 111. So $e_{m-k} e_{m-k+1} e_{m-k+2} = 000$. This is contradiction. Hence $g_{m-k-1} g_{m-k} g_{m-k+1} \neq 000$ and $g_{m-k-1} g_{m-k} \cdots g_m$ contains no 000.

2. It is trivial.

3. It is trivial since $f(1**) = 0$.

□

By lemma 6 3 we can regard CA-$14_{1-\beta}(m)$ after one step transition as CA-$14_{0-\beta}(m-1)$.

Lemma 7 *Let c be a configuration and $d = \delta(c)$. Then the following hold;*

1. *For CA-$142_{0-0}(m)$ if $c_1c_2 = 01$ then $d_1 = 1$.*
2. *For CA-$142_{1-0}(m)$ if $c_1c_2 = 01$ then $d_1d_2 = 01$.*
3. *For CA-$142_{\alpha-0}(m)$ if $c_1c_2\cdots c_i = 0^i$ then $d_1d_2\cdots d_{i-1} = 0^{i-1}$ where $2 \leq i \leq m-1$.*

Proof. It is trivial. □

With respect to transient length the following lemma hold;

Lemma 8 1. *Let $C(m,n) = \{c \mid c$ is a non-initial configuration of CA-$14_{0-0}(m)$ and $0c$ contains n subsequences $01\}$ and $h'(m,n) = \max\{h(c) \mid c \in C(m,n)\}$. Then for CA-$14_{0-0}(m)$*

$$h'(m,n) \leq m-1.$$

2. *Let $C(m,n) = \{c \mid c$ is a configuration of CA-$142_{\alpha-0}(m)$ and $\alpha c0$ contains n subsequences $01\}$ and $h'(m,n) = \max\{h(c) \mid c \in C(m,n)\}$. Then for CA-$142_{\alpha-0}(m)$*

$$h'(m,n) \leq m.$$

3. *Let c be an arbitrary configuration and $e = \delta^{m+k}(c)$ $(k \geq 0)$. Then for CA-$14_{0-1}(m)$*

$$e_1e_2\cdots e_{t+1} = (10)^*_{k+1}.$$

That is, $e_1e_2\cdots e_{t+1}$ is stable.

Proof.

1. We prove it by induction on n. Let c be an arbitrary configuration in $C(m,n+1)$, and $c_{i-1}c_i$ be nth subsequence 01 of c. By lemma 6 2 if $c_{i+1} = 0$ then the range of i is from $2n-1$ to $m-3$, and if $c_{i+1} = 1$ then it is from $2n-1$ to $m-4$. First we assume that $c_{i+1} = 0$. Then we let $c_{i+j}c_{i+j+1}$ be $(n+1)$th 01 of c, that is, $c = c_1c_2\cdots c_{i-1}c_i0^j110^{m-i-j-2}$ or $c = c_1c_2\cdots c_{i-1}c_i0^j10^{m-i-j-1}$. By lemma 6 2 we have $1 \leq j \leq m-i-2$.
   - If $j = 1$ or 2 then there exists a non negative integer $k \leq h'(i+1,n)$ such that $\delta^k(c) = 1(01)^{n-1}00110^{m-2n-3}$ by lemma 2 1, corollary 3 and lemma 4 1. So $\delta^{k+2}(c) = 1(10)^n0^{m-2n-1}$, that is, $\delta^{k+2}(c)$ is a fixed point.
   - If $j \geq 3$ then there exists a non negative integer $k \leq h'(i+1,n)$ such that $\delta^k(c) = 1(01)^{n-1}0^l110^{m-2n-l-1}$ where $l \leq j$. So $\delta^{k+l}(c) = 1(01)^n0^{m-2n-1}$.

Next we assume that $c_{i+1} = 1$. Then we let $c_{i+j+1}c_{i+j+2}$ be $(n+1)$th 01 of c, that is, $c = c_1 \cdots c_{i-1}c_i 10^j 110^{m-i-j-3}$ or $c = c_1 \cdots c_{i-1}c_i 10^j 10^{m-i-j-2}$. By lemma 6 2 we have $1 \leq j \leq m-i-3$.

- If $j = 1$ or 2 then there exists a non negative integer $k \leq h'(i+1, n)$ such that $\delta^k(c) = 1(01)^{n-1}00110^{m-2n-3}$, and $\delta^{k+2}(c)$ is a fixed point.
- If $j \geq 3$ there exists a non negative integer $k \leq h'(i+1, n)$ such that $\delta^k = 1(01)^{n-1}0^l 110^{m-2n-l-1}$ where $l \leq j$. So $\delta^{k+l}(c)$ is a fixed point.

