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出版情報：DOI Technical Report. 155, 1999-01. Department of Informatics, Kyushu University

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Abstract. Lattice structures are fundamental and useful in mathematics and theoretical computer science. It is well-known that lattice structures with meet and join operations satisfying associative, commutative and absorption laws are equivalent to lattice structures defined by ordering relations having joins and meets. This note defines a notion of lattices in Dedekind categories and tudies on some basic properties on lattice structures with element-free discussion using relational calculaus.
Keywords: lattice, Dedekind category, allegory, relational calculus.

1 Introduction

A lattice is a triple $(X, \vee, \wedge)$ of a set X and two functions $\vee : X \times X \to X$ and $\wedge : X \times X \to X$ satisfying:
Associative Law: (L1$\vee$) $(x \vee y) \vee z = x \vee (y \vee z)$ and (L1$\wedge$) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
Commutative Law: (L2$\vee$) $x \vee y = y \vee x$ and (L2$\wedge$) $x \wedge y = y \wedge x$
Absorption Law: (L3$\vee$) $x \wedge (x \vee y) = x$ and (L3$\wedge$) $x \vee (x \wedge y) = x$
for all elememts x, y and z of X. It is well-known that a lattice $(X, \vee, \wedge)$ has a natural ordering $\leq$ defined by $x \leq y$ for $x, y \in X$ iff $x \vee y = y$. Demonstrating this fact is good exercise for undergraduate students. In detail it mainly consists of checking that the ordering $\leq$ is in fact reflexive, transitive and antsymmetric, and that it has a few alternative definitions such as $x \leq y$ iff $x \wedge y = x$. Conversely, it is also well-known that the concept of lattices is obtained from ordered sets with joins and meets.

The motivation of this note is just to demonstrate these facts in categories [4]. To discuss binary operations such as $\vee : X \times X \to X$ and $\wedge : X \times X \to X$ the involved category needs to have (finite) cartesian products. Furthermore the category has to be equipped a kind of relational structures, as we treat with an ordering within it. This is a reason why categorical lattice theory is formalized in Dedekind categories [5]. The modern algebraic theory of binary relations was founded by Tarski [7], and the categorical study for relations was initiated by Mac Lane [3]. Thought Dedekind categories in this note are synonyous as allegories due to Freyd and Scedrov [1] and heterogeneous relation algebras by Schmidt and Ströhlein [6], the author adopt Dedekind categories after the historical emergence in literatures. The note is organised as follows:
In section 2 we define Dedekind categories and relational products, and review

some fundamentals on Dedekind categories. In section 3 a notion of lattices in Dedekind categories is defined, and some basic facts on a natural partial ordering on a lattice in a Dedekind category are investigated. In section 4 we try to construct a lattice from a partial ordering with joins and meets in a Dedekind category.

2 Dedekind Categories

In this section we recall the fundamentals on relation categories, which we will call elementary Dedekind categories.

Throughout this note, a morphism α from an object X into an object Y in a Dedekind category (which will be defined below) will be denoted by a half arrow $\alpha : X \rightharpoondown Y$, and the composite of a morphism $\alpha : X \rightharpoondown Y$ followed by a morphism $\beta : Y \rightharpoondown Z$ will be written as $\alpha\beta : X \rightharpoondown Z$. Also we will denote the identity morphism on X as id_X.

Definition 2.1. A Dedekind category $\mathcal{D}$ is a category satisfying the following:
D1. [Distributive Lattice] For all pairs of objects X and Y the hom-set $\mathcal{D}(X,Y)$ consisting of all morphisms of X into Y is a distributive lattice with the least morphism 0_{XY} and the greatest morphism ∇_{XY}. Its lattice structure will be denoted by

$$\mathcal{D}(X,Y) = (\mathcal{D}(X,Y), \sqsubseteq, \sqcup, \sqcap, 0_{XY}, \nabla_{XY}).$$

D2. [Converse] There is given a converse operation $\sharp : \mathcal{D}(X,Y) \to \mathcal{D}(Y,X)$. That is, for all morphisms $\alpha, \alpha' : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$, the following converse laws hold:
(a) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (b) $(\alpha^\sharp)^\sharp = \alpha$, (c) If $\alpha \sqsubseteq \alpha'$, then $\alpha^\sharp \sqsubseteq \alpha'^\sharp$
for all morphisms $\alpha, \alpha' : X \rightharpoondown Y$ and $\beta : Y \rightharpoondown Z$.
D3. [Dedekind Formula] For all morphisms $\alpha : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$ holds.
D4. [Residues] For all morphisms $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the residue (or division) $\gamma \div \beta : X \rightharpoondown Y$ is a morphism such that $\alpha\beta \sqsubseteq \gamma$ if and only if $\alpha \sqsubseteq \gamma \div \beta$ for all morphisms $\alpha : X \rightharpoondown Y$. □

The following is a basic property of Dedekind categories, which will be repeatedly used in this paper.

Proposition 2.1. *Let $\alpha, \alpha' : X \rightharpoondown Y$ and $\beta, \beta' : Y \rightharpoondown Z$ be morphisms in a Dedekind category. If $\alpha \sqsubseteq \alpha'$ and $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha'\beta'$.*

Proof. Assume $\alpha \sqsubseteq \alpha'$ and $\beta \sqsubseteq \beta'$. First we will see $\alpha\beta \sqsubseteq \alpha'\beta$. The characteristic property of residues leads $\alpha' \sqsubseteq \alpha'\beta \div \beta$ from the reflexivity $\alpha'\beta \sqsubseteq \alpha'\beta$.

Hence we have $\alpha \sqsubseteq \alpha'\beta \div \beta$ by the assumption $\alpha \sqsubseteq \alpha'$, and so $\alpha\beta \sqsubseteq \alpha'\beta$ again by the characteristic property of residues. Finally we will see $\alpha\beta \sqsubseteq \alpha\beta'$. As $\beta^\sharp \sqsubseteq \beta'^\sharp$ by D2(c), the former result shows $\beta^\sharp\alpha^\sharp \sqsubseteq \beta'^\sharp\alpha^\sharp$ and so we have $\alpha\beta = (\beta^\sharp\alpha^\sharp)^\sharp \sqsubseteq (\beta'^\sharp\alpha^\sharp)^\sharp = \alpha\beta'$ by D2(a)-(c). □

More details on fundamental properties of relational categories is referred to [2]. The following is a basic lemma [1] in Dedekind categories.

