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Numerical verification method for solutions of the perturbed Gelfand equation

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Abstract

A numerical verification method for radially symmetric solutions of the perturbed Gelfand equation is presented for the case in which this equation possesses turning points. We use Nakao's method with local uniqueness to enclose the continua of solutions and a bordering algorithm in order to treat a turning point. We describe verification procedures in detail and give a numerical example.

Keywords: Numerical verification; Perturbed Gelfand equation; Fixed point theorem; Implicit function theorem

AMS subject classifications: 65N15, 65N30

1 Introduction

We consider radially symmetric solutions of the perturbed Gelfand equation

$$\begin{cases} -\triangle u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = \{x \in R^n | |x| < 1\}(n \geq 3)$, $f(u) = \exp(u/(1+\varepsilon u))$, $\lambda \in R$, and $\varepsilon \in R^+$. This equation arises in the theory of combustion and was proposed by D.A. Frank-Kamenetskii [4]. For some fixed ε, the bifurcation diagram possesses turning points, the first of which corresponding to an explosive point. This explosive point is often denoted by λ_{FK} [2, 4]. The equation (1.1) has been discussed by several authors [3, 9, 15, 16], but a numerically verified value of λ_{FK} has not yet been obtained.

In this paper, we propose a numerical verification method for the existence and enclosure of solution curves for (1.1). If this method can be made to succeed near

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the first turning point, we can obtain a value for λ_{FK}. Briefly stated, our method consists of a combination of Nakao's method, (more precisely, it's extension to local uniqueness[19]) with linear interpolation and the implicit function theorem. By adjoining to (1.1) a suitably chosen equation characterized by a new independent parameter μ, we can produce an equation which possesses no turning points, at least locally. This is carried out by applying a "bordering algorithm" [6]. We combine the bordering algorithm with the existence and inclusion method mentioned above to obtain the desired results near turning points.

2 Change of parameters

A radially symmetric solution u of (1.1) is a function of $r = |x|$. Assuming u to be such a function, (1.1) is reduced to the ordinary differential equation

$$\begin{cases} -u_{rr} - \frac{n-1}{r}u_r - \lambda f(u) = 0 \quad \text{in } J \equiv (0,1), \\ u_r(0) = u(1) = 0. \end{cases} \tag{2.1}$$

Moreover, (2.1) can be transformed into the following integral equation:

$$u(r) = \frac{\lambda}{n-2}\int_r^1 (1 - s^{n-2})sf(u)ds + \frac{\lambda}{n-2}\int_0^r (\frac{1}{r^{n-2}} - 1)s^{n-1}f(u)ds. \tag{2.2}$$

Now, we define the operators L_1 and F_1 by

$$L_1(u,\lambda) = u, \tag{2.3}$$

and

$$F_1(u,\lambda) = \frac{\lambda}{n-2}\int_r^1 (1 - s^{n-2})sf(u)ds + \frac{\lambda}{n-2}\int_0^r (\frac{1}{r^{n-2}} - 1)s^{n-1}f(u)ds. \tag{2.4}$$

Then (2.2) can be written as

$$L_1(u,\lambda) = F_1(u,\lambda). \tag{2.5}$$

Next, we define the subset $\mathsf{R}(L_1 - F_1) \subset C[0,1] \times R$ by

$$\mathsf{R}(L_1 - F_1) \equiv \{(u,\lambda) \in C[0,1] \times R \mid D(L_1 - F_1)(u,\lambda) \text{ is onto}\}, \tag{2.6}$$

where $D(L_1 - F_1)(u,\lambda)$ represents the Fréchet derivative of $(L_1 - F_1)$ at (u,λ).

Consider some fixed $\mu \in R$ and $r_{i_c} \in J$. We define $G_1 : \mathsf{R}(L_1 - F_1) \to C[0,1] \times R$ by

$$G_1(u,\lambda) \equiv ((L_1 - F_1)(u,\lambda), u(r_{i_c}) - \mu) \tag{2.7}$$

for $(u,\lambda) \in \mathsf{R}(L_1 - F_1)$. Then we have

$$DG_1(u,\lambda)(\psi,\gamma) = (D_u(L_1 - F_1)(u,\lambda)\psi + \gamma D_\lambda(L_1 - F_1)(u,\lambda), \psi(r_{i_c})) \tag{2.8}$$

for $\gamma \in R$ and $\psi \in C[0,1]$, where D_u and D_λ denote partial derivatives with respect to u and λ, respectively.

We need a result similar to that obtained in [17]. The proof of the following lemma can be carried out by the argument appearing there.

