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Inokuchi, Shuichi
Department of Informatics Kyushu University

KAWAHARA, Yasuo
Department of Informatics Kyushu University

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Shuichi INOKUCHI and Yasuo KAWAHARA

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Department of Informatics
Kyushu University
Fukuoka 812-81, Japan

Email: inokuchi@i.kyushu-u.ac.jp Phone: +81-92-642-2697

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Shuichi INOKUCHI and Yasuo KAWAHARA
Department of Informatics, Kyushu University
Fukuoka 812-8581, Japan
e-mail:{inokuchi,kawahara}@i.kyushu-u.ac.jp

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Abstract

In this paper we propose tree expressions which express transition diagrams of finite dynamical systems by algebraic formula. And we consider the cartesian product of dynamical systems and get the product formula of tree expressions. By the product formula we easily get the tree expression of the product of dynamical systems represented by tree expressions.
Keywords : Tree Expression, Product Formula, Dynamical System

1 Introduction

In nature there exist many natural dynamical systems and many researchers investigate them. Many artificial dynamical systems are devised by engineers and scientists. They investigate dynamical systems and applied them to many fields. Generally dynamical systems are represented by a pair (X, f) of an non empty set X and a transition function f on X. It is difficult to understand behaviors of a dynamical system (X, f). So in order to understand behaviors of dynamical systems more easily we describe their transition diagrams which are graphs showing their transition. If the number of X is small then we can easily describe the transition diagram. But if the number of X is large then we cannot describe it.

So we consider special dynamical systems such that they can be separated or are composed by several dynamical systems. If a dynamical system can be separated, then analysing small separated dynamical systems is aclue to understand global behaviors. Two dynamical systems have simple behavior but often their combination behave complexly. It goes without saying that a combination of complex dynamical systems have more complex behaviors. So we investigate a product of dynamical systems.

There are several studies on a cartesian product of dynamical systems. Kumamoto and Nohmi analysed behaviors of integral affine dynamical systems[1]. In [1] they studied the cartesian product of systems, and they got that the transition diagrams of integral affine dynamical systems are the cartesian product of a system whose nodes are all on a limit cycle and a system having a fixed point. And Lee and Kawahara presented tree expressions and their product formula in 1996[2]. In [2] they investigated behaviors of cellular automata and got the result that transition diagram of some cellular automata can be represented by the cartesian product of transition diagrams of cellular automata with smaller cells. But their tree expressions can represent only dynamical systems with fixed a point. And their product formula is limited on dynamical systems such that they have a fixed point and their height is less than 2. So their product formula can't apply to dyncamical systems having a limit cycle of period lenght p and more than 3 height.

In this paper we introduce new 'tree expressions' which represent a transition diagram of dynamical systems in algebraic formulae. New tree expression can represent transition diagrams of all finite dynamical sysytems. And we investigate the cartesian product of finite dynamical systems, and present their product formula. The product formula presened in this paper can be applied to all finite dynamical systems.

2 Finite Dynamical Systems

In this section we define a finite (dynamical) system and notations.

A (dymanical) system (X, f) is a pair of a nonempty set X and a transition function $f : X \to X$. A system (X, f) is finite if X is a finite set. A dynamorphism $\varphi : (X, f) \to (Y, g)$ from a system (X, f) into another system (Y, g) is a function $\varphi : X \to Y$ such that $g\varphi = \varphi f$. A dynamorphism $\varphi : (X, f) \to (Y, g)$ is called an isomorphism of system if there exists an inverse dynamorphim $\psi : (Y, g) \to (X, f)$ such that $\varphi\psi = id_Y$ and $\psi\varphi = id_X$ where id_X and id_Y are identity function on X and Y, respectively. It is trivial that isomorphic systems have isomorphic transition diagrams.

Definition 2.1 *Let (X, f) be a system. A subsystem S of (X, f) is a subset of X such that $f(x) \in S$ for every $x \in S$. (That is, the restriction $f|S$ of f on S defines a function $S \to S$.)*

Definition 2.2 *A system (X, f) is connected if for every $x, y \in X$ there exist non-negative integers m and n such that $f^m(x) = f^n(y)$.*

We first show that every system can be decomposed into a disjoint union of connected ones.

Proposition 2.3 *Every system (X, f) is a disjoint union (coproduct) of connected subsystems. That is, $X = X_1 + \cdots + X_k$, where X_i is a connected subsystem for $i = 1, \cdots, k$.*

Proof. Let (X, f) be a system. A connection relation $\sim$ on (X, f) is a relation on X such that $x \sim y$ for $x, y \in X$ if and only if $f^m(x) = f^n(y)$ for some nonnegative integers m and n. It is clear that $\sim$ is an equivalence relation on X. Assume that the set $\{X_\lambda\}$ of all equivalence classes by $\sim$ is given. Then $f(x) \in X_\lambda$ for all $x \in X_\lambda$ and each subsystem $(X_\lambda, f|X_\lambda)$ is connected, where $f|X_\lambda : X_\lambda \to X_\lambda$ is a restriction of $f : X \to X$ onto X_λ. Therefore (X, f) has been decomposed into connected subsystems. □

Let (X, f) be a system. For $n \geq 0$ n-th images $f^n(X)$ of f is a subset of X such that $x \in f^n(X)$ if and only if there exists $z \in X$ with $x = f^n(z)$. Note that $f^n(X) \neq \emptyset$ for each $n \geq 0$ since $X \neq \emptyset$, and the following descending chain condition holds:

$$X = f^0(X) \supseteq f(X) \supseteq \cdots \supseteq f^n(X) \supseteq f^{n+1}(X) \supseteq \cdots.$$

Define $f^\infty(X) = \cap_{n \geq 0} f^n(X)$.

Lemma 2.4 *If (X, f) is a finite system, then*

1. $\exists\, n > 0 : f^\infty(X) = f^n(X)$,
2. $f^\infty(X) \neq \emptyset$,
3. $\forall\, x \in f^\infty(X)\ \exists\, n > 0 : x = f^n(x)$,
4. $\forall\, x \in f^\infty(X)\ \exists!\, y \in f^\infty(X) : x = f(y)$,
5. $x \in f^\infty(X) \iff f^{-1}(x) \cap f^\infty(X) \neq \emptyset$.