Lemma 2.1. *For two relations $\alpha : X \rightharpoondown Y$ and $\beta : Y \rightharpoondown X$ an equality $\mathrm{id}_X \sqcap (\alpha \sqcap \beta^\sharp)(\alpha^\sharp \sqcap \beta) = \mathrm{id}_X \sqcap \alpha\beta$ holds.*

Proof.

$$\begin{aligned} \mathrm{id}_X \sqcap \alpha\beta &= \mathrm{id}_X \sqcap \mathrm{id}_X \sqcap \alpha\beta \\ &\sqsubseteq \mathrm{id}_X \sqcap (\mathrm{id}_X\beta^\sharp \sqcap \alpha)(\alpha^\sharp\mathrm{id}_X \sqcap \beta) \quad \{ \text{ Dedekind Formula } \} \\ &= \mathrm{id}_X \sqcap (\alpha \sqcap \beta^\sharp)(\alpha^\sharp \sqcap \beta) \\ &\sqsubseteq \mathrm{id}_X \sqcap \alpha\beta. \end{aligned}$$

□

A morphism $f : X \rightharpoondown Y$ such that $f^\sharp f \sqsubseteq \mathrm{id}_Y$ (*univalent*) and $\mathrm{id}_X \sqsubseteq ff^\sharp$ (*total*) is called a *function* and may be introduced as $f : X \to Y$.

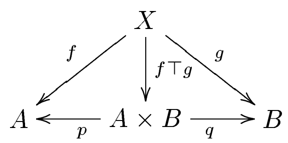
Corollary 2.1. *Let $\alpha : X \rightharpoondown Y$ and $\beta : Y \rightharpoondown X$ be relations in a Dedekind category $\mathcal{D}$. If $\alpha\beta = \mathrm{id}_X$ and $\beta\alpha = \mathrm{id}_Y$, then $\alpha = \beta^\sharp$.*

Proof. As $\alpha\beta = \mathrm{id}_X$ we have $\mathrm{id}_X = \mathrm{id}_X \sqcap \mathrm{id}_X = \mathrm{id}_X \sqcap \alpha\beta = \mathrm{id}_X \sqcap (\alpha \sqcap \beta^\sharp)(\alpha^\sharp \sqcap \beta)$ by Lemma 2.1. Hence $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$ and $\mathrm{id}_X \sqsubseteq \beta^\sharp\beta$. Also it follows from $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$ and $\beta\alpha = \mathrm{id}_Y$ that $\beta^\sharp\beta \sqsubseteq \beta^\sharp\beta\alpha\alpha^\sharp = \beta^\sharp\alpha^\sharp = (\alpha\beta)^\sharp = \mathrm{id}_X$. Therefore $\beta^\sharp\beta = \mathrm{id}_X$ and so $\beta^\sharp = \beta^\sharp\beta\alpha = \alpha$ from $\beta\alpha = \mathrm{id}_Y$. □

Definition 2.2. A Dedekind category $\mathcal{D}$ has relational products if for each pair of objects A and B there is a pair of functions $p : A \times B \to A$ and $q : A \times B \to B$ such that $p^\sharp q = \nabla_{AB}$ and $pp^\sharp \sqcap qq^\sharp = \mathrm{id}_{A\times B}$. The functions p and q will be called a pair of projections of relational products. □

Throughout the rest of the note we assume that $\mathcal{D}$ is a fixed Dedekind category with relational products.

Proposition 2.2. *Let $p : A \times B \to A$ and $q : A \times B \to B$ be a pair of projections of $A \times B$. For each pair of functions $f : X \to A$ and $g : X \to B$, a relation $f \top g = fp^\sharp \sqcap gq^\sharp : X \rightharpoondown A \times B$ is a unique function such that $(f \top g)p = f$ and $(f \top g)q = g$.*



Proof. Set $h = f\top g$. The univalency (or, single-valuedness) of h simply follows from

$$\begin{aligned} h^\sharp h &= (fp^\sharp \sqcap gq^\sharp)^\sharp(fp^\sharp \sqcap gq^\sharp) \\ &= (pf^\sharp \sqcap qg^\sharp)(fp^\sharp \sqcap gq^\sharp) \\ &\sqsubseteq pf^\sharp fp^\sharp \sqcap qg^\sharp gq^\sharp \\ &\sqsubseteq pp^\sharp \sqcap qq^\sharp \quad (\text{by } f^\sharp f \sqsubseteq \mathrm{id}_A \text{ and } g^\sharp g \sqsubseteq \mathrm{id}_B) \\ &= \mathrm{id}_{A\times B}. \end{aligned}$$

Also the totality of h comes from

$$\begin{aligned} \mathrm{id}_X \sqcap hh^\sharp &= \mathrm{id}_X \sqcap (fp^\sharp \sqcap gq^\sharp)(fp^\sharp \sqcap gq^\sharp)^\sharp \\ &= \mathrm{id}_X \sqcap (fp^\sharp)(gq^\sharp)^\sharp && \{ \text{ Lemma 2.1 } \} \\ &= \mathrm{id}_X \sqcap fp^\sharp qg^\sharp \\ &\sqsupseteq \mathrm{id}_X \sqcap ff^\sharp gg^\sharp && \{ f^\sharp g \sqsubseteq \nabla_{AB} = p^\sharp q \} \\ &= \mathrm{id}_X && \{ \mathrm{id}_X \sqsubseteq ff^\sharp \text{ and } \mathrm{id}_X \sqsubseteq gg^\sharp \} \end{aligned}$$

The uniqueness of h follows from $h = h\mathrm{id}_{A\times B} = h(pp^\sharp \sqcap qq^\sharp) = hpp^\sharp \sqcap hqq^\sharp$.
□

Remark. It is trivial that $p\top q = \mathrm{id}_{A\times B}$.

Corollary 2.2. *For a function $k : X \to A$ an equality $k(f\top g) = kf\top kg$ holds.* □

Proposition 2.3. *Let $p_{AB} : A\times B \to A$ and $q_{AB} : A\times B \to B$ be a pair of projections of $A\times B$, and $p_{BA} : B\times A \to B$ and $q_{BA} : B\times B \to A$ a pair of projections of $B\times A$. The twist function $t_{AB} : A\times B \to B\times A$ is defined as the unique function such that $t_{AB}p_{BA} = q_{AB}$ and $t_{AB}q_{BA} = p_{AB}$. (That is, $t_{AB} = q_{AB}p^\sharp_{BA} \sqcap p_{AB}q^\sharp_{BA} = q_{AB}\top p_{AB}$.) Then $t_{AB}t_{BA} = \mathrm{id}_{A\times B}$ and $t_{BA}t_{AB} = \mathrm{id}_{B\times A}$.*

Proof. By Proposition 2.2 we have

$$t_{AB}t_{BA} = t_{AB}(q_{BA}\top p_{BA}) = t_{AB}q_{BA}\top t_{AB}p_{BA} = p_{AB}\top q_{AB} = \mathrm{id}_{A\times B}.$$