Lemma 1. Let $(u,\lambda) \in \mathsf{R}(L_1 - F_1)$ and let (ψ_0, γ_0) be a basis of $KerD(L_1 - F_1)(u,\lambda)$. Then with $r_{i_c} \in J$ such that $\psi_0(r_{i_c}) \neq 0$,

$$DG_1(u,\lambda) : C[0,1] \times R \to C[0,1] \times R$$

is bijection for any $\mu \in R$.

To make a change of parameters, we define the operator $H_1 : R \times C[0,1] \times R \to C[0,1] \times R$ by

$$H_1(\mu, u, \lambda) \equiv ((L_1 - F_1)(u,\lambda), u(r_{i_c}) - \mu), \tag{2.9}$$

where r_{i_c} is taken so that

$$D_{(u,\lambda)} H_1(\mu, u, \lambda) \equiv DG_1(u,\lambda) \tag{2.10}$$

is bijection.

From (2.10) and the implicit function theorem ([20]), for any $(u,\lambda) \in \mathsf{M}_0$, there exist an $\varepsilon_0 > 0$ and a unique Fréchet differentiable map

$$(u(r_{i_c}) - \varepsilon_0, u(r_{i_c}) + \varepsilon_0) \ni \mu \mapsto (u(\mu), \lambda(\mu)) \in \mathsf{M}_0$$

such that $(u,\lambda) = (u(\mu_0), \lambda(\mu_0))$, where

$$\mathsf{M}_0 \equiv \{(u,\lambda) \in \mathsf{R}(L_1 - F_1) | (L_1 - F_1)(u,\lambda) = 0\} \text{ and } \mu_0 \equiv u(r_{i_c}).$$

Also, for any $\mu \in (u(r_{i_c}) - \varepsilon_0, u(r_{i_c}) + \varepsilon_0)$, the relation $H_1(\mu, u(\mu), \lambda(\mu)) = (0,0)$ holds. That is,

$$(L_1 - F_1)(u(\mu), \lambda(\mu)) = 0 \text{ and } u(\mu)(r_{i_c}) = \mu. \tag{2.11}$$

Below, we present a method for choosing $r_{i_c} \in J$. For this purpose, we give the following lemma.

Lemma 2 ([11]). For any $(u,\lambda) \in \mathsf{M}_0$, the tangent space $T_{(u,\lambda)}\mathsf{M}_0$ is identical to the null-space $KerD(L_1 - F_1)(u,\lambda)$.

Now, let $\Delta : r_0 < r_1 < \cdots < r_{M-1} < r_M = 1$ be a uniform partition of J into subintervals $[r_j, r_{j+1}]$ of length $h = r_{j+1} - r_j (j = 0, \cdots, M-1)$, and let $S_h \subset C[0,1]$ be a finite dimensional space depending on h which has the hat functions $\phi_j (j = 0, 1, \cdots, M-1)$ as a basis.

Then, we consider the following approximations of $\mathsf{R}(L_1 - F_1)$ and M_0:

$$\mathsf{R}_h(L_1 - F_1) = \{(u_h, \lambda_h) \in S_h \times R | D\Pi_{h0}(L_1 - F_1)(u_h, \lambda_h) \text{ is onto}\}$$

and

$$\mathsf{M}_{0h} = \{(u_h, \lambda_h) \in \mathsf{R}_h(L_1 - F_1) | \Pi_{h0}(L_1 - F_1)(u_h, \lambda_h) = 0\}.$$

Here $\Pi_{0h} : C[0,1] \to S_h$ is an interpolation operator defined by

$$\phi(r_j) = \Pi_{h0}\phi(r_j) \qquad j = 0, 1, \cdots, M-1.$$

Based on Lemma 2, we will compute the tangent vector $t_h = (\psi_{0h}(r_0), \cdots, \psi_{0h}(r_{M-1}), \gamma_h) \in T_{(u_h,\lambda_h)}\mathsf{M}_{0h}$ for $\psi_{0h} \in S_h$ and choose the continuation index i_c so that $|\psi_{0h}(r_{i_c})| = \|t_h\|_\infty$. Then, choosing $h > 0$ sufficiently small, $|\psi_{0h}(r_{i_c})|$ can be made arbitrarily close to $\|\psi_0(r_{i_c})\|$, and $\|\psi_0(r_{i_c})\| \neq 0$. Therefore , we may expect $DG_1(u, \lambda)$ is bijection for such r_{i_c} by Lemma 1.

3 Enclosure of solutions with local uniqueness for a fixed parameter λ

The arguments outlined in this section are very similar to those in [19]. We include this outline to make the present paper self-contained.