Hence from the above discussion we have

$$\begin{aligned} h'(m, n+1) \leq \quad \max\{ & \max_{2n-1 \leq i \leq m-3} \{h'(i+1, n) + 2\}, \\ & \max_{2n-1 \leq i \leq m-3} \{h'(i+1, n) + m - i - 2\}, \\ & \max_{2n-1 \leq i \leq m-4} \{h'(i+1, n) + 2\}, \\ & \max_{2n-1 \leq i \leq m-4} \{h'(i+1, n) + m - i - 3\} \ \}. \end{aligned}$$

So

$$h'(m, n+1) \leq \max_{2n-1 \leq i \leq m-3} \{h'(i+1, n) + 2, h'(i+1, n) + m - i - 2\}.$$

Now it is trivial that $h'(m, 1) = m-2 \leq m-1$ for any m. If $h'(m', n) \leq m'-1$ for any $m' \leq m$ then $h'(i+1, n)+m-i-2 \leq m-2 \leq m-1$ and $h'(i+1, n)+2 \leq i+2$. That is, if $h'(m', n) \leq i-1$ for any $m' \leq m$ then we have $h'(m, n+1) \leq m-1$.

2. we prove it by induction on n. Let c be an arbitrary configuration in $C(m, n+1)$, and $c_{i-1}c_i$ be nth subsequence 01 of $0c0$. If $c_{i+1} = 0$ then the range of i is from $2n-1$ to $m-2$, and if $c_{i+1} = 1$ then the range of i is from $2n-1$ to $m-3$. First we assume that $c_{i+1} = 0$, and we let $c_{i+j}c_{i+j+1}$ be $(n+1)$th 01 of $0c0$. That is, $c = c_1 c_2 \cdots c_i 0^j 1^l 0^{m-i-j-l}$ where $1 \leq j \leq m-i-1$ and $1 \leq l \leq m-i-j$.

   - If $j = 1$ and $l = 1$, then there exists a non negative integer $k \leq h'(i, n)$ such that $\delta^k(c) = (10)^n 0^{j'} 1^{l'} 0^{m-2n-j'-l'}$ where $0 \leq j' \leq 1$ and $1 \leq l' \leq 2$. So $\delta^{k+2}(c) = (10)^{n+1} 0^{m-2n-2}$, that is, $\delta^{k+2}(c)$ is a fixed point.
   - If $j = 1$ and $l \geq 2$, then there exists a non negative integer $k \leq h'(i, n)$ such that $\delta^k(c) = (10)^n 0^{j'} 1^{l'} 0^{m-2n-j'-l'}$ where $0 \leq j' \leq 1$ and $1 \leq l' \leq l$. So $\delta^{k+j'+l'}(c) = (10)^{n+1} 0^{m-2n-2}$, that is, $\delta^{k+j'+l'}(c)$ is a fixed point.
   - If $j \geq 2$ and $l = 1$, then there exists a non negative integer $k \leq h'(i, n)$ such that $\delta^k(c) = (10)^n 0^{j'} 1^{l'} 0^{m-2n-j'-l'}$ where $j' \leq j-1$ and $1 \leq l' \leq 2$. So $\delta^{k+j'+1}(c) = (10)^{n+1} 0^{m-2n-2}$, that is, $\delta^{k+j'+1}(c)$ is a fixed point.
   - If $j \geq 2$ and $l \geq 2$, then there exists a non negative integer $k \leq h'(i, n)$ such that $\delta^k(c) = (10)^n 0^{j'} 1^{l'} 0^{m-2n-j'-l'}$ where $0 \leq j' \leq j-1$ and $1 \leq l' \leq l$. So $\delta^{k+j'+l'}(c) = (10)^{n+1} 0^{m-2n-2}$, that is, $\delta^{k+j'+l'}(c)$ is a fixed point.

Next we assume that $c_{i+1} = 1$, and let $c_{i+s+t}c_{i+s+t+1}$ be $(n+1)$th 01 of $0c0$. That is, $c = c_1c_2\cdots c_i1^s0^t1^l0^{m-i-s-t-l}$ where $1 \le s \le m-i-2$, $1 \le t \le m-i-s-1$ and $1 \le l \le m-i-s-t$.