□

In general the pair of projections will be denoted by $p_{AB} : A\times B \to A$ and $q_{AB} : A\times B \to B$. The diagonal function $d_A : A \to A\times A$ is defined as a unique function such that $d_Ap_{AA} = \mathrm{id}_A$ and $d_Aq_{AA} = \mathrm{id}_A$. That is, $d_A = \mathrm{id}_A\top\mathrm{id}_A = p^\sharp_{AA} \sqcap q^\sharp_{AA}$. The associative function $a_{ABC} : (A\times B)\times C \to A\times(B\times C)$ is defined by

$$a_{ABC} = p_{A\times BC}p_{AB}\top(p_{A\times BC}q_{AB}\top q_{A\times BC})$$

Another associative function $b_{ABC} : A\times(B\times C) \to (A\times B)\times C$ is defined by

$$b_{ABC} = (p_{AB\times C}\top q_{AB\times C}q_{BC})\top q_{AB\times C}q_{BC}.$$

It is trivial that a_{ABC} and b_{ABC} are mutually inverses, that is, $a_{ABC}b_{ABC} = \mathrm{id}_{(A\times B)\times C}$ and $b_{ABC}a_{ABC} = \mathrm{id}_{A\times(B\times C)}$.

For a pair of functions $f : A \to X$ and $g : B \to Y$ we define a function $f\times g : A\times B \to X\times Y$ by $f\times g = p_{AB}f \top q_{AB}g (= p_{AB}fp^{\sharp}_{XY} \sqcap q_{AB}gq^{\sharp}_{XY})$. That is, $f \times g$ is a unique function such that $(f \times g)p_{XY} = p_{AB}f$ and $(f \times g)q_{XY} = q_{AB}g$.

$$\begin{array}{ccccc} A & \xleftarrow{p_{AB}} & A\times B & \xrightarrow{q_{AB}} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\times g} & & \downarrow{\scriptstyle g} \\ X & \xleftarrow[p_{XY}]{} & X\times Y & \xrightarrow[q_{XY}]{} & Y. \end{array}$$

The following lemma indicates a well-known example of pullbacks.

Lemma 2.2. *An equality $p^{\sharp}_{AB}(f \times \mathrm{id}_B) = fp^{\sharp}_{XB}$ holds for every function $f : A \to B$.*

$$\begin{array}{ccc} A\times B & \xrightarrow{f\times id_B} & X\times B \\ {\scriptstyle p_{AB}}\downarrow & & \downarrow{\scriptstyle p_{XB}} \\ A & \xrightarrow[f]{} & X \end{array}$$

Proof. Note that $p^{\sharp}_{AB}q_{AB}q^{\sharp}_{XB} = \nabla_{AB}q^{\sharp}_{XB} = \nabla_{AX\times B}$ by $p^{\sharp}_{AB}q_{AB} = \nabla_{AB}$ and the totality of q_{XB}. ($\nabla_{AX\times B} \sqsubseteq \nabla_{AX\times B}q_{XB}q^{\sharp}_{XB} \sqsubseteq \nabla_{AB}q^{\sharp}_{XB} = p^{\sharp}_{AB}q_{AB}q^{\sharp}_{XB}$.)

$$\begin{array}{lll} p^{\sharp}_{AB}(f\times \mathrm{id}_B) & = & p^{\sharp}_{AB}(p_{AB}fp^{\sharp}_{XB} \sqcap q_{AB}q^{\sharp}_{XB}) \\ & \sqsubseteq & p^{\sharp}_{AB}p_{AB}fp^{\sharp}_{XB} \\ & \sqsubseteq & fp^{\sharp}_{XB} \qquad \{\ p^{\sharp}_{AB}p_{AB} \sqsubseteq \mathrm{id}_A\ \} \\ & = & fp^{\sharp}_{XB} \sqcap p^{\sharp}_{AB}q_{AB}q^{\sharp}_{XB} \qquad \{\ p^{\sharp}_{AB}q_{AB}q^{\sharp}_{XB} = \nabla_{AX\times B}\ \} \\ & \sqsubseteq & p^{\sharp}_{AB}(p_{AB}fp^{\sharp}_{XB} \sqcap q_{AB}q^{\sharp}_{XB}) \qquad \{\ \text{Dedekind Formula}\ \} \\ & = & p^{\sharp}_{AB}(f\times \mathrm{id}_B) \end{array}$$

□

Lemma 2.3. *Let $p : A \times B \to A$, $q : A \times B \to B$ and $p_0 : A \times B' \to A$, $q_0 : A \times B' \to B'$ be projections of products $A \times B$ and $A \times B'$, respectively. Then*

(a) *$\alpha = p^{\sharp}(p\alpha \sqcap q) = p^{\sharp}p\alpha$ for every relation $\alpha : A \rightharpoondown B$.*
(b) *If relations $\delta_0, \delta_1 : A\times B \rightharpoondown B$ satisfy $\delta_0 \sqsubseteq q$ and $\delta_1 \sqsubseteq q$, then $p^{\sharp}\delta_0 \sqcap p^{\sharp}\delta_1 = p^{\sharp}(\delta_0 \sqcap \delta_1)$.*
(c) *If relations $\gamma_0, \gamma_1 : A \times B' \rightharpoondown A \times B$ satisfy $\gamma_0 \sqsubseteq p_0p^{\sharp}$ and $\gamma_1 \sqsubseteq p_0p^{\sharp}$, then $p^{\sharp}_0\gamma_0 \sqcap p^{\sharp}_0\gamma_1 = p^{\sharp}_0(\gamma_0 \sqcap \gamma_1)$ and $\gamma_0q \sqcap \gamma_1q = (\gamma_0 \sqcap \gamma_1)q$.*

Proof. (a) $\alpha = \alpha \sqcap p^{\sharp}q \sqsubseteq p^{\sharp}(p\alpha \sqcap q) \sqsubseteq p^{\sharp}p\alpha \sqsubseteq \alpha$.
(b)

$$
\begin{array}{rcll}
p^\sharp\delta_0 \sqcap p^\sharp\delta_1 & \sqsubseteq & p^\sharp(\delta_0 \sqcap pp^\sharp\delta_1) & \{\text{Dedekind Formula}\}\\
& = & p^\sharp(\delta_0 \sqcap pp^\sharp\delta_1 \sqcap q) & \{\delta_0 \sqsubseteq q\}\\
& \sqsubseteq & p^\sharp[\delta_0 \sqcap (pp^\sharp \sqcap q\delta_1^\sharp)\delta_1] & \{\text{Dedekind Formula}\}\\
& \sqsubseteq & p^\sharp[\delta_0 \sqcap (pp^\sharp \sqcap qq^\sharp)\delta_1] & \{\delta_1 \sqsubseteq q\}\\
& = & p^\sharp(\delta_0 \sqcap \delta_1). & \{pp^\sharp \sqcap qq^\sharp = \mathrm{id}_{A\times B}\}
\end{array}
$$