We define the operator $\Pi_h : C[0,1] \times R \to S_h \times R$ by

$$\Pi_h(u, \lambda) \equiv (\Pi_{h0}u, \lambda) \qquad \text{for } (u, \lambda) \in C[0,1] \times R.$$

Then $(u_h, \lambda_h) \in S_h \times R$ be an approximate solution of (2.11). Denoting (u, λ) as $u = u_h + \tilde{u}$ and $\lambda = \lambda_h + \tilde{\lambda}$ for $(\tilde{u}, \tilde{\lambda}) \in C[0,1] \times R$, (2.11) is reduced to the following fixed point form:

$$\begin{aligned}(\tilde{u}, \tilde{\lambda}) &= (F_1(u_h + \tilde{u}, \lambda_h + \tilde{\lambda}) - u_h, \tilde{\lambda} - (u_h + \tilde{u})(r_{i_c}) + \mu) \\ &\equiv F(\tilde{u}, \tilde{\lambda}).\end{aligned} \tag{3.1}$$

This equation can also be written as

$$\begin{cases} \Pi_h(\tilde{w}) &= \Pi_h F(\tilde{w}) \\ (I - \Pi_h)(\tilde{w}) &= (I - \Pi_h)F(\tilde{w}), \end{cases} \tag{3.2}$$

where $\tilde{w} = (\tilde{u}, \tilde{\lambda})$, and I represents the identity map on $C[0,1] \times R$.

Assumption 1. Suppose that restriction to $S_h \times R$ of the operator $\Pi_h[I - A']$: $C[0,1] \times R \to S_h \times R$ has the inverse

$$[I_h - A'_h]^{-1} : S_h \times R \to S_h \times R,$$

where A' is an operator close to $DF(0)$, $I_h = \Pi_h I$ and A'_h is a linear operator on $S_h \times R$, an approximation to $\Pi_h A'$.

We apply a Newton-like method to the first equation in (3.2). That is, we introduce the operator as follows:

$$\Pi_h N \equiv \Pi_h - [I_h - A'_h]^{-1}\Pi_h(I - F). \tag{3.3}$$

Then we obtain

$$\widetilde{w} = T(\widetilde{w}). \tag{3.4}$$

Here T is the operator on $C[0,1] \times R$ defined by $T \equiv \Pi_h N + (I - \Pi_h)F$. It is easy to see that $\widetilde{w} = T(\widetilde{w})$ and $\widetilde{w} = F(\widetilde{w})$ are equivalent.

We now expand the operator T at $T(0)$ and describe the verification conditions with local uniqueness by using Banach's fixed point theorem. Setting $\widetilde{w} = (\widetilde{u}, \widetilde{\lambda})$, we write $\widetilde{u_h} = \sum_{j=0}^{M-1} \widetilde{u_j}\phi_j$ and define

$$\begin{aligned} (\widetilde{w})_i &\equiv |\widetilde{u_i}| \qquad (i = 0, 1, \cdots, M-1), \\ (\widetilde{w})_M &\equiv |\widetilde{\lambda}|, \\ (\widetilde{w})_{M+1} &\equiv \|(I - \Pi_{h0})(\widetilde{u})\|_{C[0,1]}, \end{aligned} \tag{3.5}$$

where $\|v\|_{C[0,1]} \equiv \max_{r \in J} |v(r)|,$ for $v \in C[0,1]$.

Choosing a positive vector $(\widetilde{W}_0, \widetilde{W}_1, \cdots, \widetilde{W}_M, \widetilde{W}_{M+1}) \in R^{M+2}$, we define the set $\widetilde{W}$ by

$$\widetilde{W} \equiv \{\widetilde{w} \in C[0,1] \times R \mid (\widetilde{w})_i \leq \widetilde{W}_i, i = 0, 1, \cdots, M+1\}, \tag{3.6}$$

and the corresponding norm by

$$\|w\|_{\widetilde{W}} \equiv \max_{0 \leq i \leq M+1} \frac{(\widetilde{w})_i}{\widetilde{W}_i}.$$

Next, we choose the vectors

$$(\widetilde{Y}_0, \cdots, \widetilde{Y}_{M+1}) \in R^{M+1}, \widetilde{Y}_i > 0 (i = 0, \cdots, M+1),$$
$$(\widetilde{Z}_0, \cdots, \widetilde{Z}_{M+1}) \in R^{M+1}, \widetilde{Z}_i > 0 (i = 0, \cdots, M+1)$$

such that

$$\begin{aligned} (T(0))_i &\leq \widetilde{Y}_i \qquad (i = 0, \cdots, M+1), \\ (DT(\widetilde{w_1})\widetilde{w_2})_i &\leq \widetilde{Z}_i \qquad (i = 0, \cdots, M+1) \qquad \widetilde{w_1}, \widetilde{w_2} \in \widetilde{W}, \end{aligned}$$

and we define the set $\widetilde{K}$ in $C[0,1] \times R$ by

$$\widetilde{K} \equiv \{v \in C[0,1] \times R | (v)_i \leq \widetilde{Y}_i + \widetilde{Z}_i, i = 0, \cdots, M+1\}.$$

Then the verification condition is described as follows:

Theorem 1 (Local Uniqueness). If $\widetilde{K} \subset \widetilde{W}$ holds for $\widetilde{W}$ (that is, if $Y_i + Z_i \leq \widetilde{W_i}$), then there exists a solution to

$$\widetilde{w} = T(\widetilde{w})$$

in $\widetilde{K}$ and it is unique within the set $\widetilde{W}$.