Proof.

1. Because of the finiteness of X the descending chain
$$X = f^0(X) \supseteq f(X) \supseteq f^2(X) \supseteq \cdots \supseteq f^n(X) \supseteq f^{n+1}(X) \supseteq \cdots$$
should be finite, that is, $f^n(X) = f^{n+1}(X) = f^{n+2}(X) = \cdots$ and $f^\infty(X) = f^n(X)$ for some $n \geq 0$. (Neotherian Property)

2. It is a direct result of (1).

3. Let $x \in f^\infty(X)$. By means of (1) we can inductively define a sequence $\{x_0, x_1, \cdots, x_n, \cdots\}$ in $f^\infty(X)$ such that $x_0 = x$ and $f(x_{i+1}) = x_i$ for all $i \geq 0$. First note that $f^i(x_i) = x$ for all $i \geq 0$. As X is a finite set, there exist integers i and j with $x_i = x_j$ and $0 \leq i < j$. Hence $x = f^j(x_j) = f^{j-i} f^i(x_i) = f^{j-i}(x)$.

4. Assume that $f^\infty(X) = f^n(X)$. If $x \in f^\infty(X)$, then $x \in f^{n+1}(X)$ and there exists $z \in X$ such that $x = f^{n+1}(z)$. Hence $f^n(z) \in f^{-1}(x) \cap f^\infty(X)$. Conversely, if $z \in f^{-1}(x) \cap f^\infty(X)$, then $x = f(z)$ and $z \in f^n(X)$, which implies $x \in f^{n+1}(X) = f^\infty(X)$.

5. Let $x \in f^\infty(X)$. Then (1) asserts $x = f(y)$ for some $y \in f^\infty(X)$. Next assume that $x = f(y) = f(z)$ for $y, z \in f^\infty(X)$. By (3) we have $y = f^m(y)$ and $z = f^n(z)$ for some positive integers m and n ($m \leq n$). We can choice the least positive integer n such that $z = f^n(z)$. If $m < n$, then $z = f^n(z) = f^n(y) = f^{n-m}f^m(y) = f^{n-m}f(y) = f^{n-m}(z)$, which contradicts to the minimality of n. Hence $m = n$ and $y = f^m(y) = f^n(z) = z$.

□

Corollary 2.5 *Let (X, f) be a finite connected system. If $x \in f^\infty(X)$ and $m = \min\{k > 0; x = f^k(x)\}$, then $f^\infty(X) = \{x, f(x), \cdots, f^{m-1}(x)\}$ and $|f^\infty(X)| = m$.*

Proof. It is clear that $\{x, f(x), \cdots, f^{m-1}(x)\} \subseteq f^\infty(X)$. So in order to prove $f^\infty(X) = \{x, f(x), \cdots, f^{m-1}(x)\}$ we show that

$$f^\infty(X) \subset \{x, f(x), \cdots, f^{m-1}(x)\}.$$

Assume that $y \in f^\infty(X)$. By lemma 2.4 (3) $y = f^k(y)$ for some positive integer k and the connectedness of (X, f) claims $f^p(x) = f^q(y)$ for some nonnegative integers p and q. Take a positive integer r with $kr \geq q$. Then

$$y = f^{kr}(y) = f^{kr-q}f^q(y) = f^{kr-q}f^p(x),$$

which indicates $y \in \{x, f(x), \cdots, f^{m-1}(x)\}$. Next we prove $|f^\infty(X)| = m$. Assume that $f^i(x) = f^j(x)$ for integers i, j ($0 \leq i < j \leq m-1$). Then $f^{j-i}(x) = f^{j-i+m}(x) = f^{m-i}f^j(x) = f^{m-i}f^i(x) = f^m(x) = x$, which contradicts to the minimality of m. Hence $|f^\infty(X)| = m$. □

Let (X, f) be a connected finite system, $x \in f^\infty(X)$ and $|f^\infty(X)| = m$. Then $< x, f(x), \cdots, f^{m-1}(x) >$ is called a limit cycle of (X, f). In particular if $m = 1$, then $< x >$ is called a fixed point.

Definition 2.6 *Let (X, f) be a finite system. For an element $x \in X$ the height of x from limit elements $f^\infty(X)$ and the root below x are defined as follows:*

$$h_X(x) = \min\{k \geq 0; f^k(x) \in f^\infty(X)\}$$

and

$$r_X(x) = f^{h_X(x)}(x).$$

And the height $H(X, f)$ of (X, f) is defined by

$$H(X, f) = \min\{n \geq 0; f^n(X) = f^\infty(X)\}.$$

Proposition 2.7 *1. The following three conditions are equivarent;*

- $x \in f^\infty(X)$,

- $h_X(x) = 0$,
- $r_X(x) = x$.

2. If $h_X(x) = 0$, then $h_X(f(x)) = 0$.

3. If $h_X(x) > 0$, then $h_X(f(x)) = h_X(x) - 1$.

Proof.

1. ($x \in f^\infty(X) \Longrightarrow h_X(x) = 0$) Assume that $x \in f^\infty(X)$, then
$$0 \in \{k \geq 0; f^k(x) \in f^\infty(X)\}.$$
So we have
$$h_X(x) = min\{k \geq 0; f^k(x) \in f^\infty(X)\} = 0$$
($h_X(x) = 0 \Longrightarrow r_X(x) = x$) It is trivial.
($r_X(x) = x \Longrightarrow x \in f^\infty(X)$) By the definition of $r_X(x)$ and $h_X(x)$ we have $x = r_X(x) = f^{h_X(x)}(x) \in f^\infty(X)$.