(c)

$$
\begin{array}{rcll}
\gamma_0 q \sqcap \gamma_1 q & \sqsubseteq & (\gamma_0 \sqcap \gamma_1 qq^\sharp)q & \{\text{Dedekind Formula}\}\\
& = & (\gamma_0 \sqcap p_0p^\sharp \sqcap \gamma_1 qq^\sharp)q & \{\gamma_0 \sqsubseteq p_0p^\sharp\}\\
& \sqsubseteq & [\gamma_0 \sqcap \gamma_1(\gamma_1^\sharp p_0p^\sharp \sqcap qq^\sharp)]q & \{\text{Dedekind Formula}\}\\
& \sqsubseteq & [\gamma_0 \sqcap \gamma_1(pp_0^\sharp p_0p^\sharp \sqcap qq^\sharp)]q & \{\gamma_1 \sqsubseteq p_0p^\sharp\}\\
& \sqsubseteq & [\gamma_0 \sqcap \gamma_1(pp^\sharp \sqcap qq^\sharp)]q & \{p_0^\sharp p_0 \sqsubseteq \mathrm{id}_A\}\\
& = & (\gamma_0 \sqcap \gamma_1)q. & \{pp^\sharp \sqcap qq^\sharp = \mathrm{id}_{A\times B}\}
\end{array}
$$

□

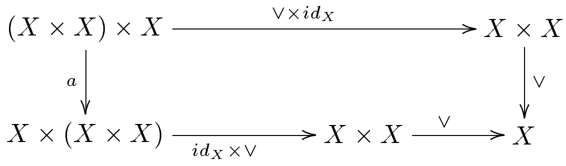
3 Lattices

In this section we will see that lattice structures with meet and join operations satisfying associative, commutative and absorption laws induce reflexive, transitive and antisymmetric relations. We will write $p = p_{XX}$, $q = q_{XX}$, $t = t_{XX}$, $d = d_X$, $a = a_{XXX}$ and $b = b_{XXX}$.

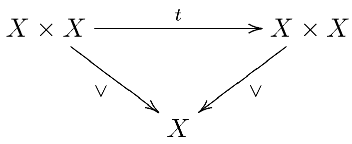
Definition 3.1. A lattice in a Dedekind category $\mathcal{D}$ is a triple $(X, \vee, \wedge)$ of an object X, and two functions (binary operations) $\vee : X \times X \to X$ and $\wedge : X \times X \to X$ satisfying
Associative Law:
(L1$\vee$) $(\vee \times \mathrm{id}_X)\vee = a(\mathrm{id}_X \times \vee)\vee$ $\{ (x \vee y) \vee z = x \vee (y \vee z) \}$
(L1$\wedge$) $(\wedge \times \mathrm{id}_X)\wedge = a(\mathrm{id}_X \times \wedge)\wedge$ $\{ (x \wedge y) \wedge z = x \wedge (y \wedge z) \}$
Commutative Law:
(L2$\vee$) $t\vee = \vee$ $\{ x \vee y = y \vee x \}$
(L2$\wedge$) $t\wedge = \wedge$ $\{ x \wedge y = y \wedge x \}$
Absorption Law:
(L3$\vee$) $(p\top\vee)\wedge = p$ $\{ x \wedge (x \vee y) = x \}$
(L3$\wedge$) $(p\top\wedge)\vee = p$ $\{ x \vee (x \wedge y) = x \}$
□

The above laws for lattices are illustrated by the following commutative diagrams:

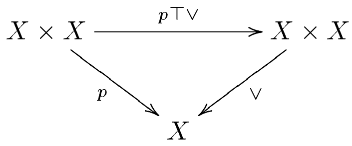
(L1$\vee$)



(L2∨)



(L3∨)



Recall $(p\top\vee)t = \vee\top p$ by the property of the twist function t and so $(\vee\top p)\wedge = (p\top\vee)t\wedge = (p\top\vee)\wedge = p$ by (L2∧) and (L3∨). Hence (L3∨′) $(\vee\top p)\wedge = p$ and (L3∧′) $(\wedge\top p)\vee = p$ are equivalent to (L3∨) and (L3∧), respectively.

Proposition 3.1. *An identity* $d\vee = \mathrm{id}_X$ *holds in every lattice* $(X, \vee, \wedge)$ *in a Dedekind category.*

Proof. First note that $d(p\top p) = dp\top dp = \mathrm{id}_X\top\mathrm{id}_X = d$ by Corollary 2.2. Hence

$$\begin{array}{rcll} d\vee & = & d(p\top p)\vee & \{\ d = d(p\top p)\ \} \\ & = & d\{(p\top\vee)p\top(p\top\vee)\wedge\}\vee & \{\ (p\top\vee)p = p \text{ and } (L3\vee)\ \} \\ & = & d(p\top\vee)(p\top\wedge)\vee & \{\ \text{Corollary 2.2}\ \} \\ & = & d(p\top\vee)p & \{\ (L3\wedge)\ \} \\ & = & dp & \{\ (p\top\vee)p = p\ \} \\ & = & \mathrm{id}_X & \end{array}$$

□

Define relations $\xi = p^\sharp(q \sqcap \vee) : X \rightharpoondown X$ and $\eta = (p \sqcap \wedge)^\sharp q : X \rightharpoondown X$. (For concrete relations: $\forall x, y \in X$, $x\xi y \iff y = x \vee y \iff x \le y$, and $x\eta y \iff x = x \wedge y \iff x \le y$.)