4 Enclosing continua of solutions $(u^{(\mu)}, \lambda^{(\mu)})_{\mu \in e_0}$ for a small interval e_0

4.1 Verification condition

We proceed to extend the results of Section 3 for continua of solutions $\widetilde{w}_{(\mu)} := (\widetilde{u}^{(\mu)}, \widetilde{\lambda}^{(\mu)})_{\mu \in e_0}$ depending smoothly on μ. Here e_0 denotes a small real interval, and we set $e_0 = [\mu_0, \mu_1]$.

Let (u_h^0, λ_h^0) and (u_h^1, λ_h^1) be the approximate solutions of (2.11) corresponding to μ_0 and μ_1, respectively. We define approximate solutions for all $\mu \in e_0$ as follows:

$$u_h^{(\mu)} = \frac{\mu_1 - \mu}{\mu_1 - \mu_0} u_h^0 + \frac{\mu - \mu_0}{\mu_1 - \mu_0} u_h^1, \qquad \lambda_h^{(\mu)} = \frac{\mu_1 - \mu}{\mu_1 - \mu_0} \lambda_h^0 + \frac{\mu - \mu_0}{\mu_1 - \mu_0} \lambda_h^1. \tag{4.1}$$

Then we define the mappings F and T for $(\mu, u_h^{(\mu)}, \lambda_h^{(\mu)}) \in e_0 \times S_h \times R$ as in Section 3, and denote these as $F_{(\mu)}$ and $T_{(\mu)}$. Moreover, we use $\widetilde{W}^{(\mu)}$ for $\mu \in e_0$ instead of $\widetilde{W}$, and define

$$W = \bigcup_{\mu \in e_0} \widetilde{W}^{(\mu)}.$$

Choosing the vectors $(Y_0, \cdots, Y_{M+1}) \in R^{M+2}$ and $(Z_0, \cdots, Z_{M+1}) \in R^{M+2}$ such that

$$(\bigcup_{\mu \in e_0} T_{(\mu)}(0))_i \leq Y_i,$$

and

$$(\bigcup_{\mu \in e_0} DT_{(\mu)}(w_1)w_2)_i \leq Z_i, \qquad w_1, w_2 \in W,$$

and defining the set K by

$$K \equiv \{v \in C[0,1] \times R | (v)_i \leq Y_i + Z_i, i = 0, 1, \cdots, M+1\},$$

the following theorem holds.

Theorem 2 (Uniqueness for small interval). If $K \subset W$, then there exists a solution to

$$\widetilde{w}_{(\mu)} = T_{(\mu)}(\widetilde{w}_{(\mu)})$$

in K for all $\mu \in e_0$, and it is unique within the set W.

4.2 Verification Procedures by Computer

Defining the set W, we estimate K in Theorem 2 by computer. For the set K, we must estimate $T_{(\mu)}(0)$ and $DT_{(\mu)}(W)W$, where $DT_{(\mu)}(W)W = \sup_{w_1 \in W} \sup_{w_2 \in W} (DT_{(\mu)}(w_1)w_2)$. First we note

$$T_{(\mu)}(0) = [I_h - A'_{h,\bar{\mu}}]^{-1} \Pi_h F_{(\mu)}(0) + (I - \Pi_h) F_{(\mu)}(0), \tag{4.2}$$

where $A'_{h,\bar{\mu}}$ denotes a linear operator on $S_h \times R$, an approximation of $\Pi_h DF_{(\bar{\mu})}(0)$ for some fixed $\bar{\mu} \in e_0$.

In order to estimate the finite part of $T_{(\mu)}(0)$, we compute the interval vector (Y_i) such that

$$[I_h - A'_{h,\bar{\mu}}]^{-1}\Pi_h F_{(\mu)}(0) \subset (\sum_{j=0}^{M-1} Y_j\phi_j, Y_M). \tag{4.3}$$

To obtain an upper bound for the infinite part, we use an error estimation for the linear interpolation:

Proposition 1 (Error estimation for linear interpolation [14]).
The relation

$$\|(I - \Pi_{h0})v\|_\infty \leq \frac{h^2}{8}\|\frac{d^2 v}{dr^2}\|_\infty$$

holds for all $v \in C[0,1] \cup C^{2,\infty}(0,1)$, where $C^{2,\infty}(0,1) := \{v \in C^2(0,1) | \|\frac{d^2 v}{dr^2}\|_\infty < \infty\}$.