2. If $h_X(x) = 0$, that is, $x \in f^\infty(X)$, then $f(x) \in f^\infty(X)$. This indicate $h_X(f(x)) = 0$.

3. Assume that $h_X(x) > 0$. Then
$$\begin{aligned} h_X(f(x)) &= \min\{k \geq 0; f^k(f(x)) \in f^\infty(X)\} \\ &= \min\{k \geq 0; f^{k+1}(x) \in f^\infty(X)\} \end{aligned}$$
Since $h_X(x) = \min\{k \geq 0; f^k(x) \in f^\infty(X)\} > 0$,
$$\begin{aligned} & \min\{k \geq 0; f^k(x) \in f^\infty(X)\} - 1 \\ = & \min\{k - 1 \geq 0; f^k(x) \in f^\infty(X)\} \\ = & \min\{h \geq 0; f^{h+1}(x) \in f^\infty(X)\} \end{aligned}$$
So we have $h_X(f(x)) = h_X(x) - 1$.

□

Definition 2.8 *Let (X, f) be a system. Define f_*^{-1} by*
$$f_*^{-1}(x) = f^{-1}(x) - f^\infty(X).$$
And for every $x \in X$ define
$$X(x) = f_*^{-1}(x) + \sum_{u \in f_*^{-1}(x)} X(u)$$
and
$$X_*(x) = \{x\} + X(x)$$

Lemma 2.9 *Let (X, f) be a system and i a positive integer. Then the following hold;*

1. *If $x \notin f^\infty(X)$, then $f_*^{-1}(x) = f^{-1}(x)$.*
2. *$x \notin X(x)$ for every $x \in X$,*
3. *If $x \in X(y)$, then $X(x) \subset X(y)$,*
4. *If $y \notin f^\infty(X)$ and $y = f^i(x)$, then $x \in X(y)$.*
5. *If $x \in X(y)$, then $X(f(x)) \subset X(y)$.*

Proof.

1. We assume that $x \notin f^\infty(X)$. Then $f^{-1}(x) \cap f^\infty(X)$. So we have
$$f_*^{-1}(x) = f^{-1}(x) - f^\infty(X) = f^{-1}(x).$$
2. It is trivial.
3. Assume that $u \in X(x)$. If $u \in f_*^{-1}(x)$ it is clear that $X(u) \subset X(x)$. Otherwise there exist $y \in X(x)$ such that $f(u) = y$. Then $X(u) \subset X(y)$. To repeat several times derive $X(u) \subset X(x)$.
4. Assume that $y \notin f^\infty(X)$ and $f^i(x) = y$ for some positive integrer i. Then if $i = 1$, then $x \in f_*^{-1}(y) \subset X(y)$. If $i \geq 2$, then we have the chain $\{x, f(x), \cdots, f^i(x) = y\}$. For any integer k $(1 \leq k \leq i)$ we have $f^{k-1}(x) \in X(f^k(x))$ and by (3) $X(f^{k-1}(x)) \subset X(f^k(x))$. So $x \in X(f(x)) \subset X(f^2(x)) \subset \cdots \subset X(f^i(x)) = X(y)$.
5. Assume that $x \in X(y)$ and $x \notin f^\infty(X)$. Then there exists a positive integer i such that $f^i(x) = y$. If $i = 1$, then $f(x) = y$. So we have $X(f(x)) \subset X(y)$. If $i \geq 2$, then $f^{i-1}(f(x)) = y$. By (4) $f(x) \in X(y)$. So we have $X(f(x)) \subset X(y)$ by (3).

□

Proposition 2.10 *Let (X, f) be a connected system with a limit cycle $< x_0, \cdots, x_{p-1} >$ of period length p. Then (X, f) is the disjoint union of $X_*(x_i)$ for $0 \leq i \leq p-1$. That is,*
$$X = \sum_{i=0}^{p-1} X_*(x_i) \quad (disjointunion).$$
In particular, $X = X_*(x_0)$ for any tree X with a limit cycle (or fixed point) $< x_0 >$ of period length 1.

Proof. It is trivial that $X \supset \Sigma_{i=0}^{p-1} X_*(x_i)$. Now we show $X \subset \Sigma_{i=0}^{p-1} X_*(x_i)$. Let x be an arbitraly element in X. If $x \in f^\infty(X)$, then $x \in X_*(x_i)$ where $x = x_i$. So we let x be an element in $X - f^\infty(X)$.Then by connectedness of (X, f) there exist non negative integers k and h such that $f^k(x_i) = f^h(x)$ for any integer i. And since $f^k(x_i) \in f^\infty(X)$ There exists non negative integer j such that $x_j = f^h(x)$. So there exists non negative integer h such that $x_j = f^h(x)$ for any j. We let $s = min\{h \geq 0; x_j = f^h(x)\ for\ any\ j\}$. Then since $f^{s-1}(x) \notin f^\infty(X)$ we have $x \in X(f^{s-1}(x))$ by lemma 2.9 4. Let $x_t = f^s(x)$, then $f^{s-1}(x) \in f_*^{-1}(x_t)$. So

$$x \in X(f^{s-1}(x)) \subset f_*^{-1}(x_t) + \sum_{u \in f_*^{-1}(x_t)} X(u) \subset X(x_t) \subset X_*(x_t).$$

Hence for any element x there exists i such that $x \in X_*(x_i)$. That is,

$$X \subset \sum_{i=0}^{p-1} X_*(x_i).$$

□

3 Tree Expressions

In this section we define tree expression which is a special dynamical system. And we present a transformation function γ which transform a dynamical system into tree expression.

Let N be the set of all positive integers, N^* the set of all finite strings of positive integers including the empty string ε, and N^+ the set of all nonempty strings in N^*. As in formal language theory we use two operations $+$ (set union) and $\cdot$ (concatenation) on subsets of N^*, that is, $S_1 + S_2 = \{w \in N^*; w \in S_1 \text{ or } w \in S_2\}$ and $S_1 \cdot S_2 = \{w_1 \cdot w_2 \in N^*; w_1 \in S_1 \text{ and } w_2 \in S_2\}$, where S_1 and S_2 are subsets of N^*. Define a function $\mu : N^* \to N^*$ by $\mu(\varepsilon) = \varepsilon$ and $\mu(wk) = w$ for all $w \in N^*$ and $k \in N$. Then (N^*, μ) is a infinite system. We often write n for a singleton set $\{n\}$.