- If $t = 1$ and $l = 1$ then there exists a non negative integer $k \le h'(i+s, n)$ such that $\delta^k(c) = (10)^n0^{t'}1^{l'}0^{m-2n-t'-l'}$ where $t' \le 1$ and $l' \le 2$. So $\delta^{k+2}(c) = (10)^{n+1}0^{m-2n-2}$.
- If $t = 1$ and $l \ge 2$ then there exists a non negative integer $k \le h'(i+s, n)$ such that $\delta^k(c) = (10)^n0^{t'}1^{l'}0^{m-2n-t'-l'}$ where $t' \le 1$ and $l' \le l$. So $\delta^{k+t'+l'}(c) = (10)^{n+1}0^{m-2n-2}$.
- If $t \ge 2$ and $l = 1$ then there exists a non negative integer $k \le h'(i+s, n)$ such that $\delta^k(c) = (10)^n0^{t'}1^{l'}0^{m-2n-t'-l'}$ where $t' \le t-1$ and $l' \le 2$. So $\delta^{k+t'+1}(c) = (10)^{n+1}0^{m-2n-2}$.
- If $t \ge 2$ and $l \ge 2$ then there exists a non negative integer $k \le h'(i+s, n)$ such that $\delta^k(c) = (10)^n0^{t'}1^{l'}0^{m-2n-t'-l'}$ where $t' \le t-1$ and $l' \le l$. So $\delta^{k+t'+l'}(c) = (10)^{n+1}0^{m-2n-2}$.

So by the above discussion we have

$$\begin{aligned} h'(m, n+1) \le \quad \max\{ & \max_{2n-1\le i\le m-2}\{h'(i,n)+2\}, \\ & \max_{2n-1\le i\le m-2}\{h'(i,n)+m-i\}, \\ & \max_{2n-1\le i\le m-2}\max_{1\le s\le m-i-2}\{h'(i+s,n)+2\}, \\ & \max_{2n-1\le i\le m-2}\max_{1\le s\le m-i-2}\{h'(i+s,n)+m-i-s\}\} \end{aligned}$$

Trivially $h'(m, 1) = m$, and if $h'(m', n) \le m'$ for $m' \le m$ then $h'(m, n+1) \le m$.

3. We prove it by induction on k. Let $d = \delta(c)$, $b = \delta^m(c)$, $e = \delta^{m+k}(c)$ and $g = \delta^{m+k+1}(c)$. First we assume that $c_1 = 1$ then $d_1 = 1$ by $f(01*) = 1$ and we have $b_1 = 1 = (10)^*_1$, and we assume that $c_1c_2\cdots c_{i+1} = 0^i1$ $(1 \le i \le m)$ then $d_1d_2\cdots d_i = 0^{i-1}1$ and $a_1 = 1$ where $a = \delta^i(c)$, so we have $b_1 = 1 = (10)^*_1$. Next we assume that $e_1e_2\cdots e_{k+1} = (10)^*_{k+1}$. If k is even then trivially $g_1g_2\cdots g_{k+2} = (10)^*_{k+2}$ since $f(1**) = 0$. Otherwise since $e_{k+1}e_{k+2}e_{k+3} \ne 000$ by lemma 6 1 we have $g_{k+2} = 1$. That is, $g_1g_2\cdots g_{k+2} = (10)^*_{k+2}$.

□

From the above discussion we have the following theorems.

Theorem 1 *1. The cellular automaton CA-*$14_{0-0}(m)$ *has only fixed points and for the number of fixed points the following formula holds;*

$$\gamma_1(m) = \left[\frac{m+3}{2}\right].$$

And its transient length is m.

2. *The cellular automaton CA-*$14_{0-1}(m)$ *has an unique fixed point. And its transient length is* $2m-1$.

3. *The cellular automaton CA-*$14_{1-0}(m)$ *has only fixed points and for the number of fixed points the following formula holds;*

$$\gamma_1(m) = \left[\frac{m+2}{2}\right].$$

   And its transient length is m.

4. *The cellular automaton CA-*$14_{1-1}(m)$ *has an unique fixed point. And its transient length is* $2m-2$.

Theorem 2 1. *The cellular automata CA-*$142_{0-0}(m)$ *and CA-*$142_{1-1}(m)$ *have only fixed points and for the number of fixed point the following holds;*

$$\gamma_1(m) = \left[\frac{m+3}{2}\right]$$

   And their transient length is m.

2. *The cellular automata CA-*$142_{1-0}(m)$ *and CA-*$142_{0-1}(m)$ *have only fixed points and for the number of fixed point the following holds;*

$$\gamma_1(m) = \left[\frac{m+2}{2}\right]$$

   And their transient length is m.