Note. It is easy to see the following basic fact on concrete lattices:

$$\begin{array}{rl} & (x, y) \in \xi = p^\sharp(q \sqcap \vee) \\ \iff & \exists(x', y') :: (x, (x', y')) \in p^\sharp \text{ and } ((x', y'), y) \in q \sqcap \vee \\ \iff & \exists(x', y') :: x = x' \text{ and } y' = y \text{ and } x' \vee y' = y \\ \iff & x \vee y = y. \end{array}$$

$$\begin{array}{ll} & (x,y) \in p^\sharp \vee \\ \Longleftrightarrow & \exists (x',y') :: (x,(x',y')) \in p^\sharp \text{ and } ((x',y'),y) \in \vee \\ \Longleftrightarrow & \exists y' :: y = x \vee y'. \end{array}$$

$$\begin{array}{ll} & (x,y) \in \eta = (p \sqcap \wedge)^\sharp q \\ \Longleftrightarrow & \exists (x',y') :: (x,(x',y')) \in (p \sqcap \wedge)^\sharp \text{ and } ((x',y'),y) \in q \\ \Longleftrightarrow & \exists (x',y') :: x = x' \text{ and } x' \wedge y' = x \text{ and } y' = y \\ \Longleftrightarrow & x \wedge y = x. \end{array}$$

$$\begin{array}{ll} & (x,y) \in \wedge^\sharp q \\ \Longleftrightarrow & \exists (x',y') :: (x,(x',y')) \in \wedge^\sharp \text{ and } ((x',y'),y) \in q \\ \Longleftrightarrow & \exists x' :: x = x' \wedge y. \end{array}$$

Proposition 3.2. *Let* $(X, \vee, \wedge)$ *be a lattice in a Dedekind category and set* $\xi = p^\sharp(q \sqcap \vee) : X \rightharpoondown X$. *Then an identity* $\mathrm{id}_X \sqcap d\vee = \mathrm{id}_X \sqcap \xi$ *holds.*

Proof.

$$\begin{array}{rcll} \mathrm{id}_X \sqcap \xi & = & \mathrm{id}_X \sqcap p^\sharp(q \sqcap \vee) & \\ & = & \mathrm{id}_X \sqcap (p \sqcap q \sqcap \vee)^\sharp (p \sqcap q \sqcap \vee) & \{ \text{ Lemma 2.1 } \} \\ & = & \mathrm{id}_X \sqcap (p \sqcap q)^\sharp \vee & \{ \text{ Lemma 2.1 } \} \\ & = & \mathrm{id}_X \sqcap d \vee . & \{ \ d = p^\sharp \sqcap q^\sharp \ \} \end{array}$$

□

Combining with Propositions 3.1 and 3.1 we have the following

Corollary 3.1. *Let* $(X, \vee, \wedge)$ *be a lattice in a Dedekind category. Then* $\xi = p^\sharp(q \sqcap \vee) : X \rightharpoondown X$ *satisfies* $\mathrm{id}_X \sqsubseteq \xi$ (reflexive). □

Proposition 3.3. *Let* $(X, \vee, \wedge)$ *be a lattice in a Dedekind category. Then* $\xi = p^\sharp(q \sqcap \vee) : X \rightharpoondown X$ *satisfies* $\xi \sqcap \xi^\sharp \sqsubseteq \mathrm{id}_X$ (antisymmetric).

Proof.

$$\begin{array}{rcll} \xi \sqcap \xi^\sharp & = & p^\sharp(q \sqcap \vee) \sqcap (q \sqcap \vee)^\sharp p & \\ & \sqsubseteq & (q \sqcap \vee)^\sharp [(q \sqcap \vee) p^\sharp \sqcap p(q \sqcap \vee)^\sharp](q \sqcap \vee) & \{ \text{ Dedekind Formula } \} \\ & \sqsubseteq & \vee^\sharp (qp^\sharp \sqcap pq^\sharp) \vee & \\ & = & \vee^\sharp t \vee & \{ \ t = qp^\sharp \sqcap pq^\sharp \ \} \\ & = & \vee^\sharp \vee & \{ \text{ (L2}\vee\text{) } \} \\ & \sqsubseteq & \mathrm{id}_X . & \{ \text{ The univalency of } \vee \ \} \end{array}$$

□

Proposition 3.4. *Let* $(X, \vee, \wedge)$ *be a lattice in a Dedekind category, and set* $\xi = p^\sharp(q \sqcap \vee) : X \rightharpoondown X$ *and* $\eta = (p \sqcap \wedge)^\sharp q : X \rightharpoondown X$. *Then an identity* $\xi = p^\sharp \vee = \wedge^\sharp q = \eta$ *holds.*

Proof. First note $p = p \sqcap p = (p\top\vee)p \sqcap (p\top\vee)\wedge = (p\top\vee)(p \sqcap \wedge)$ by (L3$\vee$) and Corollary 2.2. Hence

$$\begin{array}{lll} \xi & = & p^\sharp(q \sqcap \vee) \\ & \sqsubseteq & p^\sharp\vee \\ & = & [(p\top\vee)(p \sqcap \wedge)]^\sharp(p\top\vee)q \quad \{\ p = (p\top\vee)(p \sqcap \wedge) \text{ and } (p\top\vee)q = \vee\ \} \\ & = & (p \sqcap \wedge)^\sharp(p\top\vee)^\sharp(p\top\vee)q \\ & \sqsubseteq & (p \sqcap \wedge)^\sharp q \qquad\qquad\qquad \{\ \text{The univalency of } p\top\vee\ \} \\ & = & \eta. \end{array}$$

Similarly it follows from $p = p \sqcap p = (\wedge\top p)q \sqcap (\wedge\top p)\vee = (\wedge\top p)(q \sqcap \vee)$ by (L3$\wedge'$), Proposition 2.2 and Corollary 2.2 and so $q = tp = t(\wedge\top p)(q \sqcap \vee) = (t \wedge \top tp)(q \sqcap \vee) = (\wedge\top q)(q \sqcap \vee)$ by by (L2$\wedge$). Hence

$$\begin{array}{lll} \eta & = & (p \sqcap \wedge)^\sharp q \\ & \sqsubseteq & \wedge^\sharp q \\ & = & [(\wedge\top q)p]^\sharp(\wedge\top q)(q \sqcap \vee) \quad \{\ q = (\wedge\top q)(q \sqcap \vee) \text{ and } (\wedge\top q)p = \wedge\ \} \\ & = & p^\sharp(\wedge\top q)^\sharp(\wedge\top q)(q \sqcap \vee) \\ & \sqsubseteq & p^\sharp(q \sqcap \vee) \qquad\qquad\qquad \{\ \text{The univalency of } \wedge\top q\ \} \\ & = & \xi. \end{array}$$

□

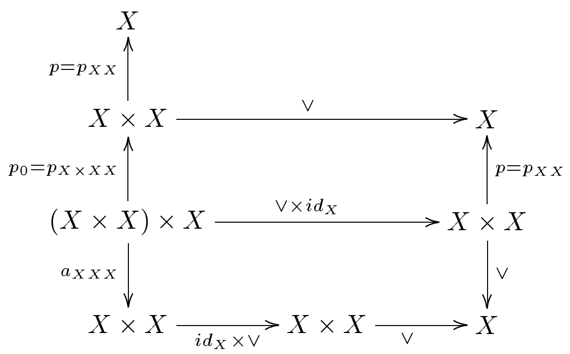
Proposition 3.5. *Let* $(X, \vee, \wedge)$ *be a lattice in a Dedekind category. Then* $\xi = p^\sharp(q \sqcap \vee) : X \rightharpoondown X$ *satisfies* $\xi\xi \sqsubseteq \xi$ (transitive).