Definition 3.1 *A p-set (pretree set) is a finite subset of N^+ defined as follows:*

1. $[0] = \emptyset$ *is a p-set,*
2. *If m is a positive integer and $E_1, E_2, \cdots, E_m$ are p-sets, then*

$$[m + \sum_{i=1}^{m} E_i] = \{1, 2, \cdots, m\} + \sum_{i=1}^{m} iE_i \ (disjoint\ union)$$

   is a p-set.

Usually a p-set $[m + m[0]]$ is denoted by $[m]$ for short.

Definition 3.2 *An ordering relation $\prec$ on the set of all p-sets is defined as follows:*

1. $[0] \prec E$ *for each p-set* E,
2. $[m + \sum_{i=1}^{m} E_i] \prec [n + \sum_{j=1}^{n} F_j]$ *if* $m < n$, *or* $m = n$ *and* $(E_1, E_2, \cdots, E_m) \prec_{\mathrm{lex}} (F_1, F_2, \cdots, F_n)$, *where* $\prec_{\mathrm{lex}}$ *is the lexicographical ordering defined by a fragment of the ordering* $\prec$ *already defined.*

The following illustrates the basic feature of the ordering defined above.

$$[0] \prec [1 + [0]] \prec [2 + [0] + [0]] \prec [2 + [0] + [1 + [0]]] \prec [2 + [1 + [0]] + [0]]$$

$$\prec [2 + [1 + [0]] + [1 + [0]]] \prec [3 + [0] + [0] + [0]] \prec \cdots$$

Also note that $E_0 \prec E_1 \prec \cdots \prec E_n \prec \cdots \prec [2 + [0] + [0]]$ for $E_0 = [1 + [0]]$ and $E_{n+1} = [1 + E_n]$.

Definition 3.3 *A normal p-set is a p-set recursively defined as follows:*

1. $[0]$ *is a normal p-set.*
2. *If* m *is a positive integer and* $E_1, E_2, \cdots, E_m$ *are normal p-sets such that* $E_1 \prec E_2 \prec \cdots \prec E_m$, *then* $[m + \sum_{i=1}^{m} E_i]$ *is a normal p-set.*

For example, $[0]$, $[1+[0]]$, $[2+[0]+[0]]$, $[2+[0]+[1+[0]]]$, $[2+[1+[0]]+[1+[0]]]$ and $[3+[0]+[0]+[0]]$ are normal, but $[2+[1+[0]]+[0]]$ and $[3+[2+[0]+[0]]+[1+[0]]+[0]]$ are not.

If $E_1 = E_2 = \cdots E_n = [0]$, then a p-set $[m+\sum_{i=1}^{m} E_i]$ is denoted by $[m+\sum_{i=n+1}^{m} E_i]$ for short.

Definition 3.4 *An equivalence relation $\sim$ on the set of all p-sets is defined as follows:*

1. $[0] \sim [0]$,
2. $[m + \sum_{i=1}^{m} E_i] \sim [n + \sum_{j=1}^{n} F_j]$ *if* $m = n$ *and there is a permutation* τ *on the set* $\{1, 2, \cdots, m\}$ *such that* $E_i \sim F_{\tau(i)}$ *for each* $i = 1, 2, \cdots, m$.

Proposition 3.5 *The following hold;*

1. *Two equivalent normal p-sets are identical.*
2. *Every p-set is equivalent to a unique normal p-set.*

Proof.

1. Assume that $[m + \sum_{i=1}^{m} E_i]$ and $[n + \sum_{j=1}^{n} F_j]$ are two equivalent p-sets. Then by the definition of equivalence we have $m = n$ and a permutation τ on the set $\{1, 2, \cdots, m\}$ such that $E_i \sim F_{\tau(i)}$ for each $i = 1, 2, \cdots, m$. But from the induction hypothesis $E_i = F_{\tau(i)}$ for each $i = 1, 2, \cdots, m$. Noticing that $1 \leq \tau(1)$ and $1 \leq \tau^{-1}(1)$ it follows that $F_1 \prec F_{\tau(1)} = E_1$ and $E_1 \prec E_{\tau^{-1}(1)} = F_1$. Hence $E_1 = F_1$. Next assume that $E_i = F_i$ for $i = 1, \cdots, k$ $(1 \leq k \leq m - 1)$. If $\tau(k+1) \geq k+1$ and $\tau^{-1}(k+1) \geq k+1$, then $F_{k+1} \prec F_{\tau(k+1)} = E_{k+1} \prec E_{\tau^{-1}(k+1)} = F_{k+1}$. If $\tau(k + 1) \leq k$, then there is an integer i such that $1 \leq i \leq k$ and $\tau(i) \geq k + 1$, which shows that $E_{k+1} = F_{\tau(k+1)} \prec F_{k+1} \prec F_{\tau(i)} = E_i \prec E_{k+1}$. If $\tau^{-1}(k+1) \leq k$, then there is an integer j such that $1 \leq j \leq k$ and $\tau^{-1}(j) \geq k+1$, which shows that $F_{k+1} = E_{\tau^{-1}(k+1)} \prec E_{k+1} \prec E_{\tau^{-1}(j)} = F_j \prec F_{k+1}$.

2. Trivially $[0]$ is itself normal. Let $E = [m + \sum_{i=1}^{m} E_i]$ be a p-set. Assume that each p-set E_i is equivalent to a normal p-set E'_i for $i = 1, \cdots, m$. We can sort $E'_1, \cdots, E'_m$ so that $E'_{s_1} \prec \cdots \prec E'_{s_m}$. Then a p-set $E' = [m + \sum_{i=1}^{m} E'_{s_i}]$ is normal one equivalent to E. The uniqueness has been shown in 1.

□

Definition 3.6 *A t-set (tree set) is a subset of N^* recursively defined as follows:*

1. $[1]_* = \{\varepsilon\}$ *is a t-set,*
2. *If m is a positive integer ≥ 2 and $E_1, E_2, \cdots, E_{m-1}$ are p-sets, then*

$$[m + \sum_{i=1}^{m-1} E_i]_* = \{\varepsilon\} + [(m - 1) + \sum_{i=1}^{m-1} E_i]$$

is a t-set.