From the above proposition we have

$$\|(I - \Pi_h)F_{(\mu)}(0)\|_{C[0,1]\times R} \leq \frac{h^2}{8}\|\frac{d^2}{dr^2}F_{(\mu)}(0)\|_{C[0,1]} =: Y_{M+1}.$$

Similarly, noting that

$$\begin{aligned} DT_{(\mu)}(W)W &= [I_h - A'_{h,\bar{\mu}}]^{-1}\Pi_h(DF_{(\mu)}(W)W - A'_{h,\bar{\mu}}\Pi_h W) \\ &+ (I - \Pi_h)DF_{(\mu)}(W)W, \end{aligned}$$

we can estimate $DT_{(\mu)}$ in the same manner as in the case of Y_i.

We now describe how to obtain a set $W = (W_0, W_1, \cdots, W_{M+1})$.

First, we take the initial value $W_{i,0}$ of W_i as

$$W_{i,0} = Y_i \qquad i = 0, 1, \cdots, M+1,$$

and apply the following procedure. Here the finite part of $DT_{(\mu)}(W)W$ is denoted by Z_i, and the infinite part by Z_{M+1}.

(i) Compute $Z_i (i = 0, 1, \cdots, M+1)$.

(ii) Check the conditions

$$Y_i + Z_i \leq W_{i,k} \qquad (i = 0, 1, \cdots, M+1),$$

where k stands for the iteration number. If the above conditions are satisfied, then stop. This means that verification is completed.

(iii) Otherwise, take

$$W_{i,k} = (1+\delta)(Y_i + Z_i) \qquad (i = 0, 1, \cdots, M+1),$$

for a certain positive number δ and return to (i).

4.3 Smoothness for a continua of solutions

We define the mapping G:int$(e_0)\times C[0,1]\times R \to C[0,1]\times R$ as

$$G(\mu,\widetilde{w}_{(\mu)}) := (I - F_{(\mu)})(\widetilde{u},\widetilde{\lambda}),$$

and consider the following equation with respect to (ψ_0,γ_0):

$$D_{(\widetilde{u},\widetilde{\lambda})}G(\mu,\widetilde{w}_{(\mu)})(\psi_0,\gamma_0) = (0,1). \tag{4.4}$$

Next we apply the verification method presented in the previous section to (4.4). That is, first, transforming (4.4) into the fixed-point form and using a Newton-Like method, (4.4) can be written as $(\psi_0,\gamma_0) = \widehat{T}_{(\mu,w)}(\psi_0,\gamma_0)$, where $\widehat{T}$ is defined in the same way as T in §3. Second, for this equation we choose a set Φ like as W in §3. Considering the vectors $\widehat{Y}$ and $\widehat{Z} \in R^{M+2}$ such that $(\bigcup_{\mu\in e_0}\bigcup_{w\in W}\widehat{T}_{(\mu,w)}(0))_i \le \widehat{Y}_i$ and $(\bigcup_{\mu\in e_0}\bigcup_{w\in W} D\widehat{T}_{(\mu,w)}(\phi_1)\phi_2)_i \le \widehat{Z}_i, \phi_1,\phi_2 \in \Phi$, we define $\widehat{K}$ by

$$\widehat{K} := \{v \in C[0,1]\times R | (v)_i \le \widehat{Y}_i + \widehat{Z}_i, i = 0,1,\cdots,M+1\}.$$

Then we have the following:

Lemma 3. If $\widehat{K} \subset \Phi$ holds for Φ, then the mapping $DG_{(\widetilde{u},\widetilde{\lambda})}(\mu,\widetilde{w}_{(\mu)})$ is a bijection on e_0.

(*Proof*)

If the assumption holds, there exists a unique solution to (4.4) in $\widehat{K}$ for all $\mu \in e_0$. From this fact, it follows that the equation

$$D_{(\widetilde{u},\widetilde{\lambda})}G(\mu,\widetilde{w}_{(\mu)})(\psi_1,\gamma_1) = (0,0)$$

has no nontrivial solution. Since the linear mapping $D_{(\widetilde{u},\widetilde{\lambda})}G(\mu,\widetilde{w}_{(\mu)})$ is a Fredholm operator with index 0 from the compactness of $D_{(\widetilde{u},\widetilde{\lambda})}F_{(\mu)}(\widetilde{u},\widetilde{\lambda})$, $D_{(\widetilde{u},\widetilde{\lambda})}G(\mu,\widetilde{w}_{(\mu)})$ is onto according to the Fredholm alternative([5]).