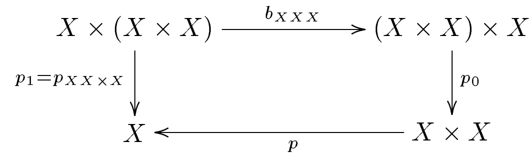
Proof.

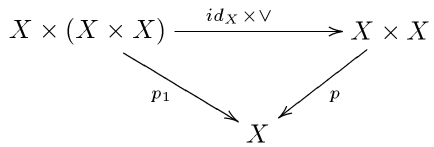
$$\begin{array}{llll} \xi\xi & = & p^\sharp \vee p^\sharp \vee & \{\ \text{Proposition 3.4}\ \} \\ & = & p^\sharp {p_0}^\sharp(\vee \times \mathrm{id}_X)\vee & \{\ \text{Lemma 2.2}\ \} \\ & = & p^\sharp {p_0}^\sharp a(\mathrm{id}_X \times \vee)\vee & \{\ (\text{L1}\vee)\ \} \\ & = & p^\sharp {p_0}^\sharp b^\sharp(\mathrm{id}_X \times \vee)\vee & \{\ a = b^\sharp\ \} \\ & = & (bp_0p)^\sharp(\mathrm{id}_X \times \vee)\vee & \\ & = & [(\mathrm{id}_X \times \vee)p]^\sharp(\mathrm{id}_X \times \vee)\vee & \{\ bp_0p = p_1 = (\mathrm{id}_X \times \vee)p\ \} \\ & = & p^\sharp(\mathrm{id}_X \times \vee)^\sharp(\mathrm{id}_X \times \vee)\vee & \\ & \sqsubseteq & p^\sharp\vee & \{\ \text{The univalency of } \mathrm{id}_X \times \vee\ \} \\ & = & \xi. & \{\ \text{Proposition 3.4}\ \} \end{array}$$

□

Note. The following three diagrams may help to understand the proof of the last proposition.







Theorem 3.1. *Let* $(X, \vee, \wedge)$ *be a lattice in a Dedekind category. Then* $\xi = p^\sharp(q \sqcap \vee) : X \rightharpoonup X$ *is reflexive, transitive and antisymmetric. Moreover,* $\xi = p^\sharp(q \sqcap \vee) = p^\sharp\vee = \wedge^\sharp q = (p \sqcap \wedge)^\sharp q$ *holds.* □

4 Orderings

In this section we will see that orderings having joins and meets induce lattice structures also in Dedekind categories. First we show a technical lemma needed later.

Lemma 4.1. *Let* $\xi : X \rightharpoonup X$ *and* $\gamma : Y \rightharpoonup X$ *be relations, and let* $h : Z \to Y$ *and* $k : Y \to X$ *be functions. Then*

(a) $h\{(\xi \div \gamma)^\sharp \sqcap \gamma\} = (\xi \div h\gamma)^\sharp \sqcap h\gamma$,
(b) *If* $\mathrm{id}_X \sqsubseteq \xi$ *and* $\gamma\xi = \gamma$, *then* $k \sqsubseteq (\xi \div \gamma)^\sharp \sqcap \gamma$ *if and only if* $k\xi = \gamma$.

Proof.
(a)

$$\begin{aligned} h\{(\xi \div \gamma)^\sharp \sqcap \gamma\} &= h(\xi \div \gamma)^\sharp \sqcap h\gamma \\ &= \{(\xi \div \gamma)h^\sharp\}^\sharp \sqcap h\gamma \\ &= \{(\xi \div \gamma) \div h\}^\sharp \sqcap h\gamma \\ &= (\xi \div h\gamma)^\sharp \sqcap h\gamma. \end{aligned}$$

(b)

$$\begin{array}{llll} k \sqsubseteq (\xi \div \gamma)^\sharp \sqcap \gamma & \iff & k \sqsubseteq (\xi \div \gamma)^\sharp \text{ and } k \sqsubseteq \gamma & \\ & \iff & k^\sharp \gamma \sqsubseteq \xi \text{ and } k \sqsubseteq \gamma & \\ & \iff & \gamma \sqsubseteq k\xi \wedge k \sqsubseteq \gamma & \{ \ k \text{ is a function } \} \\ & \iff & \gamma = k\xi & \{ \ \mathrm{id}_X \sqsubseteq \xi \text{ and } \gamma\xi = \gamma \ \} \end{array}$$

□

Definition 4.1. A relation $\xi : X \rightharpoondown X$ is an *ordering* on X if $\mathrm{id}_X \sqsubseteq \xi$ (reflexive), $\xi\xi \sqsubseteq \xi$ (transitive) and $\xi \sqcap \xi^\sharp \sqsubseteq \mathrm{id}_X$ (antisymmetric). □

For two relations $\xi, \xi' : X \rightharpoondown X$ we define relations $\xi|\xi' : X \times X \rightharpoondown X$ and $\vee_0 : X \times X \rightharpoondown X$ by $\xi|\xi' = p\xi \sqcap q\xi' (= (\xi^\sharp \top \xi'^\sharp)^\sharp)$ and $\vee_0 = (\xi \div \xi|\xi)^\sharp \sqcap \xi|\xi$. Note that this definition was suggested by Dr. Wolfram Kahl, Universität der Bundeswehr München, when he visited to Kyushu University in August, 1997.

Note. The following may give concrete meanings of relations $\xi|\xi$ and $\vee_0$.

$$\begin{array}{ll} & x \le z \text{ and } y \le z \\ \iff & (x,z) \in \xi \text{ and } (y,z) \in \xi \\ \iff & ((x,y),z) \in p\xi \text{ and } ((x,y),z) \in q\xi \\ \iff & ((x,y),z) \in p\xi \sqcap q\xi = \xi|\xi. \end{array}$$

$$\begin{array}{ll} & \forall z' :: x \le z' \text{ and } y \le z' \Rightarrow z \le z' \\ \iff & \forall z' :: ((x,y),z') \in \xi|\xi \Rightarrow (z,z') \in \xi \\ \iff & (z,(x,y)) \in \xi \div \xi|\xi \\ \iff & ((x,y),z) \in (\xi \div \xi|\xi)^\sharp. \end{array}$$

□

It is clear that if ξ is antisymmetric then $\vee_0$ is univalent.

$$\begin{array}{lll} \vee_0^\sharp \vee_0 & \sqsubseteq & (\xi \div \xi|\xi)(\xi|\xi) \sqcap (\xi|\xi)^\sharp (\xi \div \xi|\xi)^\sharp \\ & \sqsubseteq & \xi \sqcap \xi^\sharp \\ & \sqsubseteq & \mathrm{id}_X \end{array}$$

As usual we say ξ has joins (least upper bounds) if $\vee_0 = (\xi \div \xi|\xi)^\sharp \sqcap \xi|\xi$ is total, and ξ has meets (greatest lower bounds) if $\wedge_0 = (\xi^\sharp \div \xi^\sharp|\xi^\sharp)^\sharp \sqcap \xi^\sharp|\xi^\sharp$ is total.