Definition 3.7 *Define the following notation for short;*

- $[b_1 : b_2]^0 = [0]$
- $[b_1 : b_2]^k = [b_1 + b_2[b_1 : b_2]^{k-1}]$
- $[b_1 : b_2]^k_* = [b_1 + (b_2 - 1)[b_1 : b_2]^{k-1}]_*$

We call a p-set P **uniform** if $P = [b_1 : b_2]^k$. And if $T = [b_1 : b_2]^k_*$ then we call the t-set T **uniform**.

Definition 3.8 *Define $T^+ = T \cap N^+$ for a t-set T. Note that $T^+ = [(m-1) + \sum_{i=1}^{m-1} E_i]$ (a p-set) when $T = [m + \sum_{i=1}^{m-1} E_i]_*$. A t-set T is* **normal** *if a p-set T^+ is normal. Two t-sets T and T' are* **equivalent** *if two p-sets T^+ and T'^+ are equivalent.*

Proposition 3.9 *Let E be a p-set and T a t-set. Then the following hold;*

1. *If $w \in E$ and $|w| > 1$, then $\mu(w) \in E$.*

2. *For two equivalent p-sets E and F there is a bijection $f : E \to F$ such that $|f(w)| = |w|$ for all $w \in E$ and $\mu(f(w)) = f(\mu(w))$ for all $w \in E$ with $|w| > 1$.*

3. *If $w \in T$, then $\mu(w) \in T$ (that is, T is a finite subsystem of (N^*, μ)),*

4. *Two equivalent t-sets are isomorphic as systems. (That is, $T \sim T'$ implies $T \cong T'$.)*

Proof.

1. Let $E = [m + \sum_{i=1}^{m} E_i]$ $(m > 0)$ be a p-set and $w \in E$ with $|w| > 1$. Then $w = iw'$ for some integer i such that $1 \geq i \geq m$ and $w' \in E_i$. If $|w'| = 1$, then $\mu(w) = i \in \{1, 2, \cdots, m\} \subseteq E$. If $|w'| > 1$, then $\mu(w') \in E_i$ by the induction hypothesis and so $\mu(w) = i\mu(w') \in iE_i \subseteq E$.

2. Let $E = [m + \sum_{i=1}^{m} E_i]$ and $F = [m + \sum_{ip=1}^{m} F_i]$ $(m > 0)$ be equivalent p-sets and tau a permutation such that $E_i = F_{\tau(i)}$. Assume that $f_i : E_i \to F_{\tau(i)}$ is a bijection $(i = 1, \cdots, m)$ such that $\mu(f_i(w_i)) = f_i(\mu(w_i))$ for all $w_i \in E_i$ such that $|w_i| > 1$. Define $f : E \to F$ as follows: $f(\varepsilon) = \varepsilon$, $f(i) = \tau(i)$ and $f(iw_i) = \tau(i) f_i(w_i)$ for $i = 1, \cdots, m$ and $w_i \in E_i$. Finally it is easy to check that $f : E \to F$ is a desired bijection.

3. It is trivial from 1.

4. It immediately follows from 2.

□

Definition 3.10 *Let $R_0, R_1, \cdots, R_{p-1}$ be t-sets. Then a tree expression $< R_0, R_1, \cdots, R_{p-1} >$ is a subsystem $\sum_{i=0}^{p-1} iR_i$ of $(Z(p)N^*, \mu_p)$, where $\mu_p(i) = i + 1 (\mathrm{mod} p)$ and $\mu_p(iw) = i \cdot \mu(w)$ for $i \in Z(p)$ and $w \in R_i^+$. And $< R_0, R_1, \cdots, R_{p-1} >$ is called normal if t-sets $R_0, R_1, \cdots, R_{p-1}$ are normal*

The tree expression $< R_0, R_1, \cdots, R_{p-1} >$ is called uniform if for any i the t-set R_i is uniform.

Definition 3.11 *Let (X, f) be a connected system with a limit cycle $< x_0, \cdots, x_{p-1} >$ of period length p. Then a transformation function γ is defined as follows;*

$$\gamma(X) =< \gamma(X_*(x_0)), \gamma(X_*(x_1)), \cdots, \gamma(X_*(x_{p-1})) >,$$

$$\gamma(X_*(x)) = [|f_*^{-1}(x)| + 1 + \sum_{u \in f_*^{-1}(x)} \gamma(X(u))]_*$$

and

$$\gamma(X(u)) = [|f_*^{-1}(u)| + \sum_{u' \in f^{-1}(u)} \gamma(X(u'))]$$

Theorem 3.12 *Let (X, f) be a connected system with a limit cycle of period length p. Then the system $(\gamma(X), \mu_p)$ is isomorphic to (X, f), that is, $(\gamma(X), \mu_p) \cong (X, f)$.*

Proof. Let (X, f) be a connected system with a limit cycle $< x_0, x_1, \cdots, x_{p-1} >$ of period length p.

1. In the case that $H(X, f) = 0$, that is, (X, f) is a system such that $X = \{x_0, x_1, \cdots, x_{p-1}\}$ and $f(x_i) = x_{i+1(mod\ p)}$. Then we have

$$\begin{aligned} X_*(x_i) &= \{x_i\} + f_*^{-1}(x_i) + \textstyle\sum_{u \in f_*^{-1}(x_i)} X(u) \\ &= \{x_i\} \end{aligned}$$

for any i ($0 \geq i \geq p-1$). We have

$$\begin{aligned} \gamma(X) &= <\gamma(X_*(x_0)), \gamma(X_*(x_1)), \cdots, \gamma(X_*(x_{p-1})), > \\ &= <[1]_*, [1]_*, \cdots, [1]_* > \\ &= \textstyle\sum_{i=0}^{p-1} i \cdot [1]_* \\ &= \{0, 1, \cdots, p-1\} \end{aligned}$$

Therefore we have $(X, f) \cong (\gamma(X), \mu_p)$.