4 Conclusion

In this paper we analysed behaviors of CA-$14_{\alpha-\beta}(m)$ and CA-$142_{\alpha-\beta}(m)$. The difference between rule 14 and rule 142 is the image of the subsequence 111. For the boundary condition $\alpha - 0$ the behaviors after m step transition are not influenced by the difference. But for the boundary condition $\alpha - 1$ since the subsequence 111 does not disappear in CA-$142_{\alpha-1}(m)$ the differences between behaviors of CA-$14_{\alpha-1}(m)$ and CA-$142_{\alpha-1}(m)$ don't disappear.

So at boundary condition $\alpha - 0$ it is not important whether $f(111) = 0$ or $f(111) = 1$ when $f(110) = 0$, $f(101) = 0$, $f(100) = 0$, $f(011) = 1$, $f(010) = 1$, $f(001) = 1$ and $f(000) = 0$. But cellular automata with 1-bit different rule from rule 14 don't behave always like CA-$14_{0-0}(m)$ as table 3

It is guessed that the transition rule of CA-$14_{\alpha-0}(m)$ except for 111 characterize its behaviors. In the future we will analyse the behaviors of other cellular automata in the same point of view and make an universal theory of cellular automata.

	m	1	2	3	4	5	6	7	8	9	10	11	12	13
CA-$78_{0-0}(m)$	$1-cycle$	2	3	4	6	8	11	15	20	27	36	48	64	85
	$tran.len.$	0	1	3	3	5	5	7	7	9	9	11	11	13
CA-$46_{0-0}(m)$	$1-cycle$	2	2	2	2	2	2	2	2	2	2	2	2	2
	$tran.len.$	0	2	3	4	5	6	7	8	9	10	11	12	13
CA-$30_{0-0}(m)$	$1-cycle$	2	1	2	1	2	1	2	1	2	1	2	1	2
	$2-cycle$	0	1	0	1	0	1	0	1	0	1	0	1	0
	$tran.len.$	0	1	4	8	14	15	17	24	28	32	34	39	44
CA-$6_{0-0}(m)$	$1-cycle$	2	2	3	3	4	4	5	5	6	6	7	7	8
	$tran.len.$	0	2	2	6	6	10	10	14	14	18	18	22	22
CA-$10_{0-0}(m)$	$1-cycle$	1	1	1	1	1	1	1	1	1	1	1	1	1
	$tran.len.$	1	2	3	4	5	6	7	8	9	10	11	12	13
CA-$12_{0-0}(m)$	$1-cycle$	2	3	5	8	13	21	34	55	89	144	233	377	610
	$tran.len.$	0	1	1	1	1	1	1	1	1	1	1	1	1
CA-$15_{0-0}(m)$	$1-cycle$	1	1	1	1	1	1	1	1	1	1	1	1	1
	$tran.len.$	1	2	3	4	5	6	7	8	9	10	11	12	13

Table 3: Other cellular automata with 1-bit different rule from rule 14

References

[1] S. Wolfram: *Theory and Applications of Cellular Automata*, World Scientific, Singapore, 1986

[2] Chris G. Langton: *Computation at Edge of Chaos: Phase Transitions and Emergent Computation*, Phisca D, Vol. 42, 1990, pp.12-37

[3] T. Tokihiko, D. Takahashi, J. Matsukidaira and J. Satsuma: *From Soliton Equations to Integrable Cellular Automata through a Limiting Procedure*, Phys. Rev. Let., Vol. 76, No. 18, 1996, pp.3247-3250.

[4] D. Takahashi and J. Satsuma: *A Soliton Cellular Automaton*, J. Phys. Soc. Japan, Vol. 59, No. 10, 1990, pp. 3514-3519.

[5] M. Harao and S. Noguchi: *On some dynamical properties of finite cellular automaton*, IEEE Trans. Comput.C-27(1978)42.

[6] T. Sato: *On Behaviors of Cellular Automata with Rule 27*, Kyushu Journal of Mathematics, Vol. 50, 1996, pp.133-152

[7] S. Inokuchi: *On Behaviors of Cellular Automata with Rule 156*, Bulletin of Informatics and Cybernetics, Vol. 30, No.1, 1998, pp.121-131

[8] Y. Kawahara: *Existence of the characteristic numbers associated with cellular automata with local transition rule 90*, Bulletin of Informatics and Cybernetics, Vol. 24, 1991, pp.121-136

[9] Y. Kawahara, S. Kumamoto, Y. Mizoguchi, M. Nohmi, H. Ohtuka and T. Shoudai: *Period lengths of cellular automata on square lattices with rule 90*, J.Math.Phys.36(3),March 1995

[10] H. Y. Lee and Y. Kawahara: *On dynamical behaviors of cellular automata CA-60*, Bulletin of Informatics and Cybernetics, Vol. 25, 1992, pp.22-27