Theorem 4.1. *Let* $\xi : X \rightharpoondown X$ *be an ordering on* X, $\vee_0 = (\xi \div \xi|\xi)^\sharp \sqcap \xi|\xi$ *and* $\wedge_0 = (\xi^\sharp \div \xi^\sharp|\xi^\sharp)^\sharp \sqcap \xi^\sharp|\xi^\sharp$. *If* ξ *has least upper bounds and greatest lower bounds, then*

(a) $\vee_0 \xi = \xi|\xi$,

(b) $p^\sharp \vee_0 = \xi$ *and* $q^\sharp \vee_0 = \xi$,
(c) $t\vee_0 = \vee_0$ *and* $t\wedge_0 = \wedge_0$,
(d) $(p\top\vee_0)\wedge_0 = p$ *and* $(p\top\wedge_0)\vee_0 = p$,
(e) $(\vee_0 \times \mathrm{id}_X)\vee_0 = a(\mathrm{id}_X \times \vee_0)\vee_0$.

Proof. (a) By the transitivity $\xi\xi \sqsubseteq \xi$ of ξ we have $(\xi|\xi)\xi \sqsubseteq \xi|\xi$. Hence an equality $\vee_0\xi = \xi|\xi$ follows from the definition of $\vee_0$ and Lemma 4.1(b).
(b) It is trivial that $p^\sharp\vee_0 \sqsubseteq p^\sharp(\xi|\xi) \sqsubseteq p^\sharp p\xi \sqsubseteq \xi$. Recall that $\xi = p^\sharp(p\xi \sqcap q)$ by Lemma 2.3(a). So it suffices to show that $p\xi \sqcap q \sqsubseteq \vee_0$. First $p\xi \sqcap q \sqsubseteq \xi|\xi$ follows from $p\xi \sqcap q = p\xi \sqcap q(\xi \sqcap \xi^\sharp) \sqsubseteq p\xi \sqcap q\xi$. Now note that $p\xi \sqcap q \sqsubseteq (\xi \div \xi|\xi)^\sharp$ if and only if $(p\xi \sqcap q)^\sharp(\xi|\xi) \sqsubseteq \xi$. However, the latter condition follows from $(p\xi \sqcap q)^\sharp(\xi|\xi) = (p\xi \sqcap q)^\sharp(p\xi \sqcap q\xi) \sqsubseteq q^\sharp q\xi \sqsubseteq \xi$.
(c) First note that $t(\xi|\xi) = tp\xi \sqcap tq\xi = q\xi \sqcap p\xi = \xi|\xi$. By Lemma 4.1(a) we have

$$t\vee_0 = \{\xi \div t(\xi|\xi)\}^\sharp \sqcap t(\xi|\xi) = (\xi \div \xi|\xi)^\sharp \sqcap \xi|\xi = \vee_0.$$

(d) An inequality $p\xi^\sharp \sqsubseteq \vee_0\xi^\sharp$ follows from $p\xi^\sharp \sqsubseteq \vee_0\, \vee_0^\sharp\, p\xi^\sharp = \vee_0(p^\sharp\vee_0)^\sharp\xi^\sharp = \vee_0\xi^\sharp\xi^\sharp = \vee_0\xi^\sharp$ (since ξ is transitive). Then we have $(p\top\vee_0)(\xi^\sharp|\xi^\sharp) = (p\top\vee_0)p\xi^\sharp \sqcap (p\top\vee_0)q\xi^\sharp = p\xi^\sharp \sqcap \vee_0\xi^\sharp = p\xi^\sharp$, and so

$$\begin{aligned}
(p\top\vee_0)\wedge_0 &= (p\top\vee_0)\{(\xi^\sharp \div \xi^\sharp|\xi^\sharp)^\sharp \sqcap \xi^\sharp|\xi^\sharp\} \\
&= \{(\xi^\sharp \div (p\top\vee_0)(\xi^\sharp|\xi^\sharp))\}^\sharp \sqcap (p\top\vee_0)\xi^\sharp|\xi^\sharp \quad \{\text{ Lemma 4.1(a) }\} \\
&= (\xi^\sharp \div p\xi^\sharp)^\sharp \sqcap p\xi^\sharp.
\end{aligned}$$

Therefore Lemma 4.1(b) proves $p \sqsubseteq (p\top\vee_0)\wedge_0$, and so $p = (p\top\vee_0)\wedge_0$.
(e) Define two relations $\vee_1 : (X \times X) \times X \rightharpoondown X$ and $\vee_2 : X \times (X \times X) \rightharpoondown X$ by

$$\vee_1 = \{\xi \div (\xi|\xi)|\xi\}^\sharp \sqcap (\xi|\xi)|\xi \text{ and } \vee_2 = \{\xi \div \xi|(\xi|\xi)\}^\sharp \sqcap \xi|(\xi|\xi).$$

First we will prove that $(\vee_0 \times \mathrm{id}_X)\vee_0 = \vee_1$ and $(\mathrm{id}_X \times \vee_0)\vee_0 = \vee_2$, which follows from $(\vee_0 \times \mathrm{id}_X)\vee_0 \sqsubseteq \vee_1$ and $(\mathrm{id}_X \times \vee_0)\vee_0 \sqsubseteq \vee_2$, respectively, since $(\vee_0 \times \mathrm{id}_X)\vee_0$ and $(\mathrm{id}_X \times \vee_0)\vee_0$ are total functions, and $\vee_1$ and $\vee_2$ are partial functions. Hence, by Lemma 4.1(b) we have to see that $(\vee_0 \times \mathrm{id}_X)\,\vee_0\,\xi = (\xi|\xi)|\xi$ and $(\mathrm{id}_X \times \vee_0)\,\vee_0\,\xi = \xi|(\xi|\xi)$. But we have

$$\begin{aligned}
(\vee_0 \times \mathrm{id}_X)\,\vee_0\,\xi &= (\vee_0 \times \mathrm{id}_X)(\xi|\xi) \\
&= (\vee_0 \times \mathrm{id}_X)(p\xi \sqcap q\xi) \\
&= (\vee_0 \times \mathrm{id}_X)p\xi \sqcap (\vee_0 \times \mathrm{id}_X)q\xi \\
&= p_0(\xi|\xi) \sqcap q_0\xi \\
&= p_0\,\vee_0\,\xi \sqcap q_0\xi \\
&= (\xi|\xi)|\xi,
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{id}_X \times \vee_0)\,\vee_0\,\xi &= (\mathrm{id}_X \times \vee_0)(\xi|\xi) \\
&= (\mathrm{id}_X \times \vee_0)(p\xi \sqcap q\xi) \\
&= (\mathrm{id}_X \times \vee_0)p\xi \sqcap (\mathrm{id}_X \times \vee_0)q\xi \\
&= p_0\xi \sqcap q_0\,\vee_0\,\xi \\
&= \xi|(\xi|\xi).
\end{aligned}$$