□

The following theorem is a modified version of a similar result obtained in [10].

Theorem 3 (The smoothness of solutions). If $K \overset{\circ}{\subset} W$ and $\widehat{K} \subset \Phi$, then there exists a Fréchet differentiable mapping **:**

$$int(e_0) \ni \mu \longmapsto \widetilde{w}_{(\mu)} \in W$$

such that $G(\mu,\widetilde{w}_{(\mu)}) = 0$ on $\mu \in int(e_0)$.

(*Proof*)

From Lemma 3, for some arbitrarily chosen $(\bar{\mu}, \widetilde{w}_{(\bar{\mu})}) \in int(e_0) \times W$, the implicit function theorem gives an open interval $V \subset int(e_0)$ containing $\bar{\mu}$, and a Fréchet differentiable mapping

$$V \ni \mu \longmapsto \widehat{w}_{(\mu)} \in C[0,1] \times R \tag{4.5}$$

such that $G(\mu, \widehat{w}_{(\mu)}) = 0$ for $\mu \in V$ and $\widehat{w}_{(\bar{\mu})} = \widetilde{w}_{(\bar{\mu})}$. Without loss of generality, we can assume V is maximal with property (4.5).

Let $\widehat{V}$ denote the maximal subinterval of V containing $\bar{\mu}$ such that

$$\widehat{w}_{(\mu)} \in W \qquad \text{for all } \mu \in \widehat{V}, \tag{4.6}$$

that is, we take a mapping : $\widehat{V} \in \mu \mapsto \widehat{w}_{(\mu)} \in W$. If we prove $\widehat{V} = int(e_0)$, then we can conclude that $\widehat{w}_{(\mu)} = \widetilde{w}_{(\mu)}$ for all $\mu \in int(e_0)$.

Assuming that $\widehat{V} \neq int(e_0)$, there exists some $\mu^* \in int(e_0) \cap \partial\widehat{V}$. Applying the implicit function theorem at $(\mu^*, \widetilde{w}_{(\mu^*)})$, we can obtain an open interval $U \subset int(e_0)$ containing μ^* and a Fréchet differentiable mapping,

$$U \ni \mu \longmapsto \bar{w}_{(\mu)} \in C[0,1] \times R, \tag{4.7}$$

such that $G(\mu, \bar{w}_{(\mu)}) = 0$ for $\mu \in U$ and $\bar{w}_{(\mu^*)} = \widetilde{w}_{(\mu^*)}$.

From the fact that $\bar{w}_{(\mu^*)} \in T_{(\mu^*)}W \overset{\circ}{\subset} W$, there exists some open set W_ε such that

$$T_{(\mu^*)}W \overset{\circ}{\subset} T_{(\mu^*)}W + W_\varepsilon \subset W. \tag{4.8}$$

Also, the continuity of $\bar{w}_{(\mu)}$ with respect to $\mu \in U$ and (4.8) give the relation

$$\bar{w}_{(\mu)} \in T_{(\mu)}W + W_\varepsilon \subset W \qquad \text{for all } \mu \in U. \tag{4.9}$$

From (4.6), (4.7) and (4.9), we obtain

$$\widehat{w}_{(\mu)} = \bar{w}_{(\mu)} \qquad \text{for all } \mu \in \widehat{V} \cap U. \tag{4.10}$$

Then, since U is an open interval containing $\mu^* \in \partial\widehat{V}$, it follows that $\widehat{V} \overset{\circ}{\subset} \widehat{V} \cup U$.

By (4.10), $\widehat{w}_{(\mu)}$ can be extended to $\widehat{V} \cup U$ by $\bar{w}_{(\mu)}$, while $\widehat{V} \cup U \subset V$ holds due to the maximality of V. This implies the relations $\widehat{V} \overset{\circ}{\subset} \widehat{V} \cup U \subset V$, but this contradicts the maximality of $\widehat{V}$.

□

5 Verification for a large interval of parameters

In this section, we assume that the verification process on each small interval $e_i = [\mu_i, \mu_{i+1}] (i = 0, 1, \cdots, M-1)$ has succeeded.

Theorem 4. Let W_i be an enclosure set of solutions for parameters $\mu \in e_i$. If

$$K_i \subset W_{i+1}$$

or

$$K_{i+1} \subset W_i,$$

then there exist smooth solutions in $W_i \cup W_{i+1}$. Here, K_i is a set in which there exist solutions for parameters $\mu \in e_i$(see Fig.1).