2. In the case that $H(X, f) = 1$.
Then $X_*(x_i) = \{x_i\} + f_*^{-1}(x_i)$. So we have

$$\begin{aligned} \gamma(X) &= <\gamma(X_*(x_0)), \gamma(X_*(x_1)), \cdots, \gamma(X_*(x_{p-1})) > \\ &= \textstyle\sum_{i=0}^{p-1} i \cdot \gamma(X_*(x_i) \\ &= \textstyle\sum_{i=0}^{p-1} i \cdot [|f_*^{-1}(x_i)| + 1 + |f_*^{-1}(x_i)|[0]]_* \\ &= \textstyle\sum_{i=0}^{p-1} i + \sum_{i=0}^{p-1} i \cdot [|f_*^{-1}(x_i)| + |f_*^{-1}(x_i)|[0]]. \end{aligned}$$

From (1) we have

$$(f^\infty(X), f|f^\infty(X)) \cong (\{0, 1, \cdots, p-1\}, \mu_p|\{0, 1, \cdots, p-1\}).$$

Now we show

$$\begin{aligned} & (f_*^{-1}(x_i), f|f_*^{-1}(x_i)) \\ \cong\ & (i \cdot [|f_*^{-1}(x_i)| + |f_*^{-1}(x_i)|[0]], \mu_p|i \cdot [|f_*^{-1}(x_i)| + |f_*^{-1}(x_i)|[0]]) \end{aligned}$$

for any i $(0 \geq i \geq p-1)$. It is clear that

$$|f_*^{-1}(x_i)| = |i \cdot [|f_*^{-1}(x_i)| + |f_*^{-1}(x_i)|[0]]|.$$

For any $x \in f_*^{-1}(x_i)$ $f|f_*^{-1}(x_i)(x)$ is undefined and for any $y \in i \cdot [|f_*^{-1}(x_i)| + |f_*^{-1}(x_i)|[0]]$ $\mu_p|i \cdot [|f_*^{-1}(x_i)| + |f_*^{-1}(x_i)|[0]](y)$ is undefined. Thus we have

$$\begin{aligned} & (f_*^{-1}(x_i), f|f_*^{-1}(x_i)) \\ \cong\ & (i \cdot [|f_*^{-1}(x_i)| + |f_*^{-1}(x_i)|[0]], \mu_p|i \cdot [|f_*^{-1}(x_i)| + |f_*^{-1}(x_i)|[0]]). \end{aligned}$$

Now we let k and h_i be an isomorphism from $f^\infty(X)$ to $\{0, 1, \cdots, p-1\}$ and from $f_*^{-1}(x_i)$ to $i \cdot [|f_*^{-1}(x_i)| + |f_*^{-1}(x_i)|[0]]$ respectively. We define the function $h : X \longrightarrow \gamma(X)$ as follows;

$$h(x) = \begin{cases} k(x) & if \quad x \in f^\infty(X) \\ k(x_i) \cdot h_i(x) & if \quad x \in f_*^{-1}(x_i) \end{cases}$$

We show that the function h is an isomorphism. First we assume that $x \in f^\infty(X)$. Then we have

$$\begin{aligned} hf(x) &= h(f(x)) \\ &= k(f(x)) \\ &= \mu_p(k(x)) \\ &= \mu_p(h(x)) \\ &= \mu_p h(x) \end{aligned}$$

Next we assume that $x \in f_*^{-1}(x_i)$. Then we have

$$\begin{aligned} hf(x) &= h(x_i) \\ &= k(x_i) \\ &= \mu_p(k(x_i)h_i(x)) \\ &= \mu_p h(x) \end{aligned}$$

Thus the function h is an isomorphism. So we have $(X, f) \cong (\gamma(X), \mu_p)$.

3. In the case that $H(X, f) \geq 2$.
   First we show that the system $(X(x), f|X(x))$ is isomorphic to $(\gamma(X(x)), \mu|\gamma(X(x)))$ for any $x \in X - f^{-1}(f^\infty(X))$ by induction. Let x be in $X - f^{-1}(f^\infty(X))$.

   (a) Assume that $f^{-1}(x) = \emptyset$.
   Then we have $X(x) = \emptyset$ and $\gamma(X(x)) = [0] = \emptyset$. So

   $$(X(x), f|X(x)) \cong (\gamma(X(x)), \mu|\gamma(X(x)))$$

   .

   (b) Assume that $f^{-1}(x) \neq \emptyset$ and $f^{-2}(x) = f^{-1}(f^{-1}(x)) = \emptyset$.
   Then $X(x) = f_*^{-1}(x) + \sum_{u \in f_*^{-1}(x)} X(u)$ and $f|X(x)(y)$ is undefined for any $y \in X(x)$. And we have

   $$\begin{aligned} \gamma(X(x)) &= [|f_*^{-1}(x)| + |f_*^{-1}(x)|[0]] \\ &= \{0, 1, \cdots, |f_*^{-1}(x)| - 1\} \end{aligned}$$

   and $\mu|\gamma(X(x))(z)$ is undefined for any $z \in \gamma(X(x))$. So

   $$(X(x), f|X(x)) \cong (\gamma(X(x)), \mu|\gamma(X(x)))$$

   .

(c) Assume that $(X(y), f|X(y)) \cong (\gamma(X(y)), \mu|\gamma(X(y)))$ for any $y \in X - f^{-1}(f^{\infty}(X))$ such that $f_*^{-n+1}(y) = \emptyset$ and $f_*^{-n+1}(x) \neq \emptyset$ and $f_*^{-n}(x) = \emptyset$. Then

$$X(x) = f_*^{-1}(x) + \sum_{u \in f_*^{-1}(x)} X(u)$$

and

$$\gamma(X(x)) = [|f_*^{-1}(x)| + \sum_{u \in f_*^{-1}(x)} \gamma(X(u))].$$

By the induction hypothesis we have

$$(X(u), f|X(u)) \cong (\gamma(X(u)), \mu|\gamma(X(u)))$$

for any $u \in f_*^{-1}(x)$. Now we let the function $h_u : X(u) \to \gamma(X(u))$ be an isomorphism, and define the function $h : X(x) \to \gamma(X(x))$ as follows;