This proves that $(\vee_0 \times \mathrm{id}_X)\vee_0 = \vee_1$ and $(\mathrm{id}_X \times \vee_0)\vee_0 = \vee_2$. Finally we have $(\vee_0 \times \mathrm{id}_X)\vee_0 = a(\mathrm{id}_X \times \vee_0)\vee_0$ from $a\{\xi|(\xi|\xi)\} = (\xi|\xi)\xi$ and

$$\begin{aligned} a(\mathrm{id}_X \times \vee_0)\vee_0 &= a\vee_2 \\ &= a\{\xi \div \xi|(\xi|\xi)\}^\sharp \sqcap a\{\xi|(\xi|\xi)\} \\ &= [\xi \div a\{\xi|(\xi|\xi)\}]^\sharp \sqcap a\{\xi|(\xi|\xi)\} \\ &= \{\xi \div (\xi|\xi)|\xi\}^\sharp \sqcap (\xi|\xi)|\xi \\ &= \vee_1 \\ &= (\vee_0 \times \mathrm{id}_X)\vee_0, \end{aligned}$$

which completes the proof. □

Theorem 4.2. *Let $(X, \vee, \wedge)$ be a lattice in a Dedekind category $\mathcal{D}$. If $\xi = p^\sharp(\vee \sqcap q)$ and $\vee_0 = (\xi \div \xi|\xi)^\sharp \sqcap \xi|\xi$, then $\vee = \vee_0$.*

Proof. Since $\vee$ is a function and $\vee_0$ is univalent, it suffices to show that $\vee \sqsubseteq \vee_0$. To see this we have to show that $\vee \sqsubseteq \xi|\xi$ and $\xi|\xi \sqsubseteq \vee\xi$ by Lemma 4.1(b). (Note that ξ is an ordering on X by the result in Section 2.) First $\vee = \vee \sqcap \vee \sqsubseteq pp^\sharp \vee \sqcap qq^\sharp \vee = p\xi \sqcap q\xi = \xi|\xi$. Noticing that $(p \times \mathrm{id}_X)q = q_0$ and $(q \times \mathrm{id}_X)q = q_0$ and $\mathrm{id}_X \times f = p_0p^\sharp \sqcap q_0fq^\sharp \sqsubseteq p_0p^\sharp$, it follows that

$$\begin{aligned} \xi|\xi &= pp^\sharp(\vee \sqcap q) \sqcap qp^\sharp(\vee \sqcap q) \\ &= p_0^\sharp(p \times \mathrm{id}_X)(\vee \sqcap q) \sqcap p_0^\sharp(q \times \mathrm{id}_X)(\vee \sqcap q) \\ &\quad \{ \text{ Lemma 2.2 } \} \\ &= p_0^\sharp\{(p \times \mathrm{id}_X)(\vee \sqcap q) \sqcap (q \times \mathrm{id}_X)(\vee \sqcap q)\} \\ &\quad \{ \text{ Lemma 2.3(b) } \} \\ &= p_0^\sharp\{(p \times \mathrm{id}_X) \vee \sqcap (q \times \mathrm{id}_X) \vee \sqcap q_0\} \\ &= p_0^\sharp\{a(\mathrm{id}_X \times q) \vee \sqcap a(\mathrm{id}_X \times \vee)q \sqcap a(\mathrm{id}_X \times q)q\} \\ &= p_0^\sharp a\{(\mathrm{id}_X \times q) \vee \sqcap (\mathrm{id}_X \times \vee)q \sqcap (\mathrm{id}_X \times q)q\} \\ &= p_0^\sharp a\{(\mathrm{id}_X \times q) \vee \sqcap ((\mathrm{id}_X \times \vee) \sqcap (\mathrm{id}_X \times q))q\} \\ &\quad \{ \text{ Lemma 2.3(c) } \} \\ &\sqsubseteq p_0^\sharp a\{(\mathrm{id}_X \times \vee) \sqcap (\mathrm{id}_X \times q)\}\{((\mathrm{id}_X \times \vee) \sqcap (\mathrm{id}_X \times q))^\sharp(\mathrm{id}_X \times q) \vee \sqcap q\} \\ &\sqsubseteq p_0^\sharp a(\mathrm{id}_X \times \vee)\vee \\ &= p_0^\sharp(\vee \times \mathrm{id}_X)\vee \\ &\quad \{ \text{ (L1}\vee\text{) } \} \\ &= \vee p^\sharp \vee \\ &\quad \{ \text{ Lemma 2.2 } \} \\ &= \vee\xi. \end{aligned}$$

□

Note. The following four diagrams may help to understand the proof of the last theorem.

$$\begin{array}{ccc} (X \times X) \times X & \xrightarrow{f \times id_X} & X \times X \\ {\scriptstyle p_0}\downarrow & (PB) & \downarrow{\scriptstyle p} \\ X \times X & \xrightarrow[f]{} & X \end{array} \qquad \begin{array}{ccc} (X \times X) \times X & \xrightarrow{p \times id_X} & X \times X \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle id_{X \times X}} \\ X \times (X \times X) & \xrightarrow[id_X \times q]{} & X \times X \end{array}$$

$$\begin{array}{ccc} X \times (X \times X) & \xrightarrow{id_X \times g} & X \times X \\ {\scriptstyle q_1}\downarrow & (PB) & \downarrow{\scriptstyle q} \\ X \times X & \xrightarrow[g]{} & X \end{array} \qquad \begin{array}{ccc} (X \times X) \times X & \xrightarrow{q \times id_X} & X \times X \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle id_{X \times X}} \\ X \times (X \times X) & \xrightarrow[q_1]{} & X \times X \end{array}$$

$\square$

Acknowledgment

The author is grateful to Wolfram Kahl, Universität der Budeswehr München, and unknown referee for reading and commenting to this paper.

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