(*Proof*)

Let $K_i \subset W_{i+1}$, and denote the solutions for e_i and e_{i+1} as $\{w_{(\mu),i}\}_{\mu\in e_i} \subset K_i$ and $\{w_{(\mu),i+1}\}_{\mu\in e_{i+1}} \subset W_{i+1}$, respectively. By the fact that verification has succeed for e_i and e_{i+1}, there exist unique $\{w_{(\mu_{i+1}),i}\} \in K_i$ and $\{w_{(\mu_{i+1}),i+1}\} \in W_{i+1}$ which are solutions on the contact point μ_{i+1} between e_i and e_{i+1}. Then, from the assumption $K_i \subset W_{i+1}$ and the local uniqueness,

$$w_{(\mu_{i+1}),i} = w_{(\mu_{i+1}),i+1} \in K_i \subset W_{i+1}$$

holds. This implies that $\{w_{(\mu),i}\}_{\mu\in e_i} \cup \{w_{(\mu),i+1}\}_{\mu\in e_{i+1}}$ is continuous at μ_{i+1}. The remaining problem is to prove that $\{w_{(\mu),i}\}_{\mu\in e_i} \cup \{w_{(\mu),i+1}\}_{\mu\in e_{i+1}}$ is smooth at μ_{i+1}.

From Lemma 3, the implicit function theorem at $(\mu_{i+1}, w_{(\mu_{i+1}),i})$ provides an open interval $V \subset int(e_i \cup e_{i+1})$ containing μ_{i+1} and a Fréchet differentiable mapping,

$$V \ni \mu \longmapsto w_{(\mu)} \in C[0,1] \times R, \tag{5.1}$$

such that $G(\mu, w_{(\mu)}) = 0$ (i.e. $w_{(\mu)} = F(w_{(\mu)})$). Thus $w_{(\mu)} \in K_i \subset W_{i+1}$ holds on $\mu \in V$ sufficiently close to μ_{i+1}. Therefore, from the uniqueness of $w_{(\mu)}, w_{(\mu),i}$ and $w_{(\mu),i+1}$, and the property (5.1), the curve $\{w_{(\mu),i}\}_{\mu\in e_i} \cup \{w_{(\mu),i+1}\}_{\mu\in e_{i+1}}$ is smooth on μ_{i+1}.

This argument can be applied to the case of $K_{i+1} \subset W_i$ in a similar manner.

□

Finally, we state a lemma that ensures the enclosure of turning points.

Lemma 4 (Bound for turning points). Let $\mu_i, \varepsilon_i \in R (i = 1, 2, 3)$, $\mu_1 < \mu_2 < \mu_3$, and $I_i = [\lambda_h(\mu_i) - \varepsilon_i, \lambda_h(\mu_i) + \varepsilon_i]$. If $I_1 \cap I_2 = \phi, I_2 \cap I_3 = \phi$, and $I_1 \cap I_3 \neq \phi$ hold, then, there exists a turning point in $[\mu_1, \mu_3]$ (see Fig.2).

(*Proof*)

We assume that there is no turning point for any $\mu' \in (\mu_1, \mu_3)$. Without loss of generality, we may assume that $\lambda(\mu_1) \leq \lambda(\mu_2) \leq \lambda(\mu_3)$. From the fact that the verification process has succeeded, it follows that $\lambda(\mu_i) \in I_i$. Then, because $I_1 \cap I_2 = \phi$ and $\lambda(\mu_1) \leq \lambda(\mu_2)$,

$$\lambda_h(\mu_1) + \varepsilon_1 < \lambda_h(\mu_2) - \varepsilon_2.$$

Similarly, we can obtain

$$\lambda_h(\mu_2) + \varepsilon_2 < \lambda_h(\mu_3) - \varepsilon_3.$$

Therefore, we find that $\lambda_h(\mu_1) + \varepsilon_1 < \lambda_h(\mu_3) - \varepsilon_3$. However, this is a contradiction of the fact that $I_1 \cap I_3 \neq \phi$.