$$h(y) = \begin{cases} s(y) & if \quad y \in f_*^{-1}(x) \\ s(u) \cdot h_u(y) & if \quad y \in X(u) \ and \ u \in f_*^{-1}(x) \end{cases}$$

where $s : f_*^{-1}(x) \to N$ is a set function. It is an isomorphism. Thus This derive

$$(X(x), f|X(x)) \cong (\gamma(X(x)), \mu|\gamma(X(x)))$$

for any $x \in X - f_*^{-1}(f^{\infty}(X))$.
From (2)

$$\begin{aligned} & (f^{-1}(f^{\infty}(X)), f|f^{-1}(f^{\infty}(X))) \\ \cong \ & (\gamma(f^{-1}(f^{\infty}(X))), \mu_p|\gamma(f^{-1}(f^{\infty}(X)))), \end{aligned}$$

and so we let k be an isomorphism. We define the function $g : X \to \gamma(X)$ as follows;

$$g(x) = \begin{cases} k(x) & if \quad x \in f^{-1}(f^{\infty}(X)) \\ k(u) \cdot k_u(x) & if \quad x \in X(u) \ and \ u \in f^{-1}(f^{\infty}(X)) \end{cases}$$

where $k_u : X(u) \longrightarrow \gamma(X(u))$ is isomorphism for $u \in f^{-1}(f^{\infty}(X))$. Then the function g is an isomorphism.

□

4 Products of Dynamical Systems

In this section we discuss the product of connected dynamical systems, and present the product formula of tree expressions.

Let $X = (X, f)$ and $Y = (Y, g)$ be two systems. The (cartesian) product $X \times Y$ of X and Y is a system

$$(X \times Y, f \times g),$$

where $(f \times g)(x, y) = (f(x), g(y))$ for $x \in X$ and $y \in Y$.

Kumamoto and Nohmi analysed the transition diagram of integral affine dynamical systems and de cided the transition diagram of any integral affine dynamical They represented transition diagrams with the cartesian product. So they presented the product formula of transition diagrams of integral affine dynamical systems.

And we investigate the product of connected dynamical systems, and presented the product formula for tree expressions of any dynamical systems.

4.1 Product of The Integral Affine Dynamical Systems

In this subsection we introduce the results of the analysis of integral affine dynamical systems by Kumamoto and Nohmi[1]. They investigated the integral affine dynamical systems defined as follows;

Definition 4.1 *Let m be an arbitrary integer and $Z_m = \{0, 1, 2, \cdots, m-1\}$. An integral affine dynamical system $K_{m,a,b}$ is a system $K_{m,a,b} =< Z_m, f >$, where $f : Z_m \to Z_m$ is defined by $f(x) = ax + b$ (mod m).*

Kumamoto and Nohmi asserted that the transition diagram of any integral affine dynamical systems can be expressed by the product of uniform tree expressions. For the product of integral affine dynamical systems (uniform tree expressions) the following theorems hold;

Theorem 4.2 *Let $< T'_0, T'_1, \cdots, T'_{p-1} >$ and $< T''_0, T''_1, \cdots, T''_{q-1} >$ be two tree expression with a limit cycle of priod length p and q respectively, where $T'_i = [b_1 : b_2]^{k'}_*$ and $T''_j = [c_1 : c_2]^{k''}_*$. Then the product of $< T'_0, T'_1, \cdots, T'_{p-1} >$ and $< T''_0, T''_1, \cdots, T''_{q-1} >$ is isomorphic to the disjoint union of*

$$< T'_1 \times T''_{i+1}, T'_2 \times T''_{i+2}, \cdots,$$

$$T'_{n \ mod \ p} \times T''_{i+n \ mod \ q}, \cdots, T'_{lcm(p,q) \ mod \ p} \times T''_{i+lcm(p,q) \ mod \ q} >$$

where $0 \le i < gcd(p, q)$ and if $k' \ge k''$ then

$$[b_1 : b_2]^{k'}_* \times [c_1 : c_2]^{k''}_* = [b_1c_1 : b_2c_2]^{k'}_*.$$

4.2 Product of Finite Dynamical Systems and Product Formula

In this subsection we discuss the cartesian product of connected dynamical systems nad present the product formula of tree expressions.

Let $X = (X, f)$ and $Y = (Y, g)$ be two systems, and $< x_0, \cdots, x_{p-1} >$ and $< y_0, \cdots, y_{q-1} >$ be two limit cycles of X and Y with period length p and q, respectively.

Then $X \times Y$ have $gcd(p,q)$ limit cycles of period length $lcm(p,q)$. Limit cycles of $X \times Y$ can be known if limit cycles of X and Y are known, where $gcd(p,q)$ and $lcm(p,q)$ are the greatest common devisor and the least common multiple of p and q, respectively. So applying of trensformation function γ to $X \times Y$ we have its isomorphic tree expression.

Note that

$$(f \times g)_*^{-1}(x_i, y_j) = f_*^{-1}(x_i) \times g_*^{-1}(y_j) + f_*^{-1}(x_i) \times \{y_{j-1}\}) + \{x_{i-1}\} \times g_*^{-1}(y_j).$$

Then we have

$$\begin{aligned} & \gamma((X \times Y)_*(x_i, y_j)) \\ = & [|(f \times g)^{-1}(x_i, y_j)| + \sum_{(u,v) \in (f \times g)_*^{-1}(x_i, y_j)} \gamma((X \times Y)(u,v))]_* \\ = & [|f^{-1}(x_i)||g^{-1}(y_j)| + \sum_{(u,v) \in f_*^{-1}(x_i) \times g_*^{-1}(y_j)} X(u) \otimes Y(v) \\ & + \sum_{u \in f_*^{-1}(x_i)} X(u) \otimes Y(y_{j-1}) + \sum_{v \in g_*^{-1}(y_j)} X(x_{i-1}) \otimes Y(v)]_*, \end{aligned}$$

where $X(u) \otimes Y(v) = \gamma((X \times Y)(u,v))$.

For all $(u,v) \in f_*^{-1}(x) \times g_*^{-1}(y)$

$$X(u) \otimes Y(v) = [|f^{-1}(u)||g^{-1}(v)| + \sum_{(u',v') \in f^{-1}(u) \times g^{-1}(v)} X(u') \otimes Y(v')].$$

For all $u \in f_*^{-1}(x)$ we have

$$(f \times g)_*^{-1}(u, y_{j-1}) = f^{-1}(u) \times g_*^{-1}(y_{j-1}) + f^{-1}(u) \times \{y_{j-2}\}$$

and so

$$\begin{aligned} & X(u) \otimes Y(y_{j-1}) \\ = & [|f^{-1}(u)||g^{-1}(y_{j-1})| + \sum_{(u',v') \in (f \times g)_*^{-1}(u, y_{j-1})} X(u') \otimes Y(v')] \\ = & [|f^{-1}(u)||g^{-1}(y_{j-1})| + \sum_{(u',v') \in f_*^{-1}(u) \times g_*^{-1}(y_{j-1})} X(u') \otimes Y(v') \\ & + \sum_{u' \in f^{-1}(u)} X(u') \otimes Y(y_{j-2})]. \end{aligned}$$

Similarly, for all $v \in g_*^{-1}(y)$ we have

$$\begin{aligned} & X(x_{i-1}) \otimes Y(v) \\ = & [|f^{-1}(x_{i-1})||g^{-1}(v)| + \sum_{(u',v') \in f_*^{-1}(x_{i-1}) \times g^{-1}(v)} X(u') \otimes Y(v') \\ & + \sum_{v' \in g^{-1}(v)} X(x_{i-2}) \otimes Y(v')]. \end{aligned}$$

From above discussin and the transformation function γ the following formula can be found out.

Theorem 4.3 *Let* $E =< E_1, E_2, \cdots, E_p >$ *and* $D =< D_1, D_2, \cdots, D_q >$ *be two tree expressions with limit cycles* $< e_1, e_2, \cdots, e_p >$ *and* $< d_1, d_2, \cdots, d_q >$ *respectively where* $E_i = [m_i + E_i^1 + \cdots + E_i^{m_i-1}]_*$ *and* $D_j = [n_j + D_j^1 + \cdots D_j^{n_i-1}]_*$. *Then a subtree* $(E \times D)(e_i, d_j)$ *of the cartesian product* $E \times D$ *such that its root is* (e_i, d_j) *is shown recursively as follows:*

$$\begin{aligned} & (E \times D)(e_i, d_j) \\ = & [m_i n_j + m_i + n_j + 1 \\ & + \sum_{k=1}^{m_i-1} \sum_{h=1}^{n_j-1} E_i^k \otimes D_j^h + \sum_{k=1}^{m_i-1} E_i^k \otimes D_{j-1} + \sum_{h=1}^{n_j-1} E_{i-1} \otimes D_j^h]_* \end{aligned}$$

where

$$[m + \sum_{i=1}^{m} G_i] \otimes [n + \sum_{j=1}^{n} F_j] = [mn + \sum_{i=1}^{m} \sum_{j=1}^{n} G_i \otimes F_j],$$

$$[m + \sum_{i=1}^{m} F_i] \otimes D_j = [mn_j + \sum_{k=1}^{m} \sum_{h=1}^{n_j-1} F_k \otimes D_j^h + \sum_{k=1}^{m} F_k \otimes D_{j-1}],$$

$$E_i \otimes [n + \sum_{j=1}^{n} F_j] = [m_i n + \sum_{k=1}^{m_i-1} \sum_{h=1}^{n} E_i^k \otimes F_h + \sum_{h=1}^{n} E_{i-1} \otimes F_h]$$

Corollary 4.4 *For the product of tree expressions with a fixed point the following formula is acquired;*

$$< [m + \sum_{i=1}^{m-1} E_i]_* > \times < [n + \sum_{j=1}^{n-1} F_j]_* >$$

$$=< [mn + \sum_{i=1}^{m-1} E_i \otimes [n + \sum_{j=1}^{n-1} F_j]_* + \sum_{j=1}^{n-1} [m + \sum_{i=1}^{m-1} E_i]_* \otimes F_j + \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} E_i \otimes F_j]_* >$$

where

$$[m + \sum_{i=1}^{m} E_i] \otimes [n + \sum_{j=1}^{n} F_j] = [mn + \sum_{i=1}^{m} \sum_{j=1}^{n} E_i \otimes F_j],$$

$$[m + \sum_{i=1}^{m} E_i] \otimes [n + \sum_{j=1}^{n-1} F_j]_* = [mn + \sum_{i=1}^{m} E_i \otimes [n + \sum_{j=1}^{n-1} F_j]_* + \sum_{i=1}^{m} \sum_{j=1}^{n-1} E_i \otimes F_j],$$

$$[m + \sum_{i=1}^{m-1} E_i]_* \otimes [n + \sum_{j=1}^{n} F_j] = [mn + \sum_{j=1}^{n} [m + \sum_{i=1}^{m-1} E_i]_* \otimes F_j + \sum_{i=1}^{m-1} \sum_{j=1}^{n} E_i \otimes F_j].$$

5 Conclusion

In this paper we introduced tree expressions which can represent transition diagrams of finite systems, and presented their exprcite product formula. Tree expression can represent transition diagrams of all finite systems, and make us be able to deal with transition diagram of finite systems by algeblaic methods. And our pruduct formula is useful for analysis and applications of finite systems which can be separated or is the product of finite systems. But there exist several weak points. First tree expressions have redundancy. For uniform tree expressions we defined the notation, and we can represent efficiently. But for other systems the larger the number of nodes of a system is, the longer the lenght of its tree expression is. Secondly a system don't have an unique tree expression. All tree expression have the equivalent normal tree expression uniquely. But we cannot get the normal tree expression of a system. by appling the transformation function γ to it. And lastly the tree expression of the product of normal tree xpressions to be got by appling product formula is not always normal.

References

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[2] H.Lee and Y.Kawahara: *Transition Diagrams of Finite Cellular Automata*, Bull. Inform. Cybernet. 28(1996)47-69