□

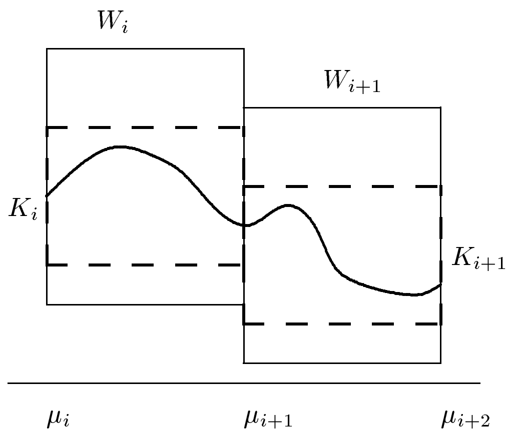

Figure 1

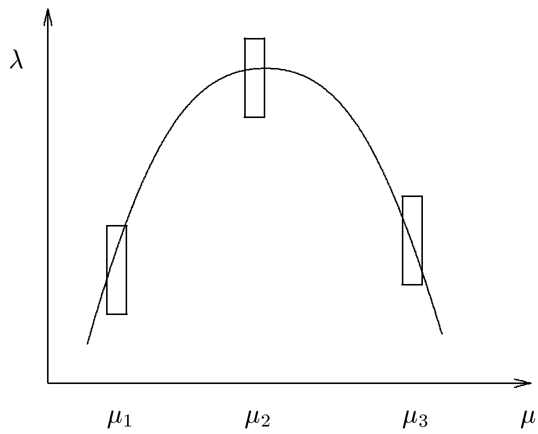

Figure 2

6 Numerical Examples

We now provide a numerical example (Table 1). Here we use $W_h := \max_{i=0,\cdots,M-1} W_{i+1}$, $K_h := \max_{i=0,\cdots,M-1}(Y_{i+1} + Z_{i+1})$, and the parameter ε is fixed $\varepsilon = 0.05$. In these computations, we used the interval library PROFIL [7], which supports the interval linear system solvers proposed by Rump [12].

Here, W_M stands for the bound $|\lambda - \lambda_h|$. Choosing $\mu_1 = 1.32, \mu_2 = 1.82$ and $\mu_3 = 2.413$, the assumptions in Lemma 4 are satisfied. This means that a certain turning point exists in $[\mu_1, \mu_3]$. From Table 1, we conclude the value 3.56341 is an upper bound of the first turning point. On the other hand, from our numerical verification without a bordering equation, it is found that there is no turning point in $\lambda = [0, 3.4]$. From this fact, Table 1 and Figure 3, it is apparent that the first turning point (the explosive point) is contained in the interval $[3.4, 3.56341]$.

μ	$\|u_h\|_\infty$	λ_h	K_h	W_h	K_M	W_M	K_{M+1}	W_{M+1}
1.32	1.32	3.39745	0.01633	0.01782	0.03057	0.03268	4.749E-6	5.022E-6
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.818	1.818	3.5267413	0.02106	0.02275	0.03482	0.03662	8.044E-6	8.281E-6
1.819	1.819	3.5267427	0.02108	0.02278	0.03484	0.03664	8.051E-6	8.288E-6
1.82	1.82	3.5267433	0.0211	0.0228	0.03486	0.03666	8.058E-6	8.296E-6
1.821	1.821	3.5267431	0.02112	0.02282	0.03488	0.03668	8.066E-6	8.303E-6
1.822	1.822	3.5267411	0.02115	0.02285	0.03491	0.03671	8.073E-6	8.311E-6
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2.413	2.413	3.417046	0.03589	0.05178	0.05291	0.06968	1.198E-5	1.346E-5

Table 1: $M = 320, \varepsilon = 0.05$

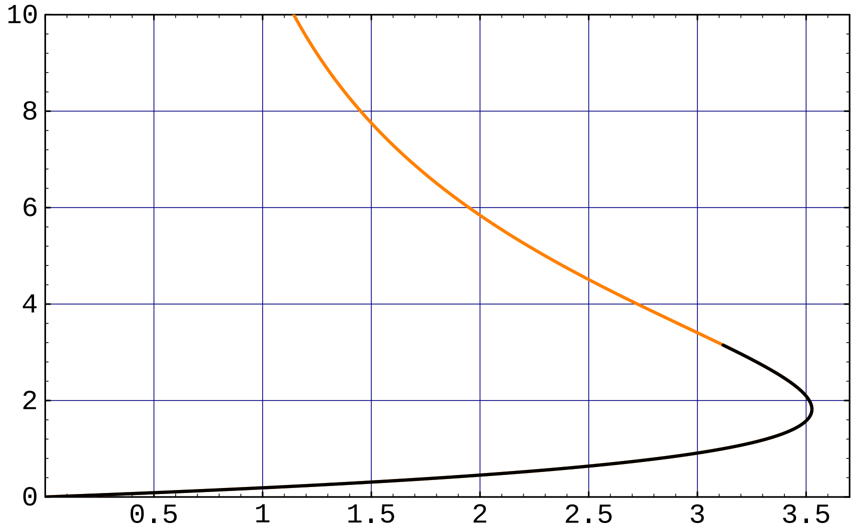

Figure 3: Bifurcation diagram for (2.11).
The value of $\|u\|_\infty$ corresponds to the ordinate axis, and λ to the abscissa. Here the black portion is the numerically verified part, and the gray portion is the approximate part